11. Building up the Standard Gauge Model of High Energy Physics

G. RAJASEKARAN
Institute of Mathematical Sciences, Madras 600 113, India

1. Introduction

The standard model based on the gauge group SU(3) × SU(2) × U(1) describes *all* that is presently-known of high energy physics. Our aim in this chapter is to build it up from the beginning.

We start with the simplest notions of Abelian gauge field theory and the breakdown of symmetry and gradually build up the various strands that make up the SU(2) × U(1) electroweak theory. We then take up the strong interaction sector and show how deep-inelastic scattering, asymptotic freedom and colour lead up to quantum chromodynamics (QCD). The renormalization group equation is shown to provide the foundation for asymptotic freedom and the justification for QCD. Combining the electroweak and QCD sectors, the complete standard model is then constructed. Its strengths and weaknesses are briefly discussed and some views beyond the standard model are presented in the final section.

This chapter is mainly intended for physicists who have not had much exposure to high energy physics although it may also benefit other beginners in high energy physics. The level is very elementary and technical details are omitted. The other chapters in this volume on (pre-gauge-theoretic) particle physics and on elements of quantum field theory may be regarded as prerequisites for the understanding of the present chapter.

2. U(1) Gauge Theory

Consider* the Lagrangian of a complex scalar field ϕ:

$$\mathscr{L} = \partial_\mu \phi^* \partial^\mu \phi - V(\phi^* \phi) \tag{1}$$

Here V, called a potential, is a function of $\phi^* \phi$ and, in particular, for a renormalizable theory, it is a quadratic function of $\phi^* \phi$. We take

$$V(\phi^* \phi) = \mu^2 \phi^* \phi + \lambda(\phi^* \phi)^2. \tag{2}$$

Then the Euler–Lagrange equation becomes $(\Box + \mu^2)\phi = -2\lambda(\phi^* \phi)\phi$.

* We use the metric $g_{00} = 1$; $g_{11} = g_{22} = g_{33} = -1$.

B. R. Iyer et al. (eds.), Gravitation, Gauge Theories and the Early Universe, 185–236.

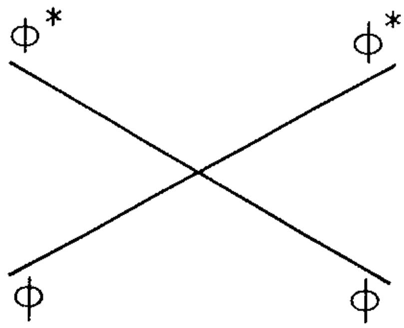


Fig. 1.

For $\lambda = 0$, this is the Klein–Gordon equation. Thus, $\mu^2\phi^*\phi$ in V represents the mass term while $\lambda(\phi^*\phi)^2$ represents the quartic interaction vertex shown in Figure 1. The quanta of the complex scalar field represent charged particles of spin zero.

The Lagrangian in (1) is invariant under the global gauge transformations:

$$\phi(x) \to e^{-i\alpha}\phi(x), \qquad \phi^*(x) \to e^{+i\alpha}\phi^*(x), \tag{3}$$

where α is an arbitrary constant. By applying Noether's theorem, one finds a conserved current:

$$j^\mu = \phi^*\partial_\mu\phi - (\partial_\mu\phi^*)\phi, \tag{4}$$

$$\partial_\mu j^\mu = 0 \quad \text{(or)} \quad \frac{d}{dt}\int d^3x j^0 = 0, \tag{5}$$

where $\int d^3x j^0$ is the total charge.

We now try to enlarge the symmetry. Instead of the constant phase α, we envisage a spacetime dependent phase $\alpha(x)$ in the transformation:

$$\phi(x) \to e^{-i\alpha(x)}\phi(x); \qquad \phi^*(x) \to e^{+i\alpha(x)}\phi^*(x). \tag{6}$$

However, the important point is that the Lagrangian in (1) is *not* invariant under this more general symmetry transformation, since the derivative term in (1) has a more complicated transformation

$$\partial^\mu\phi \to e^{-i\alpha}\partial^\mu\phi - i(\partial^\mu\alpha)e^{-i\alpha}\phi. \tag{7}$$

In order to ensure invariance, we now have to add a vector field A^μ to the system with the transformation

$$A^\mu \to A^\mu + \frac{1}{e}\partial^\mu\alpha, \tag{8}$$

where e is a constant. Using (7) and (8), we find that the combination $(\partial^\mu + ieA^\mu)\phi$ and its complex conjugate have simple transformation properties*:

$$(\partial^\mu + ieA^\mu)\phi \to e^{-i\alpha}(\partial^\mu + ieA^\mu)\phi, \tag{9}$$

$$(\partial^\mu - ieA^\mu)\phi^* \to e^{+i\alpha}(\partial^\mu - ieA^\mu)\phi^* \tag{10}$$

* Hence $(\partial^\mu + ieA^\mu)$ is called the gauge-covariant derivative.

and their product $(\partial_\mu - ieA_\mu)\phi^*(\partial^\mu + ieA^\mu)\phi$ is invariant. This product now replaces the derivative term $\partial_\mu \phi^* \partial^\mu \phi$ in the Lagrangian (1).

Since we have introduced a new field A^μ into the system, we need derivative terms (kinetic energy) for A^μ. If we define

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{11}$$

we see that $F^{\mu\nu}$ is invariant under the transformation in Equation (8). Thus, we have the complete Lagrangian:

$$\mathscr{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + (\partial_\mu - ieA_\mu)\phi^*(\partial^\mu + ieA^\mu)\phi - V(\phi^*\phi). \tag{12}$$

The beauty of this Lagrangian is that it is invariant under the transformation defined by equations (6) and (8). We have thus achieved invariance under spacetime dependent phase transformations and learnt that this necessitates the introduction of a vector field. This vector field A^μ is just the electromagnetic vector potential and $F^{\mu\nu}$ is just the electromagnetic field. So, the Lagrangian describes a charged scalar field interacting with the elecromagnetic field.

It must be noted that a mass term of the form $\tfrac{1}{2}M^2 A_\mu A^\mu$ added to the Lagrangian (12) would violate the invariance under (8). So, the 'gauge field' A_μ describes a massless vector boson and this is quite ok for the photon which is known to be massless.

The transformations of the type in Equation (3) with constant α belong to the group U(1). The more general tranformations defined through equations (6) and (8) are usually called gauge transformations. Sometimes the case of constant α is called global or rigid gauge transformation, while that of spacetime dependent α is called local gauge transformation. At this point, it is appropriate to note the analogy with relativity. Rigid transformations of the coordinates lead to special relativity while (arbitrary) x-dependent transformations lead to general relativity. Following this analogy, we shall call the transformation with constant α as *special* U(1) transformation and the case of x dependent α as *general* U(1) transformation.

3. Spontaneous Breakdown of Symmetry – Goldstone Model

The canonical momenta and Hamiltonian corresponding to the Lagrangian in Equation (1) can be easily worked out

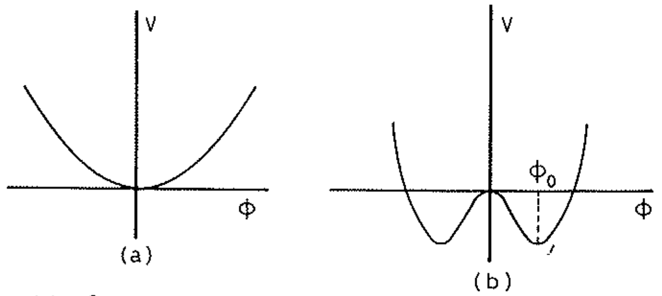
$$\pi = \frac{\partial \mathscr{L}}{\partial \dot\phi} = \dot\phi^*; \qquad \pi^* = \frac{\partial \mathscr{L}}{\partial \dot\phi^*} = \dot\phi \quad \left(\text{where } \dot\phi \equiv \frac{\partial \phi}{\partial t}\right), \tag{13}$$

$$\begin{aligned}\mathscr{H} &= \pi\dot\phi + \pi^*\dot\phi^* - \mathscr{L} \\ &= \pi^*\pi + \nabla\phi^*\cdot\nabla\phi + V(\phi^*\phi).\end{aligned} \tag{14}$$

So, the total energy of the system is

$$H = \int \mathscr{H}\, \mathrm{d}^3 x = \int \mathrm{d}^3 x \left[\pi^*\pi + \nabla\phi\cdot\nabla\phi + V(\phi^*\phi)\right]. \tag{15}$$

Consider the nature of the potential function $V(\phi^* \phi)$ given in Equation (2). The constant λ has to be positive, otherwise V will become negative for sufficiently large ϕ and, hence, the energy in Equation (15) will become unbounded from below. So, there are only two cases to be considered both of which are plotted in Figure 2.



(a) $\mu^2 > 0$
(b) $\mu^2 < 0$

Fig. 2.

In case (a), V is always positive, while in case (b) V is negative for small values of ϕ (where the $\mu^2 \phi^* \phi$ term dominates), however V becomes positive for sufficiently large values of ϕ. Case (a), corresponds to normal particles with positive (mass)^2, and there is nothing more to be said about it. Case (b) is the interesting case. Although this apparently corresponds to tachyons (particles with negative (mass)^2), this is not the correct interpretation as is clear by looking at the Figure 2. Whereas for case (a) the state with $\phi = 0$ is a state of a stable equilibrium and, hence, it is the ground state, in the case of (b), $\phi = 0$ corresponds to a maximum of the potential and, hence, is a state of unstable equilibrium. For this latter case, excitations around $\phi = 0$ have a tachyonic mass, corresponding to the negative curvature $\partial^2 V/\partial\phi\,\partial\phi^*$. The true ground state must be identified with the minimum of the potential, where ϕ has a nonvanishing value ϕ_0 (see Figure 2b). The curvature is positive here and so the tachyons do not exist. In quantum field theory, ϕ_0 must be interpreted as the vacuum expectation value of ϕ written as $\langle\phi\rangle$. This must be independent of x, otherwise Poincaré invariance of the theory will be lost.

We must now recognize that V is actually a function of two fields, the real and imaginary parts ϕ_1 and ϕ_2 of ϕ:

$$\phi = \phi_1 + i\phi_2 \tag{16}$$

So, in Figure 2 the abscissa may be regarded as ϕ_1 and the full shape of the potential V is obtained by rotating the figure around the ordinate. Thus, we obtain Figure 3 for the interesting case of (b). (We have added a constant to V so that $V \geqslant 0$.) We see that the minimum of the potential occurs all along a circle of radius ϕ_0 in the $\phi_1 - \phi_2$ plane. We can choose any one point along this circle as

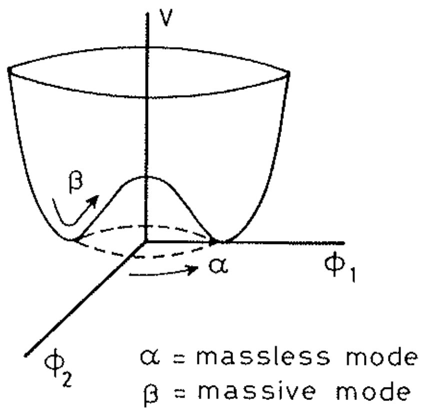


Fig. 3.

the ground state of the system; however once we choose it, the circular symmetry (which is the U(1) or SO(2) symmetry) of the system is broken. This is the mechanism of spontaneous breaking of symmetry.

An important consequence follows. Since it does not cost any energy to move around the circular trough of minimum potential, there exists a massless particle. As can be seen from Figure 3, movement along a direction normal to this circle costs positive potential energy and this corresponds to a normal particle with positive $(\text{mass})^2$. Thus, the choice of a proper ground state eliminates the two tachyonic quanta (corresponding to ϕ_1 and ϕ_2) and, instead, we end up with a massless mode and a normal massive mode. This massless mode is called the Nambu–Goldstone boson and this result is called the Goldstone theorem (proved by Goldstone, Salam and Weinberg) which states that spontaneous breakdown of any continuous symmetry is followed by the massless Nambu–Goldstone boson.

[It is worth noting that if the symmetry which is broken is a discrete symmetry, we do not get any massless Goldstone boson. Consider, for instance, the case of a single-component real field ϕ. If we choose V to be

$$V = \mu^2\phi^2 + \lambda\phi^4, \tag{17}$$

the system has a discrete reflection symmetry $\phi \to -\phi$. For case (b) illustrated by Figure 2b, there are just two possible ground states, corresponding to $\phi = \phi_0$ and $\phi = -\phi_0$. For either choice, the reflection symmetry is spontaneously broken, but there is no Goldstone boson.]

We shall now transcribe the above physical description of spontaneous symmetry breaking to analytical form. Adding a suitable constant to the potential V in Equation (2), it can be rewritten as

$$V = \lambda(\phi^*\phi - \phi_0^2)^2 \tag{18}$$

where $\phi_0^2 = -(\mu^2/2\lambda) > 0$, for case (b). We put

$$\phi = \rho\, e^{i\theta}, \tag{19}$$

where ρ and θ are real fields. In the $\phi_1 - \phi_2$ plane of Figure 3, θ corresponds to angle, while ρ corresponds to length. These correspond, respectively, to the modes along the circle, and perpendicular to it. In terms of these real fields, the Lagrangian in (1) becomes,

$$\mathscr{L} = \partial_\mu \rho \partial^\mu \rho + \rho^2 \partial_\mu \theta \partial^\mu \theta - \lambda(\rho^2 - \phi_0^2)^2. \tag{20}$$

Note that the ground-state value or the vacuum expectation value of ρ is given by

$$\langle \rho \rangle = \phi_0 = \text{constant}. \tag{21}$$

The excitations around this ground state are described by the field η defined by

$$\rho(x) - \phi_0 = \eta(x). \tag{22}$$

In view of (21), η has zero vacuum expectation value. Substituting (22) into Equation (20), we get

$$\mathscr{L} = \partial_\mu \eta \partial^\mu \eta + (\eta + \phi_0)^2 \partial_\mu \theta \partial^\mu \theta - \lambda(4\phi_0^2 \eta^2 + 4\phi_0 \eta^3 + \eta^4). \tag{23}$$

This Lagrangian describes two real scalar fields η and θ. Their masses can be read off from the coefficients of η^2 and θ^2, respectively

$$m_\eta = 2\sqrt{2\lambda}\phi_0, \qquad m_\theta = 0. \tag{24}$$

Thus, η is the massive mode and θ is the Nambu–Goldstone boson.

4. Higgs Model

This is the model for spontaneous breakdown of general U(1). We now take the Lagrangian of Equation (12) and again assume case (b) for the potential V. Using the form of V given in Equation (18), we have

$$\mathscr{L} = (\partial_\mu - ieA_\mu)\phi^*(\partial^\mu + ieA^\mu)\phi - \tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \lambda(\phi^*\phi - \phi_0^2)^2. \tag{25}$$

We again introduce the two real fields ρ and θ defined by

$$\phi = \rho\, e^{i\theta}, \tag{26}$$

but the new trick is to transform A_μ also

$$A_\mu = B_\mu - \frac{1}{e}\partial_\mu \theta. \tag{27}$$

We substitute these into the Lagrangian of Equation (25). Since θ looks like the gauge function α of equations (6) and (8) and since $\mathscr{L}$ is now gauge invariant, the result of the calculation is obvious. We get the same form of $\mathscr{L}$ as in (25) but with ρ and B_μ replacing φ and A_μ

$$\mathscr{L} = (\partial_\mu - ieB_\mu)\rho(\partial^\mu + ieB^\mu)\rho - \tfrac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 - \lambda(\rho^2 - \phi_0^2)^2. \tag{28}$$

The 'gauge function' θ does not appear and, hence, *there is no massless boson*! Again, translating the field ρ by its vacuum expectation value ϕ_v and defining η by

Equation (22), we get

$$\mathcal{L} = \partial_\mu \eta \partial^\mu \eta + e^2(\phi_0 + \eta)^2 B_\mu B^\mu - \tfrac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 - \\ - \lambda(4\phi_0^2 \eta^2 + 4\phi_0 \eta^3 + \eta^4). \tag{29}$$

The masses of the fields η and B_μ can be identified

$$m_\eta = 2\sqrt{2\lambda}\,\phi_0, \tag{30}$$

$$m_B = \sqrt{2}e\phi_0. \tag{31}$$

Thus, not only has the massless Goldstone boson disappeared, but the vector boson also has acquired mass. The original massless vector boson A_μ had only two (transverse) components; what has happened is that the A_μ swallowed the massless Goldstone boson θ and, thus, became the massive vector boson B_μ. The Goldstone boson has supplied the longitudinal component required by the massive vector boson. This is the celebrated Higgs mechanism.

The moral is that if the symmetry which is broken is a general, (i.e. gauge) symmetry then there is no Goldstone boson left in the system.

5. SU(2) Gauge Theory

The U(1) transformations considered so far form an Abelian (i.e. commuting) group of transformations. Our aim is to generalize the U(1) theory to symmetries based on non-Abelian groups such as SU(2) or SU(3). We proceed in parallel steps. We take ϕ to be a complex doublet of scalars

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{32}$$

Under an SU(2) rotation, ϕ transforms as

$$\phi \to e^{i\boldsymbol{\tau}/2\cdot\boldsymbol{\alpha}}\phi, \tag{33}$$

where $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ are the three Pauli matrices and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ are three real constants. Note that $\phi^\dagger\phi$ is invariant under the transformation in (33). A Lagrangian invariant under this '*special* SU(2) transformation' is

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - V(\phi^\dagger \phi), \tag{34}$$

where $\phi^\dagger$ refers to the Hermitian conjugate $\phi^\dagger = (\phi_1^*, \phi_2^*)$.

