

# A STUDY OF INFLATIONARY PARADIGM

*A Project Report*

*By*

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## **Abstract**

The main goal of this report is to study the theory of inflation in a general framework and to understand the origin of inhomogeneities in the universe in the inflationary paradigm. We briefly review the hot big bang model and its shortcomings, and discuss how the idea of inflation solves these problems. The formalism and some simple inflationary models are also explored. We then review the linear, cosmological perturbation theory. How inflation transforms microscopic quantum fluctuations into macroscopic seeds for the anisotropy in CMB is discussed and the primordial spectra of scalar and tensor fluctuations are calculated. We also test the results of various inflationary models with the available data.

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# Chapter 1

## Introduction

### 1.1 The hot big bang model

The big bang model is a broadly accepted theory for the structure and evolution of our universe. It postulates that the early universe, which was once very hot and dense, expanded and cooled to its present state. This theory is based on two key ideas: general relativity and the cosmological principle. The cosmological principle is the assumption that the universe is homogeneous and isotropic when averaged over very large scales. This has been confirmed by observations such as the Sloan Digital Sky Survey (SDSS) [1]. The idea of an expanding universe is based on the observations by Edwin Hubble. He observed that the spectra of the galaxies around us appear to be shifted towards longer wavelengths. The farther away they are, the larger is the redshift. This proved that the galaxies are moving away from each other. According to general relativity, this is understood as the expansion of space itself, not a motion of galaxies in space. Moreover, this model predicts the presence of a relic *Cosmic Microwave Background* (CMB) radiation with a temperature of around 2.7 Kelvin. The CMB has been found to be highly isotropic and perfectly thermal by observations such as those made by the Wilkinson Microwave Anisotropy Probe (WMAP) [2] and the Planck missions [3]. The discovery of the cosmic microwave background radiation together with the observed Hubble expansion of the universe has established hot big bang cosmology as a viable model of the universe.

## 1.2 Inflation

The hot big bang model, though rather successful, has some drawbacks such as the horizon problem and the flatness problem. These drawbacks are usually overcome by introducing an epoch of *inflation* - which refers to a brief period of accelerated expansion during the very early stages of the radiation-dominated epoch [4]. Even though the inflationary paradigm solves the puzzles of the big bang model in a simple way, the most attractive part of this scenario is its ability to provide a causal mechanism to explain the primordial perturbations.

## 1.3 Anisotropies and linear perturbation theory

Though fairly isotropic, the background contains small anisotropies of about 10 parts per million [5]. The small quantum fluctuations that had originated at the beginning of the inflationary epoch were amplified to form classical perturbations, which grew via gravitational instability into the large-scale structures (LSS). The primordial perturbations are detected today as anisotropies in the CMB and these observations help us to understand their characteristics.

Since the deviations from homogeneity are small, linear perturbation theory can be used to study these perturbations. We can split all quantities into a spatially independent homogeneous background and a spatially dependent perturbative part. At the linear order, we can classify the perturbations as scalars, vectors and tensors. The evolution of the perturbations is governed by the first-order Einstein's equations. Scalar perturbations are mainly responsible for the anisotropies in the CMB and consequently the large scale structures in the universe. The tensor perturbations generate gravitational waves, which can exist even when no sources are present.

**The report is organized as follows:** We will briefly discuss the Friedmann-Lemaître-Robertson-Walker (FLRW) model of cosmology in chapter 2. The major drawbacks of this model are also discussed in that chapter. Solving these drawbacks using the idea of inflation is discussed in the chapter 3. We will also discuss the formalism and as an illustration we will discuss some inflationary models in that chapter. In chapter 4 the linear perturbation theory is described. We will work in a particular gauge for this treatment. The generation of primordial perturbations and their evolution during the inflationary period is described in chapter 5. We will

also discuss the calculation of power spectrum in this chapter. The power spectrum, the spectral index and the tensor-to-scalar ratio can be compared with the observations of CMB which is described in chapter 6. We will summarize the report in the chapter 7.

The discussions in chapters 3, 4 and 5 are mainly based on the review article ‘An introduction to inflation and cosmological perturbation theory’ by L. Sriramkumar [6] unless otherwise cited. We follow the relations, calculations and notations as used in this article. All the relations taken from the article are verified. In addition, following the methods given in the article we discuss the small field models and some of the derivations of relevant expressions in section 3.4.2. In the section 5.3.2, the Starobinsky model is discussed and the derived expressions are independently verified and confirmed with the literature.

**Notations:** Throughout we will use the natural units

$$c = \hbar = 1. \quad (1.1)$$

We use the reduced Planck mass

$$M_{\text{P}} = (8\pi G)^{-1/2}. \quad (1.2)$$

Our metric signature is  $(+, -, -, -)$ . Greek indices will take the values  $\mu, \nu = 0, 1, 2, 3$  and Latin indices stand for  $i, j = 1, 2, 3$ . Derivatives with respect to physical time ( $t$ ) are denoted by overdots ( $\dot{\phantom{x}}$ ), while derivatives with respect to conformal time ( $\eta$ ) are indicated by primes ( $\prime$ ).



# Chapter 2

## Friedmann-Lemaître-Robertson-Walker (FLRW) model

In this chapter we apply the general theory of relativity to the study of cosmology and the evolution of the universe. Incorporating the assumptions of homogeneity and isotropy of the space lead to the *Friedmann-Lemaître-Robertson-Walker* (FLRW) cosmological model. We shall briefly discuss this model and its drawbacks.

### 2.1 The FLRW metric

As discussed in the introduction, observations suggest that our universe is homogeneous and isotropic on large scales; that is, the geometrical properties of the three-dimensional space are the same at all spatial locations and do not single out any special direction in space. The most general metric satisfying these symmetries is the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, which can be put in the following form [7]:

$$ds^2 = dt^2 - a(t)^2 \left[ \frac{dr^2}{1 - \kappa r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (2.1)$$

where  $\kappa$  can be chosen to be  $+1$ ,  $-1$ , or  $0$  for spaces of constant positive (closed), negative (open), or zero (flat) spatial curvature respectively and  $a$  is the scale factor,

which, in general, can be a function of time. We will refer to this metric as Friedmann metric throughout this thesis. However, these geometrical considerations will not allow us to find the curvature and the scale factor. They have to be determined from the Einstein's equations once the matter distribution is specified. It is useful to define the *conformal time*, which is

$$\eta = \int \frac{dt}{a(t)}. \quad (2.2)$$

In terms of conformal time, the metric becomes

$$ds^2 = a^2(\eta) \left[ d\eta^2 - \left( \frac{dr^2}{1 - \kappa r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right) \right]. \quad (2.3)$$

## 2.2 Dynamics of the Friedmann model

The unknown scale factor  $a(t)$  and the curvature constant  $\kappa$  contained in the Friedmann metric can be determined via Einstein's equations

$$G^\mu{}_\nu = R^\mu{}_\nu - \frac{1}{2} \delta^\mu{}_\nu R = 8\pi G T^\mu{}_\nu \quad (2.4)$$

where  $R^\mu{}_\nu$  and  $R$  are the Ricci tensor and Ricci scalar, which are determined from the metric and  $T^\mu{}_\nu$  is the energy-momentum tensor for the source. The assumption of isotropy and homogeneity implies that  $T^\mu{}_\nu$  is diagonal and the spatial components are equal. It is conventional to write it as the  $T^\mu{}_\nu$  for perfect fluid, which is

$$T^\mu{}_\nu = \text{diag}[\rho(t), -p(t), -p(t), -p(t)], \quad (2.5)$$

where  $\rho$  is energy density and  $p$  is pressure. Thus, from Eq. (2.4), we find:

$$H^2 + \frac{\kappa}{a^2} = \frac{8\pi G}{3} \rho, \quad (2.6a)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p), \quad (2.6b)$$

where  $H$  is the *Hubble parameter* and is defined as  $H = (\dot{a}/a)$ . These are called the *Friedmann equations*.

The evolution of the scale factor can be determined from the Einstein's equations if the equation of state is specified. From equations (2.6a) and (2.6b) we get

$$\frac{d(\rho a^3)}{da} = -3a^2 p, \quad (2.7)$$

which can also be determined from the law of conservation of energy. Given the equation of state  $p = p(\rho)$ , we can integrate Eq. (2.7) to obtain  $\rho = \rho(a)$ . Substituting this relation into Eq. (2.6a) we can determine  $a(t)$ .

Consider an equation of state of the form  $p = w\rho$  with a constant  $w$ . For the different values of  $w = 0, 1/3$  and  $-1$  we get the equations of state of matter, radiation and vacuum energy respectively. Generally Eq. (2.7) gives  $\rho \propto a^{-3(1+w)}$ ; in particular, for non-relativistic matter and radiation we get  $\rho_m \propto a^{-3}$  and  $\rho_R \propto a^{-4}$ . For  $w = -1$ ,  $\rho$  remains constant with time. For  $\kappa = 0$ , we can integrate the Friedmann equation and get

$$\begin{aligned} a(t) &\propto t^{\frac{2}{3(1+w)}} \quad (\text{for } w \neq -1) \\ &\propto \exp(\lambda t) \quad (\text{for } w = -1) \end{aligned} \quad (2.8)$$

where  $\lambda$  is a constant. For non-relativistic matter,  $a \propto t^{(2/3)}$ ; for radiation,  $a \propto t^{(1/2)}$ ; and for vacuum energy,  $a \propto \exp(\lambda t)$ .

It is useful to define critical density  $\rho_c$  and a density parameter  $\Omega(t)$  by

$$\rho_c \equiv \frac{3H^2(t)}{8\pi G}, \quad \Omega = \frac{\rho}{\rho_c}. \quad (2.9)$$

Then Eq.(2.6a) becomes

$$\frac{\kappa}{a(t)^2 H(t)^2} = \Omega(t) - 1 \equiv \Omega_\kappa \quad (2.10)$$

According to the standard big bang model, the universe has evolved through various epochs. It started as a hot primordial soup of relativistic particles and radiation, which is called the radiation-dominated epoch. As we have seen above, the radiation density falls faster than the matter density. So there was a time when these densities became equal, and after that the universe became matter-dominated. As the universe cools due to the expansion, the matter ceased to interact with radiation and hence radiation decoupled from the matter and started free streaming. This radiation

cooled down as universe expanded, and today we receive this as the cosmic microwave background (CMB) radiation at a temperature of around 2.7K. The Fig. 2.1 shows the various epochs of the universe according to the standard model along with the theory of inflation which will be discussed later.

