Statistics - Problem Set

Any queries contact me at libby@iitm.ac.in

1. The forward-backward asymmetry for $e^+e^- \to \mu^+\mu^-$ is defined as

$$A_{FB} = \frac{N(\cos\theta_{\mu^+} > 0) - N(\cos\theta_{\mu^+} < 0)}{N(\cos\theta_{\mu^+} > 0) + N(\cos\theta_{\mu^+} < 0)} \ ,$$

where θ_{μ^+} is the angle between the incoming positron direction and the outgoing μ^+ direction and $N(\cos\theta_{\mu^+} > 0)$ $[N(\cos\theta_{\mu^+} < 0)]$ are the number of events with $\cos\theta_{\mu^+} > 0$ $[\cos\theta_{\mu^+} < 0]$. Show that

$$\sigma(A_{FB}) = \sqrt{\frac{1 - A_{FB}^2}{N(\cos\theta_{\mu^+} > 0) + N(\cos\theta_{\mu^+} < 0)}} \ .$$

Hint: use binomial uncertainties.
Ans: For simplicity will define $N(\cos\theta_{\mu^+} > 0) = N_F$, $N(\cos\theta_{\mu^+} < 0) = N_B$ and $N = N_F + N_B$. Therefore,

$$A_{FB} = \frac{N_F - N_B}{N} = \frac{N_F + N_B - 2N_B}{N} = \frac{N - 2N_B}{N} = 1 - 2\frac{N_B}{N} \ .$$

N_B has binomial uncertainty

$$\sigma(N_B) = \sqrt{N \times \frac{N_B}{N} \times \left(1 - \frac{N_B}{N}\right)} \ ,$$

so the uncertainty on $\frac{N_B}{N}$ is

$$\sigma\left(\frac{N_B}{N}\right) = \sqrt{\frac{1}{N} \times \frac{N_B}{N} \times \left(1 - \frac{N_B}{N}\right)} \ .$$

and hence

$$\sigma(A_{FB}) = 2 \times \sigma\left(\frac{N_B}{N}\right) = \sqrt{\frac{4}{N} \times \frac{N_B}{N} \times \left(1 - \frac{N_B}{N}\right)} \ .$$

Now $N_B/N = \frac{1}{2}(1 - A_{FB})$ and $1 - N_B/N = \frac{1}{2}(1 + A_{FB})$, which upon substitution gives the required result.

2. $H \to \gamma\gamma$ decays happen at a rate of 10 per minute during the running of a future linear collider. What is the probability of observing less than five $H \to \gamma\gamma$ events in one minute.
Ans: Here the events follow Poisson statistics with $\lambda = 10$:

$$P(r < 5) = \Sigma_{r=0}^{4} P(r, \lambda) = e^{-\lambda} \sum_{r=0}^{4} \frac{\lambda^r}{r!} = e^{-10}\left(1 + 10 + \frac{100}{2} + \frac{1000}{6} + \frac{10000}{24}\right) = 2.9\%.$$

3. Prove that the product of two Poisson distributions with means λ_1 and λ_2 is a Poisson distribution with a mean $\lambda_1 + \lambda_2$.
Ans: The probability of observing r events is

$$\begin{aligned} P(r) &= \sum_{r_1=0}^{r} P(r_1, \lambda_1) P(r - r_1, \lambda_2) \\ &= \sum_{r_1=0}^{r} \frac{\lambda_1^{r_1} e^{-\lambda_1}}{r_1!} \frac{\lambda_2^{r-r_1} e^{-\lambda_2}}{(r - r_1)!} \\ &= \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1 + \lambda_2)^r}{r!} \sum_{r_1=0}^{r} \frac{r!}{r_1!(r-r_1)!} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{r_1} \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{r-r_1} \\ &= \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1 + \lambda_2)^r}{r!} \left(\frac{\lambda_1}{\lambda_1+\lambda_2} + \frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{r} \\ &= \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1 + \lambda_2)^r}{r!} \,. \end{aligned}$$