Next, let us try to generalize the above to the '*general* SU(2) transformation':

$$\phi \to e^{i\boldsymbol{\tau}/2\cdot\boldsymbol{\alpha}(x)}\phi, \tag{35}$$

where the $\alpha(x)$ are now functions of spacetime. In order to achieve general SU(2) invariance for our Lagrangian, quite a bit of nontrivial algebra is necessary. First note

$$\partial_\mu \phi \to e^{i\boldsymbol{\tau}/2\cdot\boldsymbol{\alpha}(x)}\partial_\mu \phi + (\partial_\mu e^{i\boldsymbol{\tau}/2\cdot\boldsymbol{\alpha}(x)})\phi. \tag{36}$$

In order to cancel the second term in (36), we have to introduce vector fields. We introduce a triplet of vector fields $W^a_\mu (a = 1, 2, 3)$ which transform as

$$W^a_\mu \to [e^{iI\cdot\alpha}]_{ab} W^b_\mu - \frac{1}{2g}\varepsilon_{abc}[(\partial_\mu e^{iI\cdot\alpha}) e^{-iI\cdot\alpha}]_{cb} \tag{37}$$

where $I_a (a = 1, 2, 3)$ are the SU(2) generators in 3×3 matrix representation given by their matrix elements

$$(I_a)_{bc} = -i\epsilon_{abc}, \tag{38}$$

where ε_{abc} is $+1$ or -1 if abc is an even or odd permutation of 123, respectively, and it is zero if any two of the indices abc are the same. Combining (36) and (37), one finds

$$\left(\partial_\mu - ig\frac{\tau}{2}\cdot \mathbf{W}_\mu\right)\phi \to e^{i\tau/2\cdot\alpha}\left(\partial_\mu - ig\frac{\tau}{2}\cdot \mathbf{W}_\mu\right)\phi, \tag{39}$$

where we have introduced the vector notation for the SU(2) triplet: $\mathbf{W}_\mu = (W^1_\mu, W^2_\mu, W^3_\mu)$.It is clear that this combination has a simpler transformation property and so can be used for forming the invariant Lagrangian.

We next need the kinetic terms for W^a_μ. Define

$$G^a_{\mu\nu} \equiv \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\varepsilon_{abc} W^b_\mu W^c_\nu. \tag{40}$$

This transforms as

$$G^a_{\mu\nu} \to [e^{iI\cdot\alpha}]_{ab} G^b_{\mu\nu}. \tag{41}$$

Hence, the generally invariant Lagrangian is*

$$\mathscr{L} = -\tfrac{1}{4}(\partial_\mu \mathbf{W}_\nu - \partial_\nu \mathbf{W}_\mu + g\mathbf{W}_\mu \times \mathbf{W}_\nu)^2 + \\ + \phi^\dagger\left(\overleftarrow{\partial}_\mu + ig\frac{\tau}{2}\cdot \mathbf{W}_\mu\right)\left(\partial_\mu - ig\frac{\tau}{2}\cdot \mathbf{W}_\mu\right)\phi - V(\phi^\dagger\phi). \tag{42}$$

Just as in the case of the Abelian gauge theory, the gauge field $\mathbf{W}_\mu$ is massless. The mass term $\frac{1}{2}m^2_W \mathbf{W}_\mu \cdot \mathbf{W}_\mu$, if added to the Lagrangian in Equation (42), would violate the general SU(2) invariance.

The theory of the non-Abelian gauge field $\mathbf{W}_\mu$ was first constructed by Yang and Mills in 1954. Note that, even in the absence of other fields such as ϕ, the Yang–Mills field $\mathbf{W}_\mu$ is self-interacting. The Lagrangian (42) contains terms cubic and quartic in $\mathbf{W}_\mu$, describing the cubic and quartic vertices of Figure 4. In this respect, the Yang–Mills field differs from the electromagnetic field and is more like gravitation. Since the gravitational field couples to everything which carries energy-momentum and since the gravitational field itself carries energy-momentum, it has to be coupled to itself. Similarly, the Yang–Mills field $\mathbf{W}_\mu$

* Henceforth, we will not be very careful in raising or lowering indices; $\mathbf{W}_\mu \cdot \mathbf{W}_\mu$ really stands for $\mathbf{W}_\mu \cdot \mathbf{W}^\mu$.

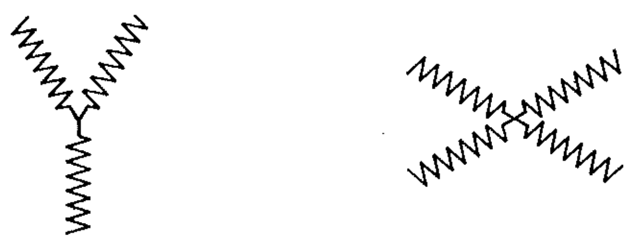


Fig. 4.

couples to everything which carries SU(2) quantum numbers and since $\mathbf{W}_\mu$ is a vector under SU(2), it has to interact with itself.

The SU(2) non-Abelian gauge theory given above can be easily generalized to any compact Lie group such as SU(n), SO(n), Sp(n) or even an exceptional group or direct products of these.

6. Spontaneous Breakdown of SU(2) Symmetry

Special SU(2): We take the potential V to be always of the (b) form. The analogue of Figure 3 must now be plotted in terms of four real fields contained in the complex doublet ϕ. We again separate these into the length type and angular type of fields by using

$$\phi = e^{i(\boldsymbol{\tau}/2)\cdot\boldsymbol{\theta}}\begin{pmatrix}0\\ \rho\end{pmatrix}, \tag{43}$$

where ρ and $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$ are four real fields, taking the place of two complex fields ϕ_1 and ϕ_2. Since V is an SU(2)-invariant function of ϕ, it depends only on ρ and not on θ. It is clear that the region of minimum potential (the analogue of the one-dimensional minimum circle of Figure 3) is now a three-dimensional manifold, corresponding to the three angles $\theta_1, \theta_2, \theta_3$. Thus, there are three massless Goldstone bosons in this case and one massive boson corresponding to ρ, or rather, to the shifted field $\rho - \langle\rho\rangle$.

General SU(2): We take the generally invariant Lagrangian of Equation (42) and make the substitution of ϕ in terms of ρ and $\boldsymbol{\theta}$ through Equation (43). We also transform $\mathbf{W}_\mu$ into $\mathbf{W}'_\mu$ with the gauge function chosen to be $\boldsymbol{\theta}$:

$$W^a_\mu = [e^{i\mathbf{I}\cdot\boldsymbol{\theta}}]_{ab} W'^b_\mu + \frac{1}{2g}\varepsilon_{abc}[(\partial_\mu e^{i\mathbf{I}\cdot\boldsymbol{\theta}}) e^{-i\mathbf{I}\cdot\boldsymbol{\theta}}]_{bc}. \tag{44}$$

As a consequence of general invariance, the Lagrangian has an identical form to that in Equation (42), except that $\mathbf{W}_\mu$ is replaced by $\mathbf{W}'_\mu$ and ϕ is replaced by $\binom{0}{\rho}$

and the 'gauge function' $\boldsymbol{\theta}$ disappears.

$$\mathscr{L} = -\tfrac{1}{4}(\partial_\mu \mathbf{W}'_\nu - \partial_\nu \mathbf{W}'_\mu + g\mathbf{W}'_\mu \times \mathbf{W}'_\nu)^2 + \\ + \left[\left(\partial_\mu - ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{W}'_\mu\right)\begin{pmatrix}0\\ \rho\end{pmatrix}\right]^\dagger \times \\ \times \left[\left(\partial_\mu - ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{W}'_\mu\right)\begin{pmatrix}0\\ \rho\end{pmatrix}\right] - \lambda(\rho^2 - \phi_0^2)^2. \tag{45}$$

Hence, the Goldstone bosons have disappeared and all the three vector bosons have become massive. The mass terms for the vector bosons are easily obtained from the relevant part of Equation (45) by the replacement of ρ by its vacuum expectation value ϕ_0:

$$\tfrac{1}{4}g^2\left[\boldsymbol{\tau}\cdot\mathbf{W}_\mu\begin{pmatrix}0\\ \phi_0\end{pmatrix}\right]^\dagger \left[\boldsymbol{\tau}\cdot\mathbf{W}_\mu\begin{pmatrix}0\\ \phi_0\end{pmatrix}\right] \\ = \tfrac{1}{4}g^2(0\ \ \phi_0)\begin{pmatrix}\mathrm{W}_3 & \mathrm{W}_1 - i\mathrm{W}_2\\ \mathrm{W}_1 + i\mathrm{W}_2 & -\mathrm{W}_3\end{pmatrix} \times \\ \times \begin{pmatrix}\mathrm{W}_3 & \mathrm{W}_1 - i\mathrm{W}_2\\ \mathrm{W}_1 + i\mathrm{W}_2 & -\mathrm{W}_3\end{pmatrix}\begin{pmatrix}0\\ \phi_0\end{pmatrix} \\ = \tfrac{1}{4}g^2\phi_0^2(\mathrm{W}_1^2 + \mathrm{W}_2^2 + \mathrm{W}_3^2). \tag{46}$$

We have ignored the Lorentz vector index μ as well as the prime on the W fields. We thus find that all the three vector fields acquire the same mass given by

$$m_\mathrm{W} = \frac{1}{\sqrt{2}}g\phi_0. \tag{47}$$

7. One More Model

In the SU(2) model considered above, the scalar field was a complex doublet field and this led to a system with all three vector bosons gaining mass after symmetry breakdown. We next consider a SU(2) model with a real triplet scalar field. In this case, all the vector bosons do not become massive.

Special SU(2)

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu\boldsymbol{\phi}\cdot\partial_\mu\boldsymbol{\phi} - V(\boldsymbol{\phi}\cdot\boldsymbol{\phi}), \tag{48}$$

where $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)$ is a triplet (vector) representation of SU(2) and it is taken to be real. We put

$$\boldsymbol{\phi} = \mathrm{e}^{iI_1\theta_1 + iI_2\theta_2}\begin{pmatrix}0\\ 0\\ \rho\end{pmatrix}, \tag{49}$$

where we have used the three-dimensional (column) matrix notation for ϕ, and I_a are the 3×3 matrix representation of the SU(2) matrices, already given in Equation (38). The fields ρ, θ_1 and θ_2 are three real fields replacing ϕ_1, ϕ_2, and ϕ_3. By following the same reasoning as before, when SU(2) symmetry is broken as a consequence of the nonvanishing vacuum expectation value of the scalar field, θ_1 and θ_2 will become massless Goldstone bosons, while ρ will become massive.

General SU(2)

$$\mathscr{L} = -\tfrac{1}{4}(\partial_\mu \mathbf{W}_\nu - \partial_\nu \mathbf{W}_\mu + g\mathbf{W}_\mu \times \mathbf{W}_\nu)^2 + \\ + \tfrac{1}{2}(\partial_\mu \boldsymbol{\phi} - g\mathbf{W}_\mu \times \boldsymbol{\phi})^2 - V(\boldsymbol{\phi}\cdot\boldsymbol{\phi}), \tag{50}$$

where we have used the SU(2) vector notation for both $\mathbf{W}_\mu$ and $\boldsymbol{\phi}$. Again, following the same argument, we see that θ_1 and θ_2 will become the longitudinal components of two of the vector bosons which will, therefore, emerge as massive vector bosons. The third vector boson will remain massless. This can be worked out from the piece $\frac{1}{2}g^2(\mathbf{W}_\mu \times \boldsymbol{\phi})^2$ contained in the above Lagrangian by replacing ϕ_a by its vacuum expectation value $\phi_0 \delta_{a3}$. Thus,

$$\frac{g^2}{2}(\mathbf{W}_\mu \times \boldsymbol{\phi})^2 = \frac{g^2}{2}\varepsilon_{abc}\varepsilon_{dfc}\phi_a\phi_d W_b W_f$$

$$\rightarrow \frac{g^2}{2}\phi_0^2\varepsilon_{3bc}\varepsilon_{3fc}W_b W_f \tag{51}$$

$$= \frac{g^2}{2}\phi_0^2(\mathrm{W}_1\mathrm{W}_1 + \mathrm{W}_2\mathrm{W}_2), \tag{52}$$

where we have dropped the Lorentz index on the vector field W for notational convenience. Thus, W_1 and W_2 have masses equal to $g\phi_0$, while W_3 remains massless.

8. General Case of Non-Abelian Symmetry Breakdown

Let us consider any compact Lie Group and work out the symmetry breaking. Let g be the number of generators of the group, which is also the number of gauge bosons and let ϕ contain n real components.

Writing ϕ in the form

$$\phi = \mathrm{e}^{i\Sigma_a I^a \theta_a}\rho, \tag{53}$$

where I^a are the generators in the representation of ϕ, we take ν nonvanishing components for ρ and $r = n - \nu$ nonvanishing components for θ:

$$\rho = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \rho_\alpha \\ \vdots \\ \rho_\nu \end{pmatrix} \left.\begin{matrix} \\ \\ \\ \\ \\ \\ \end{matrix}\right\} n, \quad \theta = (\underbrace{\underbrace{\theta_a \ldots \theta_r}_{r = n - \nu}, 0 \ldots 0}_{g}). \tag{54}$$

This split-up between the angle-type variable θ and length-type variable ρ is completely determined by the representation to which ϕ belongs. The number of length-type variables ν is, in fact, equal to the number of independent invariants one can construct out of ϕ. This number ν is called the canonical number of the representation. The θ fields are massless while the ρ fields lead to massive excitations. Hence, the number of Goldstone bosons is given by the difference: $r = n - \nu$: This is also the number of gauge bosons which will become massive and so the number of massless gauge bosons is $g - n + \nu$.

In tne examples already considered above, for the doublet ϕ, the only invariant is $\phi^\dagger \phi$ and for the real triplet also, there is only one invariant $\boldsymbol{\phi} \cdot \boldsymbol{\phi}$ and so $\nu = 1$ for both. Hence there is only one ρ field in both these examples of SU(2) breaking and the rest of the field components must be accommodated in the angle-type variables θ each leading to a Goldstone boson.

We may also write down the general mass matrix for the vector bosons, resulting from a spontaneous breakdown of symmetry

$$M^2_{ab} = g^2 I^a_{\alpha\beta} I^b_{\alpha\gamma} \langle \rho_\beta \rangle \langle \rho_\gamma \rangle. \tag{55}$$

This is a generalization of the mass calculation in Equations (51) and (52).

9. SU(2) × U(1) Model

We are now ready to face a more realistic model (needed in high energy physics), which is obtained by combining the U(1) and SU(2) models already discussed above.

We start with the Lagrangian of a scalar field ϕ, which is a doublet under SU(2), and being complex,has a nonvanishing U(1) charge also

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{56}$$

$$\mathscr{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - \lambda(\phi^\dagger \phi - \phi_0^2)^2. \tag{57}$$

This Lagrangian has *special* SU(2) × U(1) invariance. Substitution of the form

$$\phi = e^{i(\tau/2)\cdot\theta}\begin{pmatrix}0\\ \rho\end{pmatrix} \tag{58}$$

reveals the presence of three massless Goldstone bosons θ_1, θ_2 and θ_3 and a massive scalar boson ρ. (Since $\phi^\dagger\phi$ is the only invariant, the canonical number $\nu = 1$.)

To achieve *general* SU(2) × U(1) invariance, we need a triplet of SU(2) gauge bosons $\mathbf{W}_\mu$ and a singlet of U(1) gauge boson B_μ. The generally-invariant Lagrangian is

$$\begin{aligned}\mathscr{L} = &-\tfrac{1}{4}(\partial_\mu \mathbf{W}_\nu - \partial_\nu \mathbf{W}_\mu + g\mathbf{W}_\mu \times \mathbf{W}_\nu)^2 - \tfrac{1}{4}(\partial_\mu \mathrm{B}_\nu - \partial_\nu \mathrm{B}_\mu)^2 + \\ &+\phi^\dagger\left(\overleftarrow{\partial}_\mu + ig\frac{\tau}{2}\cdot\mathbf{W}_\mu - \frac{ig'}{2}B_\mu\right)\left(\partial_\mu - ig\frac{\tau}{2}\cdot\mathbf{W}_\mu - \frac{ig'}{2}B_\mu\right)\phi \\ &-\lambda(\phi^\dagger\phi - \phi_0^2)^2,\end{aligned} \tag{59}$$

which is a combination of the Lagrangians in Equations (25) and (42). We have called the SU(2) gauge coupling constant as g and the U(1) gauge coupling constant as g'. Now, since there are four gauge bosons, whereas the number of Goldstone bosons is only three, one massless gauge boson survives and along with that a general U(1) symmetry (which need not be the same one we started with) remains unbroken.