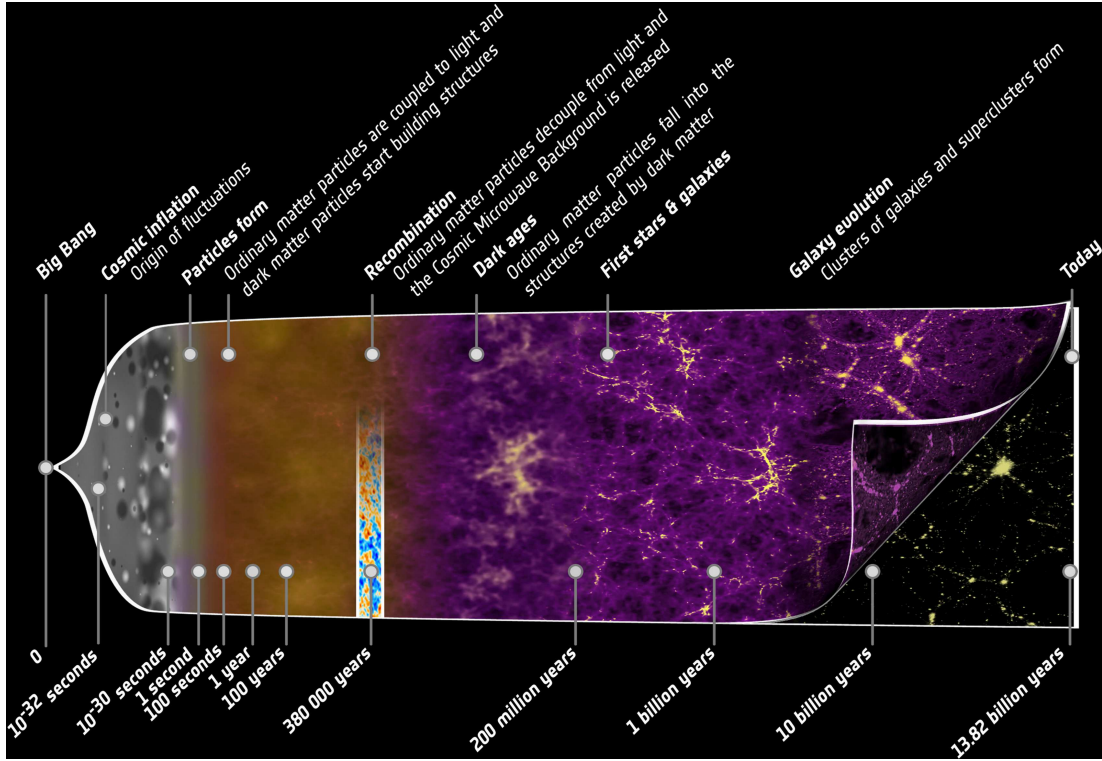


FIGURE 2.1: A schematic diagram depicting the evolution of the universe [3].  
Source: [http://www.esa.int/spaceinimages/Images/2013/03/Planck\\_history\\_of\\_Universe\\_zoom](http://www.esa.int/spaceinimages/Images/2013/03/Planck_history_of_Universe_zoom)

## 2.3 Problems of hot big bang model

### 2.3.1 Flatness problem

For a universe dominated by a fluid with equation of state  $p = w\rho$ , the term  $(aH)^{-1}$  (this term is sometimes called ‘comoving Hubble radius’) evolves as

$$(aH)^{-1} \propto a^{\frac{1}{2}(1+3w)}. \quad (2.11)$$

This means that as we go back in time, this term decreases for matter which has the constraint  $1 + 3w > 0$ . Interestingly,  $\Omega_\kappa$  is now observed to be smaller than  $10^{-2}$ . Then from Eq.(2.10), we see that at much earlier times the density parameter must have been extremely close to 1.

Why was  $\Omega_\kappa$  so small? One possibility is that the universe started with  $\Omega = 1$ . But there is no point to have a firm believe that the universe should choose such a precise state initially, but it is nevertheless a possibility. Another possibility is that at an earlier epoch, the universe was dominated by some unknown matter, so that  $aH$  was an increasing function of time. The inflation paradigm provides this possibility in a very natural way.

Since our universe is found to be extremely flat, here onwards I will use the spatially flat metric

$$ds^2 = dt^2 - a^2(t)dx^2 = a^2(\eta)(d\eta^2 - dx^2). \quad (2.12)$$

### 2.3.2 Horizon problem

Due to the finite speed of light, there exists a *horizon* beyond which we cannot see. We are able to see only a finite part of the universe from which light can reach us within time  $t_0$ . We can define such a horizon for any time  $t$ . Quantitatively, the horizon is [6]

$$h(t) = a(t) \int_0^t \frac{d\tilde{t}}{a(\tilde{t})}. \quad (2.13)$$

The hot big bang model suggests that the universe was dominated by non-relativistic matter from the time of decoupling  $t_d$  until now  $t_0$ , and before that it was radiation-dominated. From observations it is found that  $t_d \simeq 10^5$  years and  $t_0 \simeq 10^{10}$  years. For a matter-dominated universe, the physical size of the universe at the time of decoupling is

$$\ell_b(t_0, t_d) = a_d \int_{t_d}^{t_0} \frac{d\tilde{t}}{a(\tilde{t})}, \quad (2.14)$$

where  $a_d$  is the scale factor at decoupling and subscript  $b$  denotes backward light cone. Using the relation (2.8) and the fact that  $t_0 \gg t_d$ , we get

$$\ell_b \simeq 3(t_d^2 t_0)^{1/3}. \quad (2.15)$$

Similarly using Eq. (2.13), the physical size of the horizon for the radiation-dominated universe at decoupling is calculated to be

$$\ell_f \simeq 2t_d, \quad (2.16)$$

where  $f$  denotes forward light cone. If we calculate the ratio of the physical distances of the forward and backward light cones at decoupling, in this model we get the ratio to be 70 [6]. This implies that most of the regions in the last scattering surface are not causally connected. Therefore, these regions have never been able to exchange information, for example, about their temperature. But observations of the CMB tells us that the universe is extremely homogeneous at the last scattering surface with relative fluctuations of only  $(\Delta T/T) \sim 10^{-5}$ . This is the *horizon problem*.

### 2.3.3 Inhomogeneities

In the Friedmann model, we have assumed that at large scales the universe is homogeneous. But at small scales we see stars, galaxies, star clusters etc. Moreover in the CMB, a small temperature variation of the order of  $(\Delta T/T) \sim 10^{-5}$  is found. The hot big bang model cannot explain the origin and evolution of these inhomogeneities in the universe.

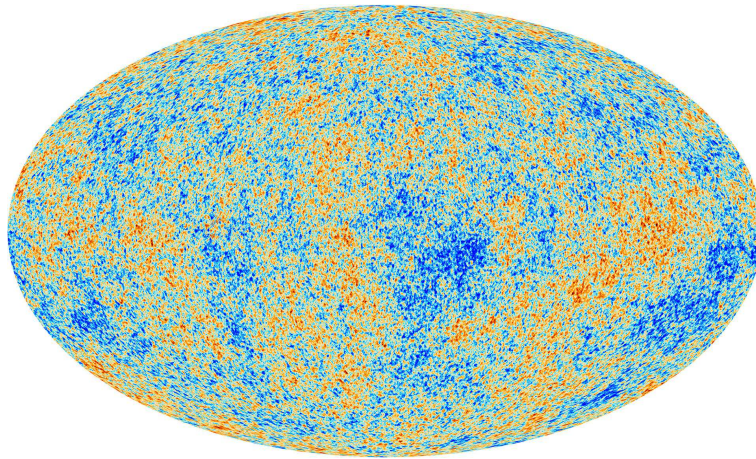


FIGURE 2.2: The anisotropies of the Cosmic microwave background (CMB) as observed by Planck. Blue spots represent directions on the sky where the CMB temperature is  $\sim 10^5$  below the mean,  $T_0 = 2.7K$ . Yellow and red indicate hot regions [3]. Source: [http://www.esa.int/spaceinimages/Images/2013/04/Planck\\_CMB\\_black\\_background](http://www.esa.int/spaceinimages/Images/2013/04/Planck_CMB_black_background)

# Chapter 3

## Inflation

As we have discussed in the previous chapter, the major drawbacks of standard big bang theory are the horizon problem and the flatness problem. In this chapter we will discuss how inflation can solve these issues. The formalism and some simple inflationary models are also described in this chapter.

### 3.1 Why do we need inflation?

In order to solve the flatness problem, we need an epoch of the universe in which  $(aH)^{-1}$  decreases with time. That is,

$$\frac{d}{dt}(aH)^{-1} < 0. \quad (3.1)$$

This implies that the universe must be accelerating during that epoch, i.e  $\ddot{a} > 0$ . From Friedmann equation (2.6b), we get

$$\ddot{a} > 0 \Rightarrow \rho + 3p < 0 \Rightarrow 1 + 3w < 0 \Rightarrow w < -1/3 \text{ if } \rho > 0 \quad (3.2)$$

in that epoch. Let us consider the physical distance  $\lambda_p$  between two points in the CMB which are causally connected now. But in standard cosmology, using Eq. (2.8) and Eq.(2.2), the scale factor goes as

$$a \sim \eta^{2/(1+3w)}. \quad (3.3)$$

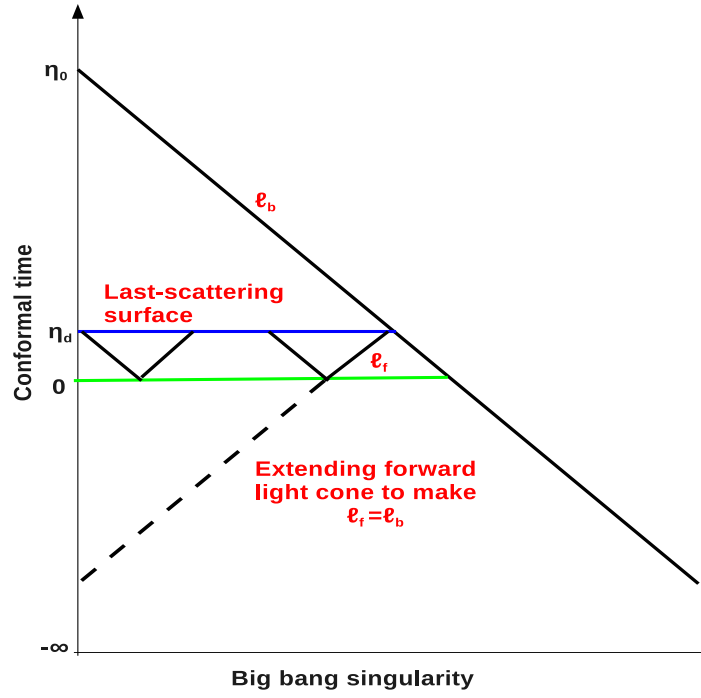


FIGURE 3.1: How inflation solves the horizon problem: the forward lightcone ( $\ell_f$ ) and the backward lightcone ( $\ell_b$ ) can be made equal by extending the conformal time to negative values.

This implies that there exists a singularity at  $\eta = 0$ . This is the source of the horizon problem. But if we have a phase of  $w < -1/3$ , then the singularity can be pushed further back and more region can be included inside the backward light cone at the time of decoupling. In general  $\eta$  can be extended to negative times so that the horizon can be made much larger than that of standard cosmology. In this way, as we go back in time, an epoch can be reached so that the  $\lambda_p$  comes inside the horizon.

In order to solve the horizon problem it is useful to know the behavior of the horizon and the *Hubble radius*  $d_H \equiv 1/H$ . Consider a scale factor of the form

$$a = a_0 t^n, \quad (3.4)$$

with  $n > 0$ . Using Eq. (2.13), the horizon size can be calculated to be

$$h(t) = \lim_{t' \rightarrow 0} \frac{t^n}{1-n} (t^{1-n} - t'^{1-n}). \quad (3.5)$$

The behavior of the scale factor is different during the period of inflation and that during radiation or matter domination. During inflation, the condition  $\ddot{a} > 0$  implies



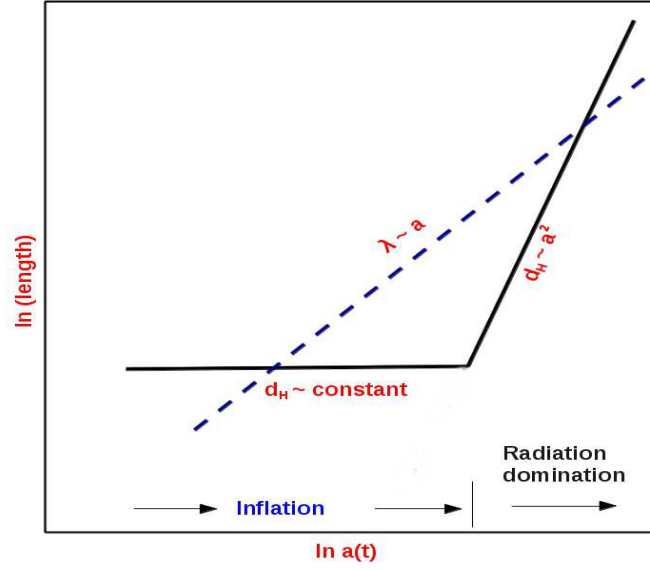


FIGURE 3.2: All scales that are relevant to cosmological observations today were larger than Hubble radius. During inflation, these lengths were brought inside the Hubble radius so that they could be causally connected.

that  $n > 1$  and during both radiation domination and matter domination,  $n < 1$ . For  $n > 1$  when  $t'$  becomes very small and the horizon size blows up. Then the horizon size is always larger than the Hubble radius  $d_H$ . For  $n < 1$ , the horizon becomes

$$h = \frac{t}{1-n} \sim d_H \quad (n < 1). \quad (3.6)$$

So during matter domination and radiation domination the Hubble radius and the horizon are often used interchangeably.

