Generation of primordial magnetic fields and imprints on gravitational waves

A project report

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Master of Science in Physics

by

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under the guidance of

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CERTIFICATE

This is to certify that the project titled **Generation of primordial magnetic fields and imprints on gravitational waves** is a bona fide record of work done by **Abhik Bhattacharjee** towards the partial fulfillment of the requirements of the Master of Science degree in Physics at the Indian Institute of Technology, Madras, Chennai 600036, India.

(L. Sriramkumar, Project supervisor)

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ABSTRACT

Magnetic fields are ubiquitous in the universe but we are still unsure about its origins. The anisotropic stress associated with magnetic fields are known to enhance the amplitude of the gravitational waves generated in the early universe. The aim of the project is to study the generation of magnetic fields during inflation (inflationary magnetogenesis) as well as to examine the imprints of the magnetic field on the two-point function of the tensor perturbations. In this report we have reviewed two models of inflationary magnetogenesis – generation of helical and non-helical magnetic fields by coupling the electromagnetic (EM) field to the inflaton, and the generation of non-helical magnetic fields by coupling the EM field to the Riemann curvature tensor. We have also outlined how to generate non-helical magnetic fields with the Riemann coupling, which we are currently working on. We have found that in a perfectly de Sitter background during inflation, scale invariant non-helical magnetic field ($B_0 \sim 0.1$nG) and helical magnetic field ($B_0 \sim 0.1\mu$G) can be generated with inflaton coupling. However that is not found to be the case with the curvature coupling case where the generated magnetic field is scale dependent. Finally we have studied the effect of the scale invariant magnetic fields on the tensor power spectrum. We find that the effect of the non-helical magnetic field is negligible while the helical magnetic field modifies the tensor power spectrum by an additive factor $\mathcal{O}(100)$.

Contents

Chapter 1

Introduction

Magnetic fields have been observed over a wide range of scales in the universe – from planets, stars to galaxies, intergalactic medium (IGM) and galaxy clusters. It can be said that the universe is magnetized at all scales with strength of magnetic fields ranging between μG (in galaxies) and $10^{14} G$ (in magnetars – highly magnetized neutron stars) [1, 2]. The origin of the magnetic fields over astrophysical scales is well understood but the same is not true over cosmological scales, say, galaxies or clusters of galaxies. The presence of cosmological plasma suggests that the discussion should be based in the language of magnetohydrodynamics (MHD). Some elaborate MHD mechanisms, known as galactic dyanamo have been proposed to amplify a very weak seed magnetic field ($\sim 10^{-21}$G to 10^{-11}G [3]) to the micro Gauss level but the efficiency of such kind of mechanisms have been called into question both by new observations and improved theoretical work [4]. This has lead us to consider a primordial origin of the galactic and extragalactic magnetic fields. Magnetic fields may have affected a number of processes (including structure formation [5]), involving electromagnetism, MHD, etc., that took place in the universe. In this report we shall study its effects on primordial gravitational waves. If primordial magnetic fields indeed affected structure formation, then they probably must have left their imprints in the temperature and polarization anisotropies of the cosmic microwave background [4].

1.1 Dynamo versus primordial

Let us start by discussing the dynamo mechanism which is based on the conversion of the kinetic energy of the turbulent motion of the conductive interstellar medium into magnetic energy. In MHD, the time evolution equation of the magnetic field is governed by the following equation [6] :

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{1}{\sigma} \nabla^2 \mathbf{B}, \tag{1.1}$$

where $\mathbf{B}$ is the magnetic field, $\mathbf{v}$ the plasma velocity and σ the electrical conductivity of plasma. But magnetic fields cannot be generated within the MHD description as the plasma does not provide source term for the magnetic fields : if $\mathbf{B}$ at $t = 0$ is zero then it is zero at all times.

Fortunately, that is not the case when we go beyond MHD, and source terms can be present [7]. The charges in standard astrophysical plasmas consists of electrons and protons, which have equal and opposite electric charge but the masses are vastly different, $m_p \approx 1837 m_e$, which implies that their Thomson interaction cross sections ($\sigma_\gamma \propto 1/m^2$) are also vastly different. Thus the prospect of electric current generation opens up when astrophysical plasmas interact with photons. For example, it has been suggested ([8]) that owing to the difference in masses and scattering cross sections of protons and electrons, a net electric current is produced which sources the magnetic fields, provided there is turbulence during cosmological recombination. This process invokes what is known as a Biermann battery, i.e. nonparallel pressure and density gradients, powered by a non-zero vorticity in the primordial fluctuation field [9]. A general prediction of the dynamo mechanism is that amplification of the seed field ends when equipartition is reached between the magnetic energy density and the kinetic energy density of the turbulent fluid motion. Once equipartition is reached, the amplification process stops before a coherent field may develop. This is one of the main arguments raised against this kind of dynamo mechanism, which makes us look for primordial origins for the magnetic fields. [4]

In this report we will be concerned with the possible origin of primordial magnetic fields in the early universe via a mechanism in the inflationary scenario. Current observations suggest that we live in an isotropic, homogeneous and spatially flat universe described by the FLRW metric,

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \tag{1.2}$$

where $a(t)$ is the scale factor. In such a background, the magnetic filed B varies as $1/a^2$ [10]. The universe is also believed to have gone through a period of accelerated called inflation, during which the universe expanded by more than 60 e-folds. Since the magnetic flux $a^2 B$ is a constant, the strength of B decays at least by e^{120} at the end of inflation. This is due to the fact that the above metric is conformally flat and the standard electromagnetic (EM) field action is conformally invariant (electrodynamics in curved spacetime is the subject of Appendix A and the conformal invariance of the EM action is discussed in Chapter 3). Therefore in order to generate magnetic fields of sufficient strength, the conformal invariance of the EM field action has to be broken.

Breaking the conformal invariance.

A number of ways have been considered for breaking the conformal invariance of the EM action during inflation. Some of them are illustrated in the action below [10]:

$$\mathcal{S} = \int \sqrt{-g} d^4x \left[-f^2(\phi, R) \left(\frac{F^{\mu\nu} F_{\mu\nu}}{4} \right) + \mathscr{L}_R + g\theta F_{\mu\nu} \mathcal{F}^{\mu\nu} - D_\mu \psi (D^\mu \psi)^* \right], \tag{1.3}$$

where $F_{\mu\nu}$ is the electromagnetic (EM) field tensor, $\mathcal{F}^{\mu\nu}$ is its dual and D_μ denotes a gauge covariant derivative. They include coupling the EM action to scalar fields (ϕ) such as the inflaton [11, 12] or the dilaton [13], coupling to a pseudo-scalar field like the axion (θ) [14], coupling to charged scalar fields (ψ) (see, for instance, [15]) and so on. The $\mathscr{L}_R$ term is proportional to $(BRF^{\mu\nu}F_{\mu\nu} + CR_{\mu\nu}F^{\mu\kappa}F^{\nu}_{\kappa} + DR_{\lambda\kappa\mu\nu}F^{\mu\nu}F^{\lambda\kappa})$. [16]

The earliest effort involved including an inflation epoch coupling $\propto e^{\alpha\phi} F^{\mu\nu} F_{\mu\nu}$, where ϕ is the inflaton and α is a parameter [11]. Similar models have been considered since then, for instance, a more general case was considered by coupling the EM field with the inflaton via a generic function $f^2(\phi)$ [12]. In order to generate helical magnetic field, some authors (for instance in Ref. [17]) added a $f(\phi)F^{\mu\nu}\mathcal{F}_{\mu\nu}/4$ term to the standard EM Lagrangian and thus broke the conformal invariance. Foe instance, in Ref. [18], the authors considered a hybrid of the inflaton coupling and the axion model with the following action :

$$\mathcal{S} = \int \sqrt{-g} d^4x \quad I^2(\eta) \left[-\frac{F^{\mu\nu}F_{\mu\nu}}{4} + \frac{\gamma}{8} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \right], \tag{1.4}$$

where I is the coupling function, γ a parameter and η the conformal time coordinate, defined later. Some of these, along with other methods of primordial magnetogenesis and observational methods are discussed in the following reviews, Refs. [19], [20] and [21].

Other mechanisms for generation of magnetic fields in the early universe

Apart from inflation, other ideas include generation of magnetic fields at the electroweak phase transition (EWPT) [22, 23], at the QCD phase transition [23, 24] and before recombination [25]. These have been summarized in Ref. [7].

1.2 Observations and methods of detection

Zeeman splitting of spectral lines (useful within our galaxy), intensity and polarization of synchotron emisson from free relativistic electrons (for intermediate distances), and Faraday rotation measurements of polarized EM radiation passing through a ionised medium (for far away galaxies) are the main observational tracers of galactic and extra-galactic magnetic fields.

The Zeeman splitting between two neighbouring energy levels is given by $\Delta = g\mu_B B_1$, where g is the Lande g-factor, B_1 is the magnetic field along the line of sight and $\mu_B = e\hbar/2m_e$ is the Bohr Magneton, which can be used to estimate the strength of the magnetic field. Since Zeeman splitting is a very weak effect and the line shift is small, this method is not very effective for high red-shift galaxies.

A linearly polarized EM wave can be decomposed into a superposition of circularly polarized components of equal amplitude but different phase. Due to this difference in phases between the right and left circularly polarized light, the polarization direction of the linearly polarized light changes upon passing through a region of magnetic field. The degree of rotation (θ) is related to the wavelength (λ) of the radiation via the rotation measure as $\theta = (RM)\lambda^2 + \theta_0$. The rotation measure (RM) (in Minkowski space) is given by [7]

$$RM \equiv \frac{\Delta(\phi)}{\lambda^2} = \frac{e^3}{2\pi m_e^2} \int d\mathbf{l} \cdot \mathbf{B} n_e, \tag{1.5}$$

where n_e is the local electron density, $\Delta(\phi)$ is the rotation angle of the linear polarization, λ is the wavelength of the observed light and the integration is along the line of sight. We now briefly summarize the observational situation.

Magnetic fields in galaxies : The interstellar magnetic field in the Milky Way has been determined using various techniques and the average field strength is found to be $3 - 4\mu G$ [1]. Such a strength corresponds to an approximate energy equipartition between magnetic field (ρ_B), small scale turbulent motion (ρ_{turb}), and the cosmic rays confined in the galaxy $(\rho_{cr}) : \rho_B \approx \rho_{turb} \approx \rho_{cr}$. Magnetic fields of similar magnitudes have been observed in a number of other spiral galaxies but equipartition threshold has not been observed in all of them. For instance, it has been observed in M33 but not in M82, where the field seems to be stronger than the equipartition threshold [4].

Magnetic fields in galaxy clusters : We have valuable information on fields in galaxy clusters, thanks to the observations made on a large number of Abel clusters, some of which have a measured X-ray emission. The magnetic field strength in the inter cluster medium has been found to be ranging from $1 - 10\mu G$ [4]. Observations of radio sources embedded in galaxy clusters have provided us with evidence of strong magnetic fields in the cluster central regions with the central field strength $\sim 10 - 30\mu G$ and peak values a large as $75\mu G$ [26].

Magnetic fields in high redshift objects : Owing to high resolution RMs of very far quasars, we have been able to probe magnetic fields in the distant past. RMs of the radio emission of the quasar 3C191, at z=1.954, are consistent with a field strength in the range of $0.4 - 4\mu G$ [27].

1.3 Qualitative constraint plot

As the name suggests, a qualitative constraint plot maps out the regions in the $\lambda - B_\lambda$ plane which are either forbidden or suggested by observations (see figure 1.1). Although a constraint plot should be used with caution [7], it can be very useful for quickly visualising the overall state of cosmological magnetic fields. Cosmological magnetic fields with $\lambda \sim$ kpc and $B_\lambda \sim 10^{-10}G$ may explain the galactic magnetic field directly (with minimal dynamo amplification). This region is denoted by the golden rectangle marked by "MW" (for MilkyWay). Big bang nucleosynthesis (BBN) constrains $B_\lambda \leq 10^{-6}G$ on all scales [28], while other observations roughly constrain $B_\lambda \leq 10^{-9}G$ on Mpc to Gpc scales [7]. Blazar spectral measurements place a lower bound $\sim 10^{-16}G$ on the strength of inergalactic magnetic fields for λ in the Mpc to Gpc range [29, 7]. Magnetic helicity measurements are uncertain, but if confirmed, they would fall within the pink rectangle [7]. In some inflationary models, the generated cosmological magnetic field is scale invariant. If such a magnetic field happens to pass through the golden rectangle, then it would correpond to the dotted horizontal line and would be subjected to several constraints (see figure 1.1). If the cosmological magnetic field has a blue k^3 or k^4 spectrum, as predicted by the EWPT, it may have a shape similar to the dashed curve [7].

1.4 Organization of the report

The report is organised as follows. In Chapter 2, we discuss inflation and the early universe. We begin with the successes of and problems in the hot big bang model. Then we discuss their solutions due to inflation before moving on to discuss how inflation is driven by scalar fields. We end the chapter by studying power law and slow roll inflation. In Chapter 3, we study the generation of magnetic fields by coupling the EM action to the inflaton. We discuss the generation of both helical

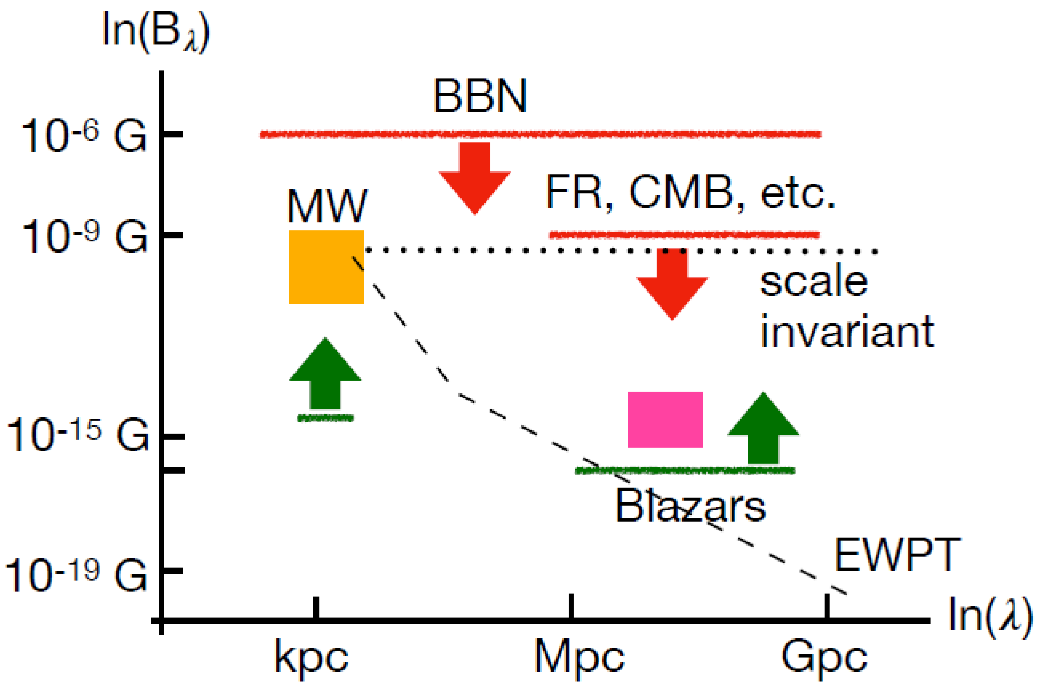


Figure 1.1: A schematic of the qualitative constraint plot.[Figure from Ref. [7]]

and non-helical magnetic fields. Chapter 4 is structured along the same line as Chapter 3. Here we discuss the generation of both helical and non-helical magnetic fields by coupling the EM action to the Riemann curvature tensor. In Chapter 5, we study the effects of the primordial magnetic fields on primordial gravitational waves. We particularly focus on the effect of the anisotropic stress of the EM field on the amplitude of the gravitational waves during inflation. Finally we conclude the report with a brief summary of the work presented in the report in Chapter 6. We have also included a couple of appendices. Appendix A deals with electrodynamics in curved space time and the subject of Appendix B is the derivation of the tensor spectra in presence of anisotropic stress, and related numerical integrations.

Notations and units

We use the metric with signature (- + + +). Greek indices going from 0 to 3 denote space-time coordinates whereas Latin indices going from 1 to 3 represent spatial coordinates. We work in natural units $c = \hbar = k_B = 1$ and have defined the Planck mass $M_{Pl} = 1/\sqrt{8\pi G}$.

Chapter 2

Inflation and the early universe

According to the hot big bang model, the universe came into being about 15 billion years ago with a homogeneous and isotropic distribution of matter at very high temperature and density, and has been expanding and cooling ever since [30]. In this model, the early universe was radiation dominated and remained so until the expansion caused the radiation density to fall sufficiently low such that the photons ceased to interact with matter, thereby leading to a matter dominated universe. Although successful to some extent, the hot big bang model has three significant drawbacks: the flatness problem, the horizon problem, and the monopole problem. These problems can be summarized by the following statements [31] : the universe is nearly flat today, and was even flatter in the past; the universe is nearly isotropic and homogeneous today, and was more so in the past; and the universe is apparently free of magnetic monopoles, respectively. Before delving into the details of these problems and how inflation solves them, let us first have a look at the successes of the hot big bang model in brief.

2.1 Successes of the hot big bang model

i) The expansion : That the universe is expanding according to the Hubble's law has been well established by observations (see, for instance, [32]). In an expanding, homogeneous and isotropic universe, at nearby distances ($d \lesssim 50$MPc), it is well approximated by [33]

$$v_H = H_0 d,$$

where H_0 is the Hubble's constant, v_H is the local "Hubble flow" velocity of a source and d is the distance to the source. From CMB anisotropy data, we get $H_0 = 67.4 \pm 0.5$ km s $^{-1}$ MPc^{-1} [34] , whereas the value from Infrared Surface Brightness Fluctuation (SBF) distances measurements is $H_0 = 73.3 \pm 0.7 \pm 2.4$ km s $^{-1}$ MPc^{-1} [35], which tallies well with the value arrived at from Type Ia supernovae data, $H_0 = 72.8 \pm 1.6 \pm 2.7$ km s $^{-1}$MPc^{-1} [36].

ii) The Cosmic Microwave Background (CMB) : The CMB is nothing but the relic radiation reaching us from the epoch of decoupling - the epoch when photons ceased to interact with matter, during the transition from a radiation dominated universe to a matter dominated one. The spectrum of the CMB is consistent with that of a blackbody at a temperature of 2.725K [37]. The energy density of radiation goes as $\rho_{rad} \propto 1/a^4$ or the temperature goes as $T \propto 1/a$ (using Stefan- Boltzmann law), implying the

universe cools as it expands. This suggests that at earlier times, the universe must have been much hotter, thus supporting the hot big bang model.
iii) Primordial nucleosynthesis : The hot big bang model has been quite successful in forecasting the primordial abundances of the light elements using only one parameter, namely the baryon-to-proton ratio, and the value required to fit these observations matches the value determined independently from the CMB anisotropies data. [38].

2.2 How inflation resolves the three problems in the hot big bang model

The Friedmann-Lemaitre-Robertson-Walker (FLRW) universe is described by the following line element

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right], \tag{2.1}$$

where t denotes the cosmic time, and the parameter κ refers to the spatial curvature and it is normalized so that $\kappa = (0, \pm 1)$. It takes on the simple form displayed in equation (1.2) for the spatially flat universe corresponding to $\kappa = 0$. In the $\kappa = 0$ case, we can define the conformal time coordinate as

$$\eta = \int \frac{dt}{a(t)}, \tag{2.2}$$

in terms of which, the above FLRW line-element simplifies to be

$$ds^2 = a(\eta)^2\,(-d\eta^2 + dx^2 + dy^2 + dz^2). \tag{2.3}$$

The time evolution of such a universe is governed by the following Friedmann equations (with $\kappa = 0$ for the spatially flat universe) :

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2} \tag{2.4}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} = -8\pi G p, \tag{2.5}$$

where the energy density ρ and the pressure p are related by the equation of state parameter w as $p = w\rho$. From equations (2.4) and (2.5), we get the acceleration equation,

$$\frac{\ddot{a}}{a} = -4\pi G\left(p + \frac{\rho}{3}\right), \tag{2.6}$$

as well as the energy conservation equation

$$\dot{\rho} = -3H(\rho + p). \tag{2.7}$$

In the chapters that follow, we are going to restrict ourselves to the $\kappa = 0$ case and work with FLRW metric expressed in terms of the conformal time coordinate η (see equation (2.3))

Inflation

Inflation is the term used to describe a period of accelerated expansion of the early universe. Hence equation (2.6) suggests that inflation corresponds to $\ddot{a} > 0$ or $3p < -\rho$. Thus inflation would have taken place if the universe were temporarily dominated by a component with $w < -1/3$.

It is often assumed that inflation was started off at time t_i and ended at some later time t_f instantaneously and the universe reverted to its former state of radiation domination. So for simplicity, the expansion factor $a(t)$ can be taken to behave as follows [31]

$$a(t) = \begin{cases} a_i\sqrt{t/t_i} & t < t_i \\ a_i \exp[H_I(t-t_i)] & t_i < t < t_f \\ a_i \exp[H_I(t_f - t_i)]\sqrt{t/t_f} & t > t_f \end{cases} \tag{2.8}$$

Thus in the time interval from t_i to t_f, the scale factor $a(t)$ increased by N e-foldings where N is defined as the natural logarithm of the ratio of $a(t_f)$ to $a(t_i)$. Thus we see from the above equation that $N = H_I(t_f - t_i)$.
Below we discuss the three problems mentioned earlier and their solutions due to inflation.

In non-inflationary cosmology, during the radiation dominated era, the Hubble radius $d_H = H^{-1}$ grows as a^2 and the physical wavelengths λ_p grow as $a \propto \sqrt{t}$. But in the inflationary model, during inflation, $\lambda_p \propto a \propto e^{H_I t}$ while the Hubble radius remains constant. This gives rise to the possibility that a given length scale can cross the Hubble radius twice in inflationary models. The situation has been summarized in figure 2.1.

2.2.1 The flatness problem

For a spatially flat universe $\kappa = 0$ and $\rho = \rho_c = 3H^2/8\pi G$, where ρ_c is the critical density. Now the dimensionless density parameter Ω is defined by $\Omega = \rho/\rho_c$, in terms of which we have the Friedmann equation as follows

$$1 - \Omega(t) = -\frac{\kappa}{a(t)^2 H(t)^2}. \tag{2.9}$$

Denoting Ω, a and H at the present time by Ω_0, a_0 and H_0 respectively, we have

$$\kappa = a_0^2 H_0^2(\Omega_0 - 1). \tag{2.10}$$

Combining the previous two equations, we get

$$(1-\Omega)H^2a^2 = (1-\Omega_0)H_0^2 a_0^2. \tag{2.11}$$

For a radiation dominated spatially flat universe, $a(t) \propto t^{1/2}$ or $\dot{a} \propto t^{-1/2}$ or $H^2a^2 \propto t^{-1}$, and for a matter dominated flat universe, $a(t) \propto t^{2/3}$ or $\dot{a} \propto t^{-1/3}$ or $H^2a^2 \propto t^{-2/3}$. So we have $|\Omega - 1| \propto t$ for radiation domination and $|\Omega - 1| \propto t^{2/3}$ for matter domination. Thus we see that in both cases, as time increases, Ω is driven further and further away from unity. This problem is overcome by inflation. Inflation reverses this state of affairs, because

$$\ddot{a} > 0 \Rightarrow \frac{d\dot{a}(t)}{dt} > 0 \Rightarrow \frac{d(aH)}{dt} > 0. \tag{2.12}$$

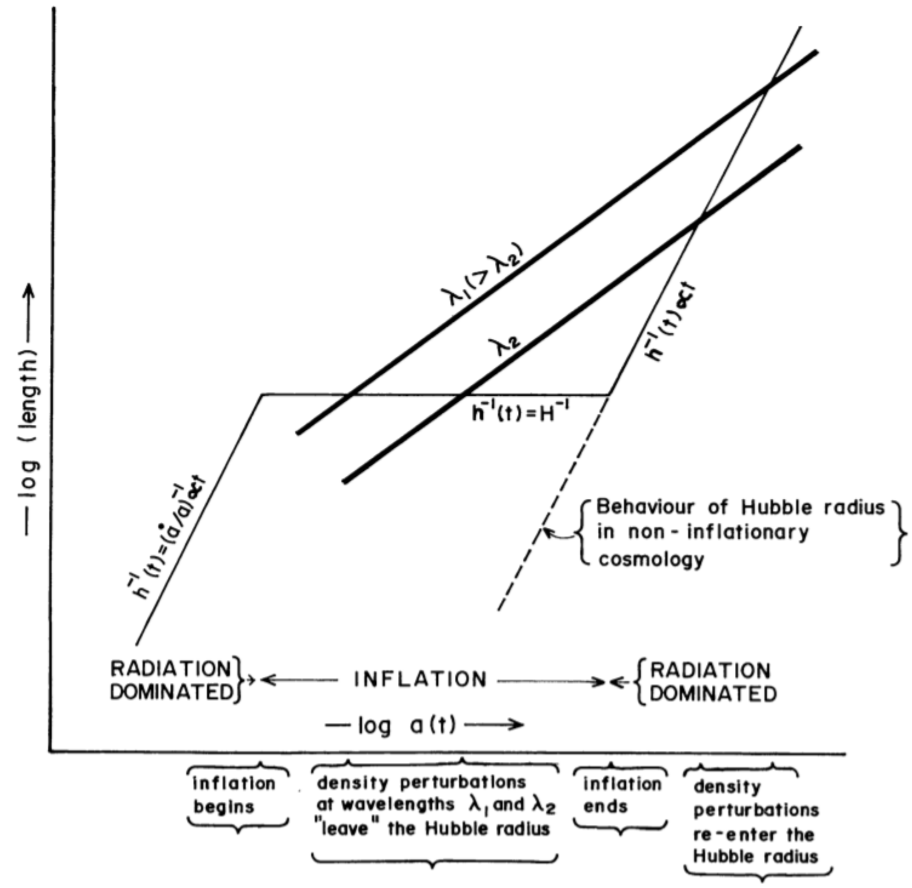


Figure 2.1: Figure illustrating the evolution of the physical wavelengths $\lambda_1, \lambda_2 \sim a$ and the Hubble radius. In this figure, h^{-1} denotes the Hubble radius and it is equal to H^{-1} during inflation. (Figure from Ref. [39].)

Thus, with $H = H_I$, $|\Omega - 1| \propto \exp(-2H_I t)$, or as time increases, the universe does indeed become more flat, which is entirely consistent with the observational data $|\Omega_0 - 1| \leq 0.2$ (this is borne out by the results of the type Ia supernova observations and the measurements of CMB anisotropy).

An estimate of the amount of inflation needed to overcome the flatness problem can be obtained by the following quantitative analysis. If $\kappa/a_i^2 H_i^2 \sim 1$ at the beginning of inflation then $\kappa/a_f^2 H_f^2 \sim e^{-2N}$ at the end of inflation, and equation (2.11) can be rewritten as

$$\Omega_0 = e^{-2N} \left(\frac{a_f H_f}{a_0 H_0} \right)^2 . \tag{2.13}$$

Thus the flatness problem is avoided if $e^N > a_f H_f / a_0 H_0$. To evaluate this we assume, for simplicity, that $a_f H_f \simeq a_r H_r$, subscript 'r' denoting the beginning of the radiation dominated era. Over the whole of the radiation and matter dominated era, the expansion rate was

$$H = \frac{H_{eq}}{\sqrt{2}} \sqrt{\left(\frac{a_{eq}}{a}\right)^3 + \left(\frac{a_{eq}}{a}\right)^4}, \tag{2.14}$$

where $a_{eq} = a_0\Omega_{rad}/\Omega_{mat}$, $H_{eq} = \sqrt{2\Omega_{mat}}H_0(a_0/a_{eq})^{3/2}$ and 'eq' stands for radiation-matter equality, 'ma for matter and 'rad' for radiation. Setting $a = a_r \ll a_{eq}$, we can obtain the following expression for H_r [40]

$$H_r = \frac{H_{eq}}{\sqrt{2}}\left(\frac{a_{eq}}{a_r}\right)^2. \tag{2.15}$$

Using this we get the following condition for N to satisfy

$$e^N > (\Omega_{mat}a_{eq})^{1/4}\sqrt{H_r/H_0} = \left(\Omega_{rad}\frac{\rho_r}{\rho_{0,c}}\right)^{1/4} = \frac{\rho_r^{1/4}}{0.037h\text{ eV}}, \tag{2.16}$$

where we have used $\rho_{0,c} = [3.00 \times 10^{-3}eV]^4h^2$, $\rho_{0,c}$ being the critical density of the universe today and $h = H_0/100$. It can be shown that $\rho_r \sim [2 \times 10^{16}GeV]^4$ [40], in which case e^N for $h = 0.72$ would have to be at least 8×10^{26}, so that $N > 62$.