4. Show that Poisson distribution tends to the Gaussian distribution when λ becomes large.
Hint: you will need Stirling's approximation of the factorial.
Ans: take the log of the Poisson probability and use Stirling's formula and $r = \lambda + x$, $x << \lambda$

$$\begin{aligned} \ln P(r; \lambda) &= -\lambda + r \ln \lambda - \ln r! \\ &\approx -\lambda + r \ln \lambda - r \ln r + r - \ln \sqrt{2\pi r} \text{ (Stirling's formula for third term)} \\ &\approx -\lambda + (\lambda + x)\left[\ln \lambda - \ln\left\{\lambda\left(1 + \frac{x}{\lambda}\right)\right\}\right] + (\lambda + x) - \ln\sqrt{2\pi\lambda\left(1 + \frac{x}{\lambda}\right)} \\ &\approx x - (x + \lambda)\left(\frac{x}{\lambda} - \frac{x^2}{2\lambda^2}\right) - \ln\sqrt{2\pi\lambda} \quad (\ln(1+\epsilon) \approx \epsilon - \epsilon^2/2) \\ &\approx \frac{x^2}{2\lambda} - \ln\sqrt{2\pi\lambda} \\ \Rightarrow P(x; \lambda) &= \frac{1}{\sqrt{2\pi\lambda}} e^{-\frac{x^2}{2\lambda}} \end{aligned}$$

which is a Gaussian width $\sqrt{\lambda}$.

5. Use the combination of error formula to show that the variance of a weighted average is $\frac{1}{\Sigma\sigma_i^2}$.
Ans: The weighted average is

$$\overline{x} = \frac{\sum x_i/\sigma_i^2}{\sum 1/\sigma_i^2};.$$

The propagation of error formula tells us that

$$
\begin{aligned}
\sigma_{\overline{x}}^2 &= \sum \left(\frac{\partial \overline{x}}{\partial x_i}\right)^2 \sigma_i^2 \\
&= \sum \left(\frac{1/\sigma_i^2}{\sum 1/\sigma_i^2}\right)^2 \sigma_i^2 \\
&= \frac{1}{(\sum 1/\sigma_i^2)^2} \sum 1/\sigma_i^2 \\
&= \frac{1}{(\sum 1/\sigma_i^2)}
\end{aligned}
$$

6. [Barlow 4.3] Find the best combined result and error from the following five measurements of c in ms^{-1}:

$$
\begin{aligned}
299\,794\,000 &\pm 3000 \\
299\,791\,000 &\pm 5000 \\
299\,770\,000 &\pm 2000 \\
299\,789\,000 &\pm 3000 \\
299\,790\,000 &\pm 4000\,.
\end{aligned}
$$

Ans: firstly the third measurement lies more than three standard deviations away from the other measurements, so discard it. Making the weighted average of the remaining four measurements yields an average $(299\ 791\ 000 \pm 2000)\ \mathrm{ms}^{-1}$.

7. A tracking chamber finds the hit coordinates in cylindrical polar coordinates (ρ, ϕ, z). The uncertainty in the ρ coordinate is negligible. Find the covariance matrix for the position in Cartesian coordinates (x, y, z).
Ans: The covariance matrix for (ρ, ϕ, z) is

$$
\mathsf{V}(\rho, \phi, z) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_\phi^2 & 0 \\ 0 & 0 & \sigma_z^2 \end{pmatrix}
$$

and

$$
\mathsf{G} = \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} ,
$$

therefore

$$
\begin{aligned}
\mathsf{V}(x, y, z) = \mathsf{G}\mathsf{V}(\rho, \phi, z)\mathsf{G}^{\mathrm{T}} &= \begin{pmatrix} \sigma_\phi^2\rho^2\sin^2\phi & -\sigma_\phi^2\rho^2\sin\phi\cos\phi & 0 \\ -\sigma_\phi^2\rho^2\sin\phi\cos\phi & \sigma_\phi^2\rho^2\cos^2\phi & 0 \\ 0 & 0 & \sigma_z^2 \end{pmatrix} \\
&= \begin{pmatrix} \sigma_\phi^2 y^2 & -\sigma_\phi^2 xy & 0 \\ -\sigma_\phi^2 xy & \sigma_\phi^2 x^2 & 0 \\ 0 & 0 & \sigma_z^2 \end{pmatrix} .
\end{aligned}
$$