Let us consider  $\lambda_p$  to be the physical distance between two points in the CMB which have the same temperature. According to standard cosmology this length enters the horizon ( $\approx d_H$  in big bang model) either during the radiation or the matter dominated epochs, and are outside the Hubble radius at earlier times. Since the points are at the same temperature, the physical length  $\lambda_p$  should be brought inside the horizon in the very early stages of the universe so that they are causally connected. This means that  $\lambda_p < h$ , which is always satisfied when  $\lambda_p < d_H$ . To achieve that we need an epoch in the early universe during which Hubble radius increases faster than the  $\lambda_p$

as we go back in time. This condition is satisfied when

$$\frac{d}{dt} \frac{d_H}{\lambda_p} < 0. \quad (3.7)$$

The  $\lambda_p \propto a$  and  $d_H = a/\dot{a}$  implies that  $\ddot{a} > 0$ , i.e., it must be accelerated expansion. Thus, a period of adequate inflation, the horizon problem is solved.

We have discussed so far about how we resolve the horizon problem using the idea of inflation. But how much inflation do we need to resolve this problem? At least we have to make the backward light cone and the forward light cone equal at the time of decoupling ( $t_d$ ). For simplicity, let us consider, during inflation an exponential expansion of scale factor which starts from  $a_i$  at  $t_i$  and ends at  $a_f$  at  $t_f$ ,

$$a = a_i \exp [H(t - t_i)]. \quad (3.8)$$

Denote  $A = a_f/a_i$  and assume that  $A \gg 1$ , then using Eq.(2.13), the horizon is evaluated to be

$$\ell_I(t_d, 0) \approx \frac{A}{H} \left( \frac{t_d}{t_f} \right)^{(1/2)}. \quad (3.9)$$

Most of the contribution to the forward light cone is from inflation. Then using Eq. (2.15), the ratio of light cones becomes

$$R_I = \frac{\ell_I}{\ell_b} \simeq \frac{A}{10^{26}}, \quad (3.10)$$

where we have used  $H \simeq 10^{13}$  GeV [6]. This number tells us that we need at least  $A = 10^{26}$  to overcome the horizon problem. Usually, the extent of inflation from  $a_i$  to some  $a$  is expressed in terms of number of e-folds, which is defined as

$$N = \ln \left( \frac{a(t)}{a_i} \right). \quad (3.11)$$

In terms of  $N$ , we need around 60 e-folds of inflation to solve the horizon problem.

## 3.2 How do we achieve inflation?

We have seen that for an accelerated expansion we need a phase of ‘matter’ with  $w < -(1/3)$ . Usually this can be achieved using scalar fields. The simplest model

involves a single scalar field  $\phi$  (the quanta of scalar fields that drives inflation is called inflaton). The action of this canonical scalar field with self- interaction  $V(\phi)$  can be written as

$$S[\phi] = \int d^4x \sqrt{-g} \left[ \left( \frac{1}{2} \right) (\partial_\mu \phi \partial^\mu \phi) - V(\phi) \right]. \quad (3.12)$$

The energy-momentum tensor for the scalar field is

$$T^\mu{}_\nu = \partial^\mu \phi \partial_\nu \phi - \delta^\mu_\nu \left[ \left( \frac{1}{2} \right) (\partial_\alpha \phi \partial^\alpha \phi) - V(\phi) \right]. \quad (3.13)$$

The homogeneity and isotropy of the background universe restricts  $\phi$  to be only time-dependent and hence the energy-momentum tensor becomes diagonal

$$T^0{}_0 = \rho = \left[ \frac{\dot{\phi}^2}{2} + V(\phi) \right], \quad (3.14a)$$

$$T^i{}_j = -p \delta^i_j = - \left[ \frac{\dot{\phi}^2}{2} - V(\phi) \right] \delta^i_j. \quad (3.14b)$$

The resulting equation of state is

$$w = \frac{p}{\rho} = \left( \frac{\dot{\phi}^2}{2} - V(\phi) \right) / \left( \frac{\dot{\phi}^2}{2} + V(\phi) \right), \quad (3.15)$$

which shows that  $w$  of the scalar field can be made as  $w = -1 < -1/3$  if the potential energy dominates over the kinetic energy, i.e we can achieve inflation using a scalar field if

$$\dot{\phi}^2 \ll V(\phi) \quad (3.16)$$

By varying the action, Eq. (3.12) we get the equation of motion of the scalar field as

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = 0. \quad (3.17)$$

The background geometry is determined by the Friedmann equations (2.6a and 2.6b) which for scalar field can be written as

$$H^2 = \left( \frac{1}{3M_p^2} \right) \left[ \frac{\dot{\phi}^2}{2} + V(\phi) \right], \quad (3.18a)$$

$$\dot{H} = - \left( \frac{1}{2M_p^2} \right) \dot{\phi}^2, \quad (3.18b)$$

where  $M_P = \sqrt{1/(8\pi G)}$  is the Planck mass.

### 3.3 Slow roll approximation

We have seen that inflation is guaranteed if we have the condition (3.16). Moreover, the equation of motion (3.17) is the same as one of a particle rolling down a potential subjected to a friction through the term  $H\dot{\phi}$ . Suppose we have a condition

$$\ddot{\phi} \ll (3H\dot{\phi}). \quad (3.19)$$

This ensures that the field is slowly rolling for a sufficiently long time. These two conditions can be expressed in terms of dimensionless parameters, called *slow roll parameters* which are described below.

The Friedmann equation (2.6b) can be rewritten as

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_P^2}(\rho + 3p) = H^2(1 - \epsilon_H), \quad (3.20)$$

where

$$\epsilon_H \equiv \frac{3}{2}(w + 1) = -\left(\frac{\dot{H}}{H^2}\right) = \frac{1}{2M_P^2} \left(\frac{\dot{\phi}}{H}\right)^2, \quad (3.21)$$

is the first *Hubble slow roll parameter*. It is clear from Eq. (3.20) that the accelerated expansion occurs if  $\epsilon_H < 1$ . Moreover from Eq.(3.21),  $\epsilon_H \rightarrow 0$  corresponds to  $w \rightarrow -1$  which is the condition for inflation, i.e., Eq (3.16).

Define second slow roll parameter as

$$\delta_H \equiv -\left(\frac{\ddot{\phi}}{H\dot{\phi}}\right) = \epsilon_H - \left(\frac{\dot{\epsilon}_H}{2H\epsilon_H}\right), \quad (3.22)$$

then the condition (3.19) corresponds to  $\delta_H \rightarrow 0$ . So we have two Hubble slow roll parameters:

$$\epsilon_H = -\left(\frac{\dot{H}}{H^2}\right) \ll 1, \quad (3.23a)$$

$$\delta_H = -\left(\frac{\ddot{\phi}}{H\dot{\phi}}\right) \ll 1. \quad (3.23b)$$

The equations (3.17) and (3.18a) can be rewritten in terms of slow roll parameters as

$$H^2 \left[ 1 - \left( \frac{\epsilon_H}{3} \right) \right] = \frac{V}{3M_P^2} \quad (3.24a)$$

$$(3H\dot{\phi}) \left[ 1 - \left( \frac{\delta_H}{3} \right) \right] = -V_\phi. \quad (3.24b)$$

In the slow roll approximation, we have the solutions

$$H^2 \simeq \left( \frac{V}{3M_P^2} \right) \quad \text{and} \quad (3H\dot{\phi}) \simeq -V_\phi. \quad (3.24c)$$

Once the slow roll conditions are satisfied, using Eq. (3.24c) we can rewrite the set of slow roll parameters in terms of potentials, so called *potential slow roll parameters*, as

$$\epsilon_H \approx \left( \frac{M_P^2}{2} \right) \left( \frac{V_\phi}{V} \right)^2 \equiv \epsilon_V \quad (3.25a)$$

$$\delta_H \approx M_P^2 \left( \frac{V_{\phi\phi}}{V} \right) - \left( \frac{M_P^2}{2} \right) \left( \frac{V_\phi}{V} \right)^2 \equiv \eta_V - \epsilon_V. \quad (3.25b)$$

where  $V_{\phi\phi} = (d^2V/d\phi^2)$ . In the slow roll limit, using Eq. (3.24c) we can express the number of e-folds from  $\phi_i(t_i)$  to  $\phi(t)$  as

$$N = \ln \left( \frac{a}{a_i} \right) = \int_{t_i}^t dt H \simeq - \left( \frac{1}{M_P^2} \right) \int_{\phi_i}^{\phi} d\phi \left( \frac{V}{V_\phi} \right). \quad (3.26)$$

Inflation ends when the condition (3.16) is violated, i.e.,  $\epsilon_H = 1$ . In slow roll approximation this condition is equivalent to  $\epsilon_V \approx 1$ .

## 3.4 Simple models

### 3.4.1 Power law inflation

If the form of the potential  $V(\phi)$  is given, from Friedmann equations (3.18a and 3.18b) and from equation of motion of the scalar field (3.17), one can solve for scale

factor  $a$  and scalar field  $\phi$ . Consider a scale factor of the form:

$$a(t) = a_1 t^q. \quad (3.27)$$

with  $q > 1$  and  $a_1$  being an arbitrary constant. This scenario in which scale factor has a power law dependence on time is referred to as *power law inflation*. Given the functional form of  $a(t)$ , from equations (3.18b) and (3.18a) the potential governing scalar field can be found as follows

$$V(t) = M_p^2 \left( 3H^2 + \dot{H} \right), \quad (3.28a)$$

$$\phi(t) = \sqrt{2} M_p \int dt \sqrt{-\dot{H}}. \quad (3.28b)$$

Now we can substitute  $a(t)$  in Eq. (3.28a) to find out the potential, then we get

$$V(\phi) = V_0 \exp - \left[ \sqrt{\frac{2}{q}} \left( \frac{\phi}{M_p} \right) \right]. \quad (3.29)$$

Similarly substituting  $a(t)$  in the Eq. (3.28b) we get

$$\left( \frac{\phi(t)}{M_p} \right) = \sqrt{2q} \ln \left[ \sqrt{\left( \frac{V_0}{(3q-1)q} \right)} \left( \frac{t}{M_p} \right) \right] \quad (3.30)$$

### 3.4.2 Large and small field models

We have seen how to solve the inflationary models exactly in the case of power law inflation. In this section we will consider two types of inflationary models and its properties in the slow roll regime. (i) Consider the potential of the form

$$V(\phi) = V_0 \phi^n \quad (3.31)$$

where  $V_0$  is a constant and  $n > 0$ . The potential slow roll parameter can be calculated as

$$\epsilon_V = \frac{n^2}{2} \left( \frac{M_P}{\phi} \right)^2, \quad (3.32)$$

which implies that when  $\epsilon_V \ll 1$ ,  $\phi \gg M_P$ . Thus inflation takes place at large values of fields. This category of potentials are called large field models. In the slow roll

limit, using Eq. (3.24c), the scalar field can be calculated to be

$$\phi^{[(4-n)/2]}(t) \simeq \phi_i^{[(4-n)/2]} + \sqrt{\frac{V_0}{3}} \left[ \frac{n(n-4)}{2} \right] M_P(t-t_i) \quad \text{for } n \neq 4, \quad (3.33a)$$

$$\phi(t) \simeq \phi_i \exp - \left[ \sqrt{(V_0/3)}(4M_P)(t-t_i) \right] \quad \text{for } n = 4. \quad (3.33b)$$

where  $\phi_i$  is the scalar field at some initial time  $t_i$ . The scale factor in terms of scalar field is found to be

$$a(t) \simeq a_i \exp - \left[ \left( \frac{1}{2n M_P^2} \right) (\phi^2(t) - \phi_i^2) \right] \quad (3.34)$$

where  $a_i$  is the value of the scale factor at  $t_i$ . Using the Eq.(3.26), we can express the scalar field and the Hubble parameter in terms of e-folds as

$$\phi^2 \simeq [\phi_i^2 - (2M_P^2 n) N], \quad (3.35a)$$

$$H^2(N) \simeq \left( \frac{V_0 M_P^{(n-2)}}{3} \right) \left[ \left( \frac{\phi_i}{M_P} \right)^2 - (2nN) \right]^{(n/2)}. \quad (3.35b)$$