2.2.2 The horizon problem

One of the most important properties of the CMB is that it is very nearly isotropic with radiation coming from all directions of the sky having almost the same temperature of 2.73K. Being at the same temperature indicates thermal equilibrium. But the radiation we see coming from one direction of the observable universe has been travelling towards us since the time of decoupling and has just reached us. Thus there has not been time for different regions of the universe to have interacted with each other, yet all regions are in thermal equilibrium. Also the CMB is not perfectly isotropic – it exhibits small fluctuations (of the order of 1 in 10^5) as detected by the COBE satellite, and subsequently by WMAP and Planck [41]. These irregularities act as seeds for structure formation in the universe. For the same reason that separated regions cannot be thermalised, irregularities can also not be created. Thus the hot big bang model does not allow the generation of seed perturbations – they have to be there already within the big bang model.

Let us get quantitative about the horizon problem. In the spatially flat FLRW universe, the horizon $h(t)$, viz., the size of a causally connected region is defined as the physical radial distance travelled by a light ray from the big bang singularity at $t = 0$ to some later time t. The horizon can be expressed in terms of the scale factor $a(t)$ as follows

$$h(t) = a(t)\int_0^t \frac{d\tilde{t}}{a(\tilde{t})}. \tag{2.17}$$

If we assume that the universe was radiation dominated from the big bang to the time of decoupling (t_{dec}) and matter dominated since then until now, then the linear dimensions of the backward and forward light cones denoted by l_B and l_F respectively are given by

$$l_B(t_0, t_{dec}) = a_{dec}\int_{t_{dec}}^{t_0} \frac{dt}{a_{MD}t^{2/3}} \approx 3\frac{a_{dec}}{a_{MD}}t_0^{1/3}, \tag{2.18}$$

where we have used the observational fact that $t_0(\approx 10^{10}$ years$) \gg t_{dec}(\approx 10^5$ years$)$ [42], and

$$l_F(t_{dec}, 0) = a_{dec}\int_0^{t_{dec}} \frac{dt}{a_{RD}t^{1/2}} = 2\frac{a_{dec}}{a_{RD}}t_{dec}^{1/2}. \tag{2.19}$$

The ratio R of the linear dimensions of the backward and the forward light cones is given by [42]

$$R \equiv \frac{l_B}{l_F} = \frac{3}{2}\left(\frac{t_0}{t_{dec}}\right)^{1/3} \approx 70, \tag{2.20}$$

where we have used $a_{dec} = a_{RD} t_{dec}^{1/2} = a_{MD} t_{dec}^{2/3}$ (a_{RD} and a_{MD} are constants of appropriate dimensions so that $a(t)$ becomes dimensionless). Inspite of this, the CMB turns out to be extraordinarily isotropic. So how does inflation solve the horizon problem?

If we consider the inflationary scenario, then the size of the horizon at the end of inflation was (using equation (2.8))

$$d_{hor}(t_f) = a_f \left(\int_0^{t_i} \frac{dt}{a_i (t/t_i)^{1/2}} + \int_{t_i}^{t_f} \frac{dt}{a_i \exp[H_I(t-t_i)]} \right).$$

If N is large, then at the end of inflation, the horizon size is (upon using $a_f = a_i e^N$)

$$d_{hor}(t_f) = e^N (2t_i + 1/H_I) \simeq 2t_i e^N, \tag{2.21}$$

(where we have used $t_i \approx 10^{-35}s \sim 10^{-11}GeV^{-1}$ [39] and $H_I \sim 10^{14}GeV$ [43]) whereas at the beginning of inflation, it was

$$d_{hor}(t_i) = a_i \int_0^{t_i} \frac{dt}{a_i \sqrt{t/t_i}} = 2t_i. \tag{2.22}$$

Thus an epoch of exponential inflation causes the horizon size to grow exponentially by a factor $\sim e^N$ over what it would have been without inflation. This indicates that a small area of the universe, small enough to be in thermal equilibrium before inflation, can expand to considerably larger volumes after inflation [44].

Estimation of N

When we consider inflation, the dominant contribution to the forward light cone comes from the exponential expansion of the universe during inflation [42]. In that case we have

$$l_{F_I}(t_{dec}, 0) = a_{dec} \int_0^{t_{dec}} \frac{dt}{a(t)} \simeq a_{dec} \int_{t_i}^{t_f} \frac{dt}{a(t)} = \frac{a_{dec}}{a_i} \frac{1-e^{-N}}{H_I} \simeq \frac{1}{H_I} \sqrt{\frac{t_{dec}}{t_f}} e^N, \tag{2.23}$$

where we have used $a_{dec}/a_i = e^N \sqrt{t_{dec}/t_f}$ (see equation (2.8)), and hence we have (upon assuming $t_f \approx 10^{-33}s$ [39])

$$R_I \equiv \left(\frac{l_B}{l_{F_I}}\right) = 3 t_0^{1/3} t_{dec}^{1/6} t_f^{1/2} \frac{H_I}{e^N} \simeq \left(\frac{10^{30}}{e^N}\right). \tag{2.24}$$

Thus for R_I to be of the order of unity, we need $a_f/a_i = e^N = 10^{30}$, which implies $N \simeq 69$. The number of inflationary e-foldings needed to solve the horizon problem essentially depends on the energy scale of inflation, as is evident from equation (2.24). Following Ref. [39], if we take $H_I \sim 10^{10}GeV$, then $N = 60$ suffices to solve the horizon problem. Similarly, inflation provides the opportunity to generate irregularities in the universe. As we shall later see, the quantum fluctuations associated with the inflation serve as the primordial seeds for the inhomogeneities.

2.2.3 The monopole problem

Inflation provides a resolution to the monopole problem as follows: If magnetic monopoles were created before or during inflation, then their number density is so much diluted due to inflation that the probability of detecting a monopole today turns out to be negligible.

At a very early time t in the hot big bang model, when the universe was radiation dominated, we had $H^2 = 8\pi G\rho_{rad}/3 \implies \dot{a}^2/a^2 \simeq 1/t^2 \simeq GT^4$. Hence the horizon size was of the order of $t \approx (GT^4)^{-1/2}$(where $G \approx (10^{19}GeV)^{-2}$ is the Newton's gravitational constant, and T the temperature), so the number density of monopoles produced at the time the temperature dropped to M ($M \approx 10^{16}GeV$ is the energy scale at which local symmetry under some simple symmetry group is spontaneously broken in Grand Unified Theories [40]) would have been of the order of $t^{-3} \approx (GM^4)^{3/2}$ which is smaller than the photon density $\approx M^3$ by a factor of $\sim (GM^2)^{3/2} = 10^{-9}$ [40].

However one monopole for every 10^9 photons is not observed today. Monopoles have been searched for in meteorites, iron ores etc. but none has been found. In ref. [45], the authors have concluded that the overall monopole/nucleon ratio in the samples is $< 1.2 \times 10^{-29}$, and hence fewer than 10^{-38} monopoles per photon [40]. Inflation resolves this paradox. As discussed in the previous subsection, inflation increases the horizon size by a factor of e^N. In order for inflation to have reduced the monopole/photon ratio to 10^{-30} the horizon size must have increased at least by a factor of 10^{10} which translates to N being greater than 23.

2.3 Driving the inflation with scalar fields

Consider a canonical scalar field ϕ described by the potential $V(\phi)$ and governed by the action

$$S[\phi] = -\int d^4x\sqrt{-g}[1/2(\partial_\lambda\phi\partial^\lambda\phi) + V(\phi)]. \tag{2.25}$$

The stress-energy tensor associated with such a scalar field is given by

$$T^\mu{}_\nu = -\partial^\mu\phi\partial_\nu\phi - \delta^\mu{}_\nu[1/2(\partial_\lambda\phi\partial^\lambda\phi) + V(\phi)]. \tag{2.26}$$

The isotropy and the homogeneity of the Friedmann universe implies that the stress-energy tensor will be diagonal with the time-time and the space-space components given by

$$T^0{}_0 = -\rho = -[\dot{\phi}^2/2 + V(\phi)], \tag{2.27}$$

$$T^i{}_j = p\delta^i{}_j = [\dot{\phi}^2/2 - V(\phi)]\delta^i{}_j. \tag{2.28}$$

So the energy conservation equation (equation (2.7)) takes the form

$$\ddot{\phi} + 3H\dot{\phi} + dV/d\phi = 0, \tag{2.29}$$

and the condition for inflation, viz. $(\rho + 3p) < 0$, reduces to

$$\dot{\phi}^2 < V(\phi). \tag{2.30}$$

Therefore, for inflation to take place, the potential term should dominate the kinetic term.

Using equations (2.27) and (2.28) for ρ and p, the Friedmann equations, with $\kappa = 0$ become

$$H^2 = \left(\frac{1}{3M_{Pl}^2}\right)(\dot{\phi}^2/2 + V(\phi)), \tag{2.31}$$

$$\dot{H} = -\frac{\dot{\phi}^2}{2M_{Pl}^2}, \tag{2.32}$$

where $M_{Pl} = 1/\sqrt{8\pi G}$, as defined earlier. These two equations can be combined to express the scalar field and the potential parametrically in terms of the cosmic time t,

$$\phi(t) = \sqrt{2}M_{Pl}\int dt\sqrt{-\dot{H}}, \tag{2.33}$$

$$V(t) = M_{Pl}^2(3H^2 + \dot{H}). \tag{2.34}$$

If we now know the scale factor $a(t)$, then we can use these equations to reverse engineer the potential from which such a scale factor can arise [42]. This point will be illustrated in the next subsection.

2.3.1 Power law inflation

This is a simple model of inflation with which we will concern ourselves in this report. Consider the power law expansion (with $q > 1$ for inflation)

$$a(t) = a_1 t^q, \tag{2.35}$$

a_1 being an arbitrary constant. On substituting this expression in equation (2.31), we obtain,

$$\frac{\phi(t)}{M_{Pl}} = \sqrt{2q}\ln\left(\frac{t}{\tilde{t}_0}\right), \tag{2.36}$$

$\tilde{t}_0$ being a constant of integration. Using equation (2.32) and denoting $V(\tilde{t}_0)$ by V_0, the above equation can be written in an alternate form as follows :

$$\frac{\phi(t)}{M_{Pl}} = \sqrt{2q}\ln\left[\sqrt{\frac{V_0}{3q(q-1)}}\left(\frac{t}{M_{Pl}}\right)\right], \tag{2.37}$$

It is then easy to obtain the potential corresponding to the given scale factor :

$$\frac{\phi}{M_{Pl}} = \sqrt{\frac{q}{2}}\ln\frac{V_0}{V} \Rightarrow V(\phi) = V_0\exp\left[-\sqrt{\frac{2}{q}}\frac{\phi}{M_{Pl}}\right]. \tag{2.38}$$

2.3.2 Slow roll inflation

In this approximation, the condition given by equation (2.30) is made stricter, thereby guaranteeing inflation : $\dot{\phi}^2 \ll V(\phi)$. Moreover, it can be ensured that the field is rolling sufficiently slowly for a long time (so as to cover at least 60 e-folds of inflation), provided $\ddot{\phi} \ll 3H\dot{\phi}$. This approximation is

usually described in terms of two slow roll parameters [42] :
i)The potential slow roll parameters(PSR), which are required to be smaller than unity, are

$$\epsilon_V \equiv \left(\frac{M_{Pl}^2}{2}\right)\left(\frac{V_\phi}{V}\right)^2 \quad ; \quad \eta_V \equiv M_{Pl}^2\left(\frac{V_{\phi\phi}}{V}\right), \tag{2.39}$$

where $V_\phi = dV/d\phi$ and $V_{\phi\phi} = d^2V/d\phi^2$.

The fact that the PSR parameters are small does not guarantee that inflation will occur–it is a necessary but not sufficient condition. This is due to the fact that these parameters only constrain the form of the potential and not the dynamics of the solution. As a result, we also require the additional condition that the scalar field moves slowly along the attractor solution determined by the equation. [42] : $3H\dot{\phi} \simeq -V_\phi$. Despite this limitation, the PSR parameters often prove handy. For example, consider the following potential :

$$V(\phi) = V_0\phi^n, \tag{2.40}$$

where V_0 is a constant and $n > 0$ and we restrict ourselves in the region $\phi > 0$ so that $V(\phi)$ is positive for all n. For this potential, $V_\phi/V = n/\phi$ and $V_{\phi\phi}/V = n(n-1)/\phi^2$. Therefore, the slow roll conditions, i.e., $\epsilon_V = M_{Pl}^2 n^2/2\phi^2 \ll 1$ and $\eta_V = M_{Pl}^2 n(n-1)/\phi^2 \ll 1$ are satisfied when $\phi \gg M_{Pl}$. Thus these parameters allows us to determine the domain and parameter of the potential that can lead to inflation.

ii)The Hubble Slow Roll parameters(HSR) turn out to be a better choice as they do not require additional conditions. The HSR parameters are so called since they are defined in terms of the Hubble parameter H which is treated as a function of ϕ.

$$\epsilon_H \equiv 2M_{Pl}^2\left(\frac{H_\phi}{H}\right)^2 \quad ; \quad \delta_H \equiv 2M_{Pl}^2\left(\frac{H_{\phi\phi}}{H}\right), \tag{2.41}$$

where $H_\phi = dH/d\phi$ and $H_{\phi\phi} = d^2H/d\phi^2$.

Equations (2.29), (2.31) and (2.32) can be used to rewrite the HSR parameters as follows:

$$\epsilon_H = \frac{3\dot{\phi}^2}{2\rho} = \frac{-\dot{H}}{H^2} \quad ; \quad \delta_H = \frac{-\ddot{\phi}}{H\dot{\phi}} = \epsilon_H - \frac{1}{2H}\left(\frac{\dot{\epsilon}_H}{\epsilon_H}\right), \tag{2.42}$$

where ρ is the energy density associated with the scalar field ϕ, which is often called the inflaton since it drives inflation. The following points are clear from these expressions–
a) $\epsilon_H \ll 1$ corresponds to neglecting the kinetic energy term,
b) $\delta_H \ll 1$ corresponds to the situation wherein the acceleration term can be ignored, and
c) the inflationary condition $\ddot{a} > 0$ exactly corresponds to $\epsilon_H < 1$.
It should be noted that since smallness of the HSR parameters ensure that $\dot{\phi}$ is small, the HSR approximation implies the PSR approximation, but the converse is not necessarily true [42].
Solutions in the slow roll approximation
We first write the equation of motion of the inflaton (equation (2.29)) and the acceleration equation in terms of the HSR parameters:

$$H^2\left(1 - \frac{\epsilon_H}{3}\right) = \frac{V}{3M_{Pl}^2}, \tag{2.43}$$

$$3H\dot{\phi}[1-\delta_H/3] = -V_\phi. \tag{2.44}$$

The slow roll approximation corresponds to the situation wherein the HSR parameters satisfy the following conditions:

$$\epsilon_H \ll 1, \quad \delta_H \ll 1 \quad \text{and} \quad \mathcal{O}[\epsilon_H^2, \delta_H^2, \epsilon_H\delta_H] \ll \epsilon_H. \tag{2.45}$$

At the leading order in slow roll approximation, equations (2.43) and (2.44) reduce to

$$H^2 \simeq V/3M_{Pl}^2 \quad \text{and} \quad 3H\dot{\phi} \simeq -V_\phi. \tag{2.46}$$

Let us now try to solve these first order equations by considering the potential in equation (2.40). Then equation (2.46) implies

$$\sqrt{\frac{3}{V_0}}\frac{1}{nM_{Pl}}\dot{\phi} = \phi^{n/2-1},$$

which can be solved to get

$$\phi^{2-n/2}(t) \simeq \phi_i^{2-n/2} + \sqrt{\frac{V_0}{3}}\left[\frac{n(n-4)}{2}\right] M_{Pl}(t-t_i) \quad ,n \neq 4 \tag{2.47}$$

$$\phi(t) \simeq \phi_i(t)\exp\left[-\sqrt{V_0/3}\times 4M_{Pl}(t-t_i)\right] \quad ,n=4 \tag{2.48}$$

where $\phi_i = \phi(t_i)$ is a constant at some initial time t_i. From the equations in (2.46), for all n, the scale factor can be expressed in terms of these solutions as follows

$$a(t) \simeq a_i \exp\left[-(\phi(t)^2 - \phi_i^2)/2nM_{Pl}^2\right], \tag{2.49}$$

with $a(t_i) = a_i$.

We can also write the scalar field and Hubble parameter in terms of the number of e-folds from $t = t_i$ to $t = t$ by noting

$$N = \ln\frac{a}{a_i} = \int_{t_i}^{t} dtH \simeq -\left(\frac{1}{M_{Pl}^2}\right)\int_{\phi_i}^{\phi} d\phi\left(\frac{V}{V_\phi}\right). \tag{2.50}$$

Thus we have

$$\phi^2(N) \simeq \phi_i^2 - (2M_{Pl}^2 n)N \quad \text{and} \quad H^2(N) \simeq \left(\frac{V_0 M_{Pl}^{(n-2)}}{3}\right)\left[(\phi_i/M_{Pl})^2 - 2nN\right]^{n/2}. \tag{2.51}$$

Inflationary attractor

Let us consider the following quadratic potential : $V(\phi) = \frac{1}{2}m^2\phi^2$. In this case equation (2.29) when supplemented by the first Friedmann equation,equation (2.31), becomes

$$\frac{d\dot{\phi}}{d\phi} = -\frac{\sqrt{12\pi G(\dot{\phi}^2 + m^2\phi^2)}\dot{\phi} + m^2\phi}{\dot{\phi}},$$

which can be studied using the phase diagram method [30]. The behaviour of the solutions in the $\phi - \dot{\phi}$ plane is shown in figure 2.2, which shows the attractor solution, determined in this case by the equation $3H\dot{\phi} = -m^2\phi \implies \dot{\phi}_{atr} \approx \pm\frac{m}{\sqrt{12\pi G}} = \pm\frac{mM_{Pl}}{\sqrt{3/2}}$. (The upper sign corresponds to $\phi < 0, \dot{\phi} > 0$ and the lower sign corresponds to $\phi > 0, \dot{\phi} < 0$.)

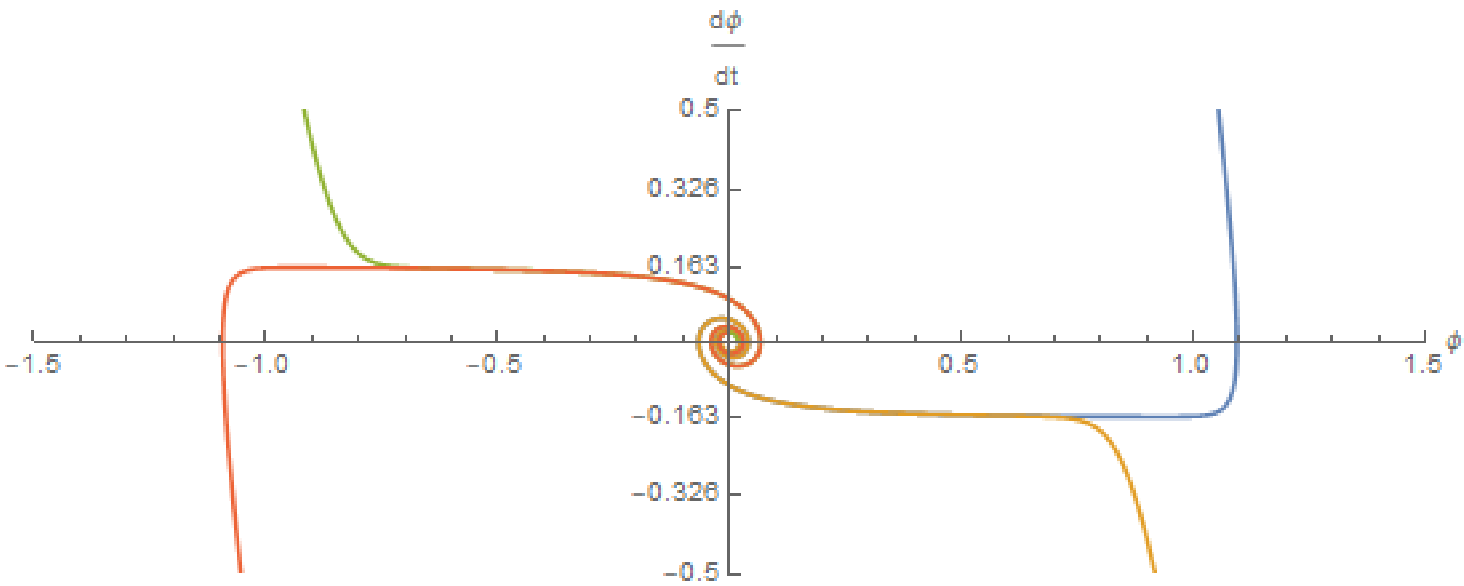


Figure 2.2: The phase space diagram for the potential $V = \frac{1}{2}m^2\phi^2$ obtained by setting $G = 1$ and $m = 1$. The phase space diagram shows the attractor solutions $\dot{\phi}_{atr} \approx \pm 1/\sqrt{12\pi} = \sqrt{2/3}M_{Pl}$.

Starobinsky model

Finally we examine a model due to Starobinsky, that, besides being historically the first inflationary model proposed, is also the one that fits best the current cosmological data [46]. Consider the potential

$$V(\phi) = \mathcal{M}^4(1 - e^{-\sqrt{2/3}\phi/M_{Pl}})^2, \tag{2.52}$$

plotted in figure 2.3. The equation of motion without the slow roll approximation is

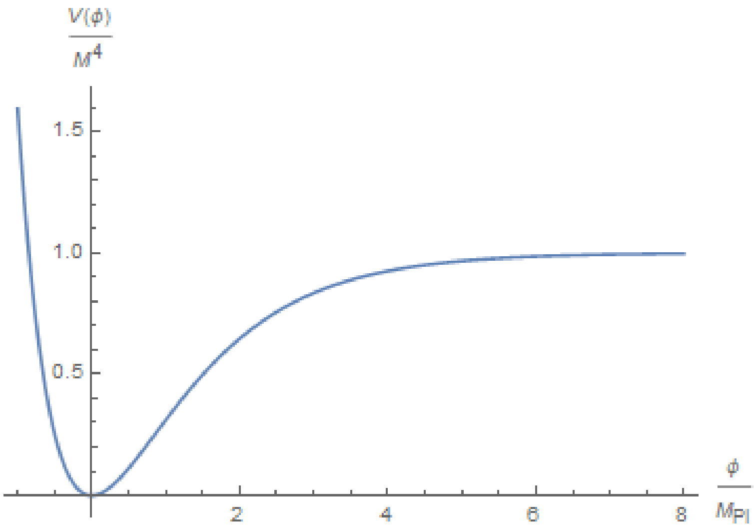


Figure 2.3: The Starobinsky potential (an example of plateau inflation)– it has a long, flat part at large values of the field where the slow roll conditions can be realized.

$$\ddot{\phi} + \frac{\sqrt{3}}{M_{Pl}}\sqrt{\dot{\phi}^2/2 + V(\phi)}\dot{\phi} + \sqrt{\frac{8}{3}}\frac{\mathcal{M}^4}{M_{Pl}}(1 - e^{-\sqrt{2/3}\phi/M_{Pl}})e^{-\sqrt{2/3}\phi/M_{Pl}} = 0, \tag{2.53}$$

and with the slow roll approximation, it becomes

$$\sqrt{3}\frac{\sqrt{V(\phi)}}{M_{Pl}}\dot{\phi} \simeq -V_\phi = -\sqrt{\frac{8}{3}}\frac{\mathcal{M}^4}{M_{Pl}}(1 - e^{-\sqrt{2/3}\phi/M_{Pl}})e^{-\sqrt{2/3}\phi/M_{Pl}},$$

or

$$\dot{\phi} \simeq -\frac{2\sqrt{2}}{3}\mathcal{M}^2 e^{-\sqrt{2/3}\phi/M_{Pl}}. \tag{2.54}$$

If we define $\varphi \equiv \sqrt{2/3}\phi/M_{Pl}$, then the potential becomes $V(\varphi) = \mathcal{M}^4(1 - e^{-\varphi})^2$ and the potential slow roll parameters can be written as

$$\epsilon_V(\varphi) = \frac{4/3}{(e^\varphi - 1)^2} \quad \text{and} \quad \eta_V = -\frac{4}{3}\frac{e^{-\varphi}(1 - 2e^{-\varphi})}{(1 - e^{-\varphi})^2}, \tag{2.55}$$

which in the large field limit behave as $\epsilon_V \simeq (4/3)e^{-2\varphi}$ and $\eta_V \simeq (-4/3)e^{-\varphi}$, so the slow roll parameters satisfy $\epsilon_V \ll |\eta_V| \ll 1$. However for small fields, the condition $|\epsilon_V| < 1$ is violated well before $\varphi \simeq 0.8$ (see figure 2.4) and we can say that the slow roll ends at $\phi = \phi_{end} \simeq M_{Pl}$.

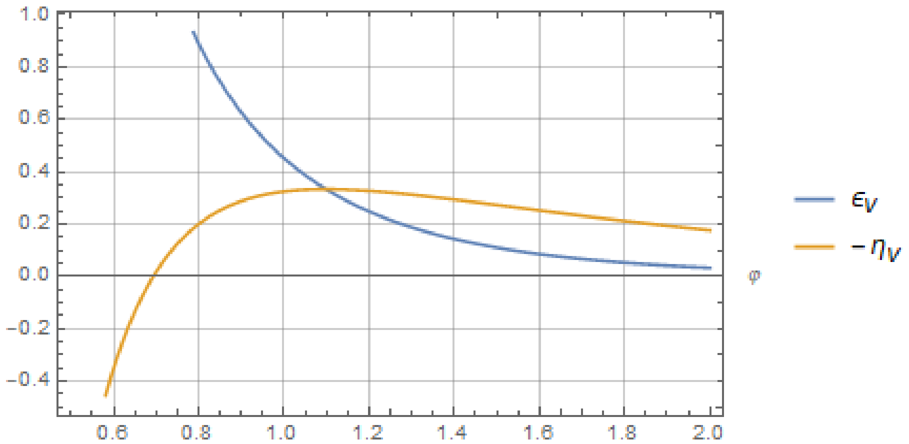


Figure 2.4: The potential slow roll parameters at small fields.