After making the gauge transformation with the gauge function $\boldsymbol{\theta}$, the field $\boldsymbol{\theta}$ gets eliminated and the final form of the Lagrangian is obtained from Equation (59) by the simple substitution

$$\phi = \begin{pmatrix}0\\ \rho\end{pmatrix}; \qquad \rho = \phi_0 + \eta, \tag{60}$$

where ϕ_0 is a constant and η is the massive scalar field. Thus, we have

$$\begin{aligned}\mathscr{L} = &-\tfrac{1}{4}(\partial_\mu \mathbf{W}_\nu - \partial_\nu \mathbf{W}_\mu + g\mathbf{W}_\mu \times \mathbf{W}_\nu)^2 - \tfrac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 + \\ &+\tfrac{1}{4}(0 \quad \phi_0)\begin{pmatrix}-g\mathrm{W}_3 + g'B & -g(\mathrm{W}_1 - i\mathrm{W}_2)\\ -g(\mathrm{W}_1 + i\mathrm{W}_2) & g\mathrm{W}_3 + g'B\end{pmatrix}^2\begin{pmatrix}0\\ \phi_0\end{pmatrix} + \\ &+\eta\text{-dependent terms.}\end{aligned} \tag{61}$$

The vector boson mass term which can be read off from this equation, is

$$\tfrac{1}{4}g^2\phi_0^2(\mathrm{W}_1 + i\mathrm{W}_2)(\mathrm{W}_1 - i\mathrm{W}_2) + \tfrac{1}{4}\phi_0^2(g\mathrm{W}_3 + g'B)^2. \tag{62}$$

We thus identify the massive fields and their masses as follows

$$\begin{aligned}\mathrm{W}_\mu^\pm &\equiv \frac{\mathrm{W}_\mu^1 \pm i\mathrm{W}_\mu^2}{\sqrt{2}}; \qquad m_\mathrm{W}^2 = \tfrac{1}{2}\phi_0^2 g^2; \\ \mathrm{Z}_\mu &\equiv \frac{g\mathrm{W}_\mu^3 + g'B_\mu}{\sqrt{g^2 + g'^2}}; \qquad m_\mathrm{Z}^2 = \tfrac{1}{2}\phi_0^2(g^2 + g'^2).\end{aligned} \tag{63}$$

The fields defined here are the normalized fields. $W_\mu^\pm$ are complex fields and correspond to massive charged vector bosons, while Z_μ is a real field and corresponds to a massive neutral vector boson. The combination orthogonal to Z_μ remains massless and so we shall identify it with the photon field A_μ (electromagnetic vector potential):

$$A_\mu \equiv \frac{-g' W_\mu^3 + g B_\mu}{\sqrt{g^2 + g'^2}}; \qquad m_A = 0. \tag{64}$$

It is convenient to define the weak mixing angle θ_W by

$$\tan \theta_W = \frac{g'}{g} \tag{65}$$

so that, part of Equations (63) and (64) can be rewritten as

$$\begin{aligned} Z_\mu &= \cos\theta_W W_\mu^3 + \sin\theta_W B_\mu, \\ A_\mu &= -\sin\theta_W W_\mu^3 + \cos\theta_W B_\mu \end{aligned} \tag{66}$$

and

$$m_W^2 = m_Z^2 \cos^2\theta_W. \tag{67}$$

This model is, in fact, the successful electroweak model of Glashow, Salam and Weinberg, which unifies the weak and electromagnetic interactions through SU(2) × U(1) gauge theory. After symmetry breakdown, a U(1) gauge symmetry remains unbroken and it is identified with electromagnetic U(1), with the corresponding massless gauge boson, namely the photon. The three massive vector bosons $W_\mu^\pm$and Z_μ mediate the short-ranged weak interactions such as β-decay.

In terms of $W_\mu^\pm$, Z_μ and A_μ, the Lagrangian becomes

$$\begin{aligned} \mathscr{L} = & -\tfrac{1}{2} W_{\mu\nu}^+ W_{\mu\nu}^- + m_W^2 W_\mu^+ W_\mu^- - \tfrac{1}{4} F_{\mu\nu}^2 - \tfrac{1}{4} Z_{\mu\nu}^2 + \tfrac{1}{2} m_Z^2 Z_\mu Z_\mu - \\ & - [2ig \sin\theta_W \{A_\mu (W_{\mu\nu}^- W_\nu^+ - W_\nu^- W_{\mu\nu}^+)\} - \\ & - g^2 \sin^2\theta_W \{A_\mu A_\nu W_\mu^+ W_\nu^- - A_\mu A_\mu W_\nu^+ W_\nu^-\}] - \\ & - 2ig \sin\theta_W F_{\mu\nu} W_\mu^+ W_\nu^- + 2ig \cos\theta_W \{Z_\mu (W_{\mu\nu}^- W_\nu^+ - W_\nu^- W_{\mu\nu}^+)\} + \\ & + g^2 \cos^2\theta_W \{Z_\mu Z_\nu W_\mu^+ W_\nu^- - Z_\mu Z_\mu W_\nu^+ W_\nu^-\} + \\ & + 2ig \cos\theta_W Z_{\mu\nu} W_\mu^+ W_\nu^- - g^2 \cos\theta_W \sin\theta_W \\ & \quad \{A_\mu Z_\nu W_\mu^+ W_\nu^- + A_\nu Z_\mu W_\mu^+ W_\nu^- - 2 A_\mu Z_\mu W_\nu^+ W_\nu^-\} + \\ & + \frac{g^2}{2} \{W_\mu^+ W_\mu^+ W_\nu^- W_\nu^- - W_\mu^+ W_\nu^+ W_\mu^- W_\nu^-\} + \eta\text{-dependent terms}, \end{aligned} \tag{68}$$

where we have put

$$\begin{aligned} W_{\mu\nu}^\pm &\equiv \partial_\mu W_\nu^\pm - \partial_\nu W_\mu^\pm; \qquad Z_{\mu\nu} \equiv \partial_\mu Z_\nu - \partial_\nu Z_\mu \\ F_{\mu\nu} &\equiv \partial_\mu A_\nu - \partial_\nu A_\mu. \end{aligned} \tag{69}$$

We make the following observations on the structure of Equation (68).

(1) The coupling of the charged vector bosons $W_\mu^\pm$ to the electromagnetic field A_μ is automatically contained in the Lagrangian, provided we identify

$$g \sin \theta_W = e. \tag{70}$$

(2) In particular, all the terms within the square brackets [...] in the above equation arise from the so-called 'minimal' electromagnetic coupling arising from the replacement

$$\partial_\mu W_\nu^\pm \to (\partial_\mu \mp ieA_\mu)W_\nu^\pm. \tag{71}$$

(3) However, there is a nonminimal term also. This is the piece $F_{\mu\nu} W_\mu^+ W_\nu^-$ which, in fact, ascribes an anomalous magnetic moment to the W bosons. The value of the anomalous magnetic moment κ_W is unity, thus giving 2 for the g-factor of the W boson:

$$g_W = 1 + \kappa_W = 2. \tag{72}$$

This feature is a consequence of the symmetry of the cubic Yang–Mills vertex between the three vector bosons and is a characteristic of any theory in which charged vector bosons are incorporated into a Yang–Mills theory.

(4) There exists a perfect $A_\mu \to Z_\mu$ symmetry. As a consequence, the charged particles are coupled to Z_μ exactly in the same manner as to A_μ, the only difference being the replacement of $g \sin \theta_W$ by $-g \cos \theta_W$ (see Figure 5).

(5) Our last comment is on the $W^+ W^+ W^- W^-$ term, which implies a direct coupling among the charged bosons without involving the electromagnetic field. It is, in fact, the presence of this term which makes this theory of massive charged vector bosons a consistent one; without such a term, the theory of massive charged vector bosons was known to be an inconsistent theory [1].

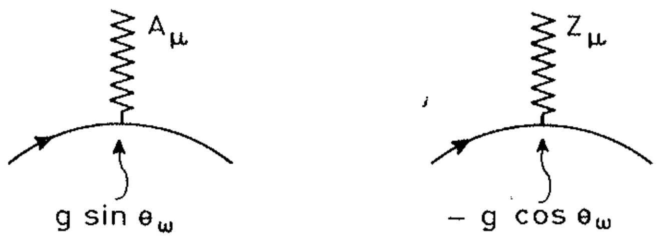


Fig. 5.

10. 'Standard Model' before Gauge Theory

Our aim is to construct the standard model of gauge theory. Before doing that, it is useful to have a brief glance at the standard model of high energy physics that existed (say, in the late 60's) before the advent of gauge theory. This pre-gauge theoretic standard model can be described by the Lagrangian

$$\begin{aligned}\mathscr{L} = & -\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu} + \bar{e}[i\gamma_\mu(\partial^\mu - ieA^\mu) - m_e]e + i\bar{\nu}_e\gamma_\mu\partial^\mu\nu_e + \\ & + \bar{\mu}[i\gamma_\lambda(\partial^\lambda - ieA^\lambda) - m_\mu]\mu + i\bar{\nu}_\mu\gamma_\lambda\partial^\lambda\nu_\mu + \\ & + \bar{u}[i\gamma_\mu(\partial^\mu + \tfrac{2}{3}ieA^\mu) - m_u]u + \bar{d}\left[i\gamma_\mu(\partial^\mu - \frac{i}{3}eA^\mu) - m_d\right]d + \\ & + \bar{s}[i\gamma_\mu(\partial^\mu - \tfrac{1}{3}ieA^\mu) - m_s]s + \frac{G_F}{\sqrt{2}}\tfrac{1}{2}\{J_\mu^+, J_\mu^-\} + \\ & + \text{strong interactions among the quarks,}\end{aligned} \tag{73}$$

where

$$\begin{aligned}J_\lambda^- = & \tfrac{1}{2}\bar{e}\gamma_\lambda(1-\gamma_5)\nu_e + \tfrac{1}{2}\bar{\mu}\gamma_\lambda(1-\gamma_5)\nu_\mu + \\ & + (\bar{d}\cos\theta_c + \bar{s}\sin\theta_c)\gamma_\lambda\frac{(1-\gamma_5)}{2}u,\end{aligned} \tag{74}$$

$$J_\lambda^+ = (J_\lambda^-)^\dagger. \tag{75}$$

This Lagrangian describes the electromagnetic and weak interactions of the quarks u, d, s, with respective electric charges $\frac{2}{3}$, $-\frac{1}{3}$ and $-\frac{1}{3}$ (in units of the electronic charge e) and the leptons e, μ, ν_e, ν_μ, with electric charges $-1, -1, 0, 0$, respectively. The existence of these quarks as the constituents of the hadrons had already been guessed from hadron spectroscopy. However, nobody knew the precise form of the 'strong interaction' among the quarks which is responsible for the binding of the quarks inside the composite hadrons. So, we have left it unspecified in Equation (73).

The weak interaction, however, was rather precisely known to be the current $\times$ current form of Feynman and Gell-Mann given in Equation (73), with the weak current being given by the $V-A$ form of Sudarshan and Marshak given in Equation (74). In this equation, the strength of the weak interaction has been distributed among the ordinary β-decay transition (described by the $\bar{d}u$ piece of the weak current) and the strangeness-changing decay (described by the $\bar{s}u$ piece of the weak current) in the proportion $\cos\theta_c$ and $\sin\theta_c$, respectively. This is called Cabibbo universality and the empirical value of the Cabibbo angle is given by

$$\sin\theta_c \approx 0.22.$$

Violation of CP invariance was experimentally well-established by that time, but not theoretically understood and so the above Lagrangian in Equation (73) does not incorporate CP violation. It is also worth pointing out that standard axioms of quantum field theory require the symmetrized form of the current $\times$ current interaction, given by the anti-commutator of currents in Equation (73) otherwise even CPT theorem will be violated [2].

What is the connection of the weak and electromagnetic interaction given in Equation (73) to the SU(2) $\times$ U(1) model developed in the earlier section? This connection is made through the algebra of the weak and electromagnetic currents which we discuss below.

11. Current Algebra and SU(2) × U(1) Charges of the Fermions

The electromagnetic interaction contained in the covariant derivatives of Equation (73) can be regrouped in the form of $eJ_\lambda^{\text{e.m.}} A_\lambda$, where the current $J_\lambda^{\text{e.m.}}$ is given by

$$J_\lambda^{\text{e.m.}} = -\bar{e}\gamma_\lambda e - \bar{\mu}\gamma_\lambda\mu + \tfrac{2}{3}\bar{u}\gamma_\lambda u - \tfrac{1}{3}\bar{d}\gamma_\lambda d - \tfrac{1}{3}\bar{s}\gamma_\lambda s. \tag{76}$$

We shall now show that the weak and electromagnetic currents of the quarks and leptons given by Equations (74)–(76) satisfy the SU(2) × U(1) algebra. To do this, let us split the currents into the leptonic and hadronic (quark) parts

$$J_\lambda^\pm = j_\lambda^\pm(\text{e}) + j_\lambda^\pm(\mu) + j_\lambda^\pm(\text{q}), \tag{77}$$

$$J_\lambda^{\text{e.m.}} = j_\lambda^{\text{e.m.}}(\text{e}) + j_\lambda^{\text{e.m.}}(\mu) + j_\lambda^{\text{e.m.}}(\text{q}) \tag{78}$$

and let us write these currents in matrix notation with the lepton pairs and quark pairs collected into doublets

$$j_\lambda^{(-)}(\text{e}) = (\bar{\nu}_e\,\bar{e})\gamma_\lambda\frac{(1-\gamma_5)}{2}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}\nu_e\\ e\end{pmatrix}, \tag{79}$$

$$j_\lambda^-(\text{q}) = (\bar{u}\,\bar{d}')\gamma_\lambda\frac{(1-\gamma_5)}{2}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}u\\ d'\end{pmatrix}, \tag{80}$$

$$j_\lambda^+(\text{e}) = (\bar{\nu}_e\,\bar{e})\gamma_\lambda\frac{(1-\gamma_5)}{2}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}\nu_e\\ e\end{pmatrix}, \tag{81}$$

$$j_\lambda^+(\text{q}) = (\bar{u}\,\bar{d}')\gamma_\lambda\frac{(1-\gamma_5)}{2}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}u\\ d'\end{pmatrix}, \tag{82}$$

$$j_\lambda^{\text{e.m.}}(\text{e}) = (\bar{\nu}_e\,\bar{e})\gamma_\lambda\begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\nu_e\\ e\end{pmatrix}, \tag{83}$$

$$j_\lambda^{\text{e.m.}}(\text{q}) = (\bar{u}\,\bar{d}')\gamma_\lambda\begin{pmatrix}\frac{2}{3} & 0\\ 0 & -\frac{1}{3}\end{pmatrix}\begin{pmatrix}u\\ d'\end{pmatrix}, \tag{84}$$

where we have defined the Cabibbo-rotated quark

$$d' \equiv d\cos\theta_c + s\sin\theta_c. \tag{85}$$

The muonic currents are similar to the electronic currents and, hence, are not written separately. We, thus, see that the weak currents involve the raising and lowering matrices of SU(2) algebra

$$\tau^+ = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \quad\text{and}\quad \tau^- = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}. \tag{86}$$

The electromagnetic current involves a diagonal matrix which is *not* the third SU(2) matrix

$$\tau^3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{87}$$

but can be written as a linear combination of τ^3 and the unit matrix. So, by taking the difference between the electric charge matrix Q and the 'weak isospin' matrix $I_3 = \frac{1}{2}\tau^3$, we get a unit matrix (multiplied by a number), which we shall call the weak hypercharge Y:

$$Y = Q - I_3. \tag{88}$$

So, for leptons, we have

$$Y = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{89}$$

while for quarks

$$Y = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{90}$$

This hypercharge matrix Y commutes with all the SU(2) matrices $\tau^\pm$ and τ^3 and, hence, can be taken to be the generator of the independent U(1) symmetry. We thus have the SU(2) × U(1). The hypercharge values for the leptonic and quark doublets can be inferred from Equations (89) and (90) to be $-\frac{1}{2}$ and $\frac{1}{6}$, respectively.

To be more precise, we must split the leptonic and quark fields into their left-handed and right-handed parts by the definition:

$$f_{\substack{L\\R}} = \tfrac{1}{2}(1 \mp \gamma_5) f \tag{91}$$

for all the fermionic fields. The weak currents involve only the left-handed fields and so these fields form the doublets under SU(2) while the right-handed fields must be regarded as singlets under SU(2). The right-handed fields have nonvanishing hypercharge, however, and their values are equal to their electric charges Q (by Equation (88)).

The SU(2) and U(1) quantum numbers of all the fermions are given in Table I. The right-handed neutrino ν_R is a singlet under SU(2) and has $Y = 0$ and so does not participate in the weak as well as the electromagnetic interactions. Hence, it has been dropped from the table. It may not even exist; in any case it has not been detected so far.

Table I.