(ii) Consider another kind of potential

$$V(\phi) = \Lambda [1 + \cos(\phi/f)], \quad (3.36)$$

where  $\Lambda$  and  $f$  are constants. With certain values of  $\Lambda$  and  $f$  this potential naturally leads to inflation for small field values (i.e.,  $\phi \ll M_P$ ) [8]. So this type of potentials are called small field models. As we have done for the case of large field model, the scalar field and the Hubble parameter are found to be

$$(\phi/f) \simeq \arccos \left[ 1 - 2 \exp \left( \frac{N M_P^2}{f^2} \right) \sin^2 (\phi_i/2f) \right], \quad (3.37a)$$

$$H^2 \simeq \frac{2\Lambda}{3M_P^2} \left[ 1 - \exp \left( \frac{N M_P^2}{f^2} \right) \sin^2 (\phi_i/2f) \right]. \quad (3.37b)$$

# Chapter 4

## Linear, cosmological perturbation theory

So far we have discussed the universe which is homogeneous and isotropic. As I mentioned in the introduction, we see that the large scale structures are distributed inhomogeneously in the universe which are evolved from the tiny fluctuations of the CMB. Cosmological perturbation theory studies the evolution of the gravitational perturbations at the cosmological scales. In this thesis we are only interested in the fluctuations of the CMB which are observed to be very small. Hence we will work with linear, cosmological perturbation theory. In this chapter we will describe the general idea of this theory such as classification of perturbations, the equations governing these perturbations and the generation of these perturbations during inflation. In this discussion, we only consider spatially flat backgrounds.

### 4.1 Linear, cosmological perturbation theory

Since the inhomogeneities in the CMB are observed to be very small, those at earlier times must have been even smaller. Then expanding the Einstein's equations at linear order in perturbations approximates the full non-linear solution to very high accuracy. So we will work with linear perturbation theory. This means that we can write all quantities as a sum of background value and linear order term. We can drop all terms which are higher order in the perturbation. Then the perturbed Einstein's



equations can be written as [9]

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu}, \quad (4.1)$$

which is the linear differential equation of the perturbed metric  $\delta g_{\mu\nu}$  in the linear theory.

### 4.1.1 Classifications of perturbations

#### 4.1.1.1 Scalar-vector-tensor decomposition

Consider the perturbed metric  $\delta g_{\mu\nu}$  in the flat Friedmann universe which can be split as

$$\delta g_{\mu\nu} = (\delta g_{00}, \delta g_{0i}, \delta g_{ij}). \quad (4.2)$$

These three components are scalar ( $\delta g_{00}$ , say  $\mathcal{A}$ ), vector ( $\delta g_{0i}$ ) and tensor ( $\delta g_{ij}$ ) which are defined according to rotations in the flat Friedmann background. As we know, we can split any 3-vector into the gradient of a scalar and a divergence free vector. Hence we can write  $\delta g_{0i}$  as

$$\delta g_{0i} = \nabla_i \mathcal{B} + \mathcal{S}_i, \quad (4.3)$$

where  $\mathcal{B}$  is a scalar and  $\nabla_i \mathcal{S}^i = 0$ . Similarly, any symmetric tensor can be decomposed as [6]

$$\delta g_{ij} = \psi \delta_{ij} + \left[ \left( \frac{1}{2} \right) (\nabla_i \nabla_j + \nabla_j \nabla_i) - \left( \frac{1}{3} \right) \delta_{ij} \nabla^2 \right] \mathcal{E} + (\nabla_i \mathcal{F}_j + \nabla_j \mathcal{F}_i) + \mathcal{H}_{ij}, \quad (4.4)$$

where  $\psi$  and  $\mathcal{E}$  are scalars,  $\nabla_i \mathcal{F}^i = 0$  and  $\mathcal{H}_{ij}$  is a symmetric traceless tensor which satisfies  $\nabla_i \mathcal{H}^{ij} = 0$ . For the metric perturbation,  $\delta g_{\mu\nu}$  has 10 components. So there would appear to be ten degrees of freedom (dof). These degrees of freedom are distributed as

- scalars:  $\mathcal{A}, \mathcal{B}, \mathcal{E}$  and  $\psi \Rightarrow 4$  dof,
- vectors:  $\mathcal{F}^i, \mathcal{S}_i$  + divergence free  $\Rightarrow 4$  dof,
- tensors:  $\mathcal{H}_{ij}$  + transverse + symmetry + traceless  $\Rightarrow 2$  dof,

in metric decomposition. However, four of them are not physical degrees of freedom, they just correspond to the freedom of choosing the four coordinates  $(x, t)$ . So there

are 6 physical degrees of freedom. This decomposition is very useful because in the perturbed Einstein's equations for scalars, vectors and tensors do not couple to each other at linear order and can therefore be treated separately.

#### 4.1.1.2 Gauge fixing

The homogeneity and isotropy of the background Friedmann universe specifies a particular observer called comoving observer. This means that the symmetry properties of the background universe fix its coordinate system. But in the case of perturbed spacetime there are no such specific coordinate system. On the other hand, for a given coordinate system in the background, there are many possible coordinate systems (called gauge) in the perturbed spacetime that we could use. There are two ways of dealing with this issue. One way to do this is to define perturbations in such a way that they do not change under a change of gauges (gauge invariant approach) [10]. The second one, which will be our approach, is to choose a particular gauge and do all calculations there.

### 4.1.2 Scalar perturbations

There are different gauges such as comoving gauge, synchronous gauge etc. are used to study the perturbations. Different gauges are good for different purposes. A convenient gauge which we will use, is known as the *longitudinal gauge* (also called the conformal Newtonian gauge). Fixing this gauge corresponds to choosing  $\mathcal{B} = \mathcal{E} = 0$ . In this gauge the Friedmann metric will be

$$ds^2 = (1 + 2\Phi) dt^2 - a^2(t) (1 - 2\Psi) dx^2, \quad (4.5)$$

where  $\Phi(\propto \mathcal{A})$  and  $\Psi(\propto \psi)$  are the two independent functions which are called *Bardeen potentials*. This metric is similar to the weak limit of the general theory, hence the name conformal Newtonian gauge. For the above metric, the perturbed

Einstein tensor  $\delta G^\mu_\nu$  can be calculated to be

$$\delta G^0_0 = -6H \left( \dot{\Psi} + H\Phi \right) + \left( \frac{2}{a^2} \right) \nabla^2 \Psi \quad (4.6a)$$

$$\delta G^0_i = 2\nabla_i \left( \dot{\Psi} + H\Phi \right), \quad (4.6b)$$

$$\delta G^j_i = -2 \left[ \ddot{\Psi} + H \left( 3\Psi + \dot{\Phi} \right) + \left( 2\dot{H} + 3H^2 \right) \Phi + \left( \frac{1}{2a^2} \right) \nabla^2 \mathcal{D} \right] \delta^i_j + \left( \frac{1}{a^2} \right) \nabla^i \nabla_j \mathcal{D}. \quad (4.6c)$$

where  $\mathcal{D} = (\Phi - \Psi)$ .

#### 4.1.2.1 Equations of perturbations

The equations of motion of perturbations can be derived from the perturbed Einsteins equations,  $\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu}$ . Let us consider a perfect fluid which does not contain any anisotropic stress. We shall see that the scalar field, which we use as a source of inflation, does not contain any anisotropic stress. In this assumption, the perturbed stress-energy tensor can be expressed as

$$\delta T^0_0 = \delta\rho, \quad \delta T^0_i = \nabla_i \delta\sigma \quad \text{and} \quad \delta T^i_j = -\delta p \delta^i_j \quad (4.7)$$

where the quantities  $\delta\rho$ ,  $\delta\sigma$  and  $\delta p$  are the perturbed energy density, momentum flux and pressure respectively. Since  $\delta T^i_j = 0$  for  $i \neq j$ , Eq. (4.6c) becomes

$$\nabla^i \nabla_j (\Phi - \Psi) = 0 \quad (i \neq j), \quad (4.8)$$

which implies that  $\Phi = \Psi$ . Substituting Eq. (4.7) into Eq. (4.6) we get the equations for the scalar perturbations:

$$-3H \left( \dot{\Phi} + H\Phi \right) + \left( \frac{1}{a^2} \right) \nabla^2 \Phi = (4\pi G) \delta\rho, \quad (4.9a)$$

$$\nabla_i \left( \dot{\Phi} + H\Phi \right) = (4\pi G) \nabla_i \delta\sigma, \quad (4.9b)$$

$$\ddot{\Phi} + 4H\dot{\Phi} + \left( 2\dot{H} + 3H^2 \right) \Phi = (4\pi G) \delta p. \quad (4.9c)$$

Note the similarity of Eq.(4.9a) with the Poisson equation. In a non-expanding universe this equation exactly coincides with the Poisson equation. Now let us define

the adiabatic speed of perturbations,  $c_A^2 \equiv (p'/\rho')$  and non-adiabatic pressure perturbation,  $\delta p^{\text{NA}}$ . Then we can write

$$\delta p = c_A^2 \delta \rho + \delta p^{\text{NA}}. \quad (4.10)$$

Using the above equation, Eq. (4.9a) and Eq. (4.9c) can be combined to write:

$$\Phi'' + 3\mathcal{H} (1 + c_A^2) \Phi' - c_A^2 \nabla^2 \Phi + [2\mathcal{H}' + (1 + 3c_A^2) \mathcal{H}^2] \Phi = (4\pi G a^2) \delta p^{\text{NA}}, \quad (4.11)$$

where  $\mathcal{H} = (a'/a)$  is the conformal Hubble parameter.