With ϕ_{end} at hand, we can estimate the number of e-folds to the end of inflation,

$$N = -\frac{1}{M_{Pl}^2}\int_\phi^{\phi_{end}} \frac{V}{V_\phi}d\phi.$$

If we set $x = exp[\sqrt{2/3}\phi/M_{Pl}]$, then $dx/d\phi = \frac{\sqrt{2/3}}{M_{Pl}}x$ and we will have the following expression for N,

$$\begin{aligned} N &= -\frac{3}{2}\int_x^{x_{end}} \frac{(1 - 1/x)^2}{2(1 - 1/x)/x^2}\frac{dx}{x^2} = -\frac{3}{4}(x - \ln x)|_x^{e^{\sqrt{2/3}}} \\ &= -\frac{3}{4}\left[e^{\sqrt{2/3}} - \sqrt{2/3} - e^{\sqrt{2/3}\phi/M_{Pl}} + \frac{\sqrt{2/3}\phi}{M_{Pl}}\right] \simeq \frac{3}{4}e^{\sqrt{2/3}\phi/M_{Pl}}, \end{aligned} \tag{2.56}$$

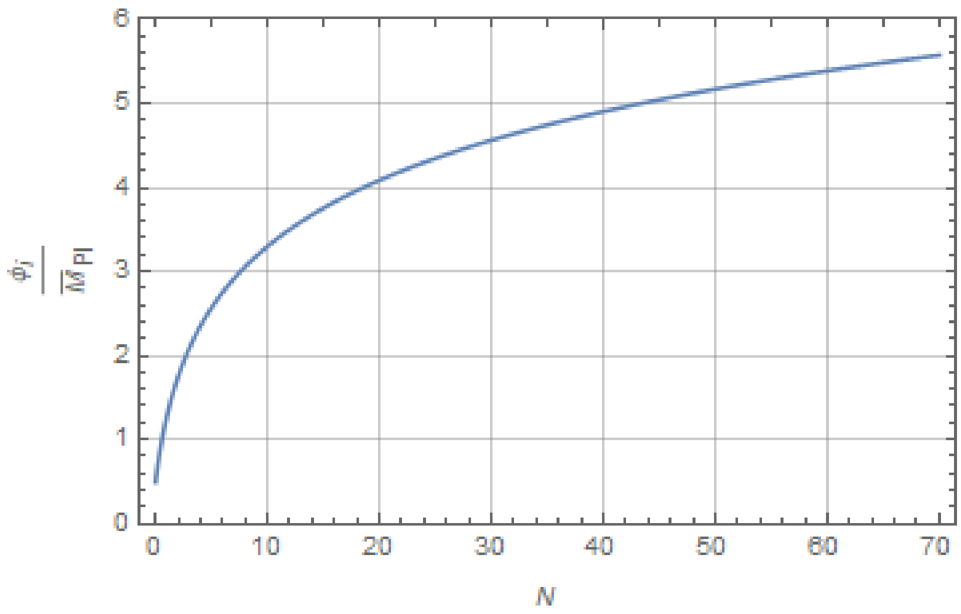


Figure 2.5: Behaviour of the initial value of the inflaton as a function of N in the Starobinsky model.

where we have used the fact that the integral is dominated by the upper integration limit since $\phi/M_{Pl} > 1$.

$$4N/3 \simeq \exp\sqrt{2/3}\phi/M_{Pl} \implies \frac{\phi(N)}{M_{Pl}} \simeq \sqrt{\frac{3}{2}}\ln\left(4N/3\right). \tag{2.57}$$

From the above expression for $\phi(N)$, we find $N(\phi_{Pl}) \simeq 1.69$ and if we consider 60 inflationary e-folds, then $\phi_i = \phi(61.69) \simeq 5.40 m_{Pl}$. If N_{act} is the actual number of inflationary e-folds then the initial field is given by

$$\phi_i \simeq M_{Pl}\left[5.40 + 1.18\ln\left(N_{act}/60\right)\right]. \tag{2.58}$$

This has been plotted in figure 2.5.

2.4 Reheating

We end this chapter by discussing in brief what happens after inflation ends. The radiation cools down dramatically as a result of the rapid expansion during inflation. The energy density of the universe (radiation) which remains locked in the scalar field (the inflaton) at the end of inflation must be released and the universe rapidly reheated to its original temperature in order to preserve the key features of the hot big bang model (such as nucleosynthesis which begins at $T \sim 1MeV$ [47]). Reheating refers to this process of transferring energy from the inflaton to radiation, allowing the universe to return to its pre-inflation temperature. [48]. In this report, we will only deal with inflation and not concern ourselves with what happens in the post-inflationary epoch or during reheating.

Chapter 3

Generation of magnetic fields I — Coupling to the inflaton

The fact that the standard electromagnetic action is conformally invariant and the spatially flat FLRW metric is conformally flat creates a serious problem when discussing magnetic field creation during inflation. Consider the electromagnetic action (in Lorentz-Heaviside units)

$$\mathcal{S}_{EM} = -\int \sqrt{-g} d^4x \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\int \sqrt{-g} d^4x \frac{1}{4} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta}. \tag{3.1}$$

Now suppose we make a conformal transformation of the metric :

$$\tilde{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}. \tag{3.2}$$

This implies

$$\sqrt{-\tilde{g}} = \Omega^4 \sqrt{-g} \qquad \text{and} \qquad \tilde{g}^{\mu\nu} = \Omega^{-2} g^{\mu\nu}. \tag{3.3}$$

Then taking $\tilde{A}_\mu = A_\mu$ and $\tilde{x}^\nu = x^\nu$ implies

$$\tilde{\mathcal{S}}_{EM} = -\int \Omega^4 \sqrt{-g} d^4x \frac{1}{4} (\Omega^{-2} g^{\mu\alpha})(\Omega^{-2} g^{\nu\beta}) F_{\mu\nu} F_{\alpha\beta} = \mathcal{S}_{EM}. \tag{3.4}$$

Thus the action of the free electromagnetic (EM) field is invariant under conformal transformations.

The FLRW metric is also conformally flat :

$$g_{\mu\nu}^{FLRW} = \Omega^2 \eta_{\mu\nu}, \tag{3.5}$$

$\eta_{\mu\nu}$ being the flat space-time Minskowski metric. This implies that one can transform the electromagnetic wave equation into its Minkowskian flat space-time version. It turns out that one cannot amplify the magnetic fields in such a FLRW universe and the field then always decreases with expansion as $1/a^2(t)$. Therefore mechanisms for magnetic field generation should involve the breaking of the conformal invariance which changes the behaviour to $B \sim 1/a^\epsilon$ ($\epsilon \ll 1$ for getting a strong field) [10]. This chapter and the following chapter deal with a couple of different ways of doing the same.

3.1 Generating non-helical magnetic fields

Consider the following action [10] in which we assume that the inflaton ϕ is the only (scalar) field that breaks conformal invariance,

$$\mathcal{S}_1 = -\frac{1}{16\pi}\int d^4x\sqrt{-g}[g^{\mu\alpha}g^{\nu\beta}f^2(\phi)F_{\mu\nu}F_{\alpha\beta}] - \int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right]. \tag{3.6}$$

The Maxwell equations now become

$$[f^2F^{\mu\nu}]_{;\nu} = 0 \quad \text{or} \quad \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}[\sqrt{-g}g^{\mu\alpha}g^{\nu\beta}f^2(\phi)F_{\alpha\beta}] = 0, \tag{3.7}$$

where $[...]_{;\nu}$ denotes covariant derivative with respect to x^ν. Assuming the electromagnetic field to be a "test" field which does not affect either the scalar field evolution or the evolution of the background FLRW universe, we take the spatially flat metric of equation (2.3). It is convenient to adopt the Coulomb gauge :

$$A_\eta(\eta,\mathbf{x}) = 0; \qquad \nabla\cdot\mathbf{A}(\eta,\mathbf{x}) = 0. \tag{3.8}$$

In this case, setting $\mu = i$ in equation (3.7) yields

$$\frac{\partial}{\partial\eta}[f^2A_i'] + \frac{\partial}{\partial x^j}[f^2(-\partial_jA_i)] = f^2A_i'' + 2ff'A_i' - f^2\partial_j\partial_jA_i = 0,$$

where overprime denotes derivative with respect to conformal time η. Dividing throughout by f^2 and using $\partial_j = g_{jk}\partial^k = a^2\partial^j$, we get

$$A_i'' + 2\frac{f'}{f}A_i' - a^2\partial_j\partial^jA_i = 0. \tag{3.9}$$

From the above equation we see that A_i satisfies the usual wave equation in η and $\mathbf{x}$ coordinates for a constant f with plane wave solutions of constant amplitude. Then the amplitude of B_i scales as $1/a$ and that of B^i scales as $1/a^3$ and so the amplitude of $\bar{B}^a$ (defined in Appendix A) scales as $1/a^2$.

3.1.1 Quantizing the electromagnetic field

We would like to quantize the EM field in the FLRW background. To do so, we first calculate the momentum Π^i conjugate to the field A_i :

$$\Pi^i = \frac{\delta\mathscr{L}_1}{\delta A_i'} = f^2a^2g^{ij}A_j', \tag{3.10}$$

where $\mathscr{L}_1 = -f^2F_{\mu\nu}F^{\mu\nu}/4$, and then promote them to operators and impose the canonical quantization condition

$$[\hat{A}^i(\eta,\mathbf{x}),\hat{\Pi}_j(\eta,\mathbf{y})] = i\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}.(\mathbf{x}-\mathbf{y})}P^i_j(\mathbf{k}) = i\delta^i_{\perp j}(\mathbf{x}-\mathbf{y}), \tag{3.11}$$

where

$$P^i_j(\mathbf{k}) = \delta^i_j - \delta_{jm}\frac{k^ik^m}{k^2}, \tag{3.12}$$

is introduced to ensure that the Coulomb gauge condition is satisfied and $\delta_\perp$ is the transverse delta function.

Decomposing the vector potential in Fourier space, we have

$$\hat{A}^i(\eta,\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \sum_{\lambda=1,2} e^i_\lambda[\hat{b}_\lambda(\mathbf{k})A(k,\eta)\exp i\mathbf{k}\cdot\mathbf{x} + \hat{b}^\dagger_\lambda(\mathbf{k})A^*(k,\eta)\exp -i\mathbf{k}\cdot\mathbf{x}], \tag{3.13}$$

where $\hat{b}^\dagger_\lambda(\mathbf{k})$ and $\hat{b}_\lambda(\mathbf{k})$ are the creation and annihilation operators, $\mathbf{k}$ is the comoving wave vector and $e^i_\lambda(\mathbf{k})$ represents the polarization vector which take care of the Coulomb gauge condition and which form a part of an orthonormal set of basis four-vectors,

$$e^\mu_0 = (1/a, \mathbf{0}), \quad e^\mu_\lambda = (0, \hat{e}^i_\lambda/a), \quad e^\mu_3 = (0, \widehat{\mathbf{k}}/a). \tag{3.14}$$

The three-vectors $\hat{e}^i_\lambda$ are unit vectors orthogonal to $\widehat{\mathbf{k}}$ and to each other. The creation and annihilation operators satisfy the following commutation relations:

$$[\hat{b}_\lambda(\mathbf{k}), \hat{b}^\dagger_{\lambda'}(\mathbf{k}')] = (2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')\delta_{\lambda\lambda'} \quad ; \quad [\hat{b}_\lambda(\mathbf{k}), \hat{b}_{\lambda'}(\mathbf{k}')] = [\hat{b}^\dagger_\lambda(\mathbf{k}), \hat{b}^\dagger_{\lambda'}(\mathbf{k}')] = 0, \tag{3.15}$$

and the orthonormal set of basis four-vectors satisfy the completeness relation :

$$\sum_{\lambda=1,2} e^i_\lambda(\mathbf{k})e_{j\lambda}(\mathbf{k}) = P^i_j(\mathbf{k}). \tag{3.16}$$

Thus if we substitute

$$\hat{A}^i(\eta,\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \sum_{\lambda=1,2} \hat{e}^i_\lambda[\hat{b}_\lambda(\mathbf{k})\bar{A}(k,\eta)\exp i\mathbf{k}\cdot\mathbf{x} + \hat{b}^\dagger_\lambda(\mathbf{k})\bar{A}^*(k,\eta)\exp -i\mathbf{k}\cdot\mathbf{x}],$$

in equation (3.9), we get the following equation for the Fourier coefficients $\bar{A} = aA(k,\eta)$

$$\bar{A}'' + 2\frac{f'}{f}\bar{A}' + k^2\bar{A} = 0. \tag{3.17}$$

The above equation can be recast to get rid of the first derivative term by defining $\mathcal{A} = afA$:

$$\mathcal{A}'' + \left(k^2 - \frac{f''}{f}\right)\mathcal{A} = 0. \tag{3.18}$$

Substituting the Fourier expansions of $\hat{A}^i$ and $\hat{\Pi}_j$ in equation (3.11) and making use of equation (3.15), we see

$$[\hat{A}^i(\eta,\mathbf{x}), \hat{\Pi}_j(\eta,\mathbf{y})] = i\int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}.(\mathbf{x}-\mathbf{y})} \sum_{\lambda=1,2} e^i_\lambda(\mathbf{k})e_{j\lambda}(\mathbf{k})W(k,\eta)f^2a^2, \tag{3.19}$$

where we have defined the complex Wronskian $W(k,\eta) = [AA'^* - A^*A']$. We can also define $\bar{W}(k,\eta) = [\bar{A}\bar{A}^{*\prime} - \bar{A}^*\bar{A}'] = a^2W$ and since $\bar{A}$ satisfies equation (3.17), we have $\bar{W}' = -(2f'/f)\bar{W}$, which upon integration gives $\bar{W} \propto (1/f^2)$. The quantization condition given by equation (3.11) is satisfied provided we set the constant of proportionality to be i such that $W = i/(f^2a^2)$.

Energy density of the EM field

The energy momentum tensor for the EM field is given by varying the Lagrangian density with respect to the metric :

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta[\sqrt{-g}\mathscr{L}_1]}{\delta g^{\mu\nu}} = -2f^2 \frac{\delta \mathscr{L}_{EM}}{\delta g^{\mu\nu}} - \frac{2f^2}{\sqrt{-g}} \mathscr{L}_{EM} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}, \tag{3.20}$$

where $\mathscr{L}_{EM}$ is the standard EM Lagrangian. Now using

$$\delta\sqrt{-g} = \frac{-1}{2\sqrt{-g}}\delta g = \frac{-1}{2\sqrt{-g}} g g^{\mu\nu}\delta g_{\mu\nu},$$

we get

$$T_{\mu\nu} = f^2 \left[g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - g_{\mu\nu} \frac{F_{\alpha\beta}F^{\alpha\beta}}{4} \right]. \tag{3.21}$$

The EM energy densities in the ground state measured by the fundamental observers having four-velocity $u^\mu = (1/a, \mathbf{0})$ are given by

$$\rho_B = \langle 0| T^B_{\mu\nu} u^\mu u^\nu |0\rangle \quad ; \quad \rho_E = \langle 0| T^E_{\mu\nu} u^\mu u^\nu |0\rangle , \tag{3.22}$$

where $|0\rangle$ denotes the vacuum state satisfying $\hat{b}_\lambda |0\rangle = 0$ and

$$T_{\mu\nu} u^\mu u^\nu = T^B_{\mu\nu} u^\mu u^\nu + T^E_{\mu\nu} u^\mu u^\nu = f^2 \frac{B^i B_i}{2} + f^2 \frac{E^i E_i}{2}.$$

In arriving at the above equation, the following definitions have been used(see Appendix A):

$$B_i \equiv \frac{1}{a} \eta_{ijk} \delta^{jm} \delta^{kn} \partial_m A_n \quad ; \quad E_i \equiv -\frac{1}{a} A_i'.$$

Thus

$$\begin{aligned}
\rho_B &= \frac{f^2}{2a^2} \langle 0| (\partial_i A_m - \partial_m A_i)(\partial_j A_l - \partial_l A_j) g^{ij} g^{ml} |0\rangle \\
&= \frac{f^2}{2a^2} \int \frac{d^3k}{(2\pi)^3} g^{ij} g^{ml} \sum_{\lambda=1,2} (\hat{e}_{\lambda m}(k) k_i - \hat{e}_{\lambda i}(k) k_m)(\hat{e}_{\lambda l}(k) k_j - \hat{e}_{\lambda j}(k) k_l) |\bar{A}(\eta, k)|^2 \\
&= \frac{f^2}{2} \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{a^4} \left[\sum_{\lambda=1,2} \hat{e}^l_\lambda(k) \hat{e}_{l\lambda}(k) - \frac{k^l k_j}{k^2} \sum_{\lambda=1,2} \hat{e}^j_\lambda(k) \hat{e}_{\lambda l}(k) \right] |\bar{A}(\eta, k)|^2 \\
&= \frac{1}{(2\pi)^2} \int dk \frac{k^4}{a^4} \left[2 - \frac{k^l k_j}{k^2} \left(\delta^j_l - \frac{k^j k_l}{k^2} \right) \right] |\mathcal{A}(\eta, k)|^2,
\end{aligned}$$

where the third line follows from equation (3.15) and the fourth from equation (3.16). Therefore we finally arrive at the following

$$\rho_B = \frac{1}{2\pi^2} \int dk \left(\frac{k}{a} \right)^4 |\mathcal{A}(\eta, k)|^2. \tag{3.23}$$

Similarly, we have

$$\rho_E = \frac{f^2}{2a^2} \langle 0| A_i' A_j' g^{ij} |0\rangle = \frac{f^2}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{a^4} \left[\sum_{\lambda=1,2} \hat{e}^i_\lambda(k) \hat{e}_{i\lambda}(k) \right] |\bar{A}'(\eta, k)|^2.$$

Therefore,

$$\rho_E = \frac{f^2}{2\pi^2}\int dk \frac{k^2}{a^4}\left|\left[\frac{\mathcal{A}(\eta,k)}{f}\right]'\right|^2. \tag{3.24}$$

Now

$$\rho_B \equiv \int \frac{dk}{k}\frac{d\rho_B}{d\ln k} \quad \text{and} \quad \rho_E \equiv \int \frac{dk}{k}\frac{d\rho_E}{d\ln k}$$

implies

$$\text{magnetic field spectral energy density}: \ \frac{d\rho_B}{d\ln k} = \frac{1}{2\pi^2}\left(\frac{k}{a}\right)^4 k|\mathcal{A}(k,\eta)|^2, \tag{3.25}$$

$$\text{electric field spectral energy density}: \ \frac{d\rho_E}{d\ln k} = \frac{f^2}{2\pi^2}\left(\frac{k}{a}\right)^4 \frac{1}{k}\left|\left[\frac{\mathcal{A}(k,\eta)}{f}\right]'\right|^2. \tag{3.26}$$

Thus one needs to calculate the evolution of the mode function in order to calculate the evolution of the energy densities.

3.1.2 Evolution of normal modes

Let us consider again the power law expansion given by equation (2.35), but this time replacing a_1 by a_0/t_0^q : $a(t) = a_0(t/t_0)^q$ with $q > 1$. Integrating $dt = ad\eta$, we get

$$\eta = -\frac{t_0}{a_0(q-1)}\left(\frac{t}{t_0}\right)^{-1/(q-1)} \quad \text{or} \quad a(\eta) = a_0\left[\frac{-a_0(q-1)\eta}{t_0}\right]^{-q/(q-1)}.$$

Therefore, defining $-q/(q-1) = 1+\beta$, we have[1]

$$a(\eta) = a_0\left|\frac{\eta}{\eta_0}\right|^{1+\beta}, \tag{3.27}$$

where $\eta_0 = (2+\beta)(t_0/a_0)$. During inflation, the conformal time lies in the range $\eta \in (-\infty, 0)$ and after the end of inflation, it lies in the range $\eta \in (0,\infty)$. Also during inflation, $\beta \le -2$ and the quantity η_0 is negative. However during matter or radiation domination $\beta = -1$ or $\beta = 0$ respectively and $\eta_0 > 0$.

Let us also take the gauge coupling function f to evolve as a power law : $f(\eta) \propto a^\alpha$. In that case, we have, with $\gamma = \alpha(1+\beta)$

$$\frac{f''}{f} = \frac{\gamma(\gamma-1)}{\eta^2}.$$

Using equation (3.18), the evolution of the mode function is then given by

$$\mathcal{A}'' + \left(k^2 - \frac{\gamma(\gamma-1)}{\eta^2}\right)\mathcal{A} = 0, \tag{3.28}$$

whose solution can be written in terms of Bessel functions,

$$\mathcal{A} = (-k\eta)^{1/2}[C_1(k)J_{\gamma-1/2}(-k\eta) + C_2(k)J_{-\gamma+1/2}(-k\eta)], \tag{3.29}$$

where the coefficients are to be set by initial conditions. Equation (3.18) of which equation (3.28) is a special case, can in general be solved in two regions – region I corresponding to $k^2 \gg f''/f$ and

[1] The case $\beta = -2$ or equivalently $q \gg 1$ corresponds to the de Sitter space-time wherein we have $a(t) \propto \exp Ht$.

region II corresponding to $k^2 \ll f''/f$. For the particular case of power law evolution of f we are considering, region I corresponds to what is called the sub-Hubble limit and region II corresponds to the super-Hubble limit which is discussed below.

The initial conditions are specified for each mode (or wavenumber k) when it is deep within the Hubble radius, given by $R_H = 1/H$, that is, when the proper length scale associated with the mode, (a/k) is much smaller than R_H, i.e. $(k/aH) \gg 1$. For such small scales, one assumes that the effects of space-time curvature are negligible. Now $H = \dot{a}/a = a'/a^2 = -q(q-1)/\eta$, and for $q \gg 1$, $aH \to -1/\eta$. Thus $k/aH = -k\eta$. A given mode is therefore within the Hubble radius for $-k\eta > 1$ and outside R_H for $-k\eta < 1$. In the asymptotic limit($|k\eta| \to \infty$), equation (3.28) admits plane wave solutions $\mathcal{A} \propto \exp \pm ik\eta$. But the assumption that the gauge field for these modes is closest to the Minkowski space vacuum state that we can have in a curved space-time leads us to pick the solution $\mathcal{A} = c_0 \exp -ik\eta$ where the constant is set to $c_0 = 1/\sqrt{2k}$ by using the Wronskian condition that $W = i/(f^2a^2)$. Thus we assume the initial condition to be $\mathcal{A} \to \exp(-ik\eta)/\sqrt{2k}$ as $(-k\eta \to \infty)$. This fixes $C_1(k)$ and $C_2(k)$ to be

$$C_1(k) = \sqrt{\frac{\pi}{4k}} \frac{\exp(-i\pi\gamma/2)}{\cos \pi\gamma} \quad ; \quad C_2(k) = \sqrt{\frac{\pi}{4k}} \frac{\exp(i\pi(\gamma+1)/2)}{\cos \pi\gamma}, \tag{3.30}$$

where we have used the asymptotic expansion

$$J_\nu(x) \to \sqrt{\frac{2}{\pi x}} \cos[x - (\nu + 1/2)\pi/2] \qquad \text{as } x \to \infty.$$

In the opposite limit $(-k\eta \to 0)$, we get from equation (3.29)

$$\mathcal{A} \to k^{-1/2}[\tilde{C}_1(\gamma)(-k\eta)^{\gamma} + \tilde{C}_2(\gamma)(-k\eta)^{(1-\gamma)}], \tag{3.31}$$

with

$$\tilde{C}_1(\gamma) = \frac{\sqrt{\pi}}{2^{\gamma+1/2}} \frac{\exp -i\pi\gamma/2}{\Gamma(\gamma+1/2)\cos(\pi\gamma)} \quad ; \quad \tilde{C}_2(\gamma) = \frac{\sqrt{\pi}}{2^{3/2-\gamma}} \frac{\exp(i\pi(\gamma+1)/2}{\Gamma(3/2-\gamma)\cos(\pi\gamma)} \tag{3.32}$$

where we have used the property that

$$J_\nu(x) \to \frac{x^\nu}{2^\nu \Gamma(\nu+1)} \qquad \text{as } x \to 0.$$

From equation (3.31) it is easy to see that $\tilde{C}_1$ term dominates for $\gamma \leq 1/2$ and $\tilde{C}_2$ term dominates for $\gamma \geq 1/2$.

3.1.3 The spectra of the electric and magnetic fields

Let us substitute equation (3.31) into equation (3.25) to calculate the spectrum of ρ_B in the late time, super Hubble limit.

$$\frac{d\rho_B}{d\ln k} = \frac{1}{2\pi^2}\left(\frac{k}{a}\right)^4 |[\tilde{C}_1(\gamma)(-k\eta)^{\gamma} + \tilde{C}_2(\gamma)(-k\eta)^{(1-\gamma)}]|^2.$$

For $\gamma \leq 1/2$ the $\tilde{C}_1(\gamma)$ dominates. Then ignoring the other term,

$$\frac{d\rho_B}{d\ln k} \approx \frac{1}{2\pi^2}\left(\frac{k}{a}\right)^4 (-k\eta)^{4+2\gamma}\frac{\pi}{2^{2\gamma+1}\Gamma^2(\gamma+1/2)\cos^2\pi\gamma},$$

and for $\gamma \geq 1/2$, considering only the contribution from dominating $\tilde{C}_2(\gamma)$ term, we have

$$\frac{d\rho_B}{d\ln k} \approx \frac{1}{2\pi^2}\left(\frac{k}{a}\right)^4 (-k\eta)^{4+2\gamma}\frac{\pi}{2^{2-2\gamma}\Gamma^2(3/2-\gamma)\cos^2\pi\gamma}.$$

Therefore, we have the magnetic spectrum as follows

$$\frac{d\rho_B}{d\ln k} \approx \frac{\mathscr{F}(n)}{2\pi^2} H_I^4 (-k\eta)^{4+2n}, \tag{3.33}$$

where $n = \gamma$ for $\gamma \leq 1/2$ and $n = 1-\gamma$ for $\gamma \geq 1/2$ and

$$\mathscr{F}(n) = \frac{\pi}{2^{2n+1}\Gamma^2(n+1/2)\cos^2(n\pi)}. \tag{3.34}$$

During slow roll inflation, H is expected to vary very slowly and the effect of $(-k\eta)^{4+2n}$ is predominant. One can see that the property of scale invariance of the spectrum (with $4+2n=0$), and having $\rho_B \sim a^0$ go together, and they require $\gamma = 3$ or $\gamma = -2$.

Similarly we can calculate the electric field spectrum. To do so, first we need to calculate $[\mathcal{A}/f]'$, for which we substitute $x = -k\eta$, $\nu = \gamma - 1/2$ and use the following properties,

$$J'_\nu - \frac{\nu}{x}J_\nu = -J_{\nu+1} \quad ; \quad J'_\nu + \frac{\nu}{x}J_\nu = -J_{\nu-1},$$

and then take the limit $x \to 0$. Finally substituting this into equation (3.26), we get

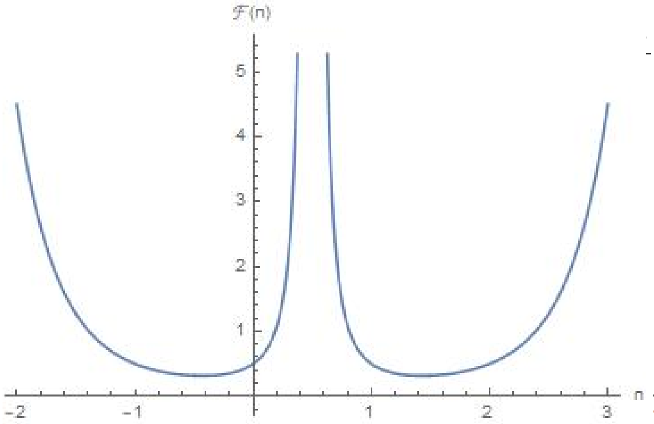
$$\frac{d\rho_E}{d\ln k} \approx \frac{\mathscr{G}(m)}{2\pi^2} H_I^4 (-k\eta)^{4+2m}, \tag{3.35}$$

where now $m = \gamma + 1$ for $\gamma \leq -1/2$ and $m = -\gamma$ for $\gamma \geq -1/2$ and

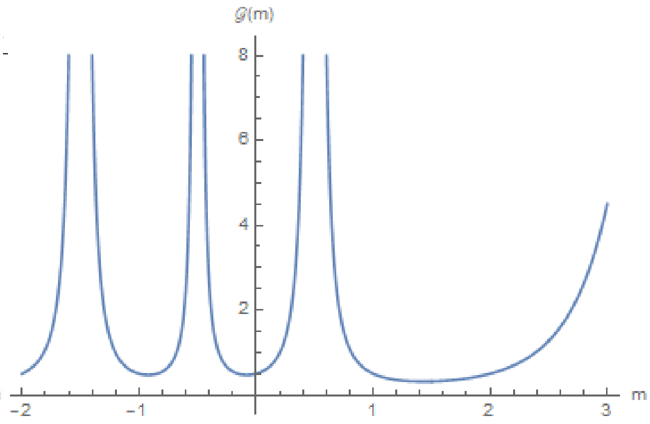
$$\mathscr{G}(m) = \frac{\pi}{2^{2m+3}\Gamma^2(m+3/2)\cos^2(m\pi)}. \tag{3.36}$$

Figure 3.1 shows the amplitude of the magnetic and electric field spectra. Now consider the case of a scale invariant magnetic spectrum. If $\gamma = 3$ then $(4+2m) = -2$ and the electric field spectrum is not scale invariant. In this case, at late times $(-k\eta) \to 0$, the electric field increases rapidly with $\rho_E \to \infty$. There is then the danger of its energy density exceeding that of the background during inflation, unless H^4 is sufficiently small! Such a value of γ is strongly constrained by the back reaction on the background expansion they imply. This problem is called the back reaction problem which, for our case at hand, has been discussed in Ref. [49]. On the other hand, consider $\gamma = -2$. In this case $(4+2m) = 2$ and $\rho_E \to 0$ as $(-k\eta) \to 0$. Thus such a value of γ is acceptable for magnetic field generation without severe back reaction effects.

