Thermal History of the Universe

A project report
submitted in partial fulfillment for the award of the degree of
Master of Science
in
Physics
by

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CERTIFICATE

This is to certify that the project titled **Thermal history of the universe** is a bona fide record of work done by **Arnab Pradhan** towards the partial fulfillment of the requirements of the Master of Science degree in Physics at the Indian Institute of Technology, Madras, Chennai 600036, India.

(L. Sriramkumar, Project supervisor)

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ABSTRACT

It is perhaps the greatest triumph of modern cosmology to present a verifiable story of our universe. This project is an attempt to fathom the thermal history of the universe. To reach our destination, we journey through background cosmology, statistical physics and particle physics. We introduce the idea of thermal decoupling and discuss it's significance in the context of the history of our universe. We study decoupling for a few selected cases, starting from first principles. We also motivate the need for non-baryonic dark matter in the universe and study the decoupling of one of the most plausible dark matter candidate called WIMP (Weakly Interacting Massive Particle).

Contents

Chapter 1

Introduction

It is to find answers to some of the most profound and fundamental questions about us and our universe, that I dwell into cosmology. The broad range of physics that cosmology draws from makes me think of it as a beautifully constructed playground with all my favorite toys. Like many others, cosmology has left me humbled and restless. This report is an attempt to make the reader feel the same. I will begin by briefly describing what modern cosmology is all about. We will then discuss about the FLRW model and thermodynamics of our universe. We then move on to the main topic of this project : thermal history of the universe. I will not be doing full justice to this topic, I will only be touching upon some of the many important events in the history of our universe, however I will try to to give a feel for what lays beyond the topics covered here.

What makes modern cosmology all the more alluring is that we have developed a consistent theoretical framework that agrees quantitatively with the vast amount of data we have gathered about our universe. To top it off, this theory makes predictions that can be tested by observations. The cornerstone of modern cosmology is the belief that the place we occupy in the universe is in no way special, it is known as the cosmological principle which forms the basis of big bang cosmology. It implies that that the universe looks the same wherever we are. It is important to realize that this is not an exact principle, rather it is an approximation that holds better and better the larger the length scales we consider, breaking down when we look at local phenomenon. Only once we get to scales of hundreds of megaparsec or more does the universe appear to be smooth, as revealed by the extremely large galaxy surveys, like the 2dF galaxy redshift survey and the Sloan Digital Sky Survey.

Observation using visible light provides us with a good picture of what's going on in the present day universe. However many other wavebands like microwave, infrared, X-rays and radio waves make vital contributions to our present understanding of the universe. For cosmology microwave is perhaps the most important waveband. The fact that earth is bathed in uniform microwave radiation, with a blackbody distribution, at a temperature of approximately 3K, from all directions, is one of the most powerful piece of information in support of big bang cosmology. This is now known as the Cosmic Microwave Background (CMB), we will be discussing it in more details in the section on recombination. Observations by the COBE (Cosmic Background Explorer) satellite as shown in figure. 1.1, has confirmed this observation to an extremely good accuracy.

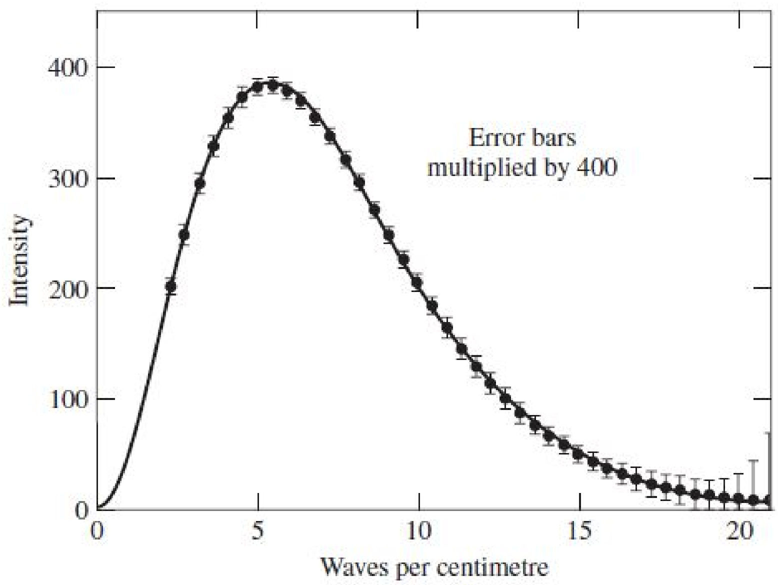

Figure 1.1: CMB spectrum as measured by the COBE satellite, the best fit is a blackbody distribution with a temperature of 2.725 K.[2]

A key piece of observational evidence in cosmology is that everything is moving away from us, and the farther away something is the more rapid it's recession seems to be. These

[2]Source : [1].

velocities are measured using the redshift of emission and absorption lines in the spectrum of galaxies, which are well known to us. Distances on the other hand are measured by studying the luminosity of standard candles. This technique was used by Hubble who realized that the velocity of recession was proportional to the distance of the galaxy from us -

$$\vec{v} = H_0 \vec{r} \tag{1.1}$$

This is known as Hubble's law and the constant of proportionality H_0 is known as Hubble's constant. Figure. 1.2 shows a plot of velocity against distance for a sample of 1355 galaxies.