Fermion	SU(2)	Y
$q_L \equiv \begin{pmatrix} u \\ d' \end{pmatrix}_L$	doublet	$\frac{1}{6}$
$l_L(e) \equiv \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$	doublet	$-\frac{1}{2}$
u_R	singlet	$\frac{2}{3}$
d'_R	singlet	$-\frac{1}{3}$
e_R	singlet	-1

We may now write the zeroth components of the leptonic and quark currents in the form

$$j_0^i(\mathrm{e}) = \tfrac{1}{2} l_{\mathrm{L}}^{\dagger} \tau^i l_{\mathrm{L}}, \tag{92}$$

$$j_0^i(\mathrm{q}) = \tfrac{1}{2} \mathrm{q}_{\mathrm{L}}^{\dagger} \tau^i \mathrm{q}_{\mathrm{L}}, \tag{93}$$

$$j_0^Y(\mathrm{e}) = -\tfrac{1}{2} l_{\mathrm{L}}^{\dagger} l_{\mathrm{L}} - \mathrm{e}_{\mathrm{R}}^{\dagger} \mathrm{e}_{\mathrm{R}}, \tag{94}$$

$$j_0^Y(\mathrm{q}) = \tfrac{1}{6} \mathrm{q}_{\mathrm{L}}^{\dagger} \mathrm{q}_{\mathrm{L}} + \tfrac{2}{3} \mathrm{u}_{\mathrm{R}}^{\dagger} \mathrm{u}_{\mathrm{R}} - \tfrac{1}{3} \mathrm{d}_{\mathrm{R}}'^{\dagger} \mathrm{d}_{\mathrm{R}}', \tag{95}$$

where $i = 1, 2, 3$ and we have defined the Cartesian components

$$j_\lambda^1 = (j_\lambda^+ + j_\lambda^-); \qquad j_\lambda^2 = -i(j_\lambda^+ - j_\lambda^-) \tag{96}$$

and also defined the hypercharge current

$$j_\lambda^Y = j_\lambda^{\mathrm{e.m.}} - j_\lambda^3. \tag{97}$$

In quantum field theory, in general one has the commutation relation

$$[\psi^\dagger(\mathbf{x}) A \psi(\mathbf{x}), \psi^\dagger(\mathbf{y}) B \psi(\mathbf{y})] = \psi^\dagger(\mathbf{x})[A, B]\psi(\mathbf{x})\delta^3(\mathbf{x} - \mathbf{y}) \tag{98}$$

which follows from the canonical equal-time anticommutation relation for Dirac fields

$$\{\psi_\alpha(\mathbf{x}), \psi_\beta^\dagger(\mathbf{y})\} = \delta^3(\mathbf{x} - \mathbf{y})\delta_{\alpha\beta}. \tag{99}$$

In equation (98), $\psi(\mathbf{x})$ is a multicomponent Dirac field, A and B are matrices and matrix multiplication is implied. Use of Equation (98) allows one to trivially verify the SU(2) $\times$ U(1) algebra for the leptonic currents and quark currents separately and also for the total currents:

$$\mathrm{J}_\lambda = j_\lambda(\mathrm{e}) + j_\lambda(\mu) + j_\lambda(\mathrm{q}), \tag{100}$$

$$[J_0^i(\mathbf{x}), J_0^j(\mathbf{y})] = i\varepsilon^{ijk} J_0^k(\mathbf{x})\delta^3(\mathbf{x} - \mathbf{y}),$$

$$[J_0^Y(\mathbf{x}), J_0^i(\mathbf{y})] = 0. \tag{101}$$

By integrating these equations over $\mathbf{x}$ and $\mathbf{y}$, we also get the algebra of charges:

$$[I^i, I^j] = i\varepsilon^{ijk} I^k, \qquad [Y, I^i] = 0, \tag{102}$$

where we have defined the SU(2) $\times$ U(1) fermionic charges:

$$I^i = \int \mathrm{d}^3 x J_0^i(\mathbf{x}) \quad (i = 1, 2, 3),$$

$$Y = \int \mathrm{d}^3 x J_0^Y(\mathbf{x}). \tag{103}$$

12. The Electroweak Gauge Theory

We are now ready to discuss the Glashow–Salam–Weinberg electroweak theory. In fact, the Lagrangian of this theory is simply obtained by adding the leptonic and quark terms to the Lagrangian of the SU(2) × U(1) model given in Equation (59). Thus, we get

$$\begin{aligned}\mathscr{L} = & -\tfrac{1}{4}(\partial_\mu \mathbf{W}_\nu - \partial_\nu \mathbf{W}_\mu + g\mathbf{W}_\mu \times \mathbf{W}_\nu)^2 - \tfrac{1}{4}(\partial_\mu \mathrm{B}_\nu - \partial_\nu \mathrm{B}_\mu)^2 + \\ & + i\bar{\mathrm{q}}_\mathrm{L}\gamma^\mu\left(\partial_\mu + ig\frac{\tau}{2}\cdot\mathbf{W}_\mu + \frac{ig'}{6}\mathrm{B}_\mu\right)\mathrm{q}_\mathrm{L} + \\ & + i\bar{u}_\mathrm{R}\gamma^\mu(\partial_\mu + \tfrac{2}{3}ig'\mathrm{B}_\mu)\mathrm{u}_\mathrm{R} - \bar{\mathrm{d}}_\mathrm{R}\gamma^\mu\left(\partial_\mu - \frac{i}{3}g'B_\mu\right)\mathrm{d}_\mathrm{R} + \\ & + i\bar{l}_\mathrm{L}\gamma^\mu\left(\partial_\mu + ig\frac{\tau}{2}\cdot\mathbf{W}_\mu - \frac{i}{2}g'\mathrm{B}_\mu\right)l_\mathrm{L} + \\ & + i\bar{\mathrm{e}}_\mathrm{R}\gamma^\mu(\partial_\mu - ig'\mathrm{B}_\mu)\mathrm{e}_\mathrm{R} + \\ & + \left|\left(\partial_\mu + ig\frac{\tau}{2}\cdot\mathbf{W}_\mu + i\frac{g'}{2}\mathrm{B}_\mu\right)\phi\right|^2 - \lambda(\phi^\dagger\phi - \phi_0^2)^2 - \\ & - (h_\mathrm{u}\bar{q}_\mathrm{L}\phi^c u_\mathrm{R} + h_\mathrm{d}\bar{q}_\mathrm{L}\phi d_\mathrm{R} + h_\mathrm{e}\bar{l}_\mathrm{L}\phi \mathrm{e}_\mathrm{R} + \text{h.c.}),\end{aligned} \tag{104}$$

where

$$\phi_c = i\tau^2\phi^* = \begin{pmatrix}\phi_2^* \\ -\phi_1^*\end{pmatrix}, \tag{105}$$

$$q_\mathrm{L} = \begin{pmatrix}u_\mathrm{L} \\ d_\mathrm{L}\end{pmatrix}; \qquad l_\mathrm{L} = \begin{pmatrix}\nu_{e\mathrm{L}} \\ e_\mathrm{L}\end{pmatrix}. \tag{106}$$

$h_\mathrm{u}, h_\mathrm{d}$ and h_e are arbitrary coupling constants and h.c. refers to Hermitian conjugate.

There are two groups of additional terms in the above Lagrangian – invariant kinetic energy terms for the quarks and leptons and terms of the type $\bar{f}f\phi$ which couple the Fermi fields f with the Higgs fields ϕ and which are called Yukawa terms. The former contain the couplings of the fermions with the gauge fields **W** and B with couplings specified by their SU(2) and U(1) quantum numbers given in Table I and, hence, are invariant under the general SU(2) × U(1) group. The Yukawa couplings with the Higgs field ϕ also are invariant under the same group. By construction, they are SU(2) scalars and they also conserve the U(1) quantum number. In fact, the Lagrangian in Equation (104) contains all the terms allowed by general SU(2) × U(1) invariance and renormalizability. The term renormalizability will be explained below.

Note an important omission: the fermionic mass terms $m\bar{f}f$ are missing. It is impossible to add any fermionic mass terms without violating SU(2) × U(1) symmetry. The only SU(2) × U(1) invariant terms are of the type $\bar{\mathrm{q}}_\mathrm{L}\mathrm{q}_\mathrm{L}, \bar{\mathrm{u}}_\mathrm{R}\mathrm{u}_\mathrm{R}$ etc. but these are zero:

$$\bar{f}_\mathrm{L} f_\mathrm{L} = \bar{f}\frac{(1+\gamma_5)}{2}\frac{(1-\gamma_5)}{2}f = 0 \tag{107}$$

and similarly for $\bar{f}_R f_R$. There are no $\bar{f}_L f_R$ type of term which conserves SU(2) × U(1) quantum numbers. Hence, all the fermions at this stage are massless. Fermion masses will be generated by the spontaneous breaking of symmetries.

The muonic terms which have been omitted in the above Lagrangian, are exactly similar to the electronic terms. Note that we have used the d quark rather than the Cabibbo-rotated quark $d' = d\cos\theta_c + s\sin\theta_c$. In effect, we have put $\theta_c = 0$ and omitted the strange quarks. This omission of the strange quark terms in the Lagrangian is deliberate. If we had used d′ instead of d, this would have led to the derivative terms in the Lagrangian

$$\bar{d}'\gamma_\mu \partial^\mu d' = \cos^2\theta_c \bar{d}\gamma_\mu \partial^\mu d + \sin^2\theta_c \bar{s}\gamma_\mu \partial^\mu s + \\ + \cos\theta_c \sin\theta_c (\bar{d}\gamma_\mu \partial^\mu s + \bar{s}\gamma_\mu \partial^\mu d), \tag{108}$$

as compared to the correct derivative terms for full-fledged Dirac particles d and s:

$$\bar{d}\gamma^\mu \partial_\mu d + \bar{s}\gamma^\mu \partial_\mu s. \tag{109}$$

This defect is due to the fact that the other orthogonal combination,

$$s' \equiv -d\sin\theta_c + s\cos\theta_c, \tag{110}$$

has so far been ignored. Including this in the Lagrangian would restore the derivative terms in full measure for the two particles d and *s*

$$\bar{d}'\gamma_\mu \partial^\mu d' + \bar{s}'\gamma_\mu \partial^\mu s' = \bar{d}\gamma_\mu \partial^\mu d + \bar{s}\gamma_\mu \partial^\mu s. \tag{111}$$

However, what about the SU(2) × U(1) invariance of $\bar{s}'\gamma_\mu \partial^\mu s'$? One simple way of ensuring this is to assume that s' is a singlet under SU(2), since there is no partner for s' to make up a doublet. Its Y value must be assumed to be equal to Q, for consistency with the relation:

$$Q = I_3 + Y. \tag{112}$$

However, we shall not pursue this rather asymmetrical assignment of quantum numbers, for there is a more serious phenomenological problem with the strange quarks, which we shall discuss soon. At that point, we shall give the correct treatment for *s* quarks. For the present, we shall carry on with $\theta_c = 0$ and ignore *s*.

The Higgs potential $\lambda(\phi^\dagger\phi - \phi_0^2)^2$ in the Lagrangian of Equation (104) implies a nonvanishing vacuum expectation value ϕ_0 for ϕ which leads to breaking of SU(2) × U(1) symmetry and generation of mass for three of the vector bosons, leaving the fourth vector boson massless, exactly as in the earlier section. For the present Lagrangian, nonvanishing ϕ_0 has one more consequence arising from the Yukawa terms $\bar{f}f\phi$. It is clear that the replacement of ϕ with its constant vacuum expectation value ϕ_0 leads to mass terms for the fermions. We have

$$
\begin{aligned}
& h_u \bar{q}_L \phi^c u_R + h_d \bar{q}_L \phi d_R + h_e \bar{l}_L \phi e_R + \text{h.c.} \\
& \to h_u(\bar{u}_L \bar{d}_L)\begin{pmatrix}\phi_0 \\ 0\end{pmatrix} u_R + h_d(\bar{u}_L \bar{d}_L)\begin{pmatrix}0 \\ \phi_0\end{pmatrix} d_R + h_e(\bar{\nu}_L \bar{e}_L)\begin{pmatrix}0 \\ \phi_0\end{pmatrix} e_R + \text{h.c.} \\
& = h_u \phi_0 \bar{u}_L u_R + h_d \phi_0 \bar{d}_L d_R + h_e \phi_0 \bar{e}_L e_R + \text{h.c.} \\
& = h_u \phi_0 (\bar{u}_L u_R + \bar{u}_R u_L) + h_d(\bar{d}_L d_R + \bar{d}_R d_L) + h_e(\bar{e}_L e_R + \bar{e}_R e_L) \\
& = h_u \phi_0 \bar{u} u + h_d \phi_0 \bar{d} d + h_e \phi_0 \bar{e} e,
\end{aligned} \tag{113}
$$

where we have assumed that ϕ_0 and the Yukawa coupling constants h are all real and used the following relations for the chiral Fermi fields f_L and f_R:

$$
\begin{aligned}
& (\bar{f}_L f_R)^\dagger = \bar{f}_R f_L, \\
& \bar{f}_L f_R = \bar{f}\frac{(1+\gamma_5)}{2}\frac{(1+\gamma_5)}{2} f = \bar{f}\frac{(1+\gamma_5)}{2} f, \\
& \bar{f}_R f_L = \bar{f}\frac{(1-\gamma_5)}{2}\frac{(1-\gamma_5)}{2} f = \bar{f}\frac{(1-\gamma_5)}{2} f, \\
& \bar{f}_L f_R + \bar{f}_R f_L = \bar{f} f.
\end{aligned} \tag{114}
$$

We thus identify the masses of the quarks and leptons:

$$
h_u \phi_0 = m_u; \qquad h_d \phi_0 = m_d; \qquad h_e \phi_0 = m_e; \qquad h_\mu \phi_0 = m_\mu, \tag{115}
$$

where the last equation has been added to make it more complete. The moral is that spontaneous breaking of SU(2) × U(I) generates masses not only for the gauge bosons, but also for the fermions.

This completes the construction of the electroweak gauge theory.

13. Consequences of the Electroweak Theory

The interactions of the quarks and leptons with the gauge bosons $\mathbf{W}_\mu$ and B_μ are all contained in the covariant derivatives occurring in the Lagrangian of Equation (104). They can be collected together and written in the alternate form:

$$
\mathscr{L}' = g\mathbf{W}_\mu \cdot \mathbf{J}_\mu + g' B_\mu (J_\mu^{\text{e.m.}} - J_\mu^3), \tag{116}
$$

where $\mathbf{J}_\mu$ and J_μ^Y are, respectively, the currents of the weak isospin group SU(2) and weak hypercharge group U(1):

$$
\mathbf{J}_\mu = (\bar{u}\,\bar{d})\gamma_\mu \frac{(1-\gamma_5)}{2}\tau\begin{pmatrix}u \\ d\end{pmatrix} + (\bar{\nu}_e \bar{e})\gamma_\mu \frac{(1-\gamma_5)}{2}\tau\begin{pmatrix}\nu_e \\ e\end{pmatrix}, \tag{117}
$$

$$
J_\mu^Y = J_\mu^{\text{e.m.}} - J_\mu^3. \tag{118}
$$

On re-expressing the fields $\mathbf{W}_\mu$ and B_μ in terms of the physical fields $W_\mu^\pm$, Z_μ and

A_μ, using Equations (63) and (66), we get

$$\mathscr{L}' = g \sin\theta_W J_\mu^{\text{e.m.}} A_\mu + \frac{g}{2\sqrt{2}}\{J_\mu^- W_\mu^+ + J_\mu^+ W_\mu^-\} +$$

$$+ \frac{g}{\cos\theta_W}\{J_\mu^3 - 2\sin^2\theta_W J_\mu^{\text{e.m.}}\} Z_\mu \qquad (119)$$

The first piece in $\mathscr{L}'$ is the familiar electromagnetic interaction with the identification already made (70)

$$g \sin\theta_w = e. \qquad (120)$$

The second piece containing interaction of the 'charged currents' $J^\pm$ with the charged bosons $W^\mp$, must be compared to the old current × current form of the weak interaction in Equation (73).

We see that the Fermi contact interaction of the old form of the weak interaction describing processes like β-decay are replaced by the W-exchange form (See Figure 6). The Fermi coupling constant G_F get related to g^2 multiplied by $1/m_w^2$ which is the propagator of W boson for small momentum-transfers. The relation is

$$\left(\frac{g}{2\sqrt{2}}\right)^2 \frac{1}{m_w^2} = \frac{G_F}{\sqrt{2}}. \qquad (121)$$

Hence, combining Equations (70), (121) and (67) and using the known values of G_F and e, we get

$$m_w = \frac{37{\cdot}4\,\text{GeV}}{\sin\theta_w}, \qquad (122)$$

$$m_z = \frac{m_w}{\cos\theta_w}. \qquad (123)$$

The third piece in Equation (119) describes the '*neutral current*' $\{J_\mu^3 - 2\sin^2\theta_w J_\mu^{\text{e.m.}}\}$ interacting with the neutral vector boson Z_μ. This is a new weak interaction predicted by the electroweak gauge theory which was not present in the old 'standard model' Lagrangian of Equation (73). This leads to the current × current form:★

$$\mathscr{L}^{\text{Neutral}}_{\text{effective}} = \frac{G_F}{\sqrt{2}} J_\mu^N J_\mu^N; \qquad J_\mu^N = J_\mu^3 - 2\sin^2\theta_w J_\mu^{\text{e.m.}}. \qquad (124)$$

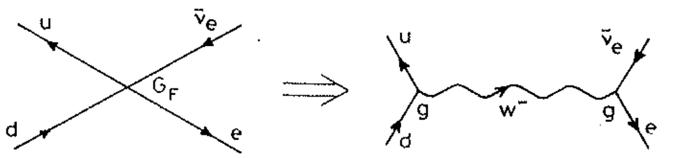


Fig. 6.

★ We have used $1/m_z^2$ for the Z propagator at low momentum-transfers and replaced $m_z^2 \cos^2\theta_w$ by m_w^2 (by Equation (123)).

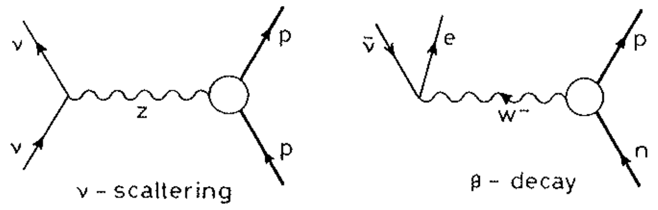


Fig. 7.

The existence of neutral-current weak interaction, which will lead to processes such as elastic ν-scattering (Figure 7) with a strength comparable to that of the usual charged current weak interaction responsible for β-decay (Figure 7), can be regarded as a natural consequence of unifying weak interaction with electrodynamics. Neutral current acts something like a bridge between conventional weak and electromagnetic phenomena. Hence, the discovery of the neutral-current weak interaction in the neutrino reactions in 1973 and the subsequent detailed studies which showed the properties of the neutral-current interaction to be exactly those predicted by the SU(2) × U(1) model, helped to confirm the model. Note that the neutral-current is *not* of the *V–A* form, the relative strengths of *V* and *A* being determined by the mixing angle θ_w. Detailed analyses have shown that all the neutral-current interactions among the leptons and quarks, so far studied, are in agreement with the predictions of the form in Equation (124), with

$$\sin^2 \theta_w \approx 0.21. \tag{125}$$

In this volume devoted to the interface between astrophysics and high energy physics, it is particularly relevant to point out the astrophysical significance of the neutral-current interaction of the neutrinos. This interaction (Figure 7) leads to coherent scattering of neutrinos on nuclei and, hence, to neutrino pressure. (Without neutral currents, such coherent scattering of neutrinos is not possible.) Possible importance of this neutrino pressure on supernova explosion has been considered in recent literature.