#### 4.1.2.2 Curvature perturbation

We can construct a quantity using the Bardeen potential and its time derivative which is conserved at the super Hubble scales for adiabatic perturbations. Such a quantity is defined as

$$\mathcal{R} \equiv \Phi + \left( \frac{2\rho}{3\mathcal{H}} \right) \left( \frac{\Phi' + \mathcal{H}\Phi}{\rho + p} \right), \quad (4.12)$$

and is called *curvature perturbation*. Curvature perturbation is very important because, as we shall see, this provides the essential link between the fluctuations created by inflation and the fluctuations that we observe in the CMB. In Fourier space, using Eq. (4.12), Eq. (2.6) and Eq. (4.11), the derivative of the curvature perturbation is calculated to be

$$\mathcal{R}'_k = \left( \frac{\mathcal{H}}{\mathcal{H}^2 - \mathcal{H}'} \right) [(4\pi G a^2) \delta p_k^{\text{NA}} - c_A^2 k^2 \Phi_k] \quad (4.13)$$

where  $k$  denotes the comoving wave number of Fourier modes of perturbations. In the super Hubble scales, the physical wave length  $\lambda = a/k$  is much larger than the Hubble radius, i.e.  $(k/aH) = (k/\mathcal{H}) \ll 1$ . Then the second term in the square bracket can be neglected. *If we assume that there is no non-adiabatic pressure, then we can say that the curvature perturbation is conserved at super-Hubble scales.*

#### 4.1.2.3 Evolution of Bardeen potential

We cannot solve Eq. (4.11) exactly but we can find out the asymptotic solutions for the long wavelength and short wavelength perturbations. It is convenient to introduce

a new variable  $\mathcal{U}$  as

$$\Phi = \left( \frac{\mathcal{H}}{a^2 \theta} \right) \mathcal{U}, \quad (4.14)$$

where

$$\theta = \left[ \frac{\mathcal{H}^2}{(\mathcal{H}^2 - \mathcal{H}') a^2} \right]^{1/2}, \quad (4.15)$$

to eliminate the ‘friction term’ that is proportional to  $\Phi'$ . Then in Fourier space, Eq. (4.11) can be written as

$$\mathcal{U}_k'' + \left[ c_A^2 k^2 - \left( \frac{\theta''}{\theta} \right) \right] \mathcal{U}_k = \left( \frac{4\pi G a^4 \theta}{\mathcal{H}} \right) \delta p_k^{\text{NA}} \quad (4.16)$$

In the absence of non-adiabatic pressure perturbations, the above equation becomes

$$\mathcal{U}_k'' + \left[ c_A^2 k^2 - \left( \frac{\theta''}{\theta} \right) \right] \mathcal{U}_k = 0. \quad (4.17)$$

This equation can be viewed either as the equation of a parametric oscillator, with the time-dependent frequency given by  $w^2(k^2, \eta) = k^2 c_A^2 - (\theta''/\theta)$ , or as Schrödinger equation with potential  $(\theta''/\theta)$  [12]. For the long wavelength modes, i.e.,  $(k/\mathcal{H}) \ll 1$ , we can neglect the term  $c_A^2 k^2$ . Obviously one of the solutions of Eq. (4.17) is  $\mathcal{U} \propto \theta$ . Using that solution one can get a second solution also and the complete solution for long wavelength modes can be written as [11]

$$\mathcal{U}_k(\eta) \simeq C_G(k) \theta(\eta) \int_{\eta}^{\eta} \frac{d\bar{\eta}}{\theta^2(\bar{\eta})} + C_D(k) \theta(\eta), \quad (4.18)$$

where the constants of integrations,  $C_G$  and  $C_D$  are functions of  $k$  that are determined by the initial conditions. Then the Bardeen potential is

$$\Phi_k(\eta) \simeq C_G(k) \left( \frac{\mathcal{H}}{a^2(\eta)} \right) \int_{\eta}^{\eta} \frac{d\bar{\eta}}{\theta^2(\bar{\eta})} + C_D(k) \left( \frac{\mathcal{H}}{a^2(\eta)} \right). \quad (4.19)$$

In the case of short wavelength modes, Eq. (4.17) becomes

$$\mathcal{U}_k'' + c_A^2 k^2 \mathcal{U}_k = 0, \quad (4.20)$$

and the solution is  $\mathcal{U}_k \propto \exp(ik\eta)$ .

Using the equation of state  $\rho = wp$  we can rewrite Eq. (4.12) as

$$\frac{2}{3}\mathcal{H}^{-1}\Phi' + \frac{5+3w}{3}\Phi = -(1+w)\mathcal{R}. \quad (4.21)$$

If  $w$  is a constant for a period in the history of the universe, and we know that  $\mathcal{R}$  is constant for adiabatic perturbations at super Hubble scales, then the above equation is a differential equation for  $\Phi$ . The general solution to this equation is [9, 11]

$$\Phi = \frac{3+3w}{5+3w}\mathcal{R} + Ca^{-\frac{5+3w}{2}}, \quad (4.22)$$

where  $C$  is the constant of integration. The second part of this solution is a decaying term, hence after a sufficient time it becomes negligible. Then we have

$$\mathcal{R}_k \simeq \left[ \frac{3w+5}{3(w+1)} \right] \Phi_k = \text{const.} \quad (4.23)$$

Assuming  $k \ll \mathcal{H}$  the whole time, the Bardeen potentials at the entry of modes in the radiation and matter domination epochs are

$$\Phi_k^R = \left( \frac{2}{3} \right) \mathcal{R}_k \quad (w = 1/3), \quad (4.24a)$$

$$\Phi_k^M = \left( \frac{3}{5} \right) \mathcal{R}_k \quad (w = 0), \quad (4.24b)$$

which implies that while the universe goes from radiation domination to matter domination,  $\Phi_k$  changes by a factor 9/10.

### 4.1.3 Vector perturbations

As in the case of scalar perturbations and here also we are going to choose a particular gauge to describe the vector perturbations. A convenient choice is that of  $\mathcal{S}_i = 0$ . In this gauge the Friedmann metric can be written as

$$ds^2 = dt^2 - a^2(t) [\delta_{ij} + (\nabla_i F_j + \nabla_j F_i)] dx^i dx^j \quad (4.25)$$

where  $F_i \propto \mathcal{F}_i$ . Using the above metric, the perturbed Einstein tensor is calculated to be

$$\delta G_0^0 = 0, \quad \delta G_i^0 = \frac{1}{2}(\nabla^2 \dot{F}_i) \quad (4.26a)$$

$$\delta G_j^i = -\left(\frac{1}{2}\right) 3H(\nabla_i \dot{F}_j + \nabla_j \dot{F}_i) + (\nabla_i \ddot{F}_j + \nabla_j \ddot{F}_i) \quad (4.26b)$$

From these relations it is evident that in the absence of vector sources, the metric perturbations  $F_i$  vanish identically. This implies that no vector perturbations are generated in the absence of vector sources. For this reason we ignore vector perturbations in this thesis.

#### 4.1.4 Tensor perturbations

Tensor perturbations are important because they are responsible for generation of the primordial gravitational waves. From the decomposition of the the Friedmann metric Eq. (4.4), the tensor part can be written as

$$ds^2 = dt^2 - a^2(t)(\delta_{ij} + h_{ij})dx^i dx^j. \quad (4.27)$$

where  $h_{ij} \propto \mathcal{H}_{ij}$ . The  $h_{ij}$  is a symmetric, traceless and transverse tensor that contains two degrees of freedom. These degrees of freedom correspond to the two polarizations of the gravitational waves. Using the above metric the perturbed Einstein tensor can be calculated to be

$$\delta G_0^0 = \delta G_i^0 = 0, \quad (4.28a)$$

$$\delta G_j^i = -\left(\frac{1}{2}\right) \left[ \ddot{h}_{ij} + 3H\dot{h}_{ij} - \left(\frac{1}{a^2}\right) \nabla^2 h_{ij} \right]. \quad (4.28b)$$

In the absence of anisotropic stresses,  $\delta T_j^i = 0$ , the above equation reduces to

$$h'' + 2\mathcal{H}h' - \nabla^2 h = 0 \quad (4.29)$$

where  $h$  is the amplitude of the gravitational wave. It should be mentioned that the *gravitational waves can be generated even in the absence of sources.*

# Chapter 5

## Generation of primordial fluctuations

In the previous chapter we have described the perturbations of metric and their evolution. As I mentioned earlier inflation provides a natural way of explaining the generation of these perturbations. According to the theory of inflation, the primordial fluctuations can be generated due to the quantum fluctuations. These fluctuations get stretched to the cosmic scale during the inflation without changing its amplitudes. The scalar fluctuations produced and evolve in this way is the cause of the inhomogeneities in the CMB and the large scale structure we see. The tensor perturbation which produce gravitational wave was predicted to be seen in CMB and appears to have been observed recently by BICEP2 experiment [13].

In this chapter we study the behavior of perturbations during inflation and calculate their resulting power spectrum. We will calculate the power spectrum for some simple models also.

### 5.1 Equation of motion for the curvature perturbation

As we discussed in the earlier sections the inflation is achieved by the scalar field  $\phi$ . Let us consider it is perturbed by a factor of  $\delta\phi$  from its average value. From Eq.



(3.14) and using the metric (4.5), the perturbed stress-energy tensor will be

$$\delta T_0^0 = \dot{\phi}\delta\phi - \dot{\phi}^2\Phi + V_\phi\delta\phi = \delta\rho, \quad (5.1a)$$

$$\delta T_i^0 = \nabla_i(\dot{\phi}\delta\phi) = \nabla_i(\delta\sigma), \quad (5.1b)$$

$$\delta T_j^i = -(\dot{\phi}\dot{\phi} - \dot{\phi}^2\Phi - V_\phi\delta\phi)\delta_j^i = -\delta p\delta_j^i. \quad (5.1c)$$

As we mentioned earlier, we are interested in the evolution of a particular quantity called curvature perturbation. In the Eq. (4.13) the field perturbation is contained in the  $\delta p^{NA}$ . From above equations one can calculate  $\delta p^{NA}$  and can substitute in Eq. (4.13). The  $\delta p^{NA}$  is calculated to be

$$\delta p^{NA} = \left( \frac{1 - c_A^2}{4\pi G a^2} \right) \nabla^2 \Phi, \quad (5.2)$$

then Eq. (4.13) reduces to

$$\mathcal{R}'_k = - \left( \frac{\mathcal{H}}{\mathcal{H}^2 - \mathcal{H}'} \right) k^2 \Phi_k \quad (5.3)$$

Now we have curvature perturbation Eq (4.12) and its derivatives in terms of Bardeen potential so we can rewrite the equation of motion of Bardeen potential (4.11) in terms of curvature perturbation as

$$\mathcal{R}''_k + 2 \left( \frac{z'}{z} \right) \mathcal{R}'_k + k^2 \mathcal{R}_k = 0 \quad (5.4)$$

where  $z$  is

$$z = \frac{a\dot{\phi}}{H} = \frac{a\phi'}{\mathcal{H}} \quad (5.5)$$

which contains the background dynamics. If we introduce another variable

$$v = \mathcal{R}z \quad (5.6)$$

called *Mukhanov-Sasaki variable*, we get an equation similar to the equation of motion of Bardeen potential Eq. (4.17) in the absence of non-adiabatic pressure. In Fourier space we get

$$v''_k + \left[ k^2 - \left( \frac{z''}{z} \right) \right] v_k = 0 \quad (5.7)$$

which is called *Mukhanov-Sasaki equation*. The Mukhanov-Sasaki equation is hard to solve most of the time in full generality for a given inflationary model. We will

work out this for the power law inflation and in the slow roll approximation.

In the case of tensor perturbation, if we write  $h = u/a$  then the Eq. (4.29) can be written as

$$u_k'' + \left[ k^2 - \left( \frac{a''}{a} \right) \right] u_k = 0 \quad (5.8)$$

which is Mukhanov-Sasaki equation for the tensor perturbation and it is similar to Eq. (5.7) with the quantity  $z$  is replaced by  $a$ .