We also encounter another problem. Consider the case $\beta = -2$, which implies $\alpha = -\gamma = 2$. In this case, the coupling function $f \propto a^2$, which implies that f will be very large at the end of inflation, but we need $f \to 1$ in order to recover classical electromagnetism. If we assume f to have attained



(a) $\mathscr{F}(n)$ vs. n



(b) $\mathscr{G}(m)$ vs. m

Figure 3.1: Amplitude of the magnetic spectrum and the electric spectrum in terms of the index α characterizing the form of the gauge coupling. We have assumed $\beta \simeq -2$, so that $\gamma \simeq -\alpha$. The divergences indicate the transition between the two branches of the spectrum.

a constant value $f_0 \sim 1$ at the end of inflation, then the effective electromagnetic charge $e_f = 1/f^2$. If we demand that e_f have the observed value at the end of inflation, then it will be very big at the beginning of inflation, rendering our perturbative analysis of field theory invalid–this problem is known as the strong coupling problem, which was first pointed out in Ref. [50].

Two different solutions to this problem have been suggested in Ref. [51] and Ref. [49]. In Ref. [51] it has been assumed that f grows as a power law during de Sitter inflation, starting with a value of one, and then decays back to its pre-inflationary value during a matter dominated phase that lasts until reheating. An alternative is to consider coupling functions with one or two transitions during inflation, with the condition $f(\eta_i) = f(\eta_f) = 1$, where η_i and η_f denote the conformal times at the beginning and end of inflation respectively [49].

Numerical estimate of the magnetic field in the scale invariant case

Using $\gamma = -2$ in equations (3.33) and (3.35), and upon using $\Gamma(-3/2) = 4\sqrt{\pi}/3$, gives

$$\frac{d\rho_B}{d\ln k} \approx \frac{9}{4\pi^2} H_I^4. \tag{3.37}$$

As mentioned earlier, $H_I \sim 10^{14}$ GeV, and $M_{Pl} \approx 2.4 \times 10^{18}$ GeV implies $H_I/M_{Pl} \sim 10^{-4}$. The present day value of the magnetic energy density is given by $\rho_{B0} = \rho_B(a_f/a_0)^4$, where a_f is the scale factor at the end of inflation and a_0 is its present day value. Assuming that the universe transited to radiation domination immediately after the end of inflation (instantaneous reheating), we have

$$3H_I^2 M_{Pl}^2 = \frac{\pi^2 g_f T_R^4}{30} \quad \text{or} \quad T_R = \left(\frac{90 H_I^2 {M_{Pl}}^2}{\pi^2 g_f}\right)^{1/4}.$$

Using entropy conservation gT^3a^3 =const., where g is the effective relativistic degrees of freedom and T the temperature of the relativistic fluid, we get [52]

$$g_f T_R^3 a_f^3 = g_0 T_0^3 a_0^3 \implies \frac{a_0}{a_f} = \frac{g_f^{1/12}}{g_0^{1/3}} \frac{\sqrt{H_I M_{Pl}}}{T_0} \left(\frac{90}{\pi^2}\right)^{1/4} = \frac{g_f^{1/12}}{g_0^{1/12}} \sqrt{H_I M_{Pl}} \left(\frac{3}{\rho_0}\right)^{1/4}, \tag{3.38}$$

where we have used $\rho_0 = g_0 T_0^4 \pi^2/30$. This implies

$$\rho_{B0} = \frac{6}{\pi}\left(\frac{g_0}{g_f}\right)^{1/3}\left(\frac{H_I}{M_{Pl}}\right)^2 \rho_0. \tag{3.39}$$

Taking $g_f \simeq 106.75$, $g_0 \simeq 3.36$ and $\rho_0 \simeq 4.3 \times 10^{-13} J/m^3$ [52] gives

$$\rho_{B0} \simeq 2.625 \times 10^{-23} \left(\frac{H_I/M_{Pl}}{10^{-4}}\right)^2 J/m^3, \tag{3.40}$$

which leads to an estimate of the present day value of magnetic field strength

$$B_0 \simeq 6.59 \times 10^{-15}\ \text{tesla(T)} \left(\frac{H_I/M_{Pl}}{10^{-4}}\right) \simeq 0.6 \times 10^{-10}\ \text{gauss(G)} \left(\frac{H_I/M_{Pl}}{10^{-4}}\right). \tag{3.41}$$

3.2 Generating helical magnetic fields

A helical magnetic field is characterized by by the quantity $\mathbf{B} \cdot \nabla \times \mathbf{B}$ and overall magnetic helicity implies that the average value of $\mathbf{B} \cdot \nabla \times \mathbf{B}$ is non-vanishing, which in turn implies a violation of parity (P) and charge+parity (CP) symmetries [7]. Magnetic helicity is defined as

$$\mathfrak{h} \equiv \int d^3x \mathbf{A} \cdot \mathbf{B}, \tag{3.42}$$

where the integral is over all space. It has been suggested that the occurrence of non-zero helicity in intergalactic magnetic fields (IGMF) is a convincing indicator for their primordial origin. The detection prospects of helical magnetic fields using TeV gamma ray blazars has been examined recently in Ref. [53].

To generate helical magnetic fields in our model, we need to add a parity breaking term to the action considered in the previous section: $\mathcal{S}_2 = \mathcal{S}_1 - f^2 F_{\mu\nu}\mathcal{F}^{\mu\nu}/4$, where

$$\mathcal{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta},$$

and $\epsilon^{\mu\nu\alpha\beta}$ is the 4D Levi-Civita symbol defined in Appendix A. Therefore the action is [54]

$$\mathcal{S}_2 = -\frac{1}{4}\int d^4x\sqrt{-g} f^2 [F_{\mu\nu}F^{\mu\nu} + F_{\mu\nu}\mathcal{F}^{\mu\nu}] - \int d^4x \sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right]. \tag{3.43}$$

Varying the action with respect to A_μ gives

$$[f^2(F^{\mu\nu} + \mathcal{F}_{\mu\nu})]_{;\nu} = 0 \quad \text{or} \quad \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}[\sqrt{-g} f^2(\phi) g^{\mu\alpha}g^{\nu\beta}(F_{\alpha\beta} + \epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta})] = 0. \tag{3.44}$$

Working again in the Coulomb gauge, for $\mu = i$,the above equation is

$$A_i'' + 2\frac{f'}{f}(A_i' + \eta_{ijk}\partial_j A_k) - a^2\partial_j\partial^j A_i = 0, \tag{3.45}$$

where η_{ijk} is the totally anti-symmetric tensor in 3 dimensions.

3.2.1 Quantizing the EM field

The conjugate momentum is

$$\Pi^i = f^2 a^2 g^{ij} A'_j. \tag{3.46}$$

The canonical quantization condition is the same as before

$$[\hat{A}^i(\eta, \mathbf{x}), \hat{\Pi}_j(\eta, \mathbf{y})] = i \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}.(\mathbf{x}-\mathbf{y})} P^i_j(\mathbf{k}) = i\delta^i_{\perp j}(\mathbf{x}-\mathbf{y}). \tag{3.47}$$

The decomposition of A^i in terms of the normal modes using the momentum space creation and annihilation operators is

$$\hat{A}^i(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \sum_{\lambda=1,2} e^i_\lambda [\hat{b}_\lambda(\mathbf{k}) A(k,\eta) \exp i\mathbf{k}\cdot\mathbf{x} + \hat{b}^\dagger_\lambda(\mathbf{k}) A^*(k,\eta) \exp -i\mathbf{k}\cdot\mathbf{x}]. \tag{3.48}$$

Defining $\bar{A}_\lambda = aA_\lambda$ and substituting equation (3.48) in equation (3.45) gives

$$\sum_\lambda \hat{b}_\lambda \left[\hat{e}_{i\lambda}\left(\bar{A}''_\lambda + 2\frac{f'}{f}\bar{A}'_\lambda + k^2\bar{A}^2_\lambda\right) + 2\frac{f'}{f}\eta_{ijm}\hat{e}_{m\lambda}k_j\bar{A}_\lambda\right] = 0. \tag{3.49}$$

To simplify the above equation, let us choose a different set of basis vectors defined as

$$\hat{e}_+ = \frac{(\hat{e}_1 + i\hat{e}_2)}{\sqrt{2}} \quad \text{and} \quad \hat{e}_- = \frac{(\hat{e}_1 - i\hat{e}_2)}{\sqrt{2}} \tag{3.50}$$

This set of basis vectors is known as helicity basis and we have $\bar{A} = \bar{A}_1\hat{e}_1 + \bar{A}_2\hat{e}_2 = \bar{A}_+\hat{e}_+ + \bar{A}_-\hat{e}_-$. Then equation (3.45) reduces to

$$\bar{A}''_\sigma + 2\frac{f'}{f}(\bar{A}'_\sigma + \sigma k\bar{A}_\sigma) + k^2\bar{A}_\sigma = 0. \tag{3.51}$$

Here $\sigma = \pm 1$ represents the helicity sign. The equation of motion in terms of $\mathcal{A}_\sigma = f\bar{A}_\sigma$ is

$$\mathcal{A}''_\sigma + \left(k^2 - \frac{f''}{f} + 2\sigma k\frac{f'}{f}\right)\mathcal{A}_\sigma = 0. \tag{3.52}$$

Energy density of the EM field

The energy momentum tensor of the EM field is

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta[\sqrt{-g}\mathscr{L}_2]}{\delta g^{\mu\nu}} = f^2\left[g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} - g_{\mu\nu}\frac{F_{\alpha\beta}F^{\alpha\beta}}{4}\right], \tag{3.53}$$

where $\mathscr{L}_2 = (-f^2/4)(F^{\mu\nu}F_{\mu\nu} + F^{\mu\nu}\mathcal{F}_{\mu\nu})$. The EM energy densities of the ground state measured by the fundamental observer with four-velocity $u^\mu = (1/a, 0, 0, 0)$ are given by,

$$\rho_B = \langle 0| T^B_{\mu\nu}u^\mu u^\nu |0\rangle = \frac{f^2}{2}\langle 0| B^i B_i |0\rangle\, ; \rho_E = \langle 0| T^E_{\mu\nu}u^\mu u^\nu |0\rangle = \frac{f^2}{2}\langle 0| E^i E_i |0\rangle\, , \tag{3.54}$$

and the spectral energy densities are now given by

$$\frac{d\rho_B}{d\ln k} = \frac{1}{2\pi^2}\left(\frac{k}{a}\right)^4 k(|\mathcal{A}_+(k,\eta)|^2 + |\mathcal{A}_-(k,\eta)|^2), \tag{3.55}$$

$$\frac{d\rho_E}{d\ln k} = \frac{f^2}{2\pi^2}\left(\frac{k}{a}\right)^4 \frac{1}{k}\left(\left|\left[\frac{\mathcal{A}_+(k,\eta)}{f}\right]'\right|^2 + \left|\left[\frac{\mathcal{A}_-(k,\eta)}{f}\right]'\right|^2\right). \tag{3.56}$$

3.2.2 Evolution of normal modes

We again choose the coupling function to be a simple power law $f \propto a^{\alpha}$. We assume that during inflation $f(\phi)$ has a form such that f evolves with a in this manner. Here we assume the background to be purely de Sitter ($a \propto 1/\eta$) and we arrive at the following

$$\frac{f'}{f} = \frac{\alpha}{\eta} \quad , \quad \frac{f''}{f} = \frac{\alpha(\alpha+1)}{\eta^2}.$$

Then equation (3.52) reduces to

$$\mathcal{A}''_\sigma + \left(k^2 - \frac{\alpha(\alpha+1)}{\eta^2} + 2\sigma k \frac{\alpha}{\eta}\right)\mathcal{A}_\sigma = 0. \tag{3.57}$$

To solve the above equation, we rewrite it by defining some new variables :

$$\mu^2 \equiv \alpha(\alpha+1) + \frac{1}{4}, \quad \kappa \equiv i\alpha\sigma \quad \text{and} \quad z \equiv 2ik\eta.$$

Then the equation takes the form

$$\frac{\partial^2 \mathcal{A}_\sigma(k,\eta)}{\partial z^2} + \left[\frac{(1/4-\mu^2)}{z^2} + \frac{\kappa}{z} - \frac{1}{4}\right]\mathcal{A}_\sigma(k,\eta) = 0. \tag{3.58}$$

The above equation admits solutions which are linear combination of the Whittaker functions,

$$\mathcal{A}_h = C_3(k) W_{\kappa,\mu}(z) + C_4(k) W_{-\kappa,\mu}(-z), \tag{3.59}$$

where the coefficients are set by initial conditions. We again take the initial condition to be $\mathcal{A}_h \to \exp(-ik\eta)/\sqrt{2k}$ as $|k\eta| \to \infty$. Now in the limit ($z \to \infty$), we have

$$W_{\kappa,\mu}(z) \to e^{-z/2} z^{\kappa} \qquad \text{as } z \to \infty$$

Therefore setting $C_4(k) = 0$, we have $\mathcal{A}_h \to C_3 \exp(-ik\eta)\exp(-i\pi\kappa/2)\exp(\kappa \ln 2k\eta)$. Therefore in the limit, $C_3(k) = e^{i\pi\kappa/2}/\sqrt{2k}$ and we have

$$\mathcal{A}_h = \frac{e^{i\pi\kappa/2}}{\sqrt{2k}} W_{\kappa,\mu}(z) = \frac{e^{-h\pi\alpha/2}}{\sqrt{2k}} W_{i\alpha h,\alpha+1/2}(2ik\eta). \tag{3.60}$$

At the end of the inflation, all the modes of cosmological interest will be outside the horizon. So we consider the other limit $(-k\eta) \to 0$, but before doing so, let us have a look at the form of the Whittaker functions [55] :

$$W_{\kappa,\mu}(z) = \frac{\Gamma(2\mu)}{\Gamma(1/2+\mu-\kappa)} M_{\kappa,-\mu}(z) + \frac{\Gamma(-2\mu)}{\Gamma(1/2-\mu-\kappa)} M_{\kappa,\mu}(z) \tag{3.61}$$

where

$$\begin{aligned} M_{\kappa,\mu}(z) &= z^{\mu+1/2} e^{-z/2} \Phi(\mu-\kappa+1/2, 2\mu+1; z), \\ M_{\kappa,-\mu}(z) &= z^{-\mu+1/2} e^{-z/2} \Phi(-\mu-\kappa+1/2, -2\mu+1; z), \end{aligned} \tag{3.62}$$

where $\Phi(p,q;z) = 1 + pz/q + \mathcal{O}(z^2)$. Thus in the limit $z \to 0$, we obtain,

$$\mathcal{A}_\sigma = \frac{c_\sigma}{\sqrt{2k}}[(-k\eta)^{-\alpha} + \sigma(-k\eta)^{1-\alpha}] + \frac{d_\sigma}{\sqrt{2k}}[(-k\eta)^{1+\alpha} + (\sigma\alpha/(1+\alpha))(-k\eta)^{2+\alpha}]. \tag{3.63}$$

Here

$$c_\sigma = e^{-\sigma\pi\alpha/2}\frac{(-2i)^{-\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha-i\sigma\alpha)} \quad ; \quad d_\sigma = e^{-\sigma\pi\alpha/2}\frac{(-2i)^{1+\alpha}\Gamma(-1-2\alpha)}{\Gamma(-\alpha-i\sigma\alpha)}. \tag{3.64}$$

3.2.3 The spectra of the electric and magnetic fields

Substituting equation (3.63) into equations (3.55) and (3.56) gives (keeping only the dominant terms)

$$\frac{d\rho_B}{d\ln k} \approx \frac{H_I^4}{8\pi^2}[(|c_+|^2 + |c_-|^2)(-k\eta)^{-2\alpha+4} + (|d_+|^2 + |d_-^2|)(-k\eta)^{2\alpha+6}], \tag{3.65}$$

$$\frac{d\rho_E}{d\ln k} \approx \frac{H_I^4}{8\pi^2}[(|c_+|^2 + |c_-|^2)(-k\eta)^{-2\alpha+4} + (|d_+|^2 + |d_-^2|)(1+2\alpha)^2(-k\eta)^{2\alpha+4}]. \tag{3.66}$$

We see that there are two branches in the above expressions. In the magnetic spectrum, the first branch dominates for $\alpha > -1/2$ and the second branch dominates for $\alpha < -1/2$, at late times. For the electric field spectrum, the first branch dominates for $\alpha > 0$ and the other branch dominates for $\alpha < 0$, at late times. As in the previous section, we find that for $\alpha = 2, -3$, the magnetic field spectrum is scale invariant. We reject the $\alpha = -3$ candidate owing to the back reaction problem as mentioned previously. For the case of $\alpha = 2$, both the magnetic and electric field spectrum are scale invariant. However, there is a problem with this value of $\alpha = -3$; the strong coupling problem as discussed in the previous section. This problem can be avoided by coupling the EM fields with the Riemann tensor, which we discuss in the next chapter.

Numerical estimate of the magnetic field in the scale invariant case

Using $\alpha = 2$ in equations (3.55) and (3.56) gives

$$\frac{d\rho_B}{d\ln k} \approx \frac{9e^{4\pi}}{320\pi^3}H_I^4 \tag{3.67}$$

To arrive at the above equation, we have used [55]

$$\Gamma(5) = 24 \quad ; \quad |\Gamma(3\pm 2i)|^2 = |(2\pm 2i)(1\pm 2i)(\pm 2i)|^2|\Gamma(\pm 2i)|^2 = 40\frac{\pi}{2\sinh 2\pi} \approx 40\pi e^{-2\pi}$$

Note that the value obtained from equation (3.67) is almost 1000 times greater than that from equation (3.37) and hence the magnitude of the generated helical magnetic field is

$$B_0 \simeq 7.52\times 10^{-12}\,\text{tesla(T)}\left(\frac{H_I/M_{Pl}}{10^{-4}}\right) \simeq 0.7\times 10^{-7}\,\text{gauss(G)}\left(\frac{H_I/M_{Pl}}{10^{-4}}\right). \tag{3.68}$$

Chapter 4

Generation of magnetic fields II — Coupling to the Riemann curvature tensor

In this chapter, we study the generation of magnetic fields by coupling the EM field to the Riemann curvature tensor. The model we shall consider invokes Effective Field Theory (EFT), which lies at the core of our modern understanding of the fundamental interactions in nature. But in this report we will not get into the details of EFT. EFTs involve a mass term M that characterizes whatever fundamental theory underlies the EFT, which we will treat as just a parameter in our model. For our case of interest, namely magnetogenesis during slow roll inflation, the value of M depends on the slow roll parameter $\varepsilon = \epsilon_H$ (see equation (2.41)); M cannot be much smaller than $\sqrt{2\varepsilon}M_{Pl}$ and we tentatively assume $M \simeq \sqrt{2\varepsilon}M_{Pl}$ [56]. According to current observations, the tensor-to-scalar ratio r is determined to be $r < 0.044$ [57]. Using the relation $\varepsilon \simeq r/16$ (see equation (5.38)), we get $\varepsilon < 2.75 \times 10^{-3}$. Therefore $M \simeq \sqrt{2\varepsilon}M_{Pl} \sim 10^{17} GeV$.

4.1 Generating helical magnetic fields

We consider the following action [58] : $\mathcal{S}_3 = \mathcal{S}_{EH} + \mathcal{S}_\phi + \mathcal{S}_{EM} + \mathcal{S}_{CB}$. Here $\mathcal{S}_{EH}$ is the Einstein-Hilbert action :

$$\mathcal{S}_{EH} = -\frac{M_{Pl}^2}{2}\int d^4x\sqrt{-g}R, \tag{4.1}$$

$\mathcal{S}_\phi$ is the action for the minimally coupled, self-interacting canonically scalar field :

$$\mathcal{S}_\phi = -\int d^4x\sqrt{-g}\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi)\right], \tag{4.2}$$

$\mathcal{S}_{EM}$ is the electromagnetic action :

$$\mathcal{S}_{EM} = -\frac{1}{4}\int d^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu}, \tag{4.3}$$

and $\mathcal{S}_{CB}$ is the conformal breaking part of the action :

$$\mathcal{S}_{CB} = \frac{-1}{M^2}\int d^4x\sqrt{-g}R_{\mu\nu}{}^{\alpha\beta}F_{\alpha\beta}\mathcal{F}^{\mu\nu} = \frac{-1}{M^2}\int d^4x\sqrt{-g}\tilde{R}^{\mu\nu\alpha\beta}F_{\alpha\beta}F_{\mu\nu}, \tag{4.4}$$

where ${R_{\mu\nu}}^{\alpha\beta}$ is the Riemann tensor and $\tilde{R}^{\mu\nu\alpha\beta} = (1/2)\epsilon^{\mu\nu\rho\sigma}{R_{\rho\sigma}}^{\alpha\beta}$ is its dual and M is the energy scale which sets the scale for the breaking of the conformal invariance [58]. Due to the Riemann tensor, in FLRW background, M appears as a time-dependent coupling. This can be seen as follows : ${R_{\mu\nu}}^{\alpha\beta} \sim a'^2/a^4$ whcih implies the coupling function is time-dependent,

$$\frac{1}{M_{eff}} \sim \frac{1}{M}\frac{a'}{a^2} = \frac{H}{M}. \tag{4.5}$$

During inflation, $H \approx H_I \sim 10^{14}$ GeV and since we are assuming $M \sim 10^{17}$ GeV, therefore, during inflation $1/M_{eff} \sim 10^{-3}$. But at the current epoch, $H = H_0 \sim 10^{-44}$ GeV [58], implying $1/M_{eff} \sim 10^{-61}$, which is tiny. Thus the coupling will have significant contribution only in the early universe.

The variation of the action with respect to A_μ gives rise to the following equation of motion

$$\partial_\mu\left[\sqrt{-g}F^{\mu\nu} + \frac{1}{M^2}\eta^{\alpha\beta\rho\sigma}{R_{\rho\sigma}}^{\mu\nu}F_{\alpha\beta} + \frac{1}{M^2}\eta^{\mu\nu\rho\sigma}{R_{\rho\sigma}}^{\alpha\beta}F_{\alpha\beta}\right] = 0, \tag{4.6}$$

where we have used $\epsilon^{\alpha\beta\gamma\delta} = \eta^{\alpha\beta\gamma\delta}/\sqrt{-g}$ and $\eta^{\alpha\beta\gamma\delta}$ is the fully anti-symmetric symbol in four dimensions whose values are ± 1 . For $\nu = i$ this reduces to

$$\partial_0\left(F_{0i} - \frac{2}{M^2}\eta^{0ijk}\frac{a''}{a^3}F_{jk}\right) + \partial_l\left(-F_{li} + \frac{4}{M^2}\eta^{0ilj}\frac{a'^2}{a^4}F_{0j}\right) = 0, \tag{4.7}$$

where we have used the following components of the Riemann tensor :

$$ {R_{ij}}^{kl} = \frac{a'^2}{a^4}(\delta^k_i\delta^l_j - \delta^l_i\delta^k_j) \quad ; \quad {R_{0i}}^{0j} = -\left(\frac{a'^2 - aa''}{a^4}\right)\delta^j_i. $$

Working again in the Coulomb gauge and using $\eta^{0ijk} = \eta_{ijk}$,(the 3-D Levi-Civita symbol) the above equation becomes

$$A''_i + \frac{4\eta_{ijk}}{M^2}\left(\frac{a'''}{a^3} - \frac{3a'a''}{a^4}\right)\partial_j A_k - a^2\partial_j\partial^j A_i = 0. \tag{4.8}$$

4.1.1 Quantizing the EM field

Decomposing the vector potential in Fourier space, we have the familiar equation :

$$\hat{A}_i(\eta,\mathbf{x}) = \int\frac{d^3k}{(2\pi)^3}\sum_{\lambda=1,2}\hat{e}_{i\lambda}[\hat{b}_\lambda(\mathbf{k})\bar{A}(k,\eta)\exp i\mathbf{k}\cdot\mathbf{x} + \hat{b}^\dagger_\lambda(\mathbf{k})\bar{A}^*(k,\eta)\exp -i\mathbf{k}\cdot\mathbf{x}], \tag{4.9}$$

with all the quantities as defined in the previous chapter. Substituting this in equation (4.8) yields

$$\sum_{\lambda=1,2}\hat{b}_\lambda\left[\hat{e}_{i\lambda}\bar{A}''_\lambda + \frac{4i}{M^2}\eta_{ijk}k_j\hat{e}_{k\lambda}\bar{A}_\lambda\left(\frac{a'''}{a^3} - \frac{3a'a''}{a^4}\right) + k^2\hat{e}_{i\lambda}\bar{A}_\lambda\right] = 0. \tag{4.10}$$

Working in the helicity basis and replacing $\eta_{ijk}k_j\hat{e}_{k\lambda}\bar{A}_\lambda\hat{b}_\lambda \to k\sum_{\sigma=\pm1}\sigma A_\sigma\hat{e}_\sigma\hat{b}_\sigma$ as before, we get

$$A''_\sigma + \left(k^2 - \frac{4k\sigma}{M^2}K(\eta)\right)A_\sigma = 0, \tag{4.11}$$

where we have defined $K(\eta) = 3a'a''/a^4 - a'''/a^3$. The EM energy densities of the ground state as observed by the fundamental observer are

$$\rho_B(k,\eta) \equiv -\frac{1}{2}\langle 0|\, B^i B_i \,|0\rangle \equiv \int \frac{dk}{k}\frac{d\rho_B}{d\ln k} = \int \frac{dk}{k}\left[\frac{1}{2\pi^2}\left(\frac{k}{a}\right)^4 k(|A_+(k,\eta)|^2 + |A_-(k,\eta)|^2)\right], \tag{4.12}$$

$$\rho_E(k,\eta) \equiv -\frac{1}{2}\langle 0|\, E^i E_i \,|0\rangle \equiv \int \frac{dk}{k}\frac{d\rho_E}{d\ln k} = \int \frac{dk}{k}\left[\frac{1}{2\pi^2}\left(\frac{k}{a}\right)^4 \frac{1}{k}(|A'_+(k,\eta)|^2 + |A'_-(k,\eta)|^2)\right], \tag{4.13}$$

and the ground state helicity density is

$$\rho_h(k,\eta) \equiv -\frac{1}{2}\langle 0|\, B^i A_i \,|0\rangle \equiv \int \frac{dk}{k}\frac{d\rho_h}{d\ln k} = \int \frac{dk}{k}\left[\frac{1}{2\pi^2}\left(\frac{k}{a}\right)^4 a(|A_+(k,\eta)|^2 - |A_-(k,\eta)|^2)\right]. \tag{4.14}$$

This helicity density is the difference between the two helicity modes and hence it is possible to maximize it by enhancing one helicity mode at the cost of the other.

4.1.2 Evolution of normal modes

We consider the power law inflation given by equation (3.27). Then we have

$$K(\eta) = \frac{2\beta(\beta+1)(\beta+2)}{a_0^2\eta_0^3},$$

which upon substitution in equation (4.11) leads to :

$$A''_\sigma + \left[k^2 - \frac{8k\sigma}{M^2}\frac{\beta(\beta+1)(\beta+2)}{a_0^2\eta_0^3}\left(\frac{\eta_0}{\eta}\right)^{2\beta+5}\right]A_\sigma = 0. \tag{4.15}$$

From this equation we see that helical magnetic fields in this model cannot be produced if the background evolution is purely de Sitter ($\beta = -2$). We will be particularly interested in the slow roll inflation with $\beta = -2 - \varepsilon$.

As in the previous chapter, we consider two scenarios–the sub-Hubble limit and the super-Hubble limit. In the sub-Hubble limit($|k\eta| \gg 1$), equation (4.15) simplifies to

$$A''_\sigma + k^2 A_\sigma \approx 0 \tag{4.16}$$

which, upon using the Bunch-Davies vacuum initial condition, admits solutions

$$A_\sigma = \frac{1}{\sqrt{2k}}e^{-ik\eta}. \tag{4.17}$$

In the super-Hubble limit($|k\eta| \ll 1$),

$$A''_\sigma + \sigma k\left(\frac{\zeta}{\eta}\right)^2 \tau^2 A_\sigma = 0, \tag{4.18}$$

where we have defined

$$\zeta^2 \equiv \frac{(2\alpha-3)(2\alpha-1)(2\alpha+1)}{M^2 a_0^2\eta_0}$$

with $\alpha = \beta + 3/2$ and $\tau \equiv |\eta_0/\eta|^\alpha$ $(0 < \tau < \infty)$. Note that $\alpha = -1/2$ corresponds to de Sitter.