Let us now go back to the expressions in Equations (122, 123) for m_w and m_z. Determination of the weak mixing angle θ_w in neutral current processes, allows us to determine the masses of *W* and *Z* bosons. Using Equation (125), we get

$$m_w \approx 82 \text{ GeV}, \tag{126}$$

$$m_z \approx 94 \text{ GeV}. \tag{127}$$

We thus see that the weak bosons are very massive, almost 100 times the mass of the nucleon. This is the reason for the apparent weakness of the weak interaction at low energies. (See Equation (121) for G_F.) At energies much larger than 100 GeV, the strength of the weak interaction is measured by g^2 and so becomes comparable to that of the electromagnetic interaction.

A proton-antiproton collider with centre-of-mass energy of 540 GeV was specially constructed for the discovery of the weak bosons W and Z and the search culminated in their actual discovery in 1983 with masses predicted in

Equations (126, 127) thus providing a spectacular confirmation of the electroweak $SU(2) \times U(1)$ gauge theory.

Let us now come back to the problem of the strange quark encountered in the last chapter. Introduction of the Cabibbo-rotated quark $d' = d \cos\theta_c + s \sin\theta_c$ into the charged currents $J_\mu^\pm$ will describe the decays of strange hadrons correctly, but the problem is in the neutral current. The contribution of d' to the neutral current J_μ^N is

$$\bar{d}' \bigcirc d' = (\bar{d}\cos\theta_c + \bar{s}\sin\theta_c) \bigcirc (d\cos\theta_c + s\sin\theta_c), \tag{128}$$

where $\bigcirc$ is some linear combination of γ_μ and $\gamma_\mu\gamma_5$. The cross-term $\bar{d}\bigcirc s$ and $\bar{s}\bigcirc d$ lead to *strangeness-changing* neutral-current weak decays such as

$$K^\pm \to \pi^\pm e^+ e^-,$$

$$\left.\begin{matrix} K^0 \\ \bar{K}^0 \end{matrix}\right\} \to \mu^+ \mu^-,$$

with the same strength as the usual charged-current weak decays. Experimentally, such strangeness-changing decays are not seen and, hence, the problem.

The solution of this phenomenological problem was provided by Glashow, Iliopoulos and Maiani (GIM). They suggested that the unused orthogonal combination $s' = -d\sin\theta_c + s\cos\theta_c$ be combined with an yet-to-be-discovered charmed quark c to form a new SU(2) doublet

$$\begin{pmatrix} c \\ s' \end{pmatrix}$$

in addition to the old SU(2) doublet of quarks:

$$\begin{pmatrix} u \\ d' \end{pmatrix}.$$

So, the neutral-current contribution from both d' and s' is

$$\bar{d}' \bigcirc d' + \bar{s}' \bigcirc s' = \bar{d} \bigcirc d + \bar{s} \bigcirc s. \tag{129}$$

This equality can be regarded as a manifestation of the invariance of the norm of the two-dimensional vector with components d and s under a two-dimensional (Cabibbo) rotation:

$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}. \tag{130}$$

The important point is that in Equation (129), the strangeness-changing pieces $\bar{d}s$ and $\bar{s}d$ have disappeared. This is the famous GIM mechanism.

But, then, where is the hypothesized charmed quark? Remarkably enough, hadrons with certain peculiar properties which could be interpreted if they were identified as bound states of charmed quark and charmed antiquark ($c\bar{c}$) (just as π, K, ϕ, etc. are bound states of the form $u\bar{u}, u\bar{s}, s\bar{s}$, etc.) were discovered in a series of exciting experiments in October 1974. Subsequent analysis established the correctness of this identification and this, in turn, established the correctness of the GIM conjecture. These $c\bar{c}$ bound states are called ψ. More will be said on ψ particles in Section (24).

Apart from the four quark 'flavours' u, d, s and c, a fifth flavour b (called 'bottom' or 'beauty') was discovered in 1977–78 by a repetition of history, namely through the observation of the bound state $b\bar{b}$. To complete the SU(2) doublet structure, one more quark flavour t (called 'top' or 'truth') must exist. If it exists then, the three quark doublets (referred to as three generations):

$$\begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} t \\ b \end{pmatrix}$$

would be in parallel with the three generations of lepton doublets which are already known to exist:

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}.$$

The τ lepton (with mass 1.78 GeV) was discovered in 1975; the existence of its associated neutrino ν_τ has been inferred indirectly from the decay properties of τ.

14. Renormalizability

The Lagrangian in Equation (104) describing electroweak theory is exactly invariant under the general SU(2) × U(1) symmetry. Of course, the physical solutions of the theory describe massive W and Z and massless photon and, hence, the general SU(2) × U(1) is broken. But the distinguishing feature of the mechanism of spontaneous breaking of symmetry through the nonvanishing vacuum expectation value of ϕ, is that although the solutions break the symmetry, the Lagrangian as well as the equations of motion remain invariant.

This symmetry is not merely a matter of aesthetics. It turns out that it is this invariance under general transformations which is directly responsible for the renormalizability of this theory.

What is renormalizability? Relativistic local quantum field theory is, in general, afflicted with ultraviolet divergences, i.e. the higher-order loop diagrams give divergent conributions from the ultraviolet end ($k \to \infty$) of the virtual momenta. However, fortunately there is a class of quantum field theories in which finite meaningful results can be obtained for physical quantities in spite of the presence of these divergences in the intermediate steps of the calculation. This is done by absorbing these divergences into a few parameters of the theory such as masses and coupling constants occurring in the theory and, thus, renormalizing these parameters. In the class of renormalizable theories, after this renormalization of the parameters, no more divergences remain; but for nonrenormalizable theories infinitely more types of divergences remain.

Examples of renormalizable theories are:

QED: $\bar{\psi}\gamma_\mu \psi A^\mu$,

Yukawa Coupling: $\bar{\psi}\psi\phi$,

Self coupling of ϕ: $\lambda\phi^4$

and examples of nonrenormalizable theories are:

Fermi theory: $\bar{\psi}\psi\bar{\psi}\psi$,

Derivative coupling: $\bar{\psi}\gamma_\mu \psi \dfrac{\partial\phi}{\partial x_\mu}$,

Massive vector boson theory: $\bar{\psi}\gamma_\mu \psi V^\mu$.

In perturbation theory, the elementary criterion of renormalizability is simply that the degree★ of divergence D of any Feynman diagram be independent of the number of vertices or of the number of internal lines. For instance, in the case of QED as well as Yukawa coupling, the degree of divergence is

$$D = 4 - \tfrac{3}{2}F_e - B_e, \tag{131}$$

where F_e and B_e are, respectively, the number of external fermion and boson lines. This is independent of the number of vertices or the number of internal lines in the diagram. After renormalization of a few simple processes with small values for F_e and B_e, for which D is positive, we see that D becomes negative for the rest of the theory, thus leading to a renormalizable theory.

On the other hand, for Fermi theory,

$$D = 4 - \tfrac{3}{4}F_e + 2V, \tag{132}$$

where V is the number of vertices in the diagram. Here D increases with the number of interaction vertices which is the same as the order of perturbation theory. A finite number of renormalizations is not enough and so this is a nonrenormalizable theory.

Our interest is in the massive vector boson theory. The coupling for this theory $\bar{\psi}\gamma_\mu \psi V^\mu$ is the same as in QED. Why is this nonrenormalizable then? The reason lies in the difference between the propagators:

$$\text{massive boson:} \quad (g_{\mu\nu} - k_\mu k_\nu/m_V^2)/(k^2 - m_V^2), \tag{133}$$

$$\text{photon:} \quad g_{\mu\nu}/k^2. \tag{134}$$

For $k \to \infty$, because of the extra term involving $k_\mu k^\nu$, the massive-boson propagator has two additional powers of momenta as compared to the photon case. Hence, for the massive boson case, we have to add two times the number of internal boson lines B_i to the degree of divergence D in QED given by Equation

★ Defined as the overall power of momenta in the numerator minus that in the denominator in the Feynman integral.

(131) and we get

$$D = 4 - \tfrac{3}{2}F_e - B_e + 2B_i. \tag{135}$$

Into any given Feynman diagram with specified numbers of external lines F_e and B_e, we can easily introduce any number of additional internal boson lines B_i which will correspond to higher-order processes. Thus, the degree of divergence again increases arbitrarily and we end up with a nonrenormalizable theory.

General invariance comes to our rescue here. In a generally-invariant theory, one can change the gauge. (We had an example of this while doing Higgs mechanism.) There exists a gauge in which the $k_\mu k_\nu$ term of the massive vector propagator can be dropped so that the propagator becomes $g_{\mu\nu}/(k^2 - m_V^2)$ whose high-momentum behaviour is the same as that of the photon propagator $g_{\mu\nu}/k^2$. Thus, the theory becomes renormalizable.

If we had added explicit mass terms $\frac{1}{2}m_W^2 \mathbf{W}_\mu \cdot \mathbf{W}_\mu$ to the Lagrangian in Equation (104), this would break the general SU(2) invariance of the Lagrangian and we would not be able to remove the $k_\mu k_\nu$ term in the propagator by a gauge tranformation. It is only because we left the Lagrangian generally-invariant and brought masses for the vector bosons through spontaneous symmetry breaking, that we are able to remove the $k_\mu k_\nu$ term and achieve renormalizability★. Hence, the importance of spontaneous symmetry breaking in the construction of the electroweak theory.

The proof of renormalizability of non-Abelian gauge theory with spontaneous symmetry breaking is not as simple as we have indicated; ours is only a heuristic argument. The proof was first given in 1971 by 't Hooft. In fact, it was 't Hooft's work which revived interest in the generally-invariant SU(2) × U(1) electroweak model, which had been ignored by most physicists although it had been constructed four years earlier. The subsequent experimental discovery of the neutral-current gave a further boost to the theory, as we have already discussed.

As mentioned above, Fermi's theory which was the basis of weak interaction physics, belongs to the class of nonrenormalizable theories and the construction of a renormalizable weak interaction theory had remained as one of the fundamental problems in high energy physics. General invariance followed by its spontaneous breaking has solved this problem.

However, there is an obstacle. The axial vector coupling of fermions which is a chief feature of weak interactions creates a quantum-field-theoretical anomaly in the higher orders of perturbation theory (See Figure 8) and destroys the renormalizability of the theory. This subject of axial vector anomaly, as well as other anomalies, has become an important topic of research in modern quantum field theory and we cannot do justice to that topic here. For our purposes, it is sufficient to note that although the anomaly exists for leptons and quarks

★ It turns out that for a massive neutral vector boson coupled to a *conserved* current, the $k_\mu k_\nu$ term can be dropped, even if the mass term $\frac{1}{2}m_V^2 V_\mu V_\mu$ is introduced explicitly (i.e. *not* by spontaneous symmetry breaking). However, for weak interactions involving charged massive vector bosons, explicit mass term would lead to a nonrenormalizable theory.

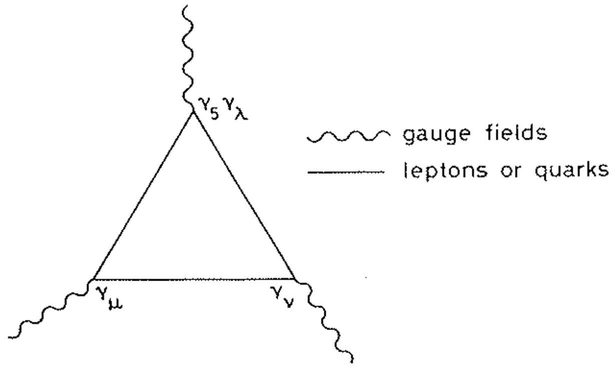


Fig. 8.

separately, it turns out that the SU(2) and U(1) quantum numbers of the leptons and quarks are so arranged that the coefficient of the anomalous term is equal and opposite for the leptons and quarks and, hence, the total contribution to the anomaly is zero. Hence, renormalizability of the theory is saved. Note that for this to be valid the exact correspondence between leptons and quarks is essential; the number of generations of the leptons and quarks has to be equal and the top quark must exist!

15. Spontaneous Symmetry Breaking and Phase Transitions

There exists a similarity between the spontaneous breakdown of symmetry and the phenomenon of phase transition. In particular, Kirzhnitz and Linde [3] in 1972 pointed out the close analogy between the Goldstone–Higgs Lagrangian of sections 3 and 4 with V chosen to be type (b) and the free energy expression in the Landau–Ginzburg phenomenological theory of phase transitions. As a consequence of this analogy, there exists a critical temperature T_c, above which the symmetry between weak and electromagnetic interactions is restored. So, a collection of leptons and quarks with conventional weak and electromagnetic interactions will behave entirely differently if their temperature is raised above T_c. The sriking physical differences are as given in Table II.

Table II.

$T < T_c$	$T > T_c$
$m_W = 82$ GeV $m_Z = 94$ GeV $m_\gamma = 0$	$\left.\begin{matrix} m_W \\ m_Z \\ m_\gamma \end{matrix}\right\} = 0$
Weak interactions weak and short ranged; Electromagnetism stronger and long-ranged	Both weak and electromagnetic interactions have same strength and are long-ranged.

However, the critical temperature T_c is of the order

$$T_c \sim \langle\phi\rangle \sim \frac{m_W}{g} \sim 500\,\text{GeV} \sim 10^{16}\,{}^0\text{K}. \tag{136}$$

This is certainly too hot for terrestrial physics, but not for the physics of the early universe. In fact, phase transitions in the early universe is now a hot topic of research where high energy physics and astrophysics come together.

Let us return to the analogy between spontaneous symmetry breaking and phase transitions and consider, in particular, phase transition of a normal metal to super-conducting state. Here, the correspondence is very close and, in fact, the Higgs Lagrangian of Equation (25) can be regarded as the relativistic generalization of the Landau–Ginzburg model for the superconductor. The superconducting state with a nonvanishing order parameter is the analogue of the broken symmetry state with nonvanishing vacuum expectation value for the Higgs field. It is known that magnetic fields cannot penetrate inside a superconductor for large distances beyond the London penetration length. This is known as the Meissner effect. An equivalent statement is that, the photon has become massive inside a superconductor (the mass being given by the inverse of the penetration length), which agrees with our result in Section 4 that the U(1) gauge boson of the Higgs model becomes massive as a result of symmetry breaking.

This analogy with superconductivity may throw further light on the mechanism of spontaneous symmetry breaking which is a crucial ingredient in our construction of the electroweak theory. In this construction, the spontaneous breaking of symmetry was facilitated by the introduction of the 'elementary' Higgs scalar field ϕ. The analogue of ϕ in superconductivity is the 'Cooper pair' formed by the composite of two electrons. Can the elementary ϕ field of the electroweak theory also be replaced by some composite $\bar{f}f$ (where f is a Fermion field)? We do not know at present.

We may also raise here a related question. Note that the electroweak theory described by the Lagrangians of Equations (104) or (68) contains the 'physical Higgs boson' η which is the remnant of ϕ. Does this η particle exist in nature? Again, we do not know the answer at present. Results of searches for η in the ongoing experiments as well as experiments projected for the future, may lead us to a better understanding of the electroweak symmetry breaking.

So much for electroweak theory. We now turn to QCD.

16. Deep Inelastic Scattering, Asymptotic Freedom and Colour SU(3)

Remember the gap in Equation (73) of Section 10, namely, the unspecified 'strong interactions among the quarks'. We now specify that these interactions are to be described by a non-Abelian gauge theory based on SU(3), the so-called colour group. The theory is known as quantum chromodynamics (QCD). According to this theory, *each* of the quarks ($u, d, \ldots$ etc.) is a triplet under colour SU(3); since

SU(3) has eight generators, there are eight colour gauge vector bosons and they are called gluons. The QCD Lagrangian is

$$\mathscr{L} = -\tfrac{1}{4}(\partial_\mu G^i_\nu - \partial_\nu G^i_\mu - g f^{ijk} G^j_\mu G^k_\nu)^2 + \\ + \bar{q}\left[i\gamma^\mu\left(\partial_\mu - ig\frac{\lambda^i}{2}G^i_\mu \right) - m \right] q, \tag{137}$$

where q is a quark field (u, d, ... etc.), G^i_μ is the gluon field, g is the gauge coupling constant, i goes over 1 to 8, f^{ijk} are the structure constants of SU(3) group and $\lambda^i/2$ are the SU(3) generators in the triplet representation of the quarks. The colour index of the quark (going over 1, 2, 3) is suppressed. The QCD Lagrangian contains the interaction vertices shown in Figure 9.

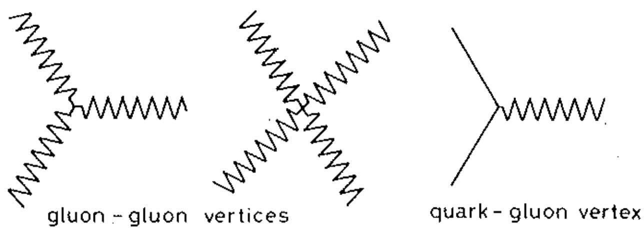


Fig. 9.