## 5.2 Quantization and power spectra

In inflation, the background field is treated classically, and only the perturbations around the mean value of the field are quantized. In quantum field theory the variable  $\mathcal{R}$  and its conjugate momentum  $\pi$  become operators  $\hat{\mathcal{R}}$  and  $\hat{\pi}$ , which satisfy equal time commutation relations [15]

$$\left[ \hat{\mathcal{R}}(\eta, \mathbf{x}), \hat{\mathcal{R}}(\eta, \mathbf{y}) \right] = [\hat{\pi}(\eta, \mathbf{x}), \hat{\pi}(\eta, \mathbf{y})] = 0; \quad \left[ \hat{\mathcal{R}}(\eta, \mathbf{x}), \hat{\pi}(\eta, \mathbf{y}) \right] = i\delta(\mathbf{x} - \mathbf{y}). \quad (5.9)$$

where we have set  $\hbar = 1$ . The curvature operator can be expanded in Fourier modes as

$$\hat{\mathcal{R}}(\eta, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left[ \hat{a}_k \mathcal{R}_k(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a}_k^\dagger \mathcal{R}_k^*(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}} \right], \quad (5.10)$$

where  $\hat{a}_k$  and  $\hat{a}_k^\dagger$  are the creation and annihilation operators which satisfy

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = [\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{k}'}^\dagger] = 0, \quad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = \delta(\mathbf{k} - \mathbf{k}'), \quad (5.11)$$

and the temporal  $\mathcal{R}(\eta)$  satisfies Eq. (5.4). We define the vacuum state  $|0\rangle$  as

$$\hat{a}_{\mathbf{k}} |0\rangle = 0. \quad (5.12)$$

Since we are dealing with the linear perturbations theory the perturbation will be Gaussian. For a Gaussian perturbation, the complete statistical information can be calculated from the two point correlation function. In cosmology, this two point correlation function is expressed by a quantity called power spectrum. The scalar and the tensor power spectra are defined to be

$$\langle 0 | \hat{\mathcal{R}}_{\mathbf{k}}(\eta) \hat{\mathcal{R}}_{\mathbf{k}'}(\eta) | 0 \rangle = \frac{(2\pi)^2}{2k^3} \mathcal{P}_S(k) \delta^{(3)}(\mathbf{k} + \mathbf{k}') \quad (5.13)$$

Using Eq. (5.10) one can calculate relation between the power spectrum and the curvature perturbation as

$$\mathcal{P}_S(k) = \left( \frac{k^3}{2\pi^2} \right) |\mathcal{R}_k|^2 = \left( \frac{k^3}{2\pi^2} \right) \left( \frac{|v_k|}{z} \right)^2. \quad (5.14)$$

In a similar way we can calculate the power spectrum of the tensor perturbation in terms of  $h_k$  and  $u_k$  as

$$\mathcal{P}_T(k) = 4 \left( \frac{k^3}{2\pi^2} \right) |h_k|^2 = 4 \left( \frac{k^3}{2\pi^2} \right) \left( \frac{|u_k|^2}{a} \right). \quad (5.15)$$

The power spectrum is calculated in the super Hubble limit where the curvature perturbation becomes constant.

We introduce a new quantity, *spectral index* which contain the information of rate of change of spectrum with respect to the comoving wavelength. Conventionally it is written as

$$n_S = 1 + \left( \frac{d \ln \mathcal{P}_S}{d \ln k} \right) \text{ and } n_T = \left( \frac{d \ln \mathcal{P}_T}{d \ln k} \right) \quad (5.16)$$

If the the spectrum remains constant at super Hubble scale we say that spectrum is scale invariant and that corresponds to  $n_S = 1$  and  $n_T = 0$ . Another quantity we are interested in is *tensor-to-scalar ratio* which is defined as

$$r(k) \equiv \frac{\mathcal{P}_T(k)}{\mathcal{P}_S(k)} \quad (5.17)$$

These two quantities are very important inflationary parameters that can be constrained by the observations.

### 5.2.1 The Bunch-Davies initial conditions

The initial conditions have to be chosen in the far past, when all comoving scales were well inside the Hubble radius,  $\eta \rightarrow -\infty$  or  $k \gg aH$ . If we consider distance and time scales much smaller than the Hubble scale, spacetime curvature does not matter and things should behave like in Minkowski space. In this limit the Eq. (5.7) becomes

$$v_k'' + k^2 v_k = 0. \quad (5.18)$$

This is the equation of simple harmonic oscillator with time-independent frequency. Then the positive-frequency solutions to these modes behave in the following form

$$\lim_{\frac{k}{\mathcal{H}} \rightarrow \infty} (v_k(\eta), u_k(\eta)) \rightarrow \left( \frac{1}{\sqrt{2k}} \right) e^{-ik\eta} \quad (5.19)$$

hence we can impose above initial condition in the limit  $k \gg aH$ . The corresponding vacuum of this mode function is referred to as Bunch-Davies vacuum.

### 5.3 Power spectra in power law and slow roll inflation

As an illustration we will calculate the power spectra for the power law inflation and for the slow roll inflation. The discussion of power law inflation is instructive because we can solve the Mukhanov-Sasaki equation exactly and it is very easy to find out the power spectrum in super Hubble limit. Once we have done this, it is not difficult to calculate the power spectra for slow roll approximation. So first we will discuss the spectrum of power law inflation.

#### 5.3.0.1 Power law inflation

In the case of power law inflation from the relation (3.30) and (3.27) we get

$$z = \left( \frac{a\dot{\phi}}{H} \right) = \sqrt{2/q} M_P a. \quad (5.20)$$

The scale factor can be expressed in terms of conformal time coordinate as

$$a(\eta) = (-\bar{\mathcal{H}}\eta)^{(\gamma+1)}, \quad (5.21)$$

where  $\gamma$  and  $\bar{\mathcal{H}}$  are constants given by

$$\gamma = - \left( \frac{2q-1}{q-1} \right) \text{ and } \bar{\mathcal{H}} = \left[ (q-1) a_1^{1/q} \right]. \quad (5.22)$$

The term  $(z''/z)$  becomes

$$\frac{z''}{z} = \frac{(\gamma+1)\gamma}{\eta^2}. \quad (5.23)$$

Then the Mukhanov-Sasaki equation becomes

$$v_k'' + \left[ k^2 - \frac{(\gamma + 1)\gamma}{\eta^2} \right] v_k = 0 \quad (5.24)$$

and the solution with the initial condition (5.19) is found to be [14]

$$v_k(\eta) = \left( \frac{-\pi\eta}{4} \right)^{1/2} e^{i[\nu + (1/2)](\pi/2)} H_\nu^{(1)}(-k\eta) \quad (5.25)$$

where  $\nu = -[\gamma + (1/2)]$  and  $H_\nu^{(1)}$  is the Hankel function of the first kind of order  $\nu$ .

For tensor perturbation we see that

$$\left( \frac{a''}{a} \right) = \left( \frac{z''}{z} \right), \quad (5.26)$$

hence the solution  $u_k$  will be the same form as the one for  $v_k$ . As we mentioned earlier the spectrum is calculated to be at the super Hubble limit. In this limit ( $k\eta \rightarrow 0$ ), the Hankel function becomes [14]

$$\lim_{-k\eta \rightarrow 0} iH_\nu^{(1)}(-k\eta) = \frac{1}{\pi} \Gamma(\nu) \left[ \frac{1}{2}(-k\eta) \right]^{-\nu}. \quad (5.27)$$

Using above limit and the Eq. (5.25) one can show that the curvature perturbation  $\mathcal{R}_k = (z/a)$  and similarly  $h_k$  becomes constant in the super Hubble limit as we expected. The scalar and tensor spectrum in the super Hubble scale is calculated to be

$$\mathcal{P}_{S(T)}(k) = A_{S(T)} \bar{\mathcal{H}}^2 \left( \frac{k}{\bar{\mathcal{H}}} \right)^{2(\gamma+2)}, \quad (5.28)$$

where

$$A_S = \frac{\gamma + 1}{16\pi^3(\gamma + 2)M_P^2} \left( \frac{|\Gamma(\nu)|^2}{2^{2\gamma+1}} \right) \quad (5.29a)$$

$$A_T = \frac{1}{\pi^3 M_P^2} \left( \frac{|\Gamma(\nu)|^2}{2^{2\gamma+1}} \right) \quad (5.29b)$$

by convention we have multiplied  $A_S$  by a factor of  $(4/M_P^2)$  [6]. Using Eq. (5.16) spectral indices  $n_S$  and  $n_T$  are calculated to be

$$n_S - 1 = n_T = 2(\gamma + 2) = - \left( \frac{2}{q - 1} \right) \quad (5.30)$$

The tensor-to-scalar ration is found to be

$$r = \frac{16(\gamma + 2)}{\gamma + 1} = \frac{16}{q}. \quad (5.31)$$

It should be noted that the spectral indices and the tensor-to-scalar are constants in the power inflation. The power spectrum becomes scale invariant when  $q \rightarrow 0$ .

### 5.3.1 Slow roll inflation

Now let us turn over our attention to the power spectra in slow roll inflation. Using Eq. (3.21),  $z$  can be calculated in terms of first Hubble slow roll parameter as

$$z = \sqrt{2}M_P(a\sqrt{\epsilon_H}). \quad (5.32)$$

Next we have to calculate the term in the Mukhanov-Sasaki equation  $z''/z$  in terms of slow roll parameters, for that we can use the relations

$$\epsilon_H = 1 - \frac{\mathcal{H}'}{\mathcal{H}^2} \text{ and } \delta_H = \epsilon_H - \frac{\epsilon'_H}{2\mathcal{H}\epsilon_H} \quad (5.33)$$

Then  $(z''/z)$  is calculated to be

$$\frac{z''}{z} = \mathcal{H}^2 \left[ 2 - \epsilon_H + (\epsilon_H - \delta_H)(3 - \delta_H) + \left( \frac{\epsilon'_H - \delta'_H}{\mathcal{H}} \right) \right] \quad (5.34)$$

For the case of tensor perturbation we can find out

$$\frac{a''}{a} = \mathcal{H}^2(2 - \epsilon_H). \quad (5.35)$$

It should be emphasized that so far we have not done any approximations. To take out the  $\eta$  dependance of the term  $(z''/z)$  from the Eq. (5.34), let us rewrite the Eq. (5.33) as

$$\eta = - \int \left( \frac{1}{1 - \epsilon_H} \right) d \left( \frac{1}{\mathcal{H}} \right) \quad (5.36)$$

Integrating the above expression by parts we get

$$\eta = - \frac{1}{(1 - \epsilon_H)\mathcal{H}} - \int \left( \frac{2\epsilon_H(\epsilon_H - \delta_H)}{(1 - \epsilon_H)^3} \right) d \left( \frac{1}{\mathcal{H}} \right) \quad (5.37)$$

The second term can be neglected at the leading order in slow roll parameters, then we get

$$\mathcal{H} \simeq -\frac{1}{(1 - \epsilon_H)\eta} \quad (5.38)$$

Now we can write the term  $(z''/z)$  and  $(a''/a)$  in terms of slow roll parameters and  $\eta$  in the leading order approximation as

$$\frac{z''}{z} \simeq \frac{2 + 6\epsilon_H - 3\delta_H}{\eta^2} \quad (5.39a)$$

$$\frac{a''}{a} \simeq \frac{2 + 3\epsilon_H}{\eta^2} \quad (5.39b)$$

The above relations show that the Mukhanov-Sasaki will be similar as we have calculated in the case of power law inflation, Eq. (5.24). Then the similar way, solution can be given in terms of the Hankel functions with different  $\nu$  values,

$$\nu_S \simeq \left[ \left( \frac{3}{2} \right) + 2\epsilon_H - \delta_H \right] \text{ and } \nu_T \simeq \left[ \left( \frac{3}{2} \right) + \epsilon_H \right]. \quad (5.40)$$

As we have done in the case of power law inflation the next task is to calculate the power spectra in the super Hubble scale. In the limit  $-k\eta \rightarrow 0$ , the scalar and tensor perturbation can be found to be

$$\mathcal{P}_S(k) = \frac{1}{32\pi^2 M_P^2 \epsilon_H} \left[ \frac{|\Gamma(\nu_S)|}{\Gamma(3/2)} \right]^2 \left( \frac{k}{a} \right)^2 \left( \frac{-k\eta}{2} \right)^{1-2\nu_S} \quad (5.41a)$$

$$\mathcal{P}_T(k) = \frac{1}{2\pi^2 M_P^2} \left[ \frac{|\Gamma(\nu_T)|}{\Gamma(3/2)} \right]^2 \left( \frac{k}{a} \right)^2 \left( \frac{-k\eta}{2} \right)^{1-2\nu_T} \quad (5.41b)$$

Then using the Eq. (5.38) above relations can be written as

$$\mathcal{P}_S(k) = \left( \frac{H^2}{2\pi\dot{\phi}} \right)_{k=aH} \left[ \frac{|\Gamma(\nu_S)|}{\Gamma(3/2)} \right]^2 2^{2\nu_S-3} (1 - \epsilon_H)^{2\nu_S-1}, \quad (5.42a)$$

$$\mathcal{P}_T(k) = \left( \frac{H^2}{2\pi^2 M_P^2} \right)_{k=aH} \left[ \frac{|\Gamma(\nu_T)|}{\Gamma(3/2)} \right]^2 2^{2\nu_T-3} (1 - \epsilon_H)^{2\nu_T-1}. \quad (5.42b)$$

At the leading order in the slow roll parameters the scalar and tensor spectrum reduce to

$$\mathcal{P}_S(k) \simeq \left( \frac{H^2}{2\pi\dot{\phi}} \right)_{k=aH} \quad (5.43a)$$

$$\mathcal{P}_T \simeq \frac{8}{M_P^2} \left( \frac{H}{2\pi} \right)^2_{k=aH} \quad (5.43b)$$

where the subscript notation signifies that the value of  $H$  for each  $k$  is to be taken at horizon exit of that particular scale. The problem has now been completely reduced to the evolution of the background scalar field and the background Hubble parameter. We just need to specify the inflation potential and calculate how the background evolves, and plug it in Eq. (5.43) to get complete information about the perturbations.