Rewriting the above equation in terms of τ, we have

$$\frac{\alpha^2\tau^2}{\eta^2}\frac{d^2A_\sigma}{d\tau^2} + \frac{\alpha(\alpha+1)}{\eta^2}\tau\frac{dA_\sigma}{d\tau} + \sigma k\zeta^2\frac{\tau^2}{\eta^2}A_\sigma = 0$$

$$\Rightarrow \tau^2\frac{d^2A_\sigma}{d\tau^2} + (1/\alpha+1)\tau\frac{dA_\sigma}{d\tau} + \sigma k\zeta^2\alpha^2\tau^2A_\sigma = 0, \tag{4.19}$$

which is a transformed version of the Bessel equation whose solutions are

$$A_+(k,\tau) = \tau^{-1/2\alpha}\left[P_+J_{1/2\alpha}\left(\frac{\zeta\sqrt{k}}{\alpha}\tau\right) + Q_+Y_{1/2\alpha}\left(\frac{\zeta\sqrt{k}}{\alpha}\tau\right)\right], \tag{4.20}$$

$$A_-(k,\tau) = \tau^{-1/2\alpha}\left[P_-J_{1/2\alpha}\left(-i\frac{\zeta\sqrt{k}}{\alpha}\tau\right) + Q_-Y_{1/2\alpha}\left(-i\frac{\zeta\sqrt{k}}{\alpha}\tau\right)\right], \tag{4.21}$$

where P_+, Q_+, P_-, Q_- are arbitrary constants which are fixed by matching A_σ and A'_σ at $k_* \sim -1/\eta_*$ where $*$ refers to quantities evaluated at horizon exit. Before proceeding further, we would like to set $\alpha = -1$. This choice is motivated by the fact that in this case $\tau \propto |\eta|$, hence the super-Hubble modes can be written in terms of η using a linear relation. Also the constants have a weak dependence on α and hence finding the values for a given α will be accurate within an order [58]. In this case, equations (4.20) and (4.21) reduce to

$$A_+(k,\tau) = \tau^{1/2}\left[P_+J_{-1/2}(-\zeta\sqrt{k}\tau) + Q_+Y_{-1/2}(-\zeta\sqrt{k}\tau)\right], \tag{4.22}$$

$$A_-(k,\tau) = \tau^{1/2}\left[P_-J_{-1/2}(i\zeta\sqrt{k}\tau) + Q_-Y_{-1/2}(i\zeta\sqrt{k}\tau)\right]. \tag{4.23}$$

Now let us look at a few properties of the Bessel functions

$$J_{-1/2}(x) = \sqrt{\frac{2}{x}}\cos x \quad , \quad J_{1/2}(x) = \sqrt{\frac{2}{x}}\sin x \quad ; \quad Y_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}.$$

Using the above equations, equations (4.22) and (4.23) can be further reduced to

$$A_+(k,\tau) = \sqrt{\frac{2}{-\zeta\sqrt{k}}}\left[P_+\cos(\zeta\sqrt{k}\tau) - Q_+\sin(\zeta\sqrt{k}\tau)\right], \tag{4.24}$$

$$A_-(k,\tau) = \sqrt{\frac{2}{i\zeta\sqrt{k}}}\left[P_-\cosh(\zeta\sqrt{k}\tau) + iQ_-\sinh(\zeta\sqrt{k}\tau)\right], \tag{4.25}$$

and it is easy to find their derivatives,

$$\frac{dA_+}{d\tau} = \sqrt{-2\zeta\sqrt{k}}[-P_+\sin(\zeta\sqrt{k}\tau) - Q_+\cos(\zeta\sqrt{k}\tau)], \tag{4.26}$$

$$\frac{dA_-}{d\tau} = \sqrt{-2i\zeta\sqrt{k}}[P_-\sinh(\zeta\sqrt{k}\tau) + iQ_-\cosh(\zeta\sqrt{k}\tau)]. \tag{4.27}$$

Thus matching the solutions and their derivatives at horizon exit, we find

$$P_+ = \frac{\sqrt{-\eta_0}e^{-i}}{2}\left(\frac{\sin\Phi}{\sqrt{\Phi}} + i\sqrt{\Phi}\cos\Phi\right) \quad , \quad Q_+ = \frac{\sqrt{-\eta_0}e^{-i}}{2}\left(\frac{\cos\Phi}{\sqrt{\Phi}} - i\sqrt{\Phi}\sin\Phi\right)$$

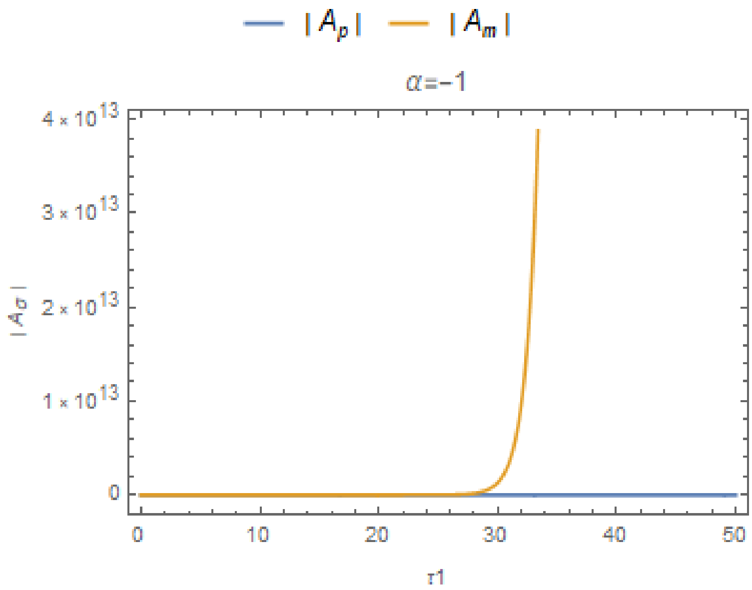


Figure 4.1: This figure shows the behaviour of $|A_\sigma|$ with τ. This graph, which has been plotted for $\alpha = -1$, shows that the negative helicity mode is decaying in conformal time. In this graph $|A_p|$ stands for $|A_+|$, $|A_m|$ stands for $|A_-|$ and $\tau 1 = 10^{-10}\tau$.

$$P_- = \frac{\sqrt{-\eta_0}e^{-i}}{2}\left(\frac{\sinh\Phi}{\sqrt{i\Phi}} + \sqrt{i\Phi}\cosh\Phi\right) \quad , \quad Q_- = \frac{\sqrt{-\eta_0}e^{-i}}{2}\left(\frac{\sinh\Phi}{\sqrt{i\Phi}} + \sqrt{i\Phi}\cosh\Phi\right)$$

where $\Phi = \sqrt{15\eta_*/M^2\eta_0^3}$ is a dimensionless constant.

To obtain the dominating helicity mode, we need to obtain the numerical values of these coefficients. To obtain these values, we take $\mathcal{H} \sim -\eta_0^{-1} \sim 10^{14}GeV$ [59] and $M \sim 10^{17}GeV$. Since at horizon exit, $k_* \sim \mathcal{H}$ and $-k_*\eta_* \approx 1$, η_* is also of the same order as η_0. This gives $\Phi \sim 10^{-3}$ which is a small value. Approximating trigonometric functions in the above equations, we get

$$|P_+| \approx |P_-| \approx \sqrt{10^{-17}}\ \mathrm{GeV}^{1/2} \quad , \quad |Q_+| \approx |Q_-| \approx \sqrt{10^{-11}}\ \mathrm{GeV}^{1/2}. \tag{4.28}$$

Using these values in equations (4.24) and (4.25), we see that $|A_+|$ is an exponentially growing solution while $|A_-|$ is a decaying one (see figure 4.1). Henceforth we set $|A_-(k,\eta)| = 0$.

Series expansion of the Bessel functions

Let us define $F(\tau) = \exp\left[\frac{i\pi}{\alpha}\left\lfloor\frac{\pi - \arg(\tau) - \arg(\sqrt{k}\zeta/\alpha)}{2\pi}\right\rfloor\right]$ where $\lfloor\ldots\rfloor$ is the floor function. Upto leading order, the series expansion of the Bessel functions are [58] :

$$J_{1/2\alpha}(\zeta\sqrt{k}\tau/\alpha) = \frac{F(\tau)}{\Gamma(1+1/2\alpha)}\left(\frac{\zeta\sqrt{k}}{2\alpha}\right)^{1/2\alpha}\tau^{1/2\alpha}\left(1 - \frac{(\zeta\sqrt{k}\tau)^2}{2\alpha(1+2\alpha)} + \mathcal{O}(\tau^3)\right), \tag{4.29}$$

$$
\begin{aligned}
Y_{1/2\alpha}(\zeta\sqrt{k}\tau/\alpha) = \frac{F(\tau)}{\pi}\left(\frac{\zeta\sqrt{k}}{2\alpha}\right)^{1/2\alpha} \tau^{1/2\alpha}\Gamma(-1/2\alpha)\cos(\pi/2\alpha)\left(-1+\frac{k\zeta^2\tau^2}{2\alpha(1+2\alpha)}+\mathcal{O}(\tau^3)\right)+ \\
+\frac{F(\tau)^{-1}}{\pi}\left(\frac{2\alpha}{\zeta\sqrt{k}}\right)^{1/2\alpha} \tau^{-1/2\alpha}\Gamma(1/2\alpha)\left(-1+\frac{k\zeta^2\tau^2}{2\alpha(-1+2\alpha)}+\mathcal{O}(\tau^3)\right),
\end{aligned}
\tag{4.30}
$$

$$
J_{1/2\alpha}(-i\zeta\sqrt{k}\tau/\alpha) = \frac{\tilde{F}(\tau)}{\Gamma(1+1/2\alpha)}\left(\frac{-i\zeta\sqrt{k}}{2\alpha}\right)^{1/2\alpha} \tau^{1/2\alpha}\left(1+\frac{(\zeta\sqrt{k}\tau)^2}{2\alpha(1+2\alpha)}+\mathcal{O}(\tau^3)\right), \tag{4.31}
$$

$$
\begin{aligned}
Y_{1/2\alpha}(-i\zeta\sqrt{k}\tau/\alpha) = -\frac{\tilde{F}(\tau)}{\pi}\left(\frac{-i\zeta\sqrt{k}}{2\alpha}\right)^{1/2\alpha} \tau^{1/2\alpha}\Gamma(-1/2\alpha)\cos(\pi/2\alpha)\left(+1+\frac{k\zeta^2\tau^2}{2\alpha(1+2\alpha)}+\mathcal{O}(\tau^3)\right) \\
-\frac{\tilde{F}(\tau)^{-1}}{\pi}\left(-\frac{2i\alpha}{\zeta\sqrt{k}}\right)^{1/2\alpha} \tau^{-1/2\alpha}\Gamma(1/2\alpha)\left(-1+\frac{k\zeta^2\tau^2}{2\alpha(-1+2\alpha)}+\mathcal{O}(\tau^3)\right).
\end{aligned}
\tag{4.32}
$$

At leading order, the helicity modes are then (cf. equations (4.22) and (4.23))

$$
\begin{aligned}
A_+(k,\tau) = F(\tau)\left(\frac{\zeta\sqrt{k}}{2\alpha}\right)^{1/2\alpha}\left[\frac{P_+}{\Gamma(1+1/2\alpha)}-\frac{Q_+}{\pi}\Gamma(-1/2\alpha)\cos(\pi/2\alpha)\right] \\
-Q_+\frac{\tilde{F}(\tau)^{-1}}{\pi}\left(\frac{2\alpha}{\zeta\sqrt{k}}\right)^{1/2\alpha}\Gamma(1/2\alpha)\tau^{-1/\alpha}
\end{aligned}
\tag{4.33}
$$

$$
\begin{aligned}
A_-(k,\tau) = \tilde{F}(\tau)\left(-\frac{i\zeta\sqrt{k}}{2\alpha}\right)^{1/2\alpha}\left[\frac{P_-}{\Gamma(1+1/2\alpha)}-\frac{Q_-}{\pi}\Gamma(-1/2\alpha)\cos(\pi/2\alpha)\right], \\
+Q_-\frac{\tilde{F}(\tau)^{-1}}{\pi}\left(-\frac{2\alpha}{i\zeta\sqrt{k}}\right)^{1/2\alpha}\Gamma(1/2\alpha)\tau^{-1/\alpha}.
\end{aligned}
\tag{4.34}
$$

If we further define the following quantities :

$$
\begin{aligned}
\mathscr{F}(\tau) = F(\tau)(\zeta/2\alpha)^{1/2\alpha} \quad , \quad \tilde{\mathscr{F}}(\tau) = \tilde{F}(\tau)(-i\zeta/2\alpha)^{1/2\alpha}, \\
C(\tau) = F(\tau)(\zeta/2\alpha)^{1/2\alpha}\left[\frac{P_+}{\Gamma(1+1/2\alpha)}-\frac{Q_+}{\pi}\Gamma(-1/2\alpha)\cos(\pi/2\alpha)\right], \\
\tilde{C}(\tau) = \tilde{F}(\tau)(-i\zeta/2\alpha)^{1/2\alpha}\left[\frac{P_-}{\Gamma(1+1/2\alpha)}-\frac{Q_-}{\pi}\Gamma(-1/2\alpha)\cos(\pi/2\alpha)\right],
\end{aligned}
$$

then we can write the positive helicity mode as follows :

$$
A_+(k,\tau) = Ck^{1/4\alpha} - Q_+\frac{\mathscr{F}^{-1}}{\pi}\Gamma(1/2\alpha)k^{-1/4\alpha}\tau^{-1/\alpha}, \tag{4.35}
$$

where $|C| \sim 10^{-5/\alpha-11/2}$ GeV$^{-1/4\alpha-1/2}$ and $|\mathscr{F}| \sim 10^{-5/\alpha}$ GeV$^{-1/4\alpha}$. The time dependence in these has been suppressed since it has been shown in Ref. [58] that $\zeta \approx 10^{-10}$ GeV$^{-1/2}$ $\implies$ $F(\tau) = 1$.

4.1.3 The spectra of the electric and magnetic fields

Substituting equation (4.35) in expression for $d\rho_B/d\ln k$, we obtain

$$\frac{d\rho_B}{d\ln k} = \frac{1}{2\pi^2}\frac{k^5}{a^4}\left(|C|^2 k^{1/2\alpha} + \left|Q_+ \frac{\mathscr{F}^{-1}}{\pi}\Gamma(1/2\alpha)\right|^2 (k\tau^4)^{-1/2\alpha}\right). \tag{4.36}$$

At horizon exit, $k = k_* \sim \mathcal{H}$ and the expression for τ is $\tau_* = \left(\frac{2\eta_0 k_*}{2\alpha-1}\right)^{\alpha}$. Therefore, at horizon exit, we have

$$\left.\frac{d\rho_B}{d\ln k}\right|_{k_*\sim\mathcal{H}} = \frac{H_I^4}{2\pi^2}\left(|C|^2 k_*^{1+1/2\alpha} + \left|Q_+ \frac{\mathscr{F}^{-1}}{\pi}\Gamma(1/2\alpha)\right|^2 \frac{(2\alpha-1)^2}{4\eta_0^2} k_*^{-1-1/2\alpha}\right). \tag{4.37}$$

Just like the expression obtained for $d\rho_B/d\ln k$ in the previous chapter for the helical magnetic field, the above expression also has two branches : The first branch (setting $Q_+ = 0$) has a scale invariant spectrum for $\alpha = -1/2$ and the second branch (setting $C = 0$) also has a scale invariant spectrum for $\alpha = -1/2$. Thus both the branches has scale invariant power spectrum for the exact de Sitter case($\alpha = -1/2$). However for the slow roll inflation ($\alpha = -1/2-\varepsilon$), the two branches scale differently : the first branch goes as $k_*^{-2\varepsilon}$ and the second branch goes as $k_*^{-6\varepsilon}$. Since $\varepsilon > 0$, for slow roll inflation, this model predicts a red magnetic spectrum.

Let us look at the electric field spectrum. Using the fact that at super-Hubble scales $\partial_\eta \sim \mathcal{H}$ [17], we obtain, at horizon exit

$$\left.\frac{d\rho_E}{d\ln k}\right|_{k_*\sim\mathcal{H}} \approx \frac{H_I^4}{2\pi^2}\left(|C|^2 k_*^{1+1/2\alpha} + \left|Q_+ \frac{\mathscr{F}^{-1}}{\pi}\Gamma(1/2\alpha)\right|^2 \frac{(2\alpha-1)^2}{4\eta_0^2} k_*^{-1-1/2\alpha}\right). \tag{4.38}$$

Thus the behaviour of the electric spectrum is similar to that of the magnetic spectrum.

The back reaction problem : The magnetic energy density is

$$\begin{aligned}\rho_B = \int_{\mathcal{H}/100}^{\mathcal{H}} \frac{dk}{k}\frac{d\rho_B}{d\ln k} &= \frac{1}{2\pi^2 a^4}\left[|C|^2 \frac{\mathcal{H}^{5+1/2\alpha}}{5+1/2\alpha} + \left|Q_+ \frac{\mathscr{F}^{-1}}{\pi}\Gamma(1/2\alpha)\right|^2 \tau^{-2/\alpha}\frac{\mathcal{H}^{5-1/2\alpha}}{5-1/2\alpha}\right] \\ \Rightarrow \rho_B|_{k_*\sim\mathcal{H}} &= \frac{H_I^4}{2\pi^2}\left(|C|^2\frac{k_*^{1+1/2\alpha}}{5+1/2\alpha} + \left|Q_+ \frac{\mathscr{F}^{-1}}{\pi}\Gamma(1/2\alpha)\right|^2 \frac{(2\alpha-1)^2}{4\eta_0^2}\frac{k_*^{-1-1/2\alpha}}{5-1/2\alpha}\right),\end{aligned} \tag{4.39}$$

where we have used $\mathcal{H}^{5+1/2\alpha} - (\mathcal{H}/100)^{5+1/2\alpha} \approx \mathcal{H}^{5+1/2\alpha}$. Similarly we have the following expression for the electric energy density at horizon exit :

$$\rho_E|_{k_*\sim\mathcal{H}} = \frac{H_I^4}{2\pi^2}\left(|C|^2\frac{k_*^{1+1/2\alpha}}{3+1/2\alpha} + \left|Q_+ \frac{\mathscr{F}^{-1}}{\pi}\Gamma(1/2\alpha)\right|^2 \frac{(2\alpha-1)^2}{4\eta_0^2}\frac{k_*^{-1-1/2\alpha}}{3-1/2\alpha}\right). \tag{4.40}$$

Thus using the above two expressions, the total energy density at the end of horizon exit is given by

$$(\rho_B + \rho_E)|_{k_*\sim\mathcal{H}} = \rho_T^{(1)} + \rho_T^{(2)}, \tag{4.41}$$

where

$$\rho_T^{(1)} = \frac{H_I^4}{2\pi^2}|C|^2\frac{4\alpha(8\alpha+1)}{(10\alpha+1)(6\alpha+1)}k_*^{1+1/2\alpha}, \tag{4.42}$$

$$\rho_T^{(2)} = \frac{H_I^4}{2\pi^2}\left|Q_+ \frac{\mathscr{F}^{-1}}{\pi}\Gamma(1/2\alpha)\right|^2 \frac{4\alpha(8\alpha-1)(2\alpha-1)^2}{(10\alpha-1)(6\alpha-1)} k_*^{-1-1/2\alpha}. \tag{4.43}$$

To identify whether these modes backreact on the metric, we define $\mathscr{R}$ as the ratio of the total energy density of the fluctuations to the background energy density during inflation :

$$\mathscr{R} = \frac{(\rho_B + \rho_E)|_{k_*\sim\mathcal{H}}}{3M_{Pl}^2 {H_I}^2}. \tag{4.44}$$

Using $M_{Pl} \sim 10^{18} GeV$ and $H_I \sim 10^{14} GeV$, the denominator is found to be $\sim 10^{65} GeV^4$. In order to estimate the numerator, we take $\mathcal{H} \sim |\eta_0|^{-1} \sim 10^{14} GeV \sim 10^{52} Mpc^{-1}$ [59] and $M \sim 10^{17} GeV$. Then using various values of α it can be seen that $\mathscr{R} \ll 1$(for instance, it is 10^{-3} for $\alpha = -3/4$ and 10^{-4} for $\alpha = -1$), implying that the back reaction of the helical modes on the background metric during inflation is negligible.

Numerical estimate of the generated helical magnetic field

In equation (3.40) if we use $g_f \sim 106.75, g_0 \simeq 3.36$ and

$$\rho_0 \simeq 4.3 \times 10^{-13} J/m^3 = \frac{4.3 \times 1.6 \times 10^{-19}(10^{-13})}{12500^3} eV^4 \simeq 3.52 \times 10^{-80} GeV^4,$$

we obtain

$$\frac{a_0}{a_f} = 0.985 \times 1.22 \times 10^{19} \times 4.29 \times 10^{19-10}\sqrt{\frac{(H_I/M_{Pl})}{10^{-5}}} \sim 10^{29}\sqrt{\frac{(H_I/M_{Pl})}{10^{-5}}}. \tag{4.45}$$

For slow roll type inflation, $\alpha = -1/2 - \varepsilon(\varepsilon \sim 10^{-3})$, and $\rho_B \sim \rho_T^{(1)} + \rho_T^{(2)} = 10^{64} GeV^4$ since (using $|C| \sim 10^{-5/\alpha - 11/2}$ and $|\mathscr{F}| \sim 10^{-5/\alpha} GeV^{-1/4\alpha}$)

$$\begin{aligned} \rho_T^{(1)} &\approx \frac{(10^{14})^4}{2\pi^2} \times 10^9 \times \frac{6}{8} \times 0.937 \sim 10^{64} GeV^4; \\ \rho_T^{(2)} &\approx \frac{(10^{14})^6}{8\pi^2} \times 10^{-25} \times \frac{40}{24} \times 0.824 \sim 10^{57} GeV^4. \end{aligned} \tag{4.46}$$

Therefore using the fact that the relevant modes left the Hubble radius with energy density $\rho_B \approx 10^{64}$ GeV^4, it can be shown that the helical magnetic fields at GPc scales is [58]

$$B_0 \simeq 10^{-40} \left(\frac{H_I/M_{Pl}}{10^{-4}}\right)^{-1} \text{GeV}^2 \sim 10^{-20} \text{ G} \left(\frac{H_I/M_{Pl}}{10^{-4}}\right)^{-1}, \tag{4.47}$$

where we have used 1 GeV$^2 = 1.95 \times 10^{-20}$ G.

4.2 Generating non-helical magnetic fields

We consider an action similar to the action S_3 in the previous section but with $\mathcal{S}_{CB}$ now given by

$$\mathcal{S}_{CB} = \frac{-1}{M^2}\int d^4x\sqrt{-g} R^{\mu\nu\alpha\beta} F_{\alpha\beta} F_{\mu\nu}.$$

(Terms of this form do actually occur in one-loop vacuum polarization calculations on a general curved background manifold, where the typical mass scale is the electron mass, $M \sim m_e$ [60]). Therefore the action in this case is

$$\begin{aligned}\mathcal{S}_4 = &-\frac{M_{Pl}^2}{2}\int d^4x\sqrt{-g}R + \int d^4x\sqrt{-g}\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi)\right] \\ &-\frac{1}{4}\int d^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu} + \frac{-1}{M^2}\int d^4x\sqrt{-g}R^{\mu\nu\alpha\beta}F_{\alpha\beta}F_{\mu\nu}\end{aligned} \tag{4.48}$$

Varying the above action with respect to A_ρ we arrive at the following equation of motion :

$$\partial_\sigma\left[\sqrt{-g}\left(F^{\rho\sigma} + \frac{4R^{\alpha\beta\rho\sigma}F_{\alpha\beta}}{M^2}\right)\right] = 0 \quad \Rightarrow \partial_\sigma\left[\sqrt{-g}\left(F^{\rho\sigma} + \frac{8R^{0k\rho\sigma}F_{0k}}{M^2} + \frac{4R^{ij\rho\sigma}F_{ij}}{M^2}\right)\right] = 0 \tag{4.49}$$

Before we can proceed further, we need the components of the Riemann tensor, which are as follows :

$$R^{\mu\nu\rho\sigma} = g^{\mu\alpha}g^{\nu\beta}R_{\alpha\beta}{}^{\rho\sigma} = g^{\mu\alpha}g^{\nu\beta}g^{\rho\lambda}g^{\sigma\tau}R_{\alpha\beta\lambda\tau}$$

$$R^{ijkl} = g^{im}g^{jn}R_{mn}{}^{kl} = \frac{a'^2}{a^8}(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}) \quad , \quad R^{0ijk} = 0 \quad , \quad R^{0i0j} = -\left(\frac{aa'' - a'^2}{a^8}\right)\delta^{ij}$$

Then for $\rho = k$, we have

$$\partial_0\left[F_{0k} + \frac{8}{M^2}\frac{aa'' - a'^2}{a^4}F_{0k}\right] + \partial_l\left[F_{kl} + \frac{4}{M^2}\frac{2a'^2}{a^4}F_{kl}\right] = 0 \tag{4.50}$$

Working in the Coulomb gauge, we arrive at the following from the above equation :

$$\left(1 + \frac{8}{M^2}\frac{aa'' - a'^2}{a^4}\right)A_k'' + \frac{8}{M^2}\left(\frac{a'''}{a^3} - 5\frac{a'a''}{a^4} + 4\frac{a'^3}{a^5}\right)A_k' - \left(1 + \frac{8}{M^2}\frac{a'^2}{a^4}\right)a^2\partial_l\partial^l A_k = 0 \tag{4.51}$$

When we take $a(\eta) = a_0(\eta/\eta_0)^{1+\beta}$ (power law inflation), equation (4.49) takes the following form [61]

$$F_1(\eta)A_k'' + F_2(\eta)A_k' - F_3(\eta)a^2\partial_l\partial^l A_k = 0, \tag{4.52}$$

where

$$\begin{aligned}F_1(\eta) &= 1 - \frac{8}{M^2}\frac{(\beta+1)}{a_0^2\eta_0^2}\left(\frac{\eta}{\eta_0}\right)^{-2\beta-4} = 1 + \frac{\mu_1}{M^2\eta_0^2}\left(\frac{\eta}{\eta_0}\right)^{-2\beta-4}; \\ F_2(\eta) &= \frac{8}{M^2}\frac{2(\beta+1)(\beta+2)}{\eta_0^3a_0^2}\left(\frac{\eta}{\eta_0}\right)^{-2\beta-5} = \frac{\mu_2}{M^2\eta_0^3}\left(\frac{\eta}{\eta_0}\right)^{-2\beta-5}; \\ F_3(\eta) &= 1 + \frac{8}{M^2}\frac{(\beta+1)^2}{a_0^2\eta_0^2}\left(\frac{\eta}{\eta_0}\right)^{-2\beta-4} = 1 + \frac{\mu_3}{M^2\eta_0^2}\left(\frac{\eta}{\eta_0}\right)^{-2\beta-4}.\end{aligned}$$

For a purely de Sitter background ($\beta = -2$), F_2 vanishes and F_1 and F_3 become independent of η.

Using equation (3.13), equation (4.52) can be written in terms of the modes $\bar{A} = aA$ as

$$F_1(\eta)\bar{A}'' + F_2(\eta)\bar{A}' + k^2F_3(\eta)\bar{A} = 0. \tag{4.53}$$

The standard quantization procedure requires the corresponding action of the field of be diagonal (recall that we defined $\mathcal{A} = f\bar{A}$ in going from equation (3.17) to (3.18)). One way this can be achieved by defining $\mathscr{A} = F(\eta)\bar{A}$, which satisfies the mode equation

$$\mathscr{A}'' + \left(\chi^2(\eta) - \frac{F''}{F}\right)\mathscr{A}, \tag{4.54}$$

where $F(\eta)$ and $\chi(\eta)$ are given by

$$2\frac{F'}{F} = \frac{F_2(\eta)}{F_1(\eta)} \quad \text{and} \quad \chi^2(\eta) = k^2\frac{F_3(\eta)}{F_1(\eta)}. \tag{4.55}$$

Although equation (4.54) looks similar to equation(3.18), solving this is not trivial due to the complicated forms of the functions $F_1(\eta), F_2(\eta)$ and $F_3(\eta)$ and the dependence of χ on η . We need to use the Bunch Davies initial condition to get the sub Hubble mode and then the super Hubble mode can be obtained by matching $\mathscr{A}$ and $\mathscr{A}'$ at horizon exit, as done in the previous section.