Colour denotes a new degree of freedom which actually had its origin in old quark physics – namely the conflict of the apparent total symmetry of the three-quark wave function in the baryonic ground state with Fermi–Dirac statistics. As a simple example, consider the baryon $\Delta^{++}(1238)$ which is a doubly-charged spin-3/2 baryon occurring as a resonance in the $p\pi^+$ system at a mass of 1238 MeV. It is made up of three u quarks each of electric charge 2/3, so that the total charge is 2. The wavefunction of the three u quarks in the ground state contains a spatial part which is symmetric, corresponding to zero relative orbital angular momenta and a spin-part which is also symmetric corresponding to total spin 3/2. There is good phenomenological support for this assumption. But then the total wavefunction of the three quarks is symmetric under interchange of their space and spin labels, thus violating the antisymmetry requirement of fermionic wave functions. Antisymmetry is restored by the invention of a new quantum number, called *colour*, which is three-valued and assigning an antisymmetric colour wavefunction for the three bound quarks. Now the total wave function made up of spatial, spin and colour parts is antisymmetric.

Why QCD?: For a long time, physicists had given up field theory as a useful approach for understanding strong interactions and taken to the *S*-matrix approach. So, what caused the resurgence of field theory in strong interaction physics and what is the reason for going for this non-Abelian gauge field theory (QCD)?

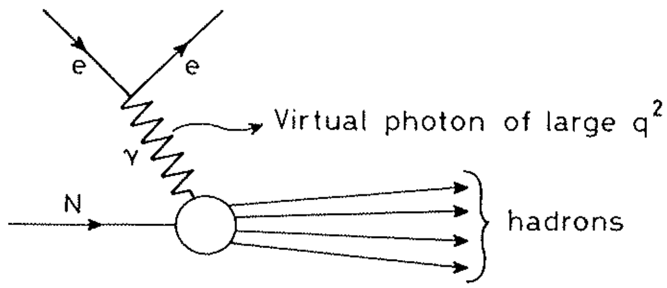


Fig. 10.

The reason comes from an experiment–the so-called deep inelastic scattering of leptons on the nucleon: (See Figure 10)

It was found that, as observed by a high q^2 probe, the nucleon behaves as if it were composed of free, point-like constituents (called partons by Feynman). The lepton scatters off each parton, elastically and incoherently. The incoherent sum of all parton cross-sections gives a very good description of the experimental results. Thus, the complete cross-section for the electron scattering off the nucleon can be written (schematically) as

$$\sigma_N \sim \sum_i \int_0^1 \mathrm{d}x\, f_i(x)\sigma_i(x), \tag{138}$$

where $\sigma_i(x)$ is the electron-scattering cross-section of the ith parton with fractional longitudinal momentum x and $f_i(x)$ is the probability for finding the ith parton with fractional longitudinal momentum x inside the nucleon. Integrating over all the fractions and summing over all the partons i incoherently, we get the electron-nucleon cross-section. It was a remarkable discovery that such a complicated process could be described by such a simple formula. This simple behaviour of deep inelastic scattering is also known as Bjorken scaling and, naively, it is related to the absence of a length-scale or momentum-scale at high energies in local quantum field theory. Similar results were found also for the neutrino-nucleon scattering processes:

$$\begin{aligned} \nu_\mu + N &\to \mu + \text{hadrons (charged-current weak interaction)} \\ &\to \nu_\mu + \text{hadrons (neutral-current weak interaction).} \end{aligned}$$

This phenomenon has a rather close resemblance to Rutherford's famous α-particle scattering experiments which led to the discovery of the nucleus inside the atom. Thomson's spread-out atomic model would lead to soft scattering (i.e. small scattering angles) only. Experimentally, Rutherford and collaborators found hard scattering (i.e. large scattering angles), thus showing the presence of the point-nucleus inside the atom. In the same way in the deep inelastic lepton-nucleon scattering, even for large q^2 (i.e. large scattering angle), scattering was observed to take place, in contrast to what would be expected for a spread out nucleon. This leads to the discovery of point-like constituents deep inside the nucleon.

More detailed study of the experimental data revealed that these partons are in fact quarks; they seemed to have the same spins and charges as expected for quarks.

Attention should now be drawn to the adjective 'free'. In addition to being point-like, the quark-partons behave as if they are free. If they are interacting, the cross-section formula would not be so simple.

Now, the quarks are bound by tremendous attractive forces to make up the nucleon. So, the interaction between quarks should really be superstrong. And yet, when observed through high q^2 probes, this superstrong interaction weakens to such an extent that the quarks behave as free particles.

For quite sometime this was a mystery. On the other hand, this provided an important clue about the nature of the strong interaction itself. We can now say that any theory of strong interactions should satisfy this property, namely, it should tend to a free particle theory or a free field theory at high q^2. Is there any such theory?

Consider nonrelativistic potential scattering, i.e. nonrelativistic particles interacting through well-defined smooth potentials. Since the total energy can be written as $E = T + V$, as the kinetic energy T increases, the potential energy V becomes less and less important in comparison, so that for high energies the theory does tend to a theory of free particles, for properly defined smooth potentials.

But, of course, this is not useful for high energy physics which has to be described by relativistic quantum mechanics. Here, particle-production dominates at high energies and potential description fails.

So, we should ask the same question in the realm of relativistic quantum field theories. Here it is renormalization group which provides the required technique. By using renormalization group, one can define a momentum-dependent coupling constant $g(q^2)$, also called effective coupling constant. So, what we need is a theory in which

$$g(q^2) \to 0 \quad \text{for } q^2 \to \infty. \tag{139}$$

Such a theory is called *asymptotically free*, i.e. the theory tends to a free field theory for asymptotic momenta.

To cut the long story short, it was soon discovered that none of the conventional field theories such as ϕ^4, Yukawa interaction $\bar{\psi}\psi\phi$ or QED $\bar{\psi}\gamma_\mu\psi A^\mu$ is asymptotically free. Of all the renormalizable quantum field theories, only non-Abelian gauge theory was found to possess the unique distinction of being asymptotically free. The characteristic triple gluon vertex shown in Figure 11 is the essential ingredient that makes this theory asymptotically free.

So, asymptotically free non-Abelian gauge theory emerged as a good choice for a theory of strong interactions. Since the colour degree of freedom with three

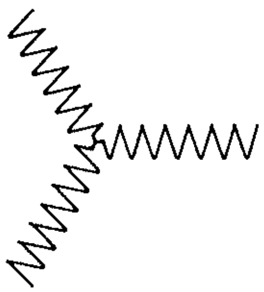


Fig. 11.

colours was already available, as explained at the beginning of this section, the gauge group was taken as the colour SU(3) and QCD was born.

In the next few sections some details on the theory of asymptotic freedom will be given.

17. The Renormalization Group Equation [4]

Consider a renormalizable field theory such as ϕ^4 theory described by the Lagrangian

$$\mathscr{L} = \frac{1}{2}\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x_\mu} - \tfrac{1}{2}\mu^2\phi^2 - g\phi^4, \tag{140}$$

where ϕ is a real scalar field. This theory is characterized by a single dimensionless coupling constant g and a single mass μ.

Let $\Gamma(p_1 \ldots p_n)$ be a *renormalized* n-point Green's function of this theory for n external particles of momenta $p_1 \ldots p_n$. Pictorially, $\Gamma(p_1 \ldots p_n)$ is represented by the sum of all the Feynman diagrams of the type indicated in Figure 12.

We take Green's function to be single-particle irreducible and external-line truncated, i.e. diagrams of the type in Figure 13 in which a single-particle line connects two parts of the diagram are not included in $\Gamma(p_1 \ldots p_n)$ and, further, the external lines are not provided with propagators.

It is possible to show that for asymptotic momenta i.e. for $p_1 \ldots p_n \to \infty$,

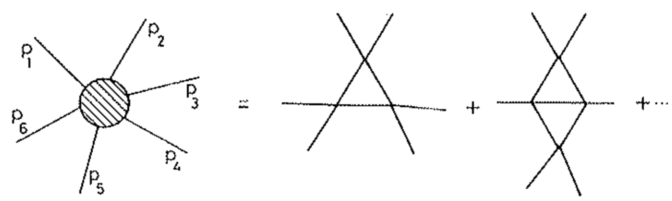


Fig. 12.

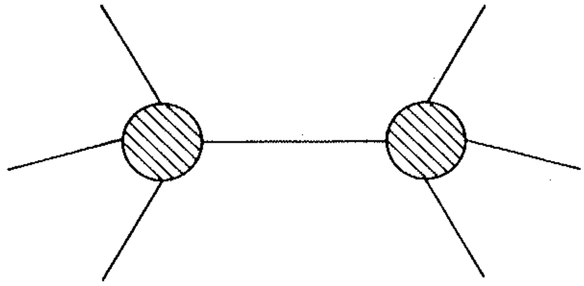


Fig. 13.

Γ satisfies the renormalization group equation

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - n\gamma(g)\right]\Gamma = 0. \tag{141}$$

(This is also the asymptotic version of the so-called Callan–Symanzik equation.) A quick derivation of this equation goes as follows: In the asymptotic region (p_1, $p_2 \ldots p_n \gg \mu$) one might be tempted to think that all memory of the actual mass μ would be lost and all Green's functions would be independent of μ. This is wrong. Γ is a Green's function for renormalized fields expressed as a function of the renormalized coupling constant. But the normalization of the field and the value of the renormalized coupling constant are defined on the mass shell. So, the Green's functions remember the mass shell, no matter how far we go into the asymptotic region. Therefore, the correct statement should be that, all memory of the actual value of μ is lost, except for that which is contained in the scale of the fields and the value of g. In other words, in the asymptotic region, a small change in mass can always be compensated for by an appropriate small change in g and an appropriate rescaling of the fields (n fields for the n-point function). Equation (141) is just the mathematical expression of this statement.

Another way of looking at the renormalization group equation is to observe that for a renormalizable theory, once the (infinite) renormalizations of the bare quantities render the theory finite, any further finite renormalizations do not change the predictive content of the theory. The renormalization group equation (141) simply expresses that fact.

For renormalizable theories, the renormalized Green's function Γ expressed as a function of the renormalized mass μ and renormalized coupling constant g is a finite function of g and μ and, hence, the coefficient functions $\beta(g)$ and $\gamma(g)$ in the partial differential equation (141) should also be finite functions of g. That they are functions of g alone, follows from dimensional argument. $\beta(g)$ is the so-called Callan–Symanzik function and it characterizes the field theory in a very important way and $\gamma(g)$ is the anomalous dimension of the field operator ϕ.

In (141), μ need not be the actual mass of the particle; more generally, it is an arbitrary mass at which fields and coupling constants are normalized. In this form, Equation (141) is applicable even to a massless theory.

18. Formal Derivation of the Renormalization Group Equation

Let g_0 be the bare coupling constant and μ the arbitrary mass at which fields and coupling constants are normalized. The renormalized coupling constant g is a function of these

$$g = g(g_0, \mu). \tag{142}$$

g is actually a function of the ultraviolet cut-off Λ also, but we shall suppress its dependence. We hold Λ fixed and do not consider variations in Λ. The unrenormalized Green's function Γ_0 is a function of g_0; expressing g_0 in terms of g using the inverse of the equation (142) and performing a multiplicative renormalization, we get the renormalized Green's function Γ (which is, in fact, independent of Λ in renormalizable ϕ^4 field theory):

$$\Gamma(g, \mu) = Z^{n/2}\Gamma_0(g_0), \tag{143}$$

where Z is the field-renormalization constant. Since the μ dependence enters only after renormalization, (we are considering either the asymptotic region or a massless theory) Γ_0 does not depend on μ. Hence,

$$\left(\frac{\partial \Gamma_0}{\partial \mu}\right)_{g_0} = 0 \tag{144}$$

Using (143) and multiplying by μ, (144) becomes

$$\mu\left(\frac{\partial Z^{-n/2}}{\partial \mu}\right)_{g_0}\Gamma + \mu Z^{-n/2}\left(\frac{\partial \Gamma}{\partial \mu}\right)_{g_0} = 0. \tag{145}$$

For the renormalized Green's function $\Gamma(g(g_0, \mu), \mu)$, we convert the partial derivative in the following way

$$\left(\frac{\partial \Gamma}{\partial \mu}\right)_{g_0} = \left(\frac{\partial \Gamma}{\partial \mu}\right)_g + \left(\frac{\partial g}{\partial \mu}\right)_{g_0}\left(\frac{\partial \Gamma}{\partial g}\right)_\mu. \tag{146}$$

Thus, we get the desired equation:

$$\left\{\mu\left(\frac{\partial}{\partial \mu}\right)_g + \beta(g)\left(\frac{\partial}{\partial g}\right)_\mu - n\gamma(g)\right\}\Gamma = 0$$

where we have defined

$$\beta(g) = \mu\left(\frac{\partial g}{\partial \mu}\right)_{g_0} \tag{147}$$

and

$$\gamma(g) = \frac{\mu}{2}\left(\frac{\partial \ln Z}{\partial \mu}\right)_{g_0}. \tag{148}$$

Thus, the above derivation has also yielded the definitions of $\beta(g)$ and $\gamma(g)$.

19. Solution of the Renormalization Group Equation

Let us rewrite the RG equation:

$$\left\{\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - n\gamma(g)\right\}\Gamma(g, \lambda p_i, \mu) = 0, \tag{149}$$

where we have put back all the dependences into Γ. p_i denotes the set of n external momenta all of which are multiplied by a scale factor λ and our aim is to determine the behaviour of Γ for large λ.

If d is the canonical (or naive) dimension* of the field (for scalar field, $d = 1$), then the mass-dimension of our n-point Green's function Γ is nd. Hence, Γ can be written as a product of, μ^{nd} and a function of dimensionless quantities only

$$\Gamma(g, p_i, \mu) = \mu^{nd} f\left(g, \frac{p_i}{\mu}\right). \tag{150}$$

So,

$$\Gamma(g, \lambda p_i, \mu) = \lambda^{nd}\left(\frac{\mu}{\lambda}\right)^{nd} f\left(g, \lambda\frac{p_i}{\mu}\right)$$

$$\equiv \lambda^{nd}\phi\left(g, \frac{\mu}{\lambda}, p_i\right), \tag{151}$$

therefore

$$\mu\frac{\partial}{\partial\mu}\Gamma(g, \lambda p_i, \mu) = \lambda^{nd}\mu\frac{\partial\phi}{\partial\mu} = -\lambda^{nd}\lambda\frac{\partial\phi}{\partial\lambda}, \tag{152}$$

where we have used the fact that the dependence of ϕ on μ and λ is through μ/λ only. We now define the variable t:

$$t \equiv \ln\lambda \tag{153}$$

Combining (149), (152) and (153), we get

$$\left\{-\frac{\partial}{\partial t} + \beta(g)\frac{\partial}{\partial g}\right\}\phi = n\gamma(g)\phi. \tag{154}$$

Equation (154) can be solved by making use of its similarity to hydrodynamic equations, t and g playing the roles of time t and position x and $\beta(g)$ playing the role of the velocity function at the point g. In hydrodynamics, one defines the moving coordinate $\bar{x}$ in terms of which the partial differential equation gets converted into a total differential equation which can then be solved. The same

*By dimension, we mean the mass-dimension which is equal to the negative of the length-dimension since $\hbar = c = 1$.

trick is used here. One defines the moving coupling constant $\bar{g}(g, t)$ by

$$\frac{\partial \bar{g}}{\partial t}(g, t) = \beta(\bar{g}); \qquad \bar{g}(g, 0) = g. \tag{155}$$

$\bar{g}(g, t)$ is also called the effective or momentum-dependent coupling constant (note that t contains the scaling factor for the momenta), and plays a very important role in renormalization group analysis. In terms of $\bar{g}(g, t)$, the solution of (154) can be directly obtained (see next section) and multiplication by λ^{nd} then gives Γ:

$$\Gamma(g, \lambda p_i, \mu) = \lambda^{nd}\Gamma(\bar{g}(g, t), p_i, \mu) \exp\left\{n \int_0^t \gamma(\bar{g}(g, t'))\, dt'\right\}. \tag{156}$$

It can be seen that the essential dependence of Γ on λ or t has been isolated; apart from the factor λ^{nd}, it is contained in the exponent. The main point is that the Γ on the right-hand side of (156) contains p_i and not the unknown dependence on λp_i. It is true that there is still a λ or t dependence of this Γ through $\bar{g}(g, t)$, but this is a mild dependence as will be clear from what follows. The crucial dependence is in the exponent.

20. Hydrodynamic Analogy

This is essentially an appendix to the last section. Consider the hydrodynamic equation

$$\frac{\partial \rho}{\partial t}(x, t) + v(x)\frac{\partial \rho}{\partial x}(x, t) = S(x)\rho(x, t), \tag{157}$$

where $\rho(x, t)$ = density of bacteria in a fluid moving in a pipe,
$v(x)$ = velocity of the fluid in the pipe,
$S(x)$ = some external influence (such as illumination) affecting the bacterial population.

To solve such an equation, one first defines the 'moving coordinate' $\bar{x}(x, t)$ by the equations:

$$\frac{\partial \bar{x}}{\partial t}(x, t) = v(\bar{x}); \; \bar{x}(x, 0) = x. \tag{158}$$

Then, Equation, (157) can be thrown into the form

$$\frac{d\rho}{dt}(\bar{x}(x, t), t) = S(\bar{x}(x, t))\, \rho(\bar{x}(x, t), t), \tag{159}$$

where the left-hand side now contains the total derivative in time. Equation (159) can be integrated to give

$$\rho(\bar{x}(x,t),t) = \rho(x,0)\exp\int_0^t S(\bar{x}(x,t'))\,dt'. \tag{160}$$

A similar technique can be used to get the result quoted in Equation (156).