The next task is to find out the spectral indices in terms of slow roll parameters in linear order. The scalar spectral index  $n_S$  is

$$n_S - 1 = \left( \frac{d \ln \mathcal{P}_S}{d \ln k} \right)_{k=aH} = \left( \frac{d \ln \mathcal{P}_S}{dt} \right) \left( \frac{dt}{d \ln a} \right) \left( \frac{d \ln a}{d \ln k} \right)_{k=aH}. \quad (5.44)$$

The Hubble constant is almost constant in slow roll inflation then we can approximate,

$$\left( \frac{d \ln a}{d \ln k} \right)_{k=aH} \simeq 1 \quad (5.45)$$

then the scalar spectrum becomes

$$n_S = 1 + \left( \frac{d \ln \mathcal{P}_S}{d \ln k} \right)_{k=aH} \simeq 1 + \frac{\dot{\mathcal{P}}_S}{\mathcal{P}_S H}, \quad (5.46)$$

which can be written in terms of slow roll parameters as

$$n_S \simeq 1 - 4\epsilon_H + 2\delta_H \text{ and } n_T \simeq -2\epsilon_H. \quad (5.47)$$

When these slow roll parameters are close to zero we get a scale invariant power spectrum in slow roll inflation. *In slow roll inflation the power spectrum is predicted to be nearly scale invariant.* The tensor-to-scalar ratio in slow roll limit is found to be

$$r \simeq 16\epsilon_H = -8n_T \quad (5.48)$$

which is often referred to as the consistency relation. In the slow roll approximation



the power spectra can be expressed in terms of the potential also. Using the slow roll equations (3.24c) Eq. (5.43) can be re written as

$$\mathcal{P}_S(k) \simeq \frac{1}{12\pi^2 M_P^6} \left( \frac{V^3}{V_\phi^2} \right)_{k=aH} \quad (5.49a)$$

$$\mathcal{P}_T(k) \simeq \frac{2}{3\pi^2} \left( \frac{V}{M_P^4} \right)_{k=aH}. \quad (5.49b)$$

which are very useful in comparing different inflationary models with datas.

### 5.3.2 $n_S$ and $r$ for some simple models

Now let us calculate  $n_S$  and  $r$  for two simple inflationary models in the slow roll approximation. First, consider the large field model that has the relation (3.31). As I mentioned earlier in the slow roll regime we have  $\epsilon_H \simeq \epsilon_V$  and  $\delta_H = (\eta_V - \epsilon_V)$ . If we have the form of potential we can calculate  $\epsilon_V$  and  $\eta_V$  from the definition of these quantities (3.25b). Then by using the relations (5.47) and (5.48) we can calculate  $n_S$  and  $r$ . Following these we get

$$n_S \simeq 1 - \left[ \frac{2(n+2)}{4N+n} \right] \text{ and } r \simeq \frac{16n}{4N+n}, \quad (5.50)$$

where  $N$  is the number of e-folds counted from the end of inflation.

Next consider the potential of the form

$$V(\phi) = \frac{3M_P^2 M^2}{4} \left[ 1 - \exp \left( -\sqrt{\frac{2}{3}} \frac{\phi}{M_P} \right) \right]^2, \quad (5.51)$$

which is called  $R^2$  inflation (or Starobinsky model) [16]. From the definitions of slow roll parameters we calculate

$$\epsilon_V = \frac{4}{3} \left[ \frac{1}{(\bar{\phi}-1)^2} \right] \text{ and } \eta_V = \frac{4}{3} \left[ \frac{2-\bar{\phi}}{(\bar{\phi}-1)^2} \right], \quad (5.52)$$

where  $\bar{\phi} = \exp \left( \sqrt{\frac{2}{3}} \frac{\phi}{M_P} \right)$ . Then  $n_S$  and  $r$  are found to be

$$n_S \simeq 1 - \left[ \frac{8(4N+9)}{(4N+3)^2} \right] \text{ and } r \simeq \frac{192}{4N+3}. \quad (5.53)$$

---

These calculations are very important since, observations can constrain these values. Then the theoretical values of these quantities can be used to test different inflationary models. The detailed discussion of this method is discussed in the next chapter.

# Chapter 6

## Comparison with observations

As I have mentioned in earlier chapters, inflation can solve many difficult problems of standard big bang cosmology. But it is not just that, this theory has made several predictions which can be tested by observations.

### 6.1 CMB power spectrum

The major test of inflation can be done by measuring the anisotropies in the CMB radiation. Since the fluctuations are on the sky, it is useful to do angular decomposition of the fluctuations in multipole space  $l$  rather than Fourier space  $k$ . In terms of spherical harmonics the fluctuations in the CMB is described as [18]

$$\frac{\Delta T(\hat{n})}{T_0} = \sum_{lm} a_{lm} Y_{lm}(\hat{n}). \quad (6.1)$$

where  $\hat{n}$  denotes the direction in the sky,  $T_0 = 2.7K$  is the background temperature and

$$a_{lm} = \int d\Omega Y_{lm}^*(\hat{n}) \frac{\Delta T(\hat{n})}{T_0}. \quad (6.2)$$

Here  $Y_{lm}$  are the standard spherical harmonics on a 2-sphere with  $l = 0, 1, \dots$  and magnetic quantum number  $m = -l, \dots, +l$ . The temperature correlation function between two positions on the sky depends only on angular separation and not orientation.

i.e., all  $m$ s are equivalent. Thus, what is observed is a quantity averaged over  $m$ :

$$C_l^{obs} \equiv \frac{1}{2l+1} \sum_m a_{lm}^* a_{lm}. \quad (6.3)$$

The anisotropies in the CMB at higher multipoles ( $\ell > 2$ ) are interpreted as being mostly the result of perturbations in the density of the early universe. As we have seen, inflation provides the mechanism of calculating these primordial perturbations. The interesting fact about inflation is that the curvature perturbations are conserved on the super Hubble scale. Early on, all of the modes are outside the Hubble radius and then the perturbations cross the Hubble radius and the universe changes from radiation domination to matter domination. The primordial Bardeen potential at the time of inflation and the Bardeen potential today can be related through so called transfer function. This transfer function contains the evolution of the Bardeen potential which depends on the background cosmological model. Since the power spectrum can be calculated from the the Bardeen potential, one can express it in terms of the primordial power spectrum generated during inflation and the transfer function. Moreover, we have to convert this power spectrum to  $C_\ell$ s to compare with the observations. There is a mathematical formalism to do this conversion which I will not discuss here. The packages such as Code for Anisotropies in the Microwave Background (CAMB) calculate  $C_\ell$ s from primordial power spectra set by inflation [19].

The anisotropies are described by an angular distribution as shown in the Fig. 6.1. Figure shows the observed spectra and the best fit curve made by theory. This is the result from Planck and it seems that the theoretical scalar power spectrum calculated by slow roll inflation is in good agreement with these observations. As we mentioned, the theoretical curve depends both on the background cosmological parameters and on the spectrum of initial fluctuations.

## 6.2 Constraints on $n_S$ and $r$

In the last chapter we have calculated the scalar spectral index  $n_S$  and the tensor-to-scalar ratio  $r$  for power law, large-field and the  $R^2$  model of inflation. Here we will discuss how these quantities are constrained by the observations. When comparing

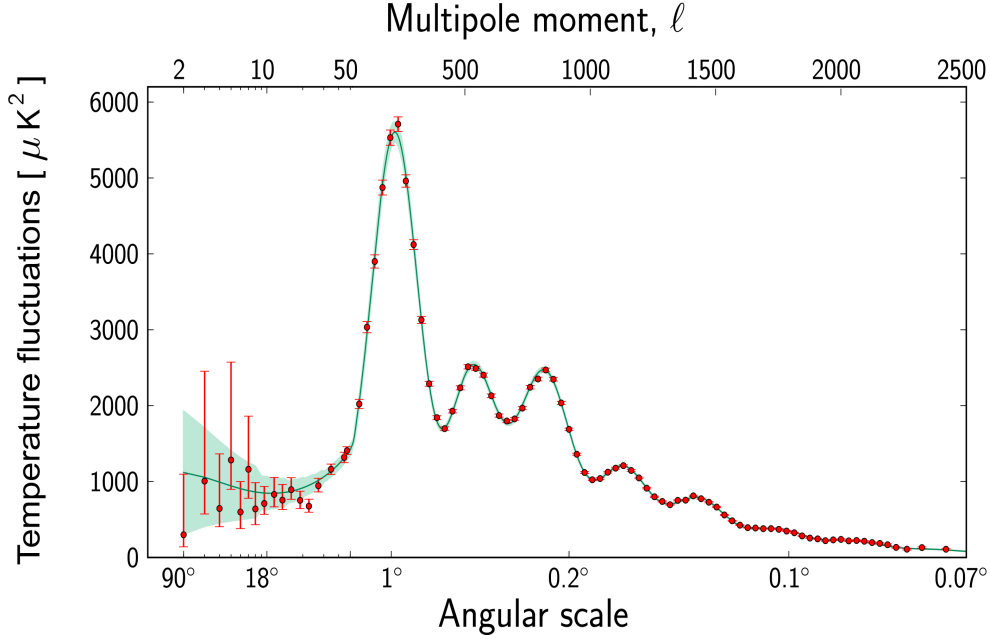


FIGURE 6.1: The temperature fluctuations in Cosmic microwave background (CMB) as observed by Planck. The red dots are measurements made by Planck. The green curve represents the best fit of the spatially flat,  $\Lambda$ CDM model with a power law primordial spectrum [3]. Source: [http://www.esa.int/spaceinimages/Images/2013/03/Planck\\_Power\\_Spectrum](http://www.esa.int/spaceinimages/Images/2013/03/Planck_Power_Spectrum)

the observations the scalar and tensor spectra are often expressed as [20]

$$\mathcal{P}_S(k) = \mathcal{A}_S \left( \frac{k}{k^*} \right)^{n_S-1} \quad \mathcal{P}_T(k) = \mathcal{A}_T \left( \frac{k}{k^*} \right)^{n_T} \quad (6.4)$$

where  $\mathcal{A}_S$  and  $\mathcal{A}_T$  are the scalar and tensor spectral amplitudes and  $k^*$  is a pivot scale at which the amplitudes are quoted. As we have discussed, the theoretical curve fit to the CMB anisotropy depends on the cosmological parameters. The primordial spectra set by inflation Eq. (6.4) contain the values of  $n_S$  and  $r$ . By knowing the rest of the cosmological parameters one can constrain the values of  $n_S$  and  $r$  by fitting the theoretical spectrum with observed spectrum. The Fig. (6.2) shows constraints on  $n_S$  and  $r$  of power law, large-field,  $R^2$  inflation and other inflationary models. The contours indicate the 68% and 95% CLs derived from the data. Figure shows that  $n = 3$  large-field model is excluded at more than 98% for  $N < 60$ . The  $n = 2$  lies inside 98% region for  $N = 60$ . Interestingly, the  $R^2$  model lies inside of 68% region