With the solution of equation (4.54) at hand we can find the electric and magnetic field spectra. Taking cue from the previous section, we do not expect them to be scale invariant in the slow roll approximation we are using ($\beta = -2 - \varepsilon$). From the magnetic energy density spectrum, the strength of the generated magnetic field can be found. We are currently working on this problem.

Chapter 5

Effects of the magnetic fields on gravitational waves

In this chapter, we study the effects of the helical and non-helical magnetic fields on the primordial gravitational waves. The anisotropic stress associated with the magnetic fields is known to enhance the amplitudes of the primordial gravitational waves. In what follows, we will examine the imprints of such effects on the two-point function of the tensor perturbations. But before that, in the following section, we will discuss cosmological perturbation theory with particular focus on tensor perturbations.

5.1 Cosmological perturbation theory–tensor perturbations

According to the CMB observations, the anisotropies at the end of decoupling are tiny, implying that the amplitude of deviations from homogeneity would have been considerably lower at earlier epochs [42]. This allows us to use linear perturbation theory to study the generation and evolution of the perturbations (until structures begin to form late in the matter dominated epoch).

The metric perturbations can be decomposed in the FLRW background based on their behaviour under local rotation of the spatial coordinates on hypersurfaces of constant time. The classification of the perturbations as scalars, vectors and tensors is based on this property. Here we will only be interested in the tensor perturbations, which describe gravitational waves. Note that they can exist even in the absence of sources.

5.1.1 Tensor perturbations

Upon the inclusion of the tensor perturbations, the Friedmann metric can be described by the line element

$$ds^2 = a^2[-d\eta^2 + (\delta_{ij} + h_{ij}^{TT})dx^i dx^j] \tag{5.1}$$

where h_{ij}^{TT} is a symmetric, transverse and traceless tensor. The last two conditions reduce the number of independent degrees of freedom of h_{ij}^{TT} is 2. Upon inclusion of the transverse and traceless conditions, the components of the perturbed Einstein tensor corresponding to the above line element simplify to [62]

$$\delta G^0_0 = \delta G^0_i = 0, \tag{5.2}$$

$$\delta G^i_j = -\frac{1}{2a^2}[(h^{TT}_{ij})'' + 2\mathcal{H}(h^{TT}_{ij})' - \nabla^2 h^{TT}_{ij}]. \tag{5.3}$$

In the absence of anisotropic stresses we obtain the following differential equation describing the amplitude h of gravitational waves :

$$h'' + 2\mathcal{H}h' - \nabla^2 h = 0 \tag{5.4}$$

where prime denotes derivative with respect to the conforml time coordinate η and $\mathcal{H} = aH$.

5.1.2 Quantizing the tensor perturbations

We can expand the Fourier transform of h^{TT}_{ij} in the basis of linear and circular polarization tensors respectively :

$$\tilde{h}^{TT}_{ij}(\eta, \mathbf{k}) = \sum_{P=T,\times} e^P_{ij}(\mathbf{k})\tilde{h}_P(\eta, \mathbf{k}) = \sum_{P=+,-} e^P_{ij}(\mathbf{k})\tilde{h}_P(\eta, \mathbf{k}), \tag{5.5}$$

where the polarization tensors follow the normalization conditions

$$e^{(T,\times)}_{ij}(\hat{\mathbf{k}})e^{(T,\times),ij}(\hat{\mathbf{k}}) = 1 \quad , \quad e^{(\pm)}_{ij}(\hat{\mathbf{k}})e^{(\mp),ij}(\hat{\mathbf{k}}) = 2 \tag{5.6}$$

For instance, in the frame where $\hat{\mathbf{k}}$ is along the $\hat{z}$ direction, we have the polarization tensors as follows :

$$e^T_{ab} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}_{ab}, e^\times_{ab} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}_{ab}; e^+_{ab} = \begin{pmatrix}1/\sqrt{2} & i/\sqrt{2}\\ i/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}_{ab}, e^-_{ab} = \begin{pmatrix}1/\sqrt{2} & -i/\sqrt{2}\\ -i/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}_{ab}$$

with $a, b = 1, 2$ spanning the (x, y) plane. Then equation (5.4) becomes

$$\tilde{h}''_P + 2\mathcal{H}\tilde{h}'_P + k^2\tilde{h}_P = 0 \quad \Rightarrow \quad u''_P + \left(k^2 - \frac{a''}{a}\right)u_P = 0 \tag{5.7}$$

where we have introduced $u_P(\eta, \mathbf{k}) = a(\eta)\tilde{h}_P(\eta, \mathbf{k})$ for convenience.

Note the similarity of equation (5.6) with the Mukhanov-Sasaki equation for scalar perturbations [42] :

$$v''_k + \left(k^2 - \frac{z''}{z}\right)v_k = 0, \tag{5.8}$$

where $z = (a\dot{\phi}/H) = (a\phi'/\mathcal{H}), v = \mathcal{R}z$ and $\mathcal{R}$ is the curvature perturbation.
Analytic solution for de Sitter : In power law inflation, $a''/a = 2/\eta^2$ and equation (5.7) becomes

$$u'' + \left(k^2 - \frac{2}{\eta^2}\right)u = 0. \tag{5.9}$$

This has solutions

$$u_1(k, \eta) = \frac{\cos(k\eta)}{k\eta} + \sin(k\eta) \quad , \quad u_2(k, \eta) = \frac{\sin(k\eta)}{k\eta} - \cos(k\eta), \tag{5.10}$$

and the general solution is a linear combination of u_1 and u_2 which satisfies the Bunch-Davies initial condition (see later).

5.1.3 The tensor power spectrum

The primordial tensor power spectrum is defined by [46]

$$\langle \tilde{h}_P(\mathbf{k})\tilde{h}^*_{P'}(\mathbf{k}')\rangle = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')\delta_{PP'}\frac{P_T(k)}{2}, \tag{5.11}$$

where $\langle ...\rangle$ denotes the vacuum state expectation value, P_T is the primordial tensor power spectrum, and the factor of 2 comes due to the following choice of normalization of the polarization tensors :

$$e^T_{ij} = \frac{(\hat{e}^1_i\hat{e}^1_j - \hat{e}^2_i\hat{e}^2_j)}{\sqrt{2}} \quad , \quad e^\times_{ij} = \frac{(\hat{e}^1_i\hat{e}^2_j + \hat{e}^2_i\hat{e}^1_j)}{\sqrt{2}}. \tag{5.12}$$

This factor will be 4 when $(P, P' = \pm)$ as we have defined

$$e^{ij}_\pm = \sqrt{2}\hat{e}^i_\pm\hat{e}^j_\pm, \tag{5.13}$$

see equation (3.50) for definitions of $\hat{e}^i_\pm$. Substituting equation (5.5) in equation (5.11) and using the normalization condition in equation (5.6) gives

$$\langle \tilde{h}^{TT}_{ij}(\mathbf{k})\tilde{h}^{TT*}_{ij}(\mathbf{k}')\rangle = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')\frac{2\pi^2}{k^3}\mathcal{P}_T(k), \tag{5.14}$$

where we have defined

$$\mathcal{P}_T(k) \equiv \frac{k^3}{2\pi^2}P_T(k). \tag{5.15}$$

In coordinate space, we get

$$\langle h^{TT}_{ij}(,\mathbf{x})h^{TT}_{ij}(,\mathbf{x}')\rangle = \int \frac{d^3k}{4\pi k^3}e^{i\mathbf{k}.(\mathbf{x}-\mathbf{x}')}\mathcal{P}_T(k) = \int_0^\infty \frac{dk}{k}\mathcal{P}_T(k) \tag{5.16}$$

The primordial tensor power spectrum is parametrized as (in a range of values of k around a reference value k_{ref} called the "pivot scale") [46]

$$\mathcal{P}_T(k) = A_T(k_{ref})(k/k_{ref})^{n_T}, \tag{5.17}$$

where A_T is the amplitude and n_T is the tilt of the tensor spectrum. The tensor power spectrum can be expressed in terms of u_P as follows [42]

$$\mathcal{P}_T(k) = 2\frac{k^3}{2\pi^2}\left(\frac{|u_P|}{a}\right)^2, \tag{5.18}$$

where the expressions on the right hand side are to be evaluated in the super-Hubble limit (the factor of 2 in this expression takes care of the two states of polarization of the gravitational waves).

The tensor power spectrum in slow roll inflation

In this subsection, we will arrive at the tensor power spectrum in slow roll inflation. But first, we'll compute the perturbation spectra in the situation of power law inflation, which will help us understand how to calculate the slow roll spectra [42].

Power law inflation
From equations (2.35) and (2.37), we have

$$z = \frac{a\dot{\phi}}{H} = \sqrt{\frac{2}{q}} M_{Pl} a. \tag{5.19}$$

Upon using the scale factor $a(\eta) = (-\mathscr{H}\eta)^{\beta+1}$, with $\mathscr{H} = [(q-1)a_1^{1/q}]$, the solution of the Mukhanov-Sasaki equation that satisfies the Bunch-Davies initial condition

$$\lim_{(k/\mathcal{H}\to\infty)} (v_k(\eta), u_P(\eta)) \to \frac{e^{-ik\eta}}{\sqrt{2k}}, \tag{5.20}$$

is found to be [63]

$$v_k(\eta) = \sqrt{\frac{-\pi\eta}{4}} \exp\left[i(\nu+1/2)\pi/2\right] H_\nu^{(1)}(-k\eta), \tag{5.21}$$

where $\nu = -(\beta+1/2)$, and $H_\nu^{(1)}$ is the Hankel function of the first kind of order ν. Since in such a power law case, $z''/z = a''/a$, u_P will have the same form as v_k. Writing $-k\eta = x$ the Hankel function can be expanded in the super-Hubble limit as

$$H_\nu^{(1)}(x) = -(i/\pi)\Gamma(\nu)(x/2)^{-\nu} \qquad \text{as } x \to 0,$$

and we obtain

$$|v_k|^2 \propto |H_\nu^{(1)}(x)|^2 \to \frac{|\Gamma(\nu)|^2}{\pi^2}\left(\frac{x}{2}\right)^{-2\nu} \quad \text{or} \quad \frac{|\Gamma(-\beta-1/2)|^2}{\pi^2}\left(\frac{-k\eta}{2}\right)^{2\beta+1}. \tag{5.22}$$

Thus the tensor power spectrum evaluated at super-Hubble scales can be written as

$$\mathcal{P}_T(k) = A_T \mathscr{H}^2 (k/\mathcal{H})^{2\beta+4}, \tag{5.23}$$

with the quantity A_T given by ($4/M_{Pl}^2$ has been put in by hand, as is the convention. [42])

$$A_T = \left(\frac{4}{\pi^3 M_{Pl}^2}\right)\frac{|\Gamma(-\beta-1/2)|^2}{2^{(2\beta+3)}}. \tag{5.24}$$

Clearly the spectral index is constant and is given by

$$n_T = [2(\beta+2)] = -\left(\frac{2}{q-1}\right). \tag{5.25}$$

The scalar power spectrum has also been calculated in Ref. [42] and it is given by $n_s = n_T + 1$, hence is also a constant. Therefore the resulting tensor-to-scalar ratio is given by

$$r = 16\frac{(\beta+2)}{(\beta+1)} = \frac{16}{q}, \tag{5.26}$$

a constant. These results suggest that the scalar and tensor spectra turn more and more scale invariant as $q \to \infty$ (exponential inflation).

Slow roll inflation

We now turn to the evaluation of the tensor power spectrum in slow roll inflation. The quantity z can be written in terms of the first Hubble slow roll parameter ϵ_H as follows

$$z = \sqrt{2} M_{Pl} (a\sqrt{\epsilon_H}). \tag{5.27}$$

Also note that equations (2.42) can be rewritten as

$$\epsilon_H = 1 - \frac{\mathcal{H}'}{\mathcal{H}^2} \quad ; \quad \delta_H = \epsilon_H - \frac{\epsilon'_H}{2\mathcal{H}\epsilon_H}. \tag{5.28}$$

Using these expressions, a''/a term appearing in equation (5.7) can be written as [64]

$$\frac{a''}{a} = \mathcal{H}^2(2 - \epsilon_H). \tag{5.29}$$

Let us rewrite equation (5.28a) above as follows:

$$\eta = -\int \frac{d(1/\mathcal{H})}{1-\epsilon_H} = \frac{-1}{(1-\epsilon_H)} - \int \left[\frac{2\epsilon_H(\epsilon_H - \delta_H)}{(1-\epsilon_H)^3}\right] d(1/\mathcal{H}). \tag{5.30}$$

The second term can be ignored at leading order in the slow roll approximation, then we have

$$\mathcal{H} \simeq \frac{-1}{(1-\epsilon_H)\eta}. \tag{5.31}$$

When we use the above expression for $\mathcal{H}$ in equation (5.29), we obtain, at leading order in slow roll approximation,

$$\frac{a''}{a} \simeq \frac{3 + 2\epsilon_H}{\eta^2}, \tag{5.32}$$

with the slow roll parameters now treated as constants. The solutions to the variable u_P will again be given in terms of Hankel functions with ν being replaced by [42, 64]

$$\nu_T \simeq [3/2 + \epsilon_H]. \tag{5.33}$$

In this case the tensor power spectrum assumes the form [64]

$$\mathcal{P}_T(k) = \left(\frac{1}{2\pi^2 M_{Pl}^2}\right)\left(\frac{|\Gamma(\nu_T)|}{\Gamma(3/2)}\right)^2 \left(\frac{k}{a}\right)^2 \left(\frac{-k\eta}{2}\right)^{-2\nu_T+1} \tag{5.34}$$

$$= \left(\frac{|\Gamma(\nu_T)|}{\Gamma(3/2)}\right)^2 \left(\frac{H}{\sqrt{2}\pi M_{Pl}}\right)^2 [2(1-\epsilon_H)]^{2\nu_T - 1}, \tag{5.35}$$

where the second equality expresses the asymptotic values in terms of the quantities at Hubble exit (when $-k\eta = (1-\epsilon_H)^{-1}$). The amplitude of the tensor power spectrum can be easily read off from the above equation [42] :

$$A_T(k) \simeq \frac{4}{M_{Pl}^2} \left(\frac{H}{\sqrt{2}\pi}\right)^2_{(k_* = \mathcal{H})}, \tag{5.36}$$

with the subscripts indicating that the quantities have to be evaluated at horizon crossing, i.e. $-k_*\eta = k_*/\mathcal{H} = 1$. From equation (5.35), we can obtain the tensor spectral index as follows :

$$n_T = -2\nu_T + 3 = -2\epsilon_H. \tag{5.37}$$

This suggests that the tensor power spectra arising in slow roll inflation will be almost scale invariant. An analysis similar to this has been done in Ref. [42] for the scalar power spectrum and the tensor-to-scalar ratio in the slow roll limit has been found to be

$$r \simeq 16\epsilon_H = -8n_T. \tag{5.38}$$

5.2 Stochastic gravitational waves

We begin this section by defining the energy density of gravitational waves in the Fourier space :

$$\rho_{GW} \equiv \int \frac{dk}{k}\frac{d\rho_{GW}}{d\ln k} = \frac{1}{16\pi G a^2}\int \frac{d^3k}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3}\langle \tilde{h}'_{ij}(\eta,\mathbf{k})\tilde{h}'^{*}_{ij}(\eta,\mathbf{k}-\mathbf{q})\rangle e^{i\mathbf{q}\cdot\mathbf{x}}. \tag{5.39}$$

Here we have dropped the "TT" for convenience. We rewrite the equation for evolution of $\tilde{h}_{ij}$ with time, now in the presence of anisotropic stresses, i.e. with $\delta G^\mu_\nu = 8\pi G\delta T^\mu_\nu \neq 0$:

$$\tilde{h}''_{ij} + 2\mathcal{H}\tilde{h}'_{ij} + k^2\tilde{h}_{ij} = 16\pi G\tilde{\tau}_{ij}, \tag{5.40}$$

where $\tilde{\tau}_{ij}$ is the transverse traceless part of the energy momentum tensor T^μ_ν of the source. Expanding $\tilde{h}_{ij}$ in accordance with equation (5.5), we are led to

$$\tilde{h}''_P + 2\mathcal{H}\tilde{h}'_P + k^2\tilde{h}_P = 16\pi G\tilde{\tau}_P, \quad (P = T, \times \text{ or } +, -). \tag{5.41}$$

We will solve the inhomogeneous equation in the de Sitter background. For doing so, we first need to find out the respective Green's function, which we do in the following subsection.

5.2.1 Green's Function for de Sitter expansion

Equation (5.7a) is the homogeneous analogue of equation (5.41), which in this case becomes equation (5.9) having two independent solutions given in equation (5.10). In general, the Green's function satisfies

$$G''(\eta,\eta') + \left(k^2 - \frac{a''}{a}\right)G(\eta,\eta') = \delta(\eta,\eta'). \tag{5.42}$$

With the two independent solutions u_1 and u_2, one can construct the Wronskian

$$W(\eta) = u_1(\eta)u'_2(\eta) - u'_1(\eta)u_2(\eta), \tag{5.43}$$

so that

$$W' = u_1u''_2 + u'_1u'_2 - u''_1u_2 - u'_1u'_2 = 0, \tag{5.44}$$

i. e. the Wronskian is a constant. In that case, we can write the (retarded) Green's function as

$$G(\eta,\eta') = \frac{-\Theta(\eta-\eta')}{W(\eta)}[u_1(\eta)u_2(\eta') - u_1(\eta')u_2(\eta)], \tag{5.45}$$

where Θ is the Heaviside function, and it can be easily checked that it satisfies equation (5.3). Therefore, the Green's function for the operator $d^2/d\eta^2 - (2/\eta)d/d\eta + k^2$ is as follows [18]:

$$G_k(\eta,\eta') = \frac{\Theta(\eta-\eta')}{k^3\eta'^2}[(1+k^2\eta\eta')\sin k(\eta-\eta') + k(\eta'-\eta)\cos k(\eta-\eta')]. \tag{5.46}$$

5.3 Energy momentum tensor of the source

The energy-momentum tensor of the electromagnetic field, obtained from the standard EM action, is given by

$$T_{\mu\nu} = g^{\alpha\beta} F_{\mu\alpha} F_{\mu\beta} - \frac{g_{\mu\nu}}{4} F^{\alpha\beta} F_{\alpha\beta}, \tag{5.47}$$

whose spatial part is given by

$$T_{ij}(\mathbf{x}, \eta) = B_i B_j + E_i E_j - \frac{g_{ij}}{2}(B_m B^m + E_m E^m), \tag{5.48}$$

where we have used

$$B_i \equiv \frac{1}{a} \eta_{ijk} \delta^{jl} \delta^{km} \partial_l A_m \quad ; \quad E_i \equiv -\frac{1}{a} A_i'. \tag{5.49}$$

After taking the Fourier transformation of the above equation, we obtain

$$\begin{aligned} \tilde{T}_{ij}(\mathbf{k}, \eta) = \int \frac{d^3q}{(2\pi)^3} B_i(\mathbf{q}, \eta) B_j^*(\mathbf{q} - \mathbf{k}, \eta) + \int \frac{d^3q}{(2\pi)^3} E_i(\mathbf{q}, \eta) E_j^*(\mathbf{q} - \mathbf{k}, \eta) \\ -\frac{g_{ij}}{2} \left[\int \frac{d^3q}{(2\pi)^3} B_m(\mathbf{q}, \eta) B^{*m}(\mathbf{q} - \mathbf{k}, \eta) + \int \frac{d^3q}{(2\pi)^3} E_m(\mathbf{q}, \eta) E^{*m}(\mathbf{q} - \mathbf{k}, \eta) \right]. \end{aligned} \tag{5.50}$$

The anisotropic stress tensor is given by the transverse-traceless projection of the spatial part of the energy-momentum tensor :

$$\tilde{\tau}_{ij}(\mathbf{k}, \eta) = \int \frac{d^3q}{(2\pi)^3} \Pi_{ij}^{mn} [E_m(\mathbf{q}, \eta) E_n^*(\mathbf{q} - \mathbf{k}, \eta) + B_m(\mathbf{q}, \eta) B_n^*(\mathbf{q} - \mathbf{k}, \eta)] = \tilde{\tau}_{ij}^{(E)} + \tilde{\tau}_{ij}^{(B)}, \tag{5.51}$$

where

$$\Pi_{ij}^{mn} = P_i^m P_j^n - \frac{P_{ij} P^{mn}}{2}, \tag{5.52}$$

is called a tensor projector and $P_{ij}(\mathbf{k}) = \delta_{ij} - \hat{k}_i \hat{k}_j$ is the usual transverse plane projector satisfying

$$P_{ij} P_{jk} = P_{ik} \quad \text{and} \quad P_{ij} \hat{k}_j = P_{ij} \hat{k}_j = 0, \tag{5.53}$$

which we encountered in Chapter **??** The second term has not been included since it does not contribute to the transverse, traceless part because

$$\Pi_{ij}^{mn} g_{ij} = a^2 \left(P_i^m P_j^n - \frac{P_{ij} P^{mn}}{2} \right) \delta_{ij} = 0. \tag{5.54}$$

In this report we will only focus on the contribution from the magnetic part. From now on, we will remove the tilde over functions to denote the quantities in Fourier space and allow the arguments $(\mathbf{x}, \eta)$ or $(\mathbf{k}, \eta)$ to solely do the job.

Neglecting the solutions of the homogeneous part of equation (5.40) for the time being, we have the following expression for $h_{ij}(\mathbf{k})$:

$$\begin{aligned} h_{ij}(\mathbf{k}) &= \frac{2}{M_{Pl}^2} \int d\eta' G_k(\eta, \eta') \tau_{ij}^{(B)}(\mathbf{k}, \eta') \\ \implies h_{ij}(\mathbf{k}) &= \frac{2}{M_{Pl}^2} \int d\eta' G_k(\eta, \eta') \int \frac{d^3q}{(2\pi)^3} \Pi_{ij}^{mn}(\mathbf{k}) [B_m(\mathbf{q}, \eta') B_n^*(\mathbf{q} - \mathbf{k}, \eta')] \end{aligned} \tag{5.55}$$

Therefore the two-point function of $h_{ij}(\mathbf{k})$ is

$$\langle h_{ij}(\mathbf{k})h^*_{ij}(\mathbf{k}')\rangle = \frac{4}{M^4_{Pl}}\left[\int d\eta' G_k(\eta,\eta')\int d\eta'' G_{k'}(\eta,\eta'')\right]\langle \tau^{(B)}_{ij}(\mathbf{k},\eta')\tau^{*(B)}_{ij}(\mathbf{k}',\eta'')\rangle \tag{5.56}$$

where

$$\begin{aligned}\langle \tau^{(B)}_{ij}(\mathbf{k},\eta')\tau^{*(B)}_{ij}(\mathbf{k}',\eta'')\rangle = \int \frac{d^3q}{(2\pi)^3a^2(\eta')}\int \frac{d^3q'}{(2\pi)^3a^2(\eta'')}\Pi^{mn}_{ij}(\mathbf{k})\Pi^{ab}_{ij}(\mathbf{k}')\\ \times \langle B_m(\mathbf{q};\eta')B^*_n(\mathbf{q}-\mathbf{k};\eta')B_a(\mathbf{q}';\eta'')B^*_b(\mathbf{q}'-\mathbf{k}';\eta'')\rangle\end{aligned} \tag{5.57}$$

For the standard EM Lagrangian without inflaton coupling, $B_i(\mathbf{q},\eta) = B_i(\mathbf{q})/a(\eta)$ and $B^i(\mathbf{q},\eta) = B^i(\mathbf{q})/a^3(\eta)$. But with the inflaton coupling, $T_{ij} \to f^2(\eta)T_{ij}$ or $B_i(\mathbf{q},\eta) = B_i(\mathbf{q})f(\eta)/a(\eta) = B_i(\mathbf{q};\eta)/a(\eta)$ and $B^i(\mathbf{q},\eta) = B^i(\mathbf{q})f(\eta)/a^3(\eta) = B^i(\mathbf{q};\eta)/a^3(\eta)$.

Using Wick's theorem, We can express the four point correlation functions, which occur in the preceding statement, in terms of the two point correlation functions as follows, since the magnetic field generated in our model is gaussian [65]:

$$\begin{aligned}\langle B_m(\mathbf{q};\eta')B^*_n(\mathbf{q}-\mathbf{k};\eta')B_a(\mathbf{q}';\eta'')B^*_b(\mathbf{q}'-\mathbf{k}';\eta'')\rangle &= \langle B_m(\mathbf{q};\eta')B^*_n(\mathbf{q}-\mathbf{k};\eta')\rangle\langle B_a(\mathbf{q}';\eta'')B^*_b(\mathbf{q}'-\mathbf{k}';\eta'')\rangle\\ &+ \langle B_m(\mathbf{q};\eta')B^*_a(\mathbf{q}';\eta'')\rangle\langle B^*_n(\mathbf{q}-\mathbf{k};\eta')B_b(\mathbf{q}'-\mathbf{k}';\eta'')\rangle\\ &+ \langle B_m(\mathbf{q};\eta')B_b(\mathbf{q}'-\mathbf{k}';\eta'')\rangle\langle B^*_n(\mathbf{q}-\mathbf{k};\eta')B^*_a(\mathbf{q}';\eta'')\rangle.\end{aligned} \tag{5.58}$$

In the above equation, we require unequal time correlation of the magnetic fields, which we write as a product of the equal time correlation, $\langle B_a(\mathbf{k};\eta')B_b(\mathbf{k}';\eta'')\rangle$ and a two-time correlation function $C_B(k,\eta',\eta'')$ [65]:

$$\langle B_a(\mathbf{k};\eta')B_b(\mathbf{k}';\eta'')\rangle = \langle B_a(\mathbf{k};\eta')B_b(\mathbf{k}';\eta'')\rangle C_B(k,\eta',\eta''), \tag{5.59}$$

with this being evident that $C_B(k,\eta,\eta) = 1$. To proceed further, we need to know the equal time correlation function of the electric and magnetic field. We will examine the cases of helical and non-helical magnetic fields separately.