8. A common resistor, with resistance, R, and a voltmeter are used to measure the potential differences across the resistor, V_1 and V_2, from two power supplies applied in turn. The uncertainty on the resistance is S_R, which is fully correlated between the measurements, and the uncertainty on the voltages comprises an uncorrelated statistical components σ_{V_1} and σ_{V_2} and a fully correlated systematic component S_V. First construct the covariance matrix for R, V_1 and V_2. Then calculate the covariance matrix for the currents I_1 and I_2 calculated from the values of R, V_1 and V_2 with Ohms law.
Ans: the covariance matrix is $\mathsf{V} = \mathsf{V}_{\text{stat}} + \mathsf{V}_{\text{syst}}$

$$\mathsf{V}(R, V_1, V_2) = \begin{pmatrix} S_R^2 & 0 & 0 \\ 0 & \sigma_1^2 + S_V^2 & S_V^2 \\ 0 & S_V^2 & \sigma_1^2 + S_V^2 \end{pmatrix}$$

and

$$\mathsf{G} = \begin{pmatrix} \frac{\partial I_1}{\partial R} & \frac{\partial I_1}{\partial V_1} & \frac{\partial I_1}{\partial V_2} \\ \frac{\partial I_2}{\partial R} & \frac{\partial I_2}{\partial V_1} & \frac{\partial I_2}{\partial V_2} \end{pmatrix} = \begin{pmatrix} -V_1/R^2 & 1/R & 0 \\ -V_2/R^2 & 0 & 1/R \end{pmatrix} ,$$

therefore

$$\mathsf{V}(I_1, I_2) = \mathsf{GV}(R, V_1, V_2)\mathsf{G}^{\mathrm{T}} = \frac{1}{R^4} \begin{pmatrix} V_1^2 S_R^2 + (\sigma_1^2 + S_V^2)R^2 & V_1 V_2 S_R^2 + S_V^2 R^2 \\ V_1 V_2 S_R^2 + S_V^2 R^2 & V_2^2 S_R^2 + (\sigma_2^2 + S_V^2)R^2 \end{pmatrix} .$$

9. Show that

$$\langle \left(\frac{d \ln L}{da} \right)^2 \rangle = -\langle \frac{d^2 \ln L}{da^2} \rangle ,$$

where L is a likelihood function dependent upon parameter a.
Ans: In class we showed that

$$\int \frac{d \ln L}{da} L d\vec{\mathbf{x}} = 0$$

which if differentiate again w.r.t. a gives

$$\begin{aligned} \int \left(\frac{d^2 \ln L}{da^2} L + \frac{d \ln L}{da} \frac{dL}{da} \right) d\vec{\mathbf{x}} &= 0 \\ \int \left(\frac{d^2 \ln L}{da^2} L + \left[\frac{d \ln L}{da} \right]^2 L \right) d\vec{\mathbf{x}} &= 0 \\ \langle \frac{d^2 \ln L}{da^2} \rangle + \langle \left(\frac{d \ln L}{da} \right)^2 \rangle = 0 \end{aligned}$$

which is the required result.

10. Ten temperatures are measured in Kelvin, each with an error of 0.2 K: 10.2, 10.4, 9.8, 10.5, 9.9, 9.8, 10.3, 10.1, 10.3, and 9.9. It is suggested that they are all the same true value, differences being due to the measurement errors. Find the number of degrees of freedom and the χ^2. What do you conclude?
How would things be different if the original suggestion were that they are all the same true value of 10.1 K?

Ans: $\overline{T} = 10.12$ K so $\chi^2 = \sum_{i=1}^{10} \left(\frac{T_i - \overline{T}}{0.2}\right)^2 = 14.9$ and number of degrees of freedom is nine. Corresponding probability is 9.4%.
For fixed $T = 10.1 \Rightarrow \chi^2 = \sum_{i=1}^{10} \left(\frac{T_i - 10.1}{0.2}\right)^2 = 15$ and number of degrees of freedom is 10. Corresponding probability is 13.2%.

11. [Barlow 7.5] An experiment studying the decay of protons observes seven events in one year in a sample of 10^6 kg of hydrogen. Give the 90% confidence interval for the number of decays, and thus the half life of the proton, assuming a) there is no background and b) that the expected background from other random processes is three events per year.
Ans: first look up one sided 95% upper and lower limits for seven observed events (i.e Table 36.3 in the Review of Particle Physics) which are 3.29 and 13.15. Therefore, the 90% C.L. is $\{3.29, 13.15\}$. There are 6.022×10^{32} protons in the sample so probability of decay per year is $\{0.55, 2.18\} \times 10^{-32}$ which when inverted gives mean lifetimes of $\{0.46, 1.83\} \times 10^{32}$ years. Multiply by $\ln 2$ to get the half life limit $\{0.32, 1.27\} \times 10^{32}$ years. With background as the mean of two Poisson distributions is $\lambda_1 + \lambda_2$ (see Problem 3) then the limit on signal plus background is the same and you subtract 3 from this to get the limit on the mean signal $\{0.29, 10.15\}$ events which leads to $\{0.41, 14.4\} \times 10^{32}$ limit on the half life.