21. Fixed Points and Asymptotic Freedom

From (155) and (156), it is clear that it is $\beta(g)$ which controls the asymptotic behaviour of Γ; for, $\beta(g)$ determines $\bar{g}$ which then is used to construct the solution in (156). Actually, it is the zeroes of $\beta(g)$ called the fixed points which control the asymptotic behaviour of Γ in a crucial way. This can be seen as follows:

For illustration, consider the example shown in Figure 14a where $\beta(g)$ is positive, but has a zero at $g = g^*$. With this form of $\beta(g)$, Equation (155) leads to the behaviour of $\bar{g}$ shown in Figure 14b. $\bar{g}$ starts with the value g at $t = 0$ (as

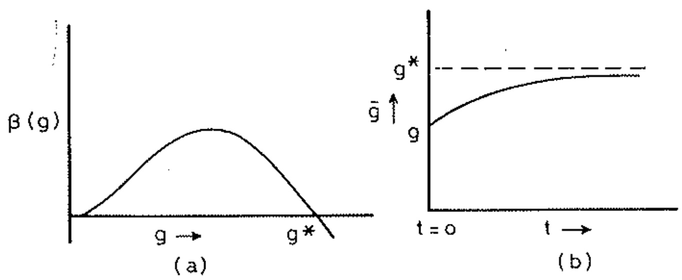


Fig. 14.

demanded by the boundary condition in (155)) and increases with t since the 'velocity' $\partial\bar{g}/\partial t = \beta(\bar{g})$ is positive. But, as g^* is approached the velocity becomes smaller and smaller and so $\bar{g}$ changes less and less. At g^*, the velocity $\beta(g^*)$ is zero and that is the asymptotic value of $\bar{g}$,

$$\bar{g}(g,t) \xrightarrow[t\to\infty]{} g^*. \tag{161}$$

As $t \to \infty$, λ and, hence, $\lambda p_i \to \infty$. So, g^* is called the ultraviolet fixed point of $\beta(g)$.

We may next consider the infrared limit $\lambda \to 0$ which corresponds to $t \to -\infty$ (see (153)). By running the above argument for negative t, one can convince oneself that

$$\bar{g}(g,t) \xrightarrow[t\to-\infty]{} 0. \tag{162}$$

Hence, in Figure 14a, the origin $g = 0$ is an infrared fixed point of $\beta(g)$.

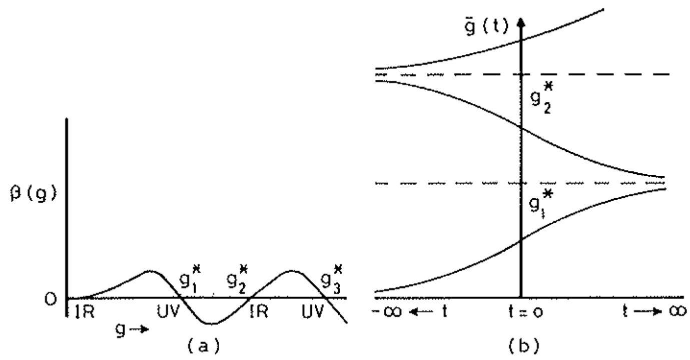


Fig. 15.

Just for fun, we may consider a theory with a number of fixed points as shown in Figure 15a. The corresponding behaviour of $\bar{g}$ is indicated in Figure 15b.

The fixed points of $\beta(g)$ alternate between ultraviolet and infrared. As shown in Figure 15b, the ultraviolet ($t \to \infty$) and infrared ($t \to -\infty$) asymptotic limits of $\bar{g}$ depend on the starting values of $\bar{g}$ at $t = 0$.

Let us now go back to (156). The ultraviolet asymptotic behaviour of Γ can be now easily obtained by making the replacement $\bar{g} \to g^*$, where g^* is an ultraviolet fixed point:

$$\Gamma(g, \lambda p_i, \mu) \xrightarrow[\lambda \to \infty]{} \lambda^{nd} \Gamma(g^*, p_i, \mu)\, e^{n\gamma(g^*)t} \sim \lambda^{n(d+\gamma(g^*))} \Gamma(g^*, p_i, \mu). \tag{163}$$

So, after all this, we recover the power-behaviour in the scale parameter λ, but the important point is that the exponent is *not* the naive or canonical dimension d, but the dynamical dimension $d + \gamma(g^*)$ evaluated at the UV fixed point g^*. The ultraviolet asymptotic behaviour of Green's function for ϕ fields is dictated by the anomalous dimension γ of the ϕ field at the UV fixed point g^* of the β function.

This anomalous dimension would spoil the Bjorken scaling of deep inelastic structure functions. But, as already noted in Section 16, experiment suggests that Bjorken scaling is valid. What is the way out? The answer is that we need a theory in which the origin $g = 0$ is an ultraviolet fixed point. In contrast to Figures 14 and 15, our β function should start negatively near the origin, as shown in Figure 16. In this case, the 'velocity' is negative and $\bar{g}$ decreases to zero asymptotically in

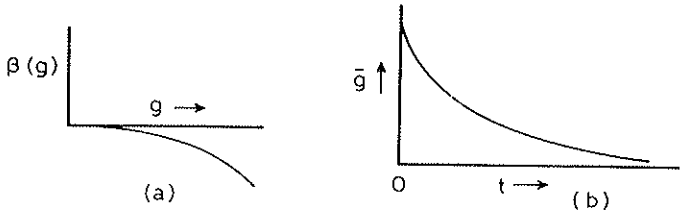


Fig. 16.

the ultraviolet region

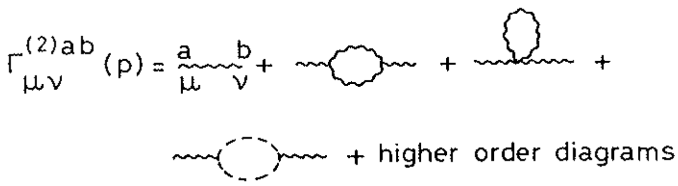
$$\bar{g}(g, t) \xrightarrow[t\to\infty]{} 0. \tag{164}$$

This is the 'asymptotically free' theory. The asymptotic behaviour of Γ is now governed by free field theory (i.e. $g = 0$). The asymptotic anomalous dimension is zero: $\gamma(0) = 0$. Such a theory can provide a framework for understanding Bjorken scaling and parton-structure.

As already mentioned, non-Abelian gauge theory alone possesses the unique distinction of being asymptotically free and, hence, QCD.

22. Asymptotic Freedom of QCD

The QCD Lagrangian is given in Equation (137). Our aim is to calculate $\beta(g)$ and $\gamma(g)$ to the lowest nontrivial order in g. Rather than use the formal definitions of (147) and (148), we proceed as follows. We ignore the quark fields first and calculate the renormalized Green's functions in perturbation theory for a few values of n, say $n = 2$ and $n = 3$ (number of external gauge boson lines). For $n = 2$, one gets*

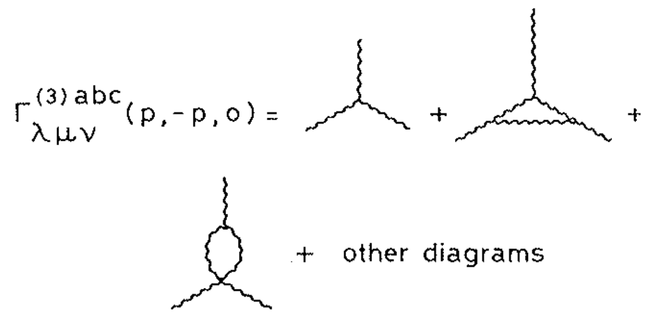


$$\xrightarrow[p^2\to\infty]{} \delta^{ab}(-g_{\mu\nu}p^2 + p_\mu p_\nu)\left\{1 + \frac{13}{3}C_G\left(\frac{g}{4\pi}\right)^2 \ln\frac{p^2}{\mu^2}\right\} + O(g^4). \tag{165}$$

Here, C_G is the quadratic Casimir operator for the adjoint representation of the group and it is defined by

$$f_{acd}f_{bcd} = 2C_G\,\delta_{ab}. \tag{166}$$

Since the Green's functions are truncated ones, two inverse propagators have been multiplied into our Green's function and so $\Gamma^{(2)}$ is actually the inverse of the propagator. For $n = 3$,



* The dotted lines in the fourth diagram represent the so-called 'ghosts' whose existence in the virtual states of non-abelian gauge theory was discovered by Faddeev and Popov. Another technical point is that the Landau gauge has been chosen in the calculations.

$$\xrightarrow[p^2\to\infty]{} f^{abc}(p_\lambda g_{\mu\nu} + p_\mu g_{\nu\lambda} - 2p_\nu g_{\lambda\mu})g \times$$
$$\times \left\{1 + \frac{17}{6} C_G \left(\frac{g}{4\pi}\right)^2 \ln\frac{p^2}{\mu^2}\right\} + O(g^5). \tag{167}$$

These perturbative expressions for $\Gamma^{(2)}$ and $\Gamma^{(3)}$ are substituted into the renormalization group equation

$$\left\{\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - n\gamma_G(g)\right\}\Gamma^{(n)} = 0 \quad (n = 2,3),$$

where $\gamma_G(g)$ is the anomalous dimension of the gauge field and the values of $\beta(g)$ and $\gamma_G(g)$ are determined from the requirement that the renormalization group equation be satisfied. The results are the following:

$$\beta = -\frac{22}{3} C_G g \left(\frac{g}{4\pi}\right)^2 + O(g^5),$$
$$\gamma_G = \frac{13}{3} C_G \left(\frac{g}{4\pi}\right)^2 + O(g^4). \tag{168}$$

Note C_G defined by (166) is positive. As already advertised, β is negative near the origin $g = 0$ and, hence, non-Abelian gauge theory is asymptotically free. It is the cubic vertex characteristic of the non-Abelian gauge field that is responsible for the negative β.

We may now include the quarks. The quark-gluon vertex is of the asymptotically nonfree type like the electron-photon vertex in QED, and adds a positive contribution to β. The results are

$$\beta = -\left\{\frac{22}{3} C_G - \frac{8}{3} C_q\right\} g \left(\frac{g}{4\pi}\right)^2 + O(g^5),$$
$$\gamma_G = \left\{\frac{13}{3} C_G - \frac{8}{3} C_q\right\} \left(\frac{g}{4\pi}\right)^2 + O(g^4),$$
$$\gamma_q = 0 + O(g^4) \tag{169}$$

where C_q is defined by

$$\mathrm{Tr}\left(\frac{\lambda^i}{2}\frac{\lambda^j}{2}\right) = 2C_q\,\delta_{ij}. \tag{170}$$

The anomalous dimension of the quark field γ_q remains zero in order g^2.

From the expression for β in (169), it follows that the condition for asymptotic freedom or ultraviolet stability of the origin is

$$C_q < \tfrac{11}{4} C_G \tag{171}$$

For the SU(3) group relevant for QCD,

$$C_G = \frac{3}{2}; \qquad C_q = \frac{N_f}{4}, \tag{172}$$

where N_f is the number of flavours (N_f enters because the trace in (170) should be taken over all the quark degrees of freedom including flavour). Hence, (171) becomes

$$N_f \leqslant 16. \tag{173}$$

In other words, as long as the number of flavour quantum numbers is less than or equal to 16, the quark contribution does not destroy the asymptotic freedom of QCD. (At present, we have three generations of quarks, which implies $N_f = 6$, and so asymptotic freedom appears to be safe.)

The situation with respect to Higgs bosons is more complicated. Once the Higgs scalar field is added to the system further independent coupling constants such as the ϕ^4 coupling constant enter the picture and the origin is generally unstable with respect to these coupling constants. So Higgs bosons are avoided in the QCD sector and we have the central dogma of high energy physics, namely $SU(3)_{\text{colour}}$ symmetry is exact. The price we have to pay for keeping asymptotic freedom is a theory with massless gluons, which leads to terrible infrared divergences. We shall come back to this a little later.

Let us write (using (169) and (172))

$$\beta(g) = -\frac{bg^3}{4\pi}; \quad b = \frac{1}{12\pi}(33 - 2N_f) > 0 \tag{174}$$

and solve the equation for the effective coupling constant

$$\frac{\partial \bar{g}(t)}{\partial t} = -\frac{b}{4\pi}\bar{g}^3(t). \tag{175}$$

To solve this, it is better to rewrite it in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\bar{g}^2(t)} = \frac{b}{2\pi}. \tag{176}$$

The solution is

$$\frac{1}{\bar{g}^2(t)} = \frac{bt}{2\pi} + \frac{1}{\bar{g}^2(0)}. \tag{177}$$

Or

$$\bar{g}^2(t) = \frac{\bar{g}^2(0)}{1 + \dfrac{b}{2\pi}\bar{g}^2(0)t}. \tag{178}$$

For deep inelastic lepton-hadron scattering, we may make the following identification

$$t = \ln \lambda = \frac{1}{2} \ln \frac{q^2}{q_0^2}, \tag{179}$$

where q^2 is the momentum transfer to the hadron and q_0^2 is a reference value. Let us also define

$$\alpha_s(q^2) \equiv \frac{\bar{g}^2(t)}{4\pi} \tag{180}$$

Then, Equation (178) can be rewritten as

$$\alpha_s(q^2) = \frac{\alpha_s(q_0^2)}{1 + b\alpha_s(q_0^2) \ln \frac{q^2}{q_0^2}}. \tag{181}$$

Thus, the effective coupling constant goes to zero for $q^2 \to \infty$, however, the approach to zero is rather slow, only logarithmic. Defining

$$\Lambda_c^2 \doteq q_0^2 \exp\left(- \frac{1}{b\alpha_s(q_0^2)} \right) \tag{182}$$

we get

$$\alpha_s(q^2) = \frac{1}{b \ln(q^2/\Lambda_c^2)}. \tag{183}$$

Since b is a known constant (apart from the slight uncertainty in the number of flavours, we see that QCD is characterized by one unknown constant Λ_c, which is to be determined by experiment. Unfortunately, there is considerable uncertainty in the empirical determinations of this parameter. A recent analysis gives

$$\Lambda_C = 150^{+150}_{-100} \, \text{MeV}. \tag{184}$$

For $\Lambda_C = 100\,\text{MeV}$,

$$\alpha_S(q^2 = 1\,\text{GeV}^2) \approx 0.2. \tag{185}$$

Thus, even at, 1 GeV, the QCD coupling constant is fairly small, thus justifying perturbative calculations.

23. Infrared Problem and Colour Confinement

It is illuminating to consider the contrasting behaviour of the effective coupling constant in QED and QCD as a function of $t = \frac{1}{2} \ln (q^2/q_0^2)$. This is illustrated in Figures 17 and 18.

In QED, the β function is positive near $e \approx 0$ and so the effective coupling constant $e(t)$ increases with t or with q^2. So, it is *not* asymptotically free. But for $q^2 \to 0$, i.e. $t \to -\infty$, the problem is very well controlled. This means that in the infrared region, there is no real difficulty with QED, as is well-known.

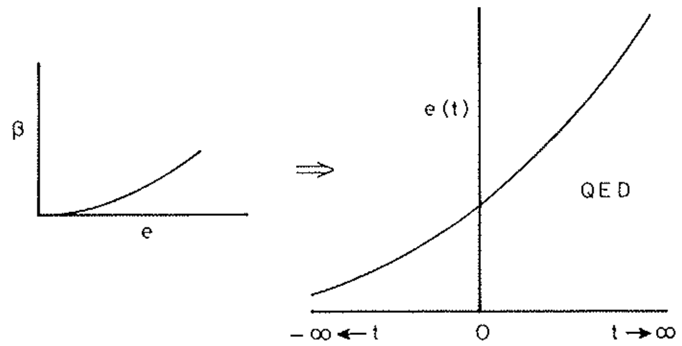


Fig. 17.

But in a non-Abelian gauge theory such as QCD, the behaviour is completely reversed. The theory is asymptotically free in the ultraviolet region and, thus, is a good candidate for a theory of strong interactions, as we have already discussed. However, the infrared region is really catastrophic for non-Abelian gauge theory. *Hence, QCD does not really exist as a theory on the mass-shell.*

It is hoped this can be turned to our advantage. The infrared catastrophe can perhaps be used to solve another problem–namely the problem of colour confinement. What the sketch in Figure 18 shows is that a single non-Abelian quantum on the mass shell ($q^2 = 0$) has infinitely large colour charge $g(-\infty) \rightarrow \infty$, and so will copiously emit virtual quanta. These virtual quanta may surround and completely screen the original quantum. So, a single non-Abelian quantum (i.e. the gluon) with $q^2 = 0$ cannot exist as a free particle outside the hadron.

The situation inside the hadron is different; short distances correspond to high q^2 for which the effective coupling goes to zero, because of asymptotic freedom and so the quanta do behave as massless particles inside hadrons.

What is described above is the infrared mechanism for colour confinement. This mechanism can also work for quarks, since the interaction of the quarks with gluons are governed by the same effective coupling constant $g(t)$.

There are many other mechanisms which have been discussed in the recent literature for confining colour. However, in spite of much work, *the dogma of colour confinement* remains an unproved hypothesis.

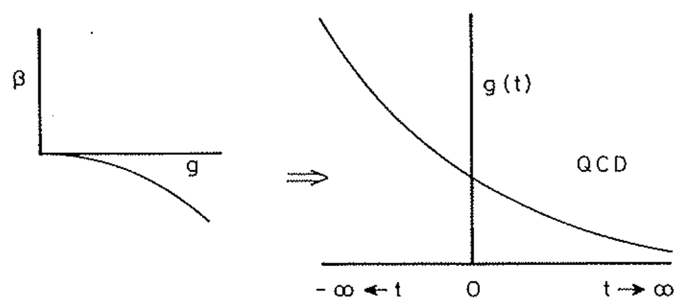


Fig. 18.