5.3.1 Non-helical magnetic field

The two point functions of the non-helical magnetic field can be written in terms of the power spectra as follows [65] :

$$\langle B_i(\mathbf{k};\eta)B^*_j(\mathbf{k}';\eta)\rangle = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')P_{ij}P_{SB}(k;\eta), \tag{5.60}$$

In writing this expression we have assumed that the generated magnetic field is homogeneous and isotropic. The delta function $\delta(\mathbf{k}-\mathbf{k}')$ arises because of the isotropy and the dependence of P_{SB} on k arises because of the homogeneity. Therefore

$$\langle \tau^{(B)}_{ij}(\mathbf{k},\eta')\tau^{*(B)}_{ij}(\mathbf{k}',\eta'')\rangle = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')f_B(k,\eta',\eta'')/a^2(\eta')a^2(\eta''), \tag{5.61}$$

where (see Appendix B for a derivation)

$$\begin{aligned}f_B(k,\eta',\eta'') = \frac{1}{(2\pi)^3}\int d^3q[P_{SB}(q;\eta')P_{SB}(|\mathbf{k}-\mathbf{q}|;\eta')\\ \times(1+\gamma^2+\beta^2+\beta^2\gamma^2)]C_B(q,\eta',\eta'')C_B(|\mathbf{k}-\mathbf{q}|,\eta',\eta''),\end{aligned} \tag{5.62}$$

with $\gamma = \hat{\mathbf{k}} \cdot \hat{\mathbf{q}}$ and $\beta = \hat{\mathbf{k}} \cdot (\mathbf{k}-\mathbf{q})/|\mathbf{k}-\mathbf{q}|$. To get the individual mode contribution, we express $\tau_{ij}^{(B)}(\mathbf{k},\eta)$ in terms of the linear polarization basis :

$$\tau_{ij}^{(B)}(\mathbf{k},\eta) = \tau_T^{(B)}(\mathbf{k},\eta)e_{ij}^T + \tau_\times^{(B)}(\mathbf{k},\eta)e_{ij}^\times. \tag{5.63}$$

Using this, we get

$$\langle \tau_{ij}^{(B)}(\mathbf{k},\eta')\tau^{(B)*ij}(\mathbf{k}',\eta'')\rangle = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')[|\tau_T^{(B)}|^2(k,\eta',\eta'',) + |\tau_\times^{(B)}|^2(k,\eta',\eta'',)]. \tag{5.64}$$

In this case, the source is such that the contribution to both the modes are equal. Therefore using equation (5.64) and equation (5.61), we get

$$|\tau_T^{(B)}|^2(k,\eta',\eta'') = |\tau_\times^{(B)}|^2(k,\eta',\eta'') = f_B(k,\eta',\eta'')/2a^2(\eta')a^2(\eta''). \tag{5.65}$$

Therefore

$$\begin{aligned}\langle h_P(\mathbf{k})h_P(\mathbf{k}')\rangle &= \frac{4}{M_{Pl}^4}\left[\int d\eta' G_k(\eta,\eta')\int d\eta'' G_k(\eta,\eta'')\right] \times (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')\left(|\tau_P^{(B)}|^2(k,\eta',\eta'')\right)\\ &= (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')\frac{4}{M_{Pl}^4}\left[\int d\eta' \frac{G_k(\eta,\eta')}{a^2(\eta')}\int d\eta'' \frac{G_k(\eta,\eta'')}{a^2(\eta'')}\right]\frac{f_B(k,\eta',\eta'')}{2}.\end{aligned} \tag{5.66}$$

We need to calculate f_B for which we need P_{SB} and $C_B(k,\eta,\eta')$. We define $P_{SB}(k;\eta)$ as

$$P_{SB}(k;\eta) = \frac{2\pi^2}{k^3}\frac{d\tilde{\rho}_B(k;\eta)}{d\ln k}, \tag{5.67}$$

where $d\tilde{\rho}_B(k;\eta)/d\ln k = a^4(\eta)d\rho_B(k,\eta)/d\ln k$. Using some definitions given in Chapter 3, this can be easily obtained as follows :

$$\begin{aligned}\langle B_i(\mathbf{q};\eta')B_j^*(\mathbf{p};\eta')\rangle &= \eta_{iab}\eta_{jcd}q_a p_c\langle \mathcal{A}_b(\mathbf{q},\eta')\mathcal{A}_d^*(\mathbf{p},\eta')\rangle\\ &= qp\mathcal{A}(q,\eta')\mathcal{A}^*(p,\eta')\langle \hat{b}_1(\mathbf{q})\hat{e}_{1i}\hat{b}_1^\dagger(\mathbf{p})\hat{e}_{1j} + \hat{b}_2(\mathbf{q})\hat{e}_{2i}\hat{b}_2^\dagger(\mathbf{p})\hat{e}_{2j}\rangle\\ &= qp\mathcal{A}(q,\eta')\mathcal{A}^*(p,\eta') \times P_{ij} \times \langle \hat{b}_\lambda(\mathbf{q})\hat{b}_\lambda^\dagger(\mathbf{p})\rangle\\ \implies \langle B_i(\mathbf{q};\eta')B_j^*(\mathbf{p};\eta')\rangle &= (2\pi)^3\delta(\mathbf{q}-\mathbf{p})P_{ij} \times q^2|\mathcal{A}(q,\eta')|^2 = (2\pi)^3\delta(\mathbf{q}-\mathbf{p})P_{ij}P_{SB}(q;\eta).\end{aligned} \tag{5.68}$$

Comparing with the expression for $d\rho_B/d\ln k$ in equation (3.25), we obtain equation (5.67).

Now the expression for $d\rho_B(k,\eta)/d\ln k$ in this case is

$$\frac{d\rho_B}{d\ln k} \approx \frac{\mathscr{F}(n)}{2\pi^2}H_I^4\left(\frac{k}{aH_I}\right)^{4+2n} \implies \frac{d\tilde{\rho}_B}{d\ln k} = \frac{\mathscr{F}(n)}{2\pi^2}k^4(-k\eta)^{2n} \tag{5.69}$$

For a scale invariant magnetic spectrum, $n=-2$ (this choice also takes care of the back-reaction problem), therefore

$$\frac{d\tilde{\rho}_B}{d\ln k} = \frac{1}{\eta^4}\frac{1}{2\pi^2}\frac{8\pi}{16\pi/9} \implies P_{SB}(k;\eta) = \frac{9}{2k^3\eta^4} \tag{5.70}$$

Therefore

$$\begin{aligned} f_B(k,\eta',\eta'') &= \frac{81}{4(2\pi)^3}\int d^3q\left[\frac{1}{q^3\eta'^4}\frac{1}{|\mathbf{k}-\mathbf{q}|^3\eta'^4}\right]\\ &\times(1+\gamma^2+\beta^2+\beta^2\gamma^2)C_B(q,\eta',\eta'')C_B(|\mathbf{k}-\mathbf{q}|,\eta',\eta''),\end{aligned} \tag{5.71}$$

We can obtain the unequal time correlator as follows : $d\tilde{\rho}_B/d\ln k \propto \eta^{2n} \implies B \propto \eta^n$. Therefore $C_B(\eta_1, \eta_2) = (\eta_2/\eta_1)^n$,$(\eta_1, \eta_2 < 0)$ which will be equal to $(\eta_1/\eta_2)^2$ in the scale invariant case we are considering.[1] Therefore equation (5.71) can be written as

$$f_B(k, \eta', \eta'') = \frac{81}{4(2\pi)^3} \int d^3q \left[\frac{1}{q^3\eta'^4} \frac{1}{|\mathbf{k}-\mathbf{q}|^3\eta''^4} \right] \times (1 + \gamma^2 + \beta^2 + \beta^2\gamma^2), \tag{5.72}$$

Therefore at the end of inflation ($\eta \simeq 0$), equation (5.66) becomes (with $a(\eta) = -1/H_I\eta$)

$$\langle h_P(\mathbf{k}) h_P(\mathbf{k}'))\rangle = \frac{H_I^4}{2M_{Pl}^4} \left(\int d\eta' \frac{G_k(0,\eta')}{\eta'^2} \right)^2 \times (2\pi)^3 \delta(\mathbf{k}-\mathbf{k}') \frac{81}{(2\pi)^3} \int d^3q \frac{(1+\gamma^2+\beta^2+\beta^2\gamma^2)}{q^3(k^2+q^2-2kq\gamma)^{3/2}}, \tag{5.73}$$

where

$$G_k(0, \eta') = \frac{\Theta(-\eta')}{k^3\eta'^2} [-\sin k\eta' + k\eta' \cos k\eta'],$$

and

$$1 + \gamma^2 + \beta^2 + \gamma^2\beta^2 = \left(1 + \left[\frac{k - q\cos\theta}{|\mathbf{k}-\mathbf{q}|} \right]^2 \right) (1 + \cos^2\theta).$$

Therefore the $\mathbf{q}$ integral becomes (with $\mathbf{k}$ directed along the z-axis, $\mathbf{k} = k\hat{\mathbf{z}}$)

$$\int d^3q \frac{(1+\gamma^2+\beta^2+\beta^2\gamma^2)}{q^3(k^2+q^2-2kq\gamma)^{3/2}} = 2\pi \int dq d\gamma \frac{(1+\gamma^2)(2k^2+q^2(1+\gamma^2)-4kq\gamma)}{q(k^2+q^2-2kq\gamma)^{5/2}}$$

Now, following [66] ,we re-scale the integration variable q by a factor k ($q \to kq$), and obtain

$$\int d^3q \frac{(1+\gamma^2+\beta^2+\beta^2\gamma^2)}{q^3(k^2+q^2-2kq\gamma)^{3/2}} = \frac{2\pi}{k^3} \int dp d\gamma \frac{(1+\gamma^2)(2+p^2(1+\gamma^2)-4p\gamma)}{p(1+p^2-2p\gamma)^{5/2}} \tag{5.74}$$

In the super Hubble limit, $G_k(0, \eta') \simeq \eta'/3$ and the η' integral then becomes $\int d\eta'/3\eta'$, which we shall compute in the next section.

5.3.2 Helical magnetic field

The two-point function of the helical magnetic field can be written in terms of the power spectra as follows [67] :

$$\langle B_i(\mathbf{k};\eta) B_j^*(\mathbf{k}';\eta)\rangle = (2\pi)^3 \delta(\mathbf{k}-\mathbf{k}')(P_{ij} P_{SB}(k;\eta) + i\eta_{ijl}\hat{k}_l P_{AB}(k;\eta)), \tag{5.75}$$

with $P_{SB/AB}$ given by [68] :

$$P_{SB/AB}(k, \eta) = k^2(|\mathcal{A}_+(k,\eta)|^2 \pm |\mathcal{A}_-(k,\eta)|^2). \tag{5.76}$$

Note the additional anti-symmetric contribution to the two-point correlation function in equation (5.75). An analysis similar to that carried out in the previous subsection can be done in this case.

[1] This form of the function also makes sense, since it renders $f_B(k, \eta_1, \eta_2)$ symmetric in the two conformal time arguments, as it should be.

However we directly use the expression for the two-point function of the h_P graviton ($P = \pm$) obtained in [18] or recently in [69] :

$$\langle h_P(\mathbf{k})h_P(\mathbf{k}')\rangle = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')\frac{H_I^4}{2M_{Pl}^4}\sum_{\rho,\sigma=\pm}\int\frac{d^3q}{(2\pi)^3}q^2|\mathbf{k}-\mathbf{q}|^2(1-\rho(P)\gamma)^2(1-\sigma(P)\beta)^2 \times\left|\int d\eta'\eta'^2 G_k(\eta,\eta')\mathcal{A}_\rho(q,\eta')\mathcal{A}_\sigma(|\mathbf{k}-\mathbf{q}|,\eta')\right|^2 \tag{5.77}$$

How this can be derived has been discussed in Appendix B. In the super Hubble limit ($-k\eta \ll 1$), from equation (3.63), viz.

$$\mathcal{A}_\sigma = \frac{c_\sigma}{\sqrt{2k}}[(-k\eta)^{-\alpha} + h(-k\eta)^{1-\alpha}] + \frac{d_\sigma}{\sqrt{2k}}[(-k\eta)^{1+\alpha} + (h\alpha/(1+\alpha))(-k\eta)^{2+\alpha}],$$

with c_σ and d_σ being given by equation (3.64), viz.

$$c_\sigma = e^{-\sigma\pi\alpha/2}\frac{(-2i)^{-\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha-i\sigma\alpha)} \quad ; \quad d_\sigma = e^{-\sigma\pi\alpha/2}\frac{(-2i)^{1+\alpha}\Gamma(-1-2\alpha)}{\Gamma(-\alpha-i\sigma\alpha)},$$

we see that for the scale invariant case we are interested in, i.e. $\alpha = 2$, we have (keeping only the dominant terms)

$$\mathcal{A}_\sigma \approx \frac{c_\sigma}{\sqrt{2k}}[(-k\eta)^{-2}] = \frac{c_\sigma}{\sqrt{2}}\frac{1}{k^{5/2}\eta^2}, \tag{5.78}$$

and

$$c_+ = e^{-\pi}\frac{(-0.25)24}{\Gamma(3-2i)} \quad , \quad c_- = e^{\pi}\frac{(-0.25)24}{\Gamma(3+2i)}.$$

Thus we see that the negative helicity mode is amplified by a factor $e^{2\pi}$ compared to the positive helicity mode. So we take $\rho,\sigma = -$. As negative helicity gravitons are produced much less abundantly, we take $P = +$, in which case equation (5.77) becomes

$$\langle h_+(\mathbf{k})h_+(\mathbf{k}')\rangle = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')\frac{H_I^4}{2M_{Pl}^4}\int\frac{d^3q}{(2\pi)^3}q^2|\mathbf{k}-\mathbf{q}|^2(1+\gamma)^2(1+\beta)^2 \times\left|\int d\eta'\eta'^2 G_k(\eta,\eta')\mathcal{A}_-(q,\eta')\mathcal{A}_-(|\mathbf{k}-\mathbf{q}|,\eta')\right|^2, \tag{5.79}$$

with

$$\left|\int d\eta'\eta'^2 G_k(\eta,\eta')\mathcal{A}_-(q,\eta')\mathcal{A}_-(|\mathbf{k}-\mathbf{q}|,\eta')\right|^2 = \left(\frac{|c_-|^2}{2}\right)^2\frac{1}{q^5}\frac{1}{|\mathbf{k}-\mathbf{q}|^5}\left(\int d\eta'\frac{G_k(\eta,\eta')}{\eta'^4}\right)^2 = \left(\frac{e^{2\pi}18}{80\pi e^{-2\pi}}\right)^2\frac{1}{q^5}\frac{1}{|\mathbf{k}-\mathbf{q}|^5}\left(\int d\eta'\frac{G_k(\eta,\eta')}{\eta'^2}\right)^2.$$

Therefore, at the end of inflation ($\eta \simeq 0$), equation (5.79) becomes

$$\langle h_+(\mathbf{k})h_+(\mathbf{k}')\rangle = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')\frac{H_I^4}{2M_{Pl}^4}\int\frac{d^3q}{(2\pi)^3}\frac{(1+\gamma)^2(1+\beta)^2}{q^3|\mathbf{k}-\mathbf{q}|^3}\times\frac{81e^{8\pi}}{1600\pi^2}\left(\int d\eta'\frac{G_k(0,\eta')}{\eta'^2}\right)^2 \tag{5.80}$$

The $\mathbf{q}$ integral is

$$\int d^3q\frac{(1+\gamma)^2(1+\beta)^2}{q^3|\mathbf{k}-\mathbf{q}|^3} = \frac{2\pi}{k^3}\int dp d\gamma\frac{(1+\gamma)^2(\sqrt{1+p^2-2p\gamma}+1-p\gamma)^2}{p(p^2+1-2p\gamma)^{5/2}}, \tag{5.81}$$

and in the super Hubble limit we are considering, the η' integral is $\int d\eta'/3\eta'$, same as in the non-helical case.

5.4 Correction to the tensor power spectrum

With $\epsilon_H = 0$, equation (5.35) becomes

$$\mathcal{P}_T^0(k) = \left(\frac{H_I}{\sqrt{2}\pi M_{Pl}}\right)^2 \times 4 \simeq \frac{2}{\pi^2}10^{-8} \approx 2.026 \times 10^{-9}, \tag{5.82}$$

with the superscript $'0'$ indicating "no anisotropic stress" and we have used $H_I/M_{Pl} \simeq 10^{-4}$ as mentioned earlier. We will now compare this with the tensor power spectra in the presence of anisotropic stress due to the magnetic field. We will denote the power spectra in the non-helical and helical case by $\mathcal{P}_T^{nh}(k)$ and $\mathcal{P}_T^h(k)$ respectively.

Non-helical magnetic field

We evaluate the $d\eta'$ integral between the limits η_{min} and η_{max} which we obtain as follows. Since in approximating the Green's function we had assumed $|k\eta'| \ll 1$ (super Hubble limit), let us set $|k\eta'| = \delta' \ll 1$. Then η_{min} corresponds to δ'/k_{max} and η_{max} corresponds to δ'/k_{min}. If we consider CMB scales then $k_{min} = 10^{-5}$ MPc^{-1} and $k_{max} = 1$ MPc^{-1}. Therefore

$$\int \frac{d\eta'}{3\eta'} = \frac{1}{3}\ln\left(\frac{k_{min}}{k_{max}}\right) = -3.83. \tag{5.83}$$

The integral in equation (5.74) has been computed numerically using *Mathematica* (see Appendix B). The integral diverges at 0 and 1, so the integral was broken into two parts : one from $(0+\delta)$ to $(1-\delta)$ and another from $(1+\delta)$ to infinity. δ was chosen to be of $\sim 10^{-6}$ and it was found that the first part yields a value $\mathcal{O}(100)$ and the second part yields a value $\mathcal{O}(10)$, and consequently the contribution from the second part was ignored. Therefore equation (5.73) becomes

$$\begin{aligned}\langle h_T(\mathbf{k})h_T(\mathbf{k}'))\rangle = \langle h_\times(\mathbf{k})h_\times(\mathbf{k}'))\rangle &\simeq (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')\frac{H_I^4}{2M_{Pl}^4}(14.67) \times \frac{81}{(2\pi)^3}\frac{2\pi}{k^3}(108) \\ &\approx (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')\frac{2\pi^2}{k^3}(8.23\times 10^{-15}).\end{aligned} \tag{5.84}$$

We therefore get the following correction to the power spectrum : $\mathcal{P}_T^{nh}(k) \sim 10^{-14}$, which implies $\mathcal{P}_T^{nh}/\mathcal{P}_T^0 \approx 10^{-5}$.

Helical magnetic field

The $d\eta'$ integral is the same. The integral in equation (5.81) is again calculated numerically, but now with $\delta \sim 10^{-4}$. Therefore equation (5.80) becomes

$$\begin{aligned}\langle h_+(\mathbf{k})h_+(\mathbf{k}')\rangle &\simeq (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')\frac{2\pi^2}{k^3}\left[\frac{H_I^4}{2M_{Pl}^4}\frac{81e^{8\pi}}{8\cdot 1600\pi^6}(14.67\times 194)\right] \\ &\approx (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')\frac{2\pi^2}{k^3}(7.7\times 10^{-8}).\end{aligned} \tag{5.85}$$

We therefore get the following correction to the power spectrum : $\mathcal{P}_T^h(k) \sim 10^{-7}$, which implies $\mathcal{P}_T^h/\mathcal{P}_T^0 \approx 38$. We can compare our result with that obtained in Ref. [66], where the author has studied the effect on the tensor power spectrum using a different model (involving a parameter $\xi \gtrsim 3$) of generating helical magnetic fields. Using the expression obtained for $\mathcal{P}_T^h(k)$ in the paper, it can be shown that $\mathcal{P}_T^h(k)/\mathcal{P}_T^0(k)$ is atleast 0.278 (corresponding to $\xi = 3$) and increases considerably with increase in ξ. For instance, it is of the order of 10^9 for $\xi = 5$.

Therefore when we consider the anisotropic stress due scale invariant magnetic field generated in the inflaton coupling model discussed in Chapter 3, we find that the modification to the tensor power spectrum is negligible for non-helical fields and it is modified by an additive factor $\mathcal{O}(100)$ for helical fields.

Chapter 6

Conclusions

In this chapter we will present a summary of the work done in the report and mention future prospects of work. The work presented here is largely a review of inflation, particularly in the slow roll approximation, and inflationary magnetogeneis and re-derivation of the results. We shall summarize our work in the following paragraphs.

We have introduced the topic of inflationary magnetogenesis in the first chapter. There we encountered the problem related to inflationary magnetogenesis,viz. the standard EM action is conformally invariant and the spatially flat FLRW metric is conformally flat. We saw that this led us to break the conformal invariance of the EM action in order to generate magnetic fields during inflation and a few ways in which it is done has been illustrated in equation (1.3). Then we discussed in brief about observations and methods of detection of magnetic fields in the universe. We ended the chapter with a discussion on the qualitative constraint plot from Ref. [7].

In the second chapter we discussed in brief the successes of the hot big bang model before going on to discuss three major problems with the model and their solutions due to inflation. We then discussed how inflation can be driven with scalar fields, with particular focus on the slow roll approximation of inflation. We ended the chapter by studying the Starobinsky model of inflation followed by a brief discussion on reheating.

The third and fourth chapters deal with two specific models of generating magnetic fields during inflation. In the third chapter, we studied the inflaton coupling model. It was found that for a power law form of the coupling function and in a perfectly de Sitter background during inflation, the generated helical and non-helical magnetic fields are scale invariant. The values of the magnetic field are given in the following table.

Type of magnetic field	Strength of magnetic field
Non-helical	$B_0 \sim 0.1\text{nG}$
Helical	$B_0 \sim 0.1\mu\text{G}$

In the fourth chapter we studied the Riemann curvature coupling model. It was found that the generated magnetic field is not scale invariant and it is of the order of 10^{-20}G at GPc scales. We have also presented an overview of how to perform a similar analysis for non-helical magnetic fields, which has not yet been done, to the best of our knowledge. We are working on this at present.

Finally, in the fifth chapter we studied the effects of the anisotropic stress due to the generated magnetic fields on the two-point function of the tensor perturbations. This in turn allowed us to study the effects of the magnetic fields on the tensor power spectra. The results obtained are summarized in the following table.

Type of magnetic field	Effect on $\mathcal{P}_T(k)$
Non-helical	$\frac{\mathcal{P}_T^{nh}(k)}{\mathcal{P}_T^0(k)} \approx 10^{-5}$ (negligible)
Helical	$\frac{\mathcal{P}_T^{h}(k)}{\mathcal{P}_T^0(k)} \approx 38$

Apart from the generation of non-helical magnetic fields from our model of curvature coupling, which we are currently working on, possibilities of future work include studying the effects of the magnetic fields generated via the curvature coupling on the tensor power spectrum and also studying the imprints of the primordial magnetic fields on the three-point function of the tensor perturbations.

Appendix A

Electrodynamics in curved spacetime

Consider the action for the electromagnetic fields interacting with charged particles :

$$\mathcal{S}_{EM} = -\int \sqrt{-g} d^4x \left[\frac{F^{\mu\nu}F_{\mu\nu}}{16\pi} - A_\mu J^\mu \right]. \tag{A.1}$$

Here A_μ is the four-vector potential, J^μ the four-vector current and $F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu} = A_{\nu,\mu} - A_{\mu,\nu}$. Upon variation with respect to A_μ, this action yields the Maxwell's equations

$$F^{\mu\nu}{}_{;\nu} = J^\mu \quad \text{and} \quad \mathcal{F}^{\mu\nu}{}_{;\nu} = 0, \tag{A.2}$$

where $\mathcal{F}^{\mu\nu} = (1/2)\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ and $\epsilon^{\mu\nu\alpha\beta}$ is the 4-dimensional Levi-Civita tensor in curved space-time which is given by $1/\sqrt{-g}$ for even permutation of indices, $(-1/\sqrt{-g})$ for odd permutation of indices and 0 for repeated indices. We would like to define the corresponding electric and magnetic fields from the tensor $F^{\mu\nu}$. For this one needs to isolate a time direction which can be done using a family of observers with 4-velocities described by the four-vector [10]

$$u^\mu = \frac{dx^\mu}{ds} \quad ; \quad u^\mu u_\nu = -1.$$

We can also define the projection tensor as

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu,$$

and then the line element can be rewritten as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(u_\mu dx^\mu)^2 + h_{\mu\nu}dx^\mu dx^\nu. \tag{A.3}$$

The electric and magnetic field four-vectors can then be written as

$$E_\mu = F_{\mu\nu}u^\nu \quad ; \quad B_\mu = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}u^\nu F^{\alpha\beta} = \mathcal{F}_{\mu\nu}u^\nu. \tag{A.4}$$

From these above definitions and remembering that $F^{\mu\nu}$ is antisymmetric, we see that these four-vectors have purely spatial components and are effectively three-vectors in the space orthogonal to u^μ : $E_\mu u^\mu = B_\mu u^\mu = 0$.

We now invert equations (A.4) to write the EM field tensor and its dual in terms of the magnetic and electric field four-vectors :

$$F_{\mu\nu} = u_\mu E_\nu - u_\nu E_\mu + \epsilon_{\mu\nu\alpha\beta} B^\alpha u^\beta, \tag{A.5}$$

$$\mathcal{F}^{\alpha\beta} = \epsilon^{\alpha\beta\mu\nu} u_\mu E_\nu + u^\alpha B^\beta - u^\beta B^\alpha. \tag{A.6}$$

Now consider the projection of the source-free Maxwell's equation from equation (A.2) on u_α. We have

$$u_\alpha(\mathcal{F}^{\alpha\beta}{}_{;\beta}) = 0 = (u_\alpha \mathcal{F}^{\alpha\beta})_{;\beta} - u_{\alpha;\beta}\mathcal{F}^{\alpha\beta}. \tag{A.7}$$

Substituting equation (A.6) into equation (A.7) and using $u_\alpha \epsilon^{\alpha\beta\mu\nu} u_\mu = 0, u_\alpha B^\alpha = 0$ and $u_{\alpha;\beta} u^\alpha = 0$, and defining the four-acceleration $a_\beta = u^\alpha u_{\beta;\alpha}$ we get

$$B^\beta{}_{;\beta} - B^\beta a_\beta + u_{\alpha;\beta}\epsilon^{\alpha\beta\mu\nu} u_\mu E^\nu = 0. \tag{A.8}$$

The $u_{\alpha;\beta}$ appearing in the above equation can be decomposed into shear, expansion, vorticity and acceleration parts as follows [10] :

$$u_{\alpha;\beta} = h^\mu_\alpha h^\nu_\beta u_{\mu;\nu} - a_\alpha u_\beta = \Theta_{\alpha\beta} + \omega_{\alpha\beta} - a_\alpha u_\beta. \tag{A.9}$$

In the above equation, $\Theta_{\mu\nu}$ and $\omega_{\mu\nu}$ are purely spatial, symmetric expansion tensor and antisymmetric vorticity tensor respectively. Now $\Theta_{\mu\nu}$ can be further decomposed into a shear tensor(trace-free part) and its trace : $\Theta_{\mu\nu} = \tau_{\mu\nu} + (1/3)\Theta h_{\mu\nu}$. Thus we have

$$u_{\alpha;\beta} = \sigma_{\alpha\beta} + (1/3)\Theta h_{\alpha\beta} + \omega_{\alpha\beta} - a_\alpha u_\beta. \tag{A.10}$$

Defining the spatial projection of the covariant derivative as

$$D_\beta B^\alpha = h^\mu_\beta h^\alpha_\nu B^\nu_{;\mu},$$

we have

$$D_\beta B^\beta = B^\mu_{;\mu} - u^\mu u_{\nu;\mu} B^\nu, \tag{A.11}$$

and equation (A.8) reduces to (with $\omega^\nu = -(1/2)\omega_{\alpha\beta}\epsilon^{\alpha\beta\mu\nu}u_\mu$)

$$D_\beta B^\beta = 2\omega^\beta E_\beta, \tag{A.12}$$

which is a generalization of the flat space equation $\nabla \cdot \mathbf{B} = 0$ to a general curved spacetime. The vorticity of the relative motion of the observers measuring the EM field causes the right hand side of the above equation to act as an effective magnetic charge.

The projection of the source-free Maxwell's equations on h^κ_α gives the generalization of the Faraday law in curved spacetime [10] :

$$h^\kappa_\alpha(\mathcal{F}^{\alpha\beta}{}_{;\beta}) = (\delta^\kappa_\alpha + u^\kappa u_\alpha)(\epsilon^{\alpha\beta\mu\nu} u_\mu E_\nu + u^\alpha B^\beta - u^\beta B^\alpha{}_{;\beta}) = 0,$$

$$\Rightarrow h^\kappa_\alpha \dot{B}^\alpha = [u^\kappa_{;\beta} B^\beta - u^\beta_{;\beta} B^\kappa] + h^\kappa_\alpha \bar{\epsilon}^{\alpha\mu\nu} a_\mu E_\nu - \bar{\epsilon}^{\kappa\beta\nu} E_{\nu;\beta}.$$

Therefore

$$h^\kappa_\alpha \dot{B}^\alpha = [\sigma^\kappa_\beta + \omega^\kappa_\beta - (2/3)\Theta\delta^\kappa_\beta]B^\beta - \bar{\epsilon}^{\kappa\mu\nu}a_\mu E_\nu - \mathrm{Curl}(E^\kappa), \tag{A.13}$$

where $\bar{\epsilon}^{\alpha\beta\nu} \equiv \epsilon^{\alpha\beta\nu\mu}u_\mu$, $\mathrm{Curl}(E^\kappa) \equiv \bar{\epsilon}^{\kappa\beta\nu}E_{\nu;\beta}$ and $[u^\kappa_{;\beta}B^\beta - u^\beta_{;\beta}B^\kappa] = [u^\kappa_{;\beta} - \Theta\delta^\kappa_\beta]B^\beta = [\sigma^\kappa_\beta + \omega^\kappa_\beta - (2/3)\Theta\delta^\kappa_\beta]B^\beta$ have been used.