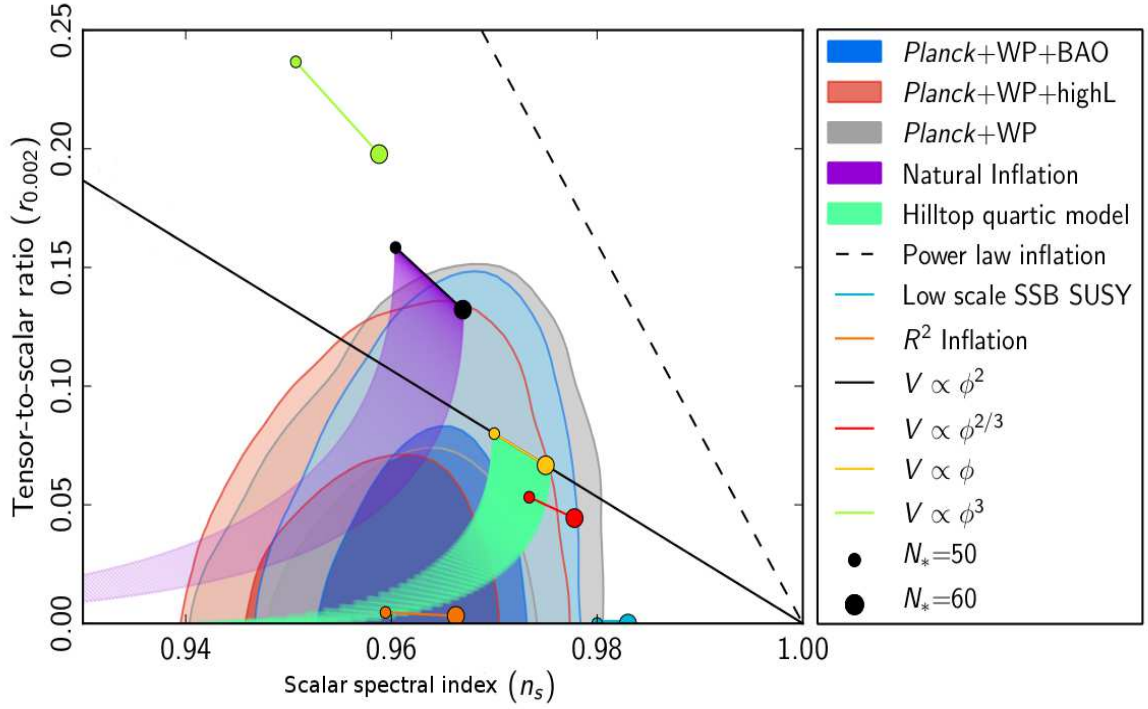
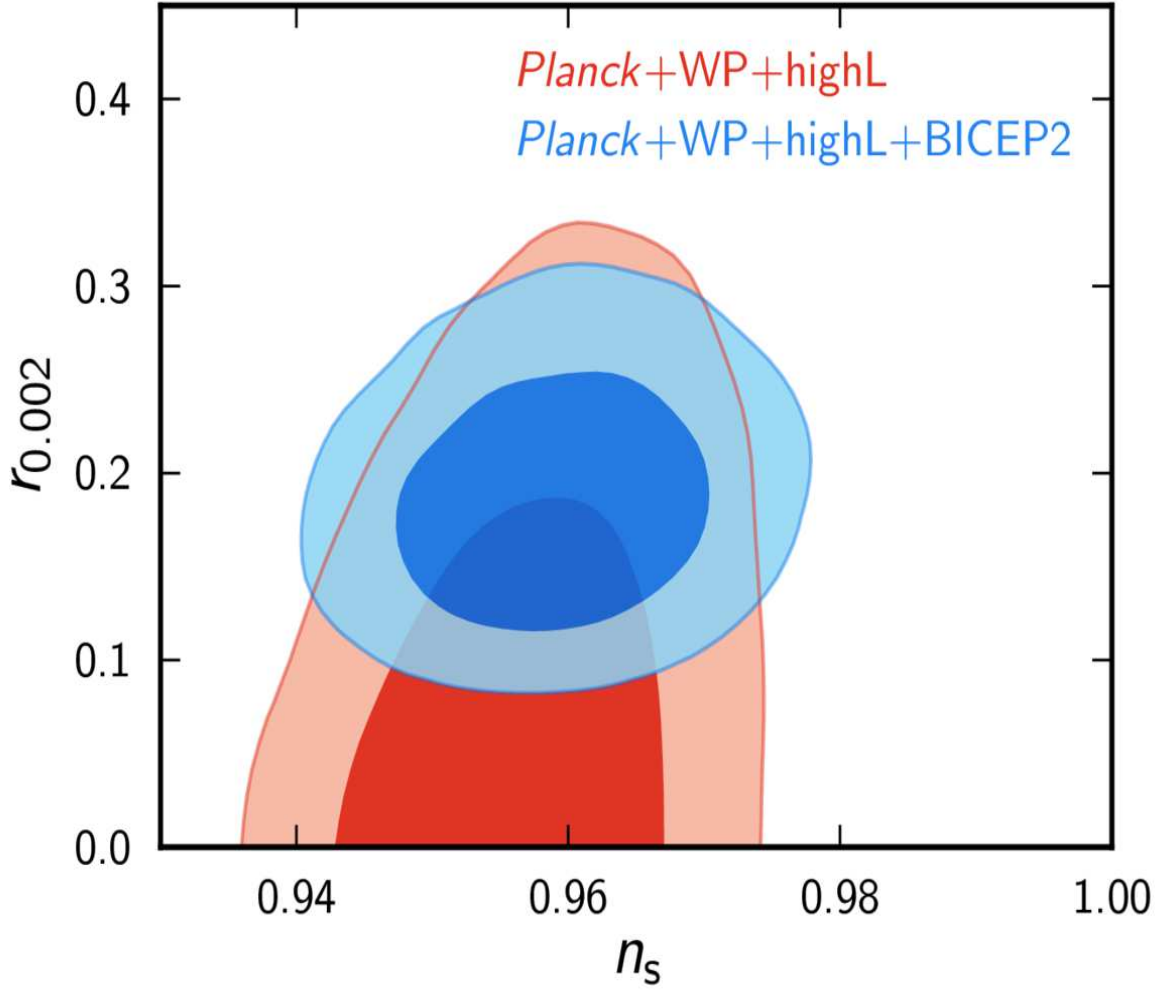


FIGURE 6.2: Constraints on  $n_s$  and  $r$  according to Planck with other data sets and compared to theoretical predictions [3].

for both  $N = 50$  and  $60$ . According to the Planck result,  $R^2$  model is considered to be a better model of inflation compared to the others.

Recent data from BICEP2 shows that the 68%  $r$  is lifted up and concentrated around  $0.1 < r < 0.3$ . The previous results show that  $r$  can be close to zero but according to this result it should not be so. The BICEP2 results exclude  $R^2$  model at more than 98% for both  $N = 50$  and  $60$ , in contradiction to the Planck result.

The BICEP2 result is very important in that it claims the detection of gravitational waves from the B-mode polarizations of the CMB spectrum. We are not interested in the details of the polarizations of the CMB. So we will not discuss about this result. But it is very important because the amplitude of the tensor perturbations also depends directly on the Hubble parameter during inflation, so it will provide a measurement of the energy scale of inflation.

FIGURE 6.3: Constraints on  $n_s$  and  $r$  according to BICEP2 2014 [21].

### 6.3 Energy scale of inflation

In the slow-roll approximation, for a scale-invariant spectrum  $n_s = 1$ , using equations (6.4) and (5.49b), the tensor-to-scalar ratio can be approximated as

$$r \simeq \frac{2V}{3\pi^2 M_P^4} \frac{1}{\mathcal{A}_S}, \quad (6.5)$$

where  $\mathcal{A}_S$  is the amplitude of the scalar perturbations. The cosmic microwave background anisotropies seen by COBE gives the amplitude of the initial power spectrum as  $\mathcal{A}_S \simeq 2.14 \times 10^{-9}$ , which is often referred to as the COBE normalization [6]. This implies

$$V^{(1/4)} \simeq 3.2 \times 10^{16} r^{1/4}. \quad (6.6)$$

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From BICEP2 result,  $r \simeq 0.2$ . Using this, we get  $V^{(1/4)} \simeq 2.1 \times 10^{16} \text{GeV}$ . This is a remarkably large energy scale, the energy scale of GUT! This is very interesting since inflation is found to connect classical energy scale to GUT energy scale.



# Chapter 7

## Summary

At large scales, our universe can be approximated to be homogeneous and isotropic. In this approximation, the geometry of the universe is expressed in terms of the FRW metric. The standard big bang model suggests that the universe started its expansion as a hot primordial soup with relativistic particles and radiation. At that epoch, the universe was radiation-dominated. Since the density of radiation falls faster than that of non-relativistic matter, as the universe cooled down, the normal matter ceased to interact with radiation. The radiation decoupled from matter at this stage and the photons started to propagate freely. That is what we see today as the CMB radiation. After decoupling, the universe became matter-dominated. At present, the universe is known to be dominated by vacuum energy. Even though the big bang model predicts expansion of the universe, formation of various elements, CMB spectrum etc., it has some drawbacks such as the flatness problem and the horizon problem.

Inflation solves the puzzles of the standard big bang model in an elegant and simple way by introducing an epoch of the universe which was dominated by particles with negative pressure. We have shown that such an epoch can be achieved by using scalar fields whose potential energy dominates over their kinetic energy. Moreover, to get sufficient inflation, we have introduced the idea of slow-roll inflation and slow-roll parameters.

The idea of inflation was first postulated to resolve the puzzles of big bang model. Soon after it was realized that inflation can also be the mechanism for the generation of primordial perturbations, which lead to the inhomogeneities in the CMB and the large scale structures of the universe. At the linear order, we have shown that

the perturbed Einstein's equations can be separated into scalar, vector and tensor perturbations. Fixing a gauge is an important part of calculating the perturbations; so we have chosen a particular gauge rather than working in a gauge-invariant way. The scalar perturbation is mainly responsible for the inhomogeneities in the CMB and the large scale structures in the universe, and the tensor perturbation gives the primordial gravitational waves. We have shown that the curvature perturbation  $\mathcal{R}$  does not evolve at super-Hubble scales  $k \ll aH$ , unless non-adiabatic pressure is significant. This fact is crucial for relating the initial conditions from inflation to late-time observables.

According to the theory of inflation, the source of the perturbations is the quantum fluctuations which were present in the initial stages of inflation. As a result of inflation, these were stretched to the cosmological scale with nearly same amplitude. We have calculated the power spectrum in the super Hubble scale for power law and slow-roll inflation. In the case of slow-roll inflation, we got nearly scale-invariant power spectrum.

The power spectrum calculated theoretically can be compared directly with the observations of CMB. From Planck data, it is shown that the power spectrum calculated using the slow-roll approximation fits well with the observations. Inflation predicts a nearly scale-invariant power spectrum, which has actually been seen in the Planck and BICEP2 results. The BICEP2 observations are more important because they observed the primordial gravitational waves, the existence of which is a major prediction of the theory of inflation. For the scale-invariant power spectrum the COBE normalization gives the energy scale of the inflation, which is found to be the energy scale of GUT.

Inflation is really a beautiful idea. Not only could it explain the origin of large scale structures of the universe, but also these are generated by physics at very high energy scales. These are energy scales that we would really like to be able to explore, but unfortunately that will probably never be possible in the particle accelerators. We expect very interesting new physics to lie there; new particles, possibly GUT theories, and maybe even string theory. We can now explore them with cosmological observations. Current observations are in beautiful agreement with the basic inflationary predictions, that the universe is flat with a scale-invariant spectrum, the existence of primordial gravitational waves, and Gaussian and adiabatic density

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fluctuations. We are now looking forward to future experiments that can provide further tests of cosmic inflation.

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