24. Tests of QCD

This is still an important area of experimental and theoretical activity in high energy physics, but we shall be very brief.

(i) *Parton Model and Scaling.* As already mentioned in Section 16, the motivation for QCD came from observed scaling in deep inelastic lepton-hadron scattering. Quantum chromodynamics via asymptotic freedom provides the theoretical foundation for scaling and the parton model. So, the many early successes achieved in the confrontation of the parton model with experimental data, can be regarded as tests of QCD.

(ii) *Logarithmic Corrections.* Since the approach of the QCD coupling constant to zero for asymptotic momenta is logarithmic (see Equation (183)), there are logarithmic corrections to the parton model and scaling. Such corrections appear to have received experimental support. However, by their very nature, logarithmic variations are hard to see clearly, as is evidenced by the large uncertainties in the experimental determination of the QCD scale parameter Λ_c occurring in the logarithm (see Equation (184)).

(iii) *Narrow Widths of ψ and Υ.* As remarked in Section 13, the charmed quark c was discovered through a certain peculiar property observed for the particle which was being interpreted as a bound state of c and $\bar{c}$. This peculiar property is the strikingly narrow decay width observed for ψ:

$$\Gamma_\psi = 60\,\text{KeV} \tag{186}$$

which is in sharp contrast to the large widths expected for strongly interacting hadrons. For instance, ρ meson has width

$$\Gamma_\rho \simeq 150\,\text{Mev.} \tag{187}$$

Since the mass of ψ which is 3.1 GeV, is much higher than the mass of ρ which is 770 MeV, the available phase space is much more for ψ decay and, hence, the expected width of ψ is several hundred MeV. This was the puzzle of the ψ particle. It was resolved by asymptotic freedom; the momentum-dependent coupling constant of QCD evaluated at 3.1 GeV is small enough to provide an explanation for the small width of ψ. For, decay width or decay probability is a product of the coupling constant and phase space apart from other kinematic factors.

Thus, the correct interpretation of ψ and its properties not only requires a crucial ingredient of electroweak theory, namely the existence of a new quark c, but also asymptotic freedom which is a characteristic of QCD.

The phenomenon of small width repeats itself for $b\bar{b}$ bound states called Υ (upsilon), occurring at mass $\sim 10\,\text{GeV}$, where the width is

$$\Gamma_y \approx 42\,\text{KeV.} \tag{188}$$

The same phenomenon may occur with a vengeance for $t\bar{t}$ bound states (called toponium), whose mass is expected to be very high (>80 GeV). At such high energies or momenta, the strong QCD coupling constant would have become so small that weak and electromagnetic decays may dominate over strong decays!

(iv) *Jets and Gluon Radiation.* At high energies, electron-positron annihilation is known to produce hadrons in the form of two jets and this has been understood to be due to the production of a quark-antiquark pair which subsequently materializes in the form of a pair of jets made up of hadrons (see Figure 19a). If QCD is right, one must also see events with three jets, the third jet coming from a gluon radiated away from a quark or an antiquark (see Figure 19b). Such

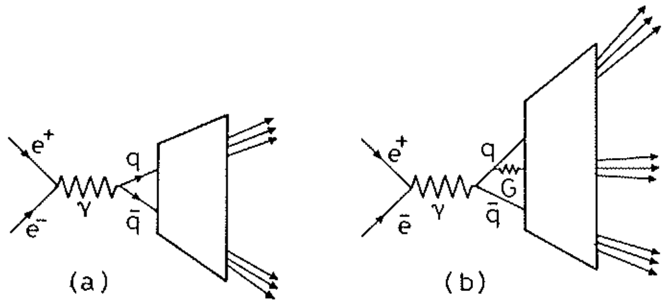


Fig. 19.

a three-jet phenomenon was discovered in the e^+e^- collider, PETRA, at Hamburg (with a c.m. energy $\sim$ 30 GeV or higher). This is generally taken to be the evidence for the existence of the gluon which, in turn, supports QCD. After the advent of the $p\bar{p}$ collider at CERN (with a c.m. energy of the order of 500 GeV or higher), jet phenomena and gluon physics have received further experimental support.

However, one must keep in mind that all the above tests of QCD are indirect and are to be contrasted with the direct test of electroweak theory, such as the discovery of the neutral current or the discovery of W and Z bosons. In fact, *because of the dogma of colour confinement, QCD is doomed to indirect verification only.*

25. The Standard Model of High Energy Physics

We have now built up all the elements of the standard model and we assemble them here. The standard model is based on the gauge group SU(3) × SU(2) × U(1). While SU(3) leads to quantum chromodynamics (QCD) and decribes strong interactions among the quarks, SU(2) × U(1) leads to quantum flavour dynamics (QFD) and describes electroweak interactions among the quarks and leptons. The gauge bosons of QCD are the eight gluons $G^i_\mu (i = 1 \dots 8)$. The gauge bosons of QFD are $W^a_\mu (a = 1, 2, 3)$ and B_μ.

The colour symmetry SU(3) is supposed to be unbroken, leaving the gluons massless, paving the way for colour confinement. The electroweak symmetry SU(2) × U(1), on the other hand, is broken and the breaking is presumed to be induced by the nonvanishing vacuum expectation value of the Higgs scalar field

ϕ which is chosen to be a doublet under SU(2). The vacuum expectation value $\langle\phi\rangle$ is chosen to be $\sim$300 GeV in order to obtain consistency with the observed value of G_F and Equations (63) and (121).

The particle sector comprising leptons and quarks is taken to be the three generations of fermions

$$\underbrace{\begin{pmatrix}\nu_e\\ e^-\end{pmatrix}\begin{pmatrix}u_\alpha\\ d_\alpha\end{pmatrix}}_{1}\quad \underbrace{\begin{pmatrix}\nu_\mu\\ \mu^-\end{pmatrix}\begin{pmatrix}c_\alpha\\ s_\alpha\end{pmatrix}}_{2}\quad \underbrace{\begin{pmatrix}\nu_\tau\\ \tau^-\end{pmatrix}\begin{pmatrix}t_\alpha\\ b_\alpha\end{pmatrix}}_{3}$$

In the above, α denotes the colour index of the quarks. Among these fermions, the top quark, t, which is the heaviest, has remained elusive, although the UAI experiment at the p$\bar{\text{p}}$ collider has obtained some evidence for its existence around a mass of about 40 GeV.

Since the weak interaction is helicity-dependent, it is necessary to separate the helicities of fermions, as already explained. Counting the L and R helicities as distinct particles, we have 15 particles for each generation. For the first generation, they are the same as before

$$\begin{pmatrix}\nu_e\\ e^-\end{pmatrix}_L, \begin{pmatrix}u_\alpha\\ d_\alpha\end{pmatrix}_L, e_R^-, u_{\alpha R}, d_{\alpha R} \quad (\alpha = 1, 2, 3),$$

where the doublets and singlets under SU(2) are explicitly indicated.

Now let us write down the Lagrangian of the standard model, which is obtained by combining our Lagrangians of QCD and QFD (Equations (137) and (104)).

$$\begin{aligned}
\mathscr{L} = & -\tfrac{1}{4}(\partial_\mu G_\nu^i - \partial_\nu G_\mu^i - g_3 f^{ijk} G_\mu^j G_\nu^k)^2 - \\
& -\tfrac{1}{4}(\partial_\mu W_\nu^a - \partial_\nu W_\mu^a - g_2 \varepsilon^{abc} W_\mu^b W_\nu^c)^2 - \tfrac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 - \\
& -\sum_n \bar{q}_{nL}\gamma^\mu\left(\partial_\mu + ig_3\frac{\lambda^i}{2}G_\mu^i + ig_2\frac{\tau^a}{2}W_\mu^a + i\frac{g_1}{6}B_\mu\right)q_{nL} - \\
& -\sum_n \bar{u}_{nR}\gamma^\mu\left(\partial_\mu + ig_3\frac{\lambda^i}{2}G_\mu^i + i\tfrac{2}{3}g_1 B_\mu\right)u_{nR} - \\
& -\sum_n \bar{d}_{nR}\gamma^\mu\left(\partial_\mu + ig_3\frac{\lambda^i}{2}G_\mu^i - i\frac{g_1}{3}B_\mu\right)d_{nR} - \\
& -\sum_n \bar{l}_{nL}\gamma^\mu\left(\partial_\mu + ig_2\frac{\tau^a}{2}W_\mu^a - \frac{ig_1}{2}B_\mu\right)l_{nL} - \\
& -\sum_n \bar{e}_{nR}\gamma^\mu(\partial_\mu - ig_1 B_\mu)e_{nR} + \\
& +\left|\left(\partial_\mu + ig_2\frac{\tau^a}{2}W_\mu^a + i\frac{g_1}{2}B_\mu\right)\phi\right|^2 - \lambda(\phi^\dagger\phi - \phi_0^2)^2 - \\
& -\sum_{m,n}(\Gamma_{mn}^u \bar{q}_{mL}\phi^c u_{nR} + \Gamma_{mn}^d \bar{q}_{mL}\phi d_{nR} + \Gamma_{mn}^e \bar{l}_{mL}\phi e_{nR} + \text{h.c.}).
\end{aligned} \qquad (189)$$

The first three terms describe the pure gauge field part of the SU(3) × SU(2) × U(1) non-Abelian gauge theory. We now use g_3, g_2 and g_1 to denote the gauge coupling constants for these three gauge groups, respectively. In the fermionic terms, the SU(3) colour and the SU(2) flavour indices have been suppressed and, instead, the index n is used to denote the generation number. The left-handed SU(2) doublet quark of the nth generation is denoted by $\mathrm{q}_{n\mathrm{L}}$ and the corresponding right-handed SU(2) singlets are denoted by $\mathrm{u}_{n\mathrm{R}}$ and $\mathrm{d}_{n\mathrm{R}}$. For the leptons, $\mathrm{l}_{n\mathrm{L}}$ is the doublet while $\mathrm{e}_{n\mathrm{R}}$ is the singlet.

The last group of terms describes the Higgs field ϕ and its interactions with itself, with the gauge bosons and with the fermions which are, respectively, responsible for the spontaneous symmetry breaking, generation of W and Z masses and generation of the masses for the quarks and leptons. Note that

$$\phi^c = i\tau_2 \phi^* = \begin{pmatrix} \phi_2^* \\ -\phi_1^* \end{pmatrix}. \tag{190}$$

The masses of the quarks and leptons arise from the Yukawa couplings of ϕ given in the last part of the Lagrangian in Equation (189), as briefly explained in an earlier section. In contrast to the rest of the Lagrangian, the Yukawa coupling constants $\Gamma^u_{mn}, \Gamma^d_{mn}$ and Γ^e_{mn} mix the generations. In fact, the mass terms are nondiagonal with respect to parity, as well as flavour quantum numbers such as strangeness, charm etc. These mass matrices can be diagonalized★, but then the mixing between the generations enters through the charged-current weak interactions (mediated by the $W^\pm$ bosons).The mixing is described by a unitary matrix V called the Cabibbo–Kobayashi–Maskawa matrix which is a 3 × 3 generalization of the 2 × 2 Cabibbo rotation matrix already introduced earlier:

$$\begin{pmatrix} \cos\theta_c & -\sin\theta_c \\ \sin\theta_c & \cos\theta_c \end{pmatrix}. \tag{191}$$

For the three-generation case, V can be written in terms of three rotation angles θ_1, θ_2 and θ_3 and a CP violating phase δ. Such a CP-violating phase exists only if the number of generations is $\geqslant 3$, as was first pointed out by Kobayashi and Maskawa. Thus, CP violation also can be introduced into the standard model.

The charged-current weak interaction is therefore modified to

$$\bar{U}\gamma_\mu(1-\gamma_5)VDW^\mu + \text{h.c.}, \tag{192}$$

where $\bar{U}$ stands for $(\bar{\mathrm{u}}\,\bar{\mathrm{c}}\,\bar{\mathrm{t}})$ and D stands for

$$\begin{pmatrix} \mathrm{d} \\ \mathrm{s} \\ \mathrm{b} \end{pmatrix}$$

and the elements of the mixing matrix V control the various flavour-changing charged-current weak transitions among the quarks. A similar mixing matrix will

★ For details of this diagonalization procedure, see for instance reference [5].

also exist in the leptonic sector if neutrinos have masses. For massless left-handed neutrinos, suitable redefinition of the neutrino fields removes leptonic mixing (see [5]).(If neutrinos have masses, they will mix and this will lead to neutrino oscillations.)

The neutral-current weak interaction, on the other hand, is not modified and remains diagonal in the generation space, because of the unitarity of the mixing matrix. This is a generalization of the original Glashow–Iliopoulos–Maiani mechanism which achieved the cancellation of the strangeness-changing neutral current sector.

The diagonalization procedure finally yields expressions for the mixing matrix V and the diagonal mass matrices in terms of the Yukawa coupling matrices occurring in the standard model Lagrangian. So far, there is no theoretical framework for fixing the values of the Yukawa coupling constants and, hence, there exists no theoretical understanding of the values of the elements of the mixing matrix or the diagonal mass matrices. They are purely empirically determined.

As already mentioned, the standard model Lagrangian given in Equation (189) is supposed to describe *all that is known in high energy physics.* That is the achievement of two decades of work (the 60's and 70's).

Dirac, referring to his relativistic wave equation of the electron, is supposed to have said that it describes all of chemistry and almost all of physics. In the same vein, we are tempted to say that the standard model lagrangian describes all of physics except gravitation. However, note the contrast in complexity. Whereas the Dirac equation can be written down on one single line and there is no adjustable constant, the standard model Lagrangian occupies almost half a page and, further, there are more than 20 constants to be fixed by experiment. The model lacks the simplicity which is the hall mark of any truly fundamental theory. This supplies the chief motivation for going beyond the standard model.

26. Beyond the Standard Model

Let us first spell out in more detail the standard reasons usually given for attempting to go beyond the standard model.

(i) *Too many parameters:* Counting the coupling constants, the boson masses, the various quark and lepton masses and the quark mixing parameters, the total number of independent parameters in the standard model is about 20. They are all empirically determined and there is no fundamental theoretical understanding of these numbers. As we have already mentioned, this is one of the weakest points of the standard model and the strongest motivation for considering possible next steps.

(ii) *Generation puzzle:* The standard model contains no explanation for the existence of several generations of quarks and leptons, nor any clue as to the actual number of generations existing in nature.

(iii) *Pattern within one generation:* The model does not explain why quark and lepton charges are quantized in a related way: why does integer charge come with colour singlets and noninteger charge with colour triplets? Also, the model does not explain the apparent quark-lepton universality: why do the quarks and leptons possess identical SU(2) properties? Since the same pattern repeats at least three times (for the three generations), there must be a particularly good reason for this pattern.

(iv) *Unification:* Theoretical physicists have an innate urge for unification. It is felt that, in nature, the three interactions must be unified in some manner so that the three gauge coupling constants g_1, g_2 and g_3 are replaced by a single unified coupling constant. The energy scale of this so-called grand unification turns out to be $\sim 10^{14}$ GeV.

(v) *Inclusion of gravitation:* Unification of other forces with gravitation is, of course, an important aim of physics. This becomes all the more compelling if the other forces are already unified and that unification scale (10^{14} GeV) is so close to the gravitational scale, given by Planck mass (10^{19} GeV).

(vi) *Hierarchy problem:* Assuming that there is an important energy scale beyond the standard model such as the grand unification scale or the Planck scale, it is difficult to understand how particles with masses corresponding to the low energy scales of the standard model can survive the enormous self-energy correction. Vector bosons and fermions may be protected from such corrections by gauge symmetry or chiral symmetry, respectively. Scalars and their vacuum expectation values (which generate the masses of the vector bosons and fermions) are not generally protected. In the presence of the large energy scale ($> 10^{14}$ GeV), the small scale of the standard model (~ 100 GeV) cannot be maintained.

The various avenues open to high energy physicists in going beyond the standard model are the following

(a) Grand unification,
(b) Preons,
(c) Induced gravity,
(d) Supersymmetry and supergravity,
(e) Higher dimensional unification,
(f) Superstrings.

For a brief introduction to these ideas, see, for instance, [6].

Grand unification solves problems (iii) and (iv) mentioned above. Preons do not solve any of the problems (nevertheless, they may turn out to be the correct next step!). Induced Gravity may provide a revolutionary solution to problem (v); it claims that gravity and the geometry of spacetime may be derived from the quantum effects of matter interacting through the other forces of nature (weak, electromagnetic, strong, etc.), quite the opposite to what Einstein strove to achieve.

The chief virtue of supersymmetry is that it provides an elegant solution to problem (vi). Supergravity coupled with grand unification is capable of solving problems (iii), (iv), (v) and (vi). Higher dimensions offer a beautiful geometrical understanding of the forces contained in the standard model; gravitation in a 11-dimensional spacetime unifies four-dimensional gravitation with four-dimensional SU(3) × SU(2) × U(1) forces.

Finally, superstrings in 10 dimensions offer the tantalizing hope of achieving a finite or renormalizable theory of gravity, in which case superstring theory may turn out to be the correct theory of quantum gravity. Current advertisements claim that as a bonus, superstrings may solve all the problems of high-energy physics (i)–(vi) mentioned at the beginning of this section.

To conclude, we must bear in mind that everything *beyond the standard model* is a speculative idea. None of these ideas has an iota of experimental support at present. In fact, many of these theories beyond the standard model have a bearing on the super high-energy scales 10^{14}–10^{19} GeV and so their direct experimental confrontation is not expected soon. This is very unfortunate. However, indirect clues coming from lower-energy experiments in the immediate future may be of great value in deciding the future course of the subject.

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