The other two Maxwell equations, involving the source terms, can be obtained by the following transformations : $\mathbf{E} \to -\mathbf{B}$, $\mathbf{B} \to \mathbf{E}$ and $J^\mu \to -J^\mu$. Therefore we get the remaining two equations,

$$D_\beta E^\beta = 4\pi\rho_q - 2\omega^\beta B_\beta, \tag{A.14}$$

$$h^\kappa_\alpha \dot{E}^\alpha = [\sigma^\kappa_\beta + \omega^\kappa_\beta - (2/3)\Theta\delta^\kappa_\beta]E^\beta + \bar{\epsilon}^{\kappa\mu\nu}a_\mu B_\nu + \mathrm{Curl}(B^\kappa) - 4\pi j^\kappa, \tag{A.15}$$

where we have defined $\rho_q \equiv -J^\nu u_\nu$ and $j^\kappa \equiv J^\mu h^\kappa_\mu$.

The expanding universe

Consider the spatially flat FLRW metric : $ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$. In this case, we have

$$a^\alpha = 0, \quad \omega_{\alpha\beta} = 0, \quad \sigma_{\alpha\beta} = 0, \quad \Theta = 3\dot{a}/a,$$

and the Maxwell equations reduce to

$$B^\beta_{;\beta} = 0, \quad E^\beta_{;\beta} = 4\pi\rho_q, \quad u^\alpha B^\beta_{;\alpha} + (2/3)B^\beta = -\mathrm{Curl}(E^\beta), \quad u^\alpha E^\beta_{;\alpha} + (2/3)E^\beta = \mathrm{Curl}(B^\beta) - 4\pi j^\beta, \tag{A.16}$$

which can be further reduced to (with η_{ilm} being the 3D Levi-Civita symbol)

$$B^i_{,i} = 0, \quad E^i_{,i} = 4\pi\rho_q, \quad \frac{1}{a^3}\partial_t[a^3 B^i] = -\frac{1}{a}\eta_{ilm}\partial_m E^l, \quad \frac{1}{a^3}\partial_t[a^3 E^i] = \frac{1}{a}\eta_{ilm}\partial_m B^l - 4\pi j^i, \tag{A.17}$$

upon using the Christoffel connections for the spatially flat FLRW metric :

$$\Gamma^0_{00} = \Gamma^0_{0i} = \Gamma^i_{jk} = 0; \quad \Gamma^0_{ij} = \delta_{ij}a\dot{a}; \quad \Gamma^i_{0j} = \delta_{ij}\dot{a}/a.$$

The electric and magnetic field components used above are referred to a coordinate basis, where the space-time metric is of the FLRW form. Now consider, for example, the case when the plasma in the universe has no peculiar velocity and is also highly conducting, $\sigma \to \infty$. Then $E^\nu = 0$ and The Faraday's law in equation (A.17), we have $B^i \propto 1/a^3$. However, there is a simple result that can be deduced in flat space-time: in a highly conducting fluid, magnetic flux through a surface that moves in tandem with the fluid is constant (flux freezing). Given that all proper surfaces in the expanding universe increase as a^2, the strength of the "proper" magnetic field is expected to decrease as $1/a^2$. This naively seems to be at conflict with $B^i \propto 1/a^3$ and hence $B_i \propto 1/a$. This can be addressed as follows [10].

Let us refer all tensor quantities to a set of orthonormal basis vectors, known as tetrads. Any observer can be thought of carrying along his/her worldline a set of 4 orthonormal vectors $e_{(a)}$, where $a = 0, 1, 2, 3$ such that $g_{\mu\nu}e^\mu_{(a)}e^\nu_{(b)} = \gamma_{ab}$ and $\gamma^{ab}e^\mu_{(a)}e^\nu_{(b)} = g^{\mu\nu}$. Here γ_{ab} has the form of a flat space metric. For the present case, we have the components of the tetrads as $e^\mu_{(0)} = \delta^\mu_0 (= u^\mu)$ and $e^\mu_{(i)} = \delta^\mu_i/a$. Note that fundamental observers move along geodesics and such observers parallel transport their tetrads along their trajectories,i.e., $u^\mu e^\nu_{(a);\mu} = 0$. The magnetic and electric field components can be represented as its projection along the four orthonormal tetrads using

$$\bar{B}^a = g_{\mu\nu} B^\mu e^{\nu(a)}, \qquad \bar{E}^a = g_{\mu\nu} E^\mu e^{\nu(a)}, \tag{A.18}$$

which gives

$$\bar{B}^0 = 0, \quad \bar{E}^0 = 0 \quad ; \quad \bar{B}^i = a(t) B^i, \quad \bar{E}^i = a(t) E^i. \tag{A.19}$$

If we define $\bar{B}_a = \gamma_{ab} \bar{B}^b$, then numerically $\bar{B}^i = \bar{B}_i$ and $\bar{B}^0 = -\bar{B}_0$, etc. In the FLRW universe, $B^i \propto 1/a^3$, $B_i \propto 1/a$, but we see that $\bar{B}^i = \bar{B}_i \propto 1/a^2$. This suggests that the magnetic field components defined within this tetrad formalism seem to be the natural quantities to be used as the physical components of the magnetic field. The Maxwell's equations then become

$$\bar{B}^i_{,i} = 0, \quad (a^2 \bar{E}^i)_{,i} = 4\pi \rho_q a^2, \quad \partial_t (a^2 \bar{B}^i) = -\eta_{ilm} \partial_m (a^2 \bar{E}^l), \quad \partial_t (a^2 \bar{E}^i) = \eta_{ilm} \partial_m (a^2 \bar{E}^l) - 4\pi \bar{j}^i a^2, \tag{A.20}$$

with $\bar{j}^i = \rho_q v^i + \sigma[\bar{E}^i + \eta^{ilm} v_l \bar{B}_m]$ (in the limit of non relativistic fluid velocity). This concludes our discussion of electrodynamics in curved space-time.

Appendix B

Tensor power spectra in the presence of anisotropic stress

In this appendix we will give the derivation of equation (5.71) for non-helical magnetic field and equation (5.77) for helical magnetic field. We will also attach the *Mathematica* notebooks for the numerical integrals used in section 5.4.

B.1 Non-helical magnetic field

We have seen that the calculation of the tensor power spectra involves the computation of the four point correlation function of the magnetic field, which has been written in terms of the two point (unequal time)correlation functions in equation (5.67). In equation (5.68), we have written the unequal time correlation function of the magnetic fields as a product of equal time correlation functions and a two-time correlator $C_B(k, \eta', \eta'')$. Using equation (5.69), viz.

$$\langle B_i(\mathbf{k}; \eta) B_j^*(\mathbf{k}'; \eta)\rangle = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') P_{ij} P_{SB}(k; \eta),$$

we can write the four point correlation function as follows :

$$\begin{aligned}
\langle B_m(\mathbf{q}; \eta') B_n^*(\mathbf{q} - \mathbf{k}; \eta') B_a(\mathbf{q}'; \eta'') B_b^*(\mathbf{q}' - \mathbf{k}'; \eta'')\rangle &= \langle B_m(\mathbf{q}; \eta') B_n^*(\mathbf{q} - \mathbf{k}; \eta')\rangle \langle B_a(\mathbf{q}'; \eta'') B_b^*(\mathbf{q}' - \mathbf{k}'; \eta'')\rangle \\
&+ \langle B_m(\mathbf{q}; \eta') B_a^*(\mathbf{q}'; \eta'')\rangle \langle B_n^*(\mathbf{q} - \mathbf{k}; \eta') B_b(\mathbf{q}' - \mathbf{k}'; \eta'')\rangle \\
&+ \langle B_m(\mathbf{q}; \eta') B_b(\mathbf{q}' - \mathbf{k}'; \eta'')\rangle \langle B_n^*(\mathbf{q} - \mathbf{k}; \eta') B_a^*(\mathbf{q}'; \eta'')\rangle
\end{aligned}$$

$$\begin{aligned}
&\implies \langle B_m(\mathbf{q}; \eta') B_n^*(\mathbf{q} - \mathbf{k}; \eta') B_a(\mathbf{q}'; \eta'') B_b^*(\mathbf{q}' - \mathbf{k}'; \eta'')\rangle = (2\pi)^6 ([\delta(\mathbf{k}) P_{mn} P_{SB}(q, \eta') \delta(\mathbf{k}') P_{ab} P_{SB}(q', \eta'')] \\
&+ [\delta(\mathbf{q} - \mathbf{q}') P_{ma} P_{SB}(q, \eta') C_B(q', \eta', \eta'') \delta((\mathbf{q} - \mathbf{k}) - (\mathbf{q}' - \mathbf{k}')) P_{nb} P_{SB}(|\mathbf{q} - \mathbf{k}|, \eta'') C_B(|\mathbf{q} - \mathbf{k}|, \eta', \eta'')] \\
&+ [\delta(\mathbf{q} - \mathbf{q}' + \mathbf{k}') P_{mb} P_{SB}(q, \eta') C_B(q', \eta', \eta'') \delta(\mathbf{q} - \mathbf{k} - \mathbf{q}') P_{na} P_{SB}(|\mathbf{q} - \mathbf{k}|, \eta'') C_B(|\mathbf{q} - \mathbf{k}|, \eta', \eta'')])
\end{aligned} \tag{B.1}$$

In order to calculate the transverse traceless part, we take the tensor projection and integrate the above equation with respect to q' to get [65]

$$\begin{aligned}
&\int \frac{d^3 q'}{(2\pi)^3} \Pi_{ij}^{mn}(\mathbf{k}) \Pi_{ij}^{ab}(\mathbf{k}') \langle B_m(\mathbf{q}; \eta') B_n^*(\mathbf{q} - \mathbf{k}; \eta') B_a(\mathbf{q}'; \eta'') B_b^*(\mathbf{q}' - \mathbf{k}'; \eta'')\rangle \\
&= (2\pi)^3 \Pi_{ij}^{mn}(\mathbf{k}) \Pi_{ij}^{ab}(\mathbf{k}') (P_{ma}(\mathbf{q}) P_{nb}(\mathbf{q} - \mathbf{k}) + P_{mb}(\mathbf{q}) P_{na}(\mathbf{q} - \mathbf{k})) \\
&\times P_{SB}(q; \eta') P_{SB}(|\mathbf{q} - \mathbf{k}|; \eta') C_B(q, \eta' \eta'') C_B(|\mathbf{q} - \mathbf{k}|, \eta', \eta'') \delta(\mathbf{k} - \mathbf{k}').
\end{aligned} \tag{B.2}$$

Let $\mathbf{p} = \mathbf{q} - \mathbf{k}$. Then

$$\begin{aligned}&(P_{ma}(\mathbf{q})P_{nb}(\mathbf{q}-\mathbf{k}) + P_{mb}(\mathbf{q})P_{na}(\mathbf{q}-\mathbf{k})) = (P_{ma}(\mathbf{q})P_{nb}(\mathbf{p}) + P_{mb}(\mathbf{q})P_{na}(\mathbf{p}))\\ &= [(\delta_{ma} - \hat{q}_m\hat{q}_a)(\delta_{nb} - \hat{p}_n\hat{p}_b) + (\delta_{mb} - \hat{q}_m\hat{q}_b)(\delta_{na} - \hat{p}_n\hat{p}_a)].\end{aligned} \tag{B.3}$$

Also $\Pi_{ij}^{mn}(\mathbf{k})\Pi_{ij}^{ab}(\mathbf{k}')\delta(\mathbf{k}-\mathbf{k}') = \Pi_{ij}^{mn}(\mathbf{k})\Pi_{ij}^{ab}(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}')$ and we have

$$\begin{aligned}\Pi_{ij}^{mn}(\mathbf{k})\Pi_{ij}^{ab}(\mathbf{k}) &= \left[P_i^m P_j^n - \frac{P^{mn}P_{ij}}{2}\right]\left[P_i^a P_j^b - \frac{P^{ab}P_{ij}}{2}\right] = P^{ma}(\mathbf{k})P^{nb}(\mathbf{k}) - \frac{P^{mn}(\mathbf{k})P^{ab}(\mathbf{k})}{2}\\ &= (\delta^{ma} - \hat{k}^m\hat{k}^a)(\delta^{nb} - \hat{k}^n\hat{k}^b) - \frac{1}{2}(\delta^{mn} - \hat{k}^m\hat{k}^n)(\delta^{ab} - \hat{k}^a\hat{k}^b).\end{aligned} \tag{B.4}$$

Then the product of equations (B.3) and (B.4) gives $(1+\gamma^2)(1+\beta^2)$, where $\hat{\mathbf{k}}\cdot\hat{\mathbf{q}} = \gamma$ and $\hat{\mathbf{k}}\cdot\hat{\mathbf{p}} = \beta$ and we are finally led to

$$\begin{aligned}&\int \frac{d^3q'}{(2\pi)^3}\Pi_{ij}^{mn}(\mathbf{k})\Pi_{ij}^{ab}(\mathbf{k}')\langle B_m(\mathbf{q};\eta')B_n^*(\mathbf{q}-\mathbf{k};\eta')B_a(\mathbf{q}';\eta'')B_b^*(\mathbf{q}'-\mathbf{k}';\eta'')\rangle\\ &= (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')(1+\gamma^2)(1+\beta^2)P_{SB}(q;\eta')P_{SB}(|\mathbf{q}-\mathbf{k}|;\eta')C_B(q,\eta'\eta'')C_B(|\mathbf{q}-\mathbf{k}|,\eta',\eta'').\end{aligned} \tag{B.5}$$

Therefore,

$$\begin{aligned}f_B(k,\eta',\eta'') &= \frac{1}{(2\pi)^3}\int d^3q[P_{SB}(q;\eta')P_{SB}(|\mathbf{k}-\mathbf{q}|;\eta')\\ &\times(1+\gamma^2+\beta^2+\beta^2\gamma^2)]C_B(q,\eta',\eta'')C_B(|\mathbf{k}-\mathbf{q}|,\eta',\eta''),\end{aligned} \tag{B.6}$$

B.2 Helical magnetic field

We start from equation (5.55), viz.

$$h_{ij}(\mathbf{k}) = \frac{2}{M_{Pl}^2}\int d\eta' G_k(\eta,\eta')\int\frac{d^3q}{(2\pi)^3}\Pi_{ij}^{mn}(\mathbf{k})[B_m(\mathbf{q},\eta')B_n^*(\mathbf{q}-\mathbf{k},\eta')].$$

Using $B_a(\mathbf{k},\eta) = i\eta_{abc}k_b\mathcal{A}_c(\mathbf{k},\eta)/a$ and $\mathbf{p} = \mathbf{q}-\mathbf{k}$, we get

$$h_{ij}(\mathbf{k}) = \frac{-2H_I^2}{M_{Pl}^2}\int d\eta' G_k(\eta,\eta')\eta'^2\int\frac{d^3q}{(2\pi)^3}\Pi_{ij}^{mn}(\mathbf{k})\eta_{mab}\eta_{ncd}q_a p_c\mathcal{A}_b(\mathbf{q},\eta')\mathcal{A}_d^*(\mathbf{p},\eta') \tag{B.7}$$

Now we introduce polarization tensors $\Pi_{\pm}^{ij}(\mathbf{k})$, so that $h_\pm(\mathbf{k}) = \Pi_\pm^{ij}(\mathbf{k})h_{ij}(\mathbf{k})$ [66]. Comparing with the second equality in equation (5.5), we obtain

$$h_\pm(\mathbf{k}) = \frac{1}{2}e_\mp^{ij}h_{ij}(\mathbf{k}) \implies \Pi_\pm^{ij}(\mathbf{k}) = \frac{e_\mp^{ij}(\mathbf{k})}{2} = \frac{\hat{e}_\mp^i(\mathbf{k})\hat{e}_\mp^j(\mathbf{k})}{\sqrt{2}}. \tag{B.8}$$

Thus the expression for the helicity-P ($P=\pm1$) garviton is given by

$$h_P(\mathbf{k}) = \frac{-H_I^2}{M_{Pl}^2}\int d\eta' G_k(\eta,\eta')\eta'^2\int\frac{d^3q}{(2\pi)^3}\hat{e}_P^{mn}(\mathbf{k})\eta_{mab}\eta_{ncd}q_a p_c\mathcal{A}_b(\mathbf{q},\eta')\mathcal{A}_d^*(\mathbf{p},\eta') \tag{B.9}$$

Using the following property of the helicity basis vectors,

$$i\eta_{abc}k_b\hat{e}_c^\pm = \pm k\hat{e}_a^\pm,$$

we get

$$h_P(\mathbf{k}) = \frac{H_I^2}{M_{Pl}^2}\int d\eta' G_k(\eta,\eta')\eta'^2 \int \frac{d^3q}{(2\pi)^3} e_P^{mn}(\mathbf{k}) \sum_{\rho,\sigma=\pm} qp\hat{e}_m^{\rho}\hat{e}_n^{\sigma}\mathcal{A}_\rho(q,\eta')\mathcal{A}_\sigma(p,\eta'). \tag{B.10}$$

Similarly

$$h_P(\mathbf{k}') = \frac{H_I^2}{M_{Pl}^2}\int d\eta'' G_{k'}(\eta,\eta'')\eta''^2 \int \frac{d^3q'}{(2\pi)^3} e_P^{ab}(\mathbf{k}') \sum_{\rho,\sigma=\pm} q'p'\hat{e}_a^{\rho}\hat{e}_b^{\sigma}\mathcal{A}_\rho(q',\eta'')\mathcal{A}_\sigma(p',\eta'') \tag{B.11}$$

Therefore

$$\begin{aligned}\langle h_P(\mathbf{k})h_P(\mathbf{k}')\rangle = \frac{H_I^4}{M_{Pl}^4}\int d\eta' G_k(\eta,\eta')\eta'^2 \int d\eta'' G_{k'}(\eta,\eta'')\eta''^2 \times \\ \times \int \frac{d^3q}{(2\pi)^3}\int \frac{d^3q'}{(2\pi)^3} e_P^{mn}(\mathbf{k})e_P^{ab}(\mathbf{k}') \sum_{\rho,\sigma=\pm} qpq'p'\hat{e}_m^{\rho}(\mathbf{q})\hat{e}_n^{\sigma}(\mathbf{p})\hat{e}_a^{\rho}(\mathbf{q}')\hat{e}_b^{\sigma}(\mathbf{p}')\times \\ \times\langle\mathcal{A}_\rho(q',\eta'')\mathcal{A}_\sigma(p',\eta'')\mathcal{A}_\rho(q,\eta')\mathcal{A}_\sigma(p,\eta')\rangle.\end{aligned} \tag{B.12}$$

This equation can be simplified using Wick's theorem and then upon using the following property of the helicity projectors [18]

$$|\hat{e}_P^i(\mathbf{p_1})\hat{e}_{P'}^i(\mathbf{p_2})|^2 = \frac{1}{4}\left(1-(PP')\frac{\mathbf{p}.\mathbf{p_2}}{p_1p_2}\right)^2, \tag{B.13}$$

we are led to equation (5.77) viz.,

$$\begin{aligned}\langle h_P(\mathbf{k})h_P(\mathbf{k}')\rangle = \delta(\mathbf{k}-\mathbf{k}')\frac{H_I^4}{2M_{Pl}^4}\left|\int d\eta' G_k(\eta,\eta')\eta'^2\right|^2 \sum_{\rho,\sigma=\pm}\int d^3q(1-\rho(P)\gamma)^2(1-\sigma(P)\beta)^2 \\ \times q^2p^2|\mathcal{A}_\rho(q,\eta')\mathcal{A}_\sigma(p,\eta')|^2.\end{aligned}$$

The following pages contain the *Mathematica* notebooks pertaining to the numerical computation of the integrals in equations (5.74) and (5.81).

Non-helical Magnetic Field

In[1]:=
```
ClearAll;
```

In[2]:=
```
g[q_, γ_] =
  (1 + γ^2) * (2 * k^2 + q^2 + q^2 * γ^2 - 4 * k * q * γ) / (q * ((k^2 + q^2 - 2 * k * q * γ) ^ (5 / 2)))
```

Out[2]= $$\frac{(1+\gamma^2)\,(2\,k^2+q^2-4\,k\,q\,\gamma+q^2\,\gamma^2)}{q\,(k^2+q^2-2\,k\,q\,\gamma)^{5/2}}$$

In[3]:=
```
k = 1;
```

In[4]:=
```
g1[q_, γ_] = Integrate[g[q, γ], γ]
```

Out[4]= $$-\frac{1}{15\,q^4\,(1+q^2-2\,q\,\gamma)^{3/2}}\Big(-12+8\,q^8+36\,q\,\gamma-24\,q^7\,\gamma+4\,q^5\,\gamma\,(-3+\gamma^2)+12\,q^6\,(1+\gamma^2)-4\,q^3\,\gamma\,(-27+4\,\gamma^2)-2\,q^2\,(29+9\,\gamma^2)+q^4\,(-37-6\,\gamma^2+3\,\gamma^4)\Big)$$

In[5]:=
```
g2[q_] = FullSimplify[g1[q, 1] - g1[q, -1]]
```

Out[5]= $$\frac{\frac{12+4\,q\,(-3+2\,q\,(5+q^3-q^4))}{\sqrt{(-1+q)^2}}+\frac{4\,(-3+q\,(-3+2\,q\,(-5+q^3+q^4)))}{\sqrt{(1+q)^2}}}{15\,q^4}$$

In[6]:=
```
NIntegrate[g2[q], {q, 0.000001, 0.999999}, AccuracyGoal → 3]
```

Out[6]= 107.619

In[7]:=
```
NIntegrate[g2[q], {q, 1.000001, Infinity}, AccuracyGoal → 2]
```

Out[7]= 32.8231

Helical Magnetic Field

In[1]:=
```
ClearAll;
```

In[2]:=
```
f[q_, γ_] = (1 + γ) ^2 *
  (Sqrt[k^2 + q^2 - 2 * k * q * γ] + k - q * γ) ^2 / (q * ((k^2 + q^2 - 2 * k * q * γ) ^ (5 / 2)))
```

Out[2]= $$\frac{(1+\gamma)^2 \left(k - q\gamma + \sqrt{k^2+q^2-2kq\gamma}\right)^2}{q\left(k^2+q^2-2kq\gamma\right)^{5/2}}$$

In[3]:=
```
k = 1;
```

In[4]:=
```
f1[q_, γ_] = Integrate[f[q, γ], γ]
```

Out[4]= $$-\frac{1}{16q^4}\left(4q(1+q)(1+3q)\gamma - \frac{(-1+q)^2(1+q)^6}{3\left(1+q^2-2q\gamma\right)^{3/2}} + \frac{2(-1+q)(1+q)^5}{1+q^2-2q\gamma} + \frac{4(1+q)^4\left(-1-q+q^2\right)}{\sqrt{1+q^2-2q\gamma}} + 2(1+q)^2\left(-5+3q^2\right)\sqrt{1+q^2-2q\gamma} - \frac{4}{3}\left(-1+q+q^2\right)\left(1+q^2-2q\gamma\right)^{3/2} + \left(1+q^2-2q\gamma\right)^2 + \frac{1}{5}\left(1+q^2-2q\gamma\right)^{5/2} + 2(1+q)^3(-1+3q)\,\mathrm{Log}\left[1+q^2-2q\gamma\right]\right)$$

In[5]:=
```
f1[q, 1]
```

Out[5]= $$-\frac{1}{16q^4}\left(4q(1+q)(1+3q) - \frac{(-1+q)^2(1+q)^6}{3\left(1-2q+q^2\right)^{3/2}} + \frac{2(-1+q)(1+q)^5}{1-2q+q^2} + \left(1-2q+q^2\right)^2 + \frac{1}{5}\left(1-2q+q^2\right)^{5/2} + \frac{4(1+q)^4\left(-1-q+q^2\right)}{\sqrt{1-2q+q^2}} - \frac{4}{3}\left(1-2q+q^2\right)^{3/2}\left(-1+q+q^2\right) + 2(1+q)^2\sqrt{1-2q+q^2}\left(-5+3q^2\right) + 2(1+q)^3(-1+3q)\,\mathrm{Log}\left[1-2q+q^2\right]\right)$$

In[6]:=
```
f1[q, -1]
```

Out[6]= $$-\frac{1}{16q^4}\left(-4q(1+q)(1+3q) - \frac{(-1+q)^2(1+q)^6}{3\left(1+2q+q^2\right)^{3/2}} + \frac{2(-1+q)(1+q)^5}{1+2q+q^2} + \frac{4(1+q)^4\left(-1-q+q^2\right)}{\sqrt{1+2q+q^2}} - \frac{4}{3}\left(-1+q+q^2\right)\left(1+2q+q^2\right)^{3/2} + \left(1+2q+q^2\right)^2 + \frac{1}{5}\left(1+2q+q^2\right)^{5/2} + 2(1+q)^2\sqrt{1+2q+q^2}\left(-5+3q^2\right) + 2(1+q)^3(-1+3q)\,\mathrm{Log}\left[1+2q+q^2\right]\right)$$

In[7]:= **f2[q_] = %5 - %6**

Out[7]=
$$-\frac{1}{16 q^4}\left(4 q (1+q) (1+3 q)-\frac{(-1+q)^2 (1+q)^6}{3 \left(1-2 q+q^2\right)^{3/2}}+\frac{2 (-1+q) (1+q)^5}{1-2 q+q^2}+\left(1-2 q+q^2\right)^2+\frac{1}{5} \left(1-2 q+q^2\right)^{5/2}+\frac{4 (1+q)^4 \left(-1-q+q^2\right)}{\sqrt{1-2 q+q^2}}-\frac{4}{3} \left(1-2 q+q^2\right)^{3/2} \left(-1+q+q^2\right)+2 (1+q)^2 \sqrt{1-2 q+q^2} \left(-5+3 q^2\right)+2 (1+q)^3 (-1+3 q) \text{Log}\left[1-2 q+q^2\right]\right)+\frac{1}{16 q^4}\left(-4 q (1+q) (1+3 q)-\frac{(-1+q)^2 (1+q)^6}{3 \left(1+2 q+q^2\right)^{3/2}}+\frac{2 (-1+q) (1+q)^5}{1+2 q+q^2}+\frac{4 (1+q)^4 \left(-1-q+q^2\right)}{\sqrt{1+2 q+q^2}}-\frac{4}{3} \left(-1+q+q^2\right) \left(1+2 q+q^2\right)^{3/2}+\left(1+2 q+q^2\right)^2+\frac{1}{5} \left(1+2 q+q^2\right)^{5/2}+2 (1+q)^2 \sqrt{1+2 q+q^2} \left(-5+3 q^2\right)+2 (1+q)^3 (-1+3 q) \text{Log}\left[1+2 q+q^2\right]\right)$$

In[8]:= **Simplify[%7]**

Out[8]=
$$\frac{1}{16 q^4}\left(-(-1+q)^4-\frac{1}{5} \left((-1+q)^2\right)^{5/2}+2 (-1+q) (1+q)^3+(1+q)^4-\frac{2 (1+q)^5}{-1+q}+\frac{(1+q)^6}{3 \sqrt{(-1+q)^2}}-\frac{(-1+q)^2 (1+q)^4}{3 \sqrt{(1+q)^2}}+\frac{1}{5} \left((1+q)^2\right)^{5/2}-8 q (1+q) (1+3 q)-\frac{4 (1+q)^4 \left(-1-q+q^2\right)}{\sqrt{(-1+q)^2}}+\frac{4 (1+q)^4 \left(-1-q+q^2\right)}{\sqrt{(1+q)^2}}+\frac{4}{3} \left((-1+q)^2\right)^{3/2} \left(-1+q+q^2\right)-\frac{4}{3} \left((1+q)^2\right)^{3/2} \left(-1+q+q^2\right)-2 \sqrt{(-1+q)^2} (1+q)^2 \left(-5+3 q^2\right)+2 \left((1+q)^2\right)^{3/2} \left(-5+3 q^2\right)-2 (1+q)^3 (-1+3 q) \text{Log}\left[(-1+q)^2\right]+2 (1+q)^3 (-1+3 q) \text{Log}\left[(1+q)^2\right]\right)$$

In[9]:= **NIntegrate[f2[q], {q, 0.0001, 0.9999}, AccuracyGoal → 3]**

Out[9]= 194.188

In[10]:= **NIntegrate[f2[q], {q, 1.0001, Infinity}, AccuracyGoal → 2]**

Out[10]= 11.2699

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