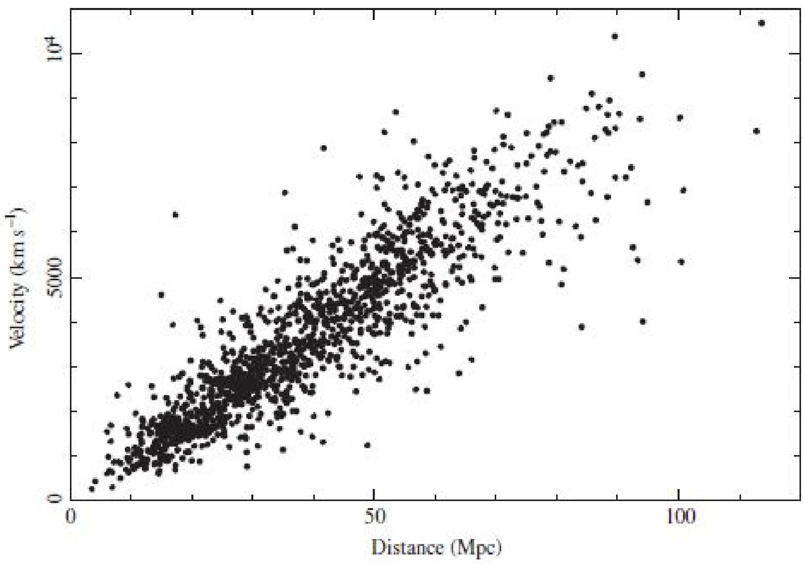

Figure 1.2: Hubble diagram - a plot of velocity vs. estimated distance for a sample of 1355 galaxies, a straight line relation implies Hubble's law. [4]

The cosmological principle together with Hubble's law describes an expanding universe that remains isotropic and homogeneous at all times. We can try to visualize it in two dimensions by imagining a grid that is expanding with time, and the galaxies are fixed in the coordinate system defined by the grids, while the actual physical distance changes with time. This is depicted in figure. 1.3, and will be accounted for in the next chapter.

[4]Source : [1].

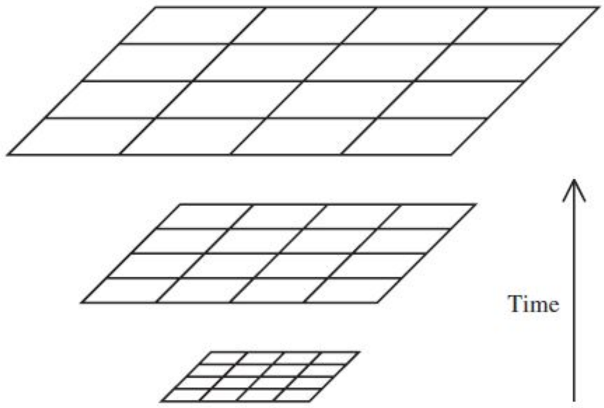

Figure 1.3: The comoving coordinate system carried along with the expansion.[5]

Everything in the universe is made of fundamental particles, and the behavior of the universe as a whole depends on the properties of these particles. All the ordinary matter we see around us is termed as baryons. Of all the possible baryons only the protons and neutrons are stable, so these are thought to be the only types of baryonic particles significantly present in the universe. Our visual perception of the universe comes from electromagnetic radiation, a quantum mechanical description of which introduces the idea of photons. Photons can interact with matter via scattering processes. Neutrinos on the other hand are extremely weakly interacting with matter and comes in three flavors, which together with photons makes up the relativistic material in our present day universe. There is also considerable support for the existence of non-baryonic dark matter, it will be discussed in more details in chapter 4. The simplest evidence comes from galaxy rotation curves, which shows the velocity of matter rotating in a spiral as a function of the radius from the center. If a galaxy has mass M(R) within a radius R, then the balance between the centrifugal force and gravitational pull demands that -

$$\frac{v^2}{R} = \frac{GM(R)}{R^2} \tag{1.2}$$

which implies that -

$$v = \sqrt{\frac{GM(R)}{R}} \tag{1.3}$$

[5]Source : [1]

The mass outside R does not contribute to the gravitational pull. At large distances, enclosing most of the visible part of the galaxy, we expect the total mass to remain constant and hence the velocity should fall of as square root of R. Instead it is found to remain more or less constant as shown in figure. 1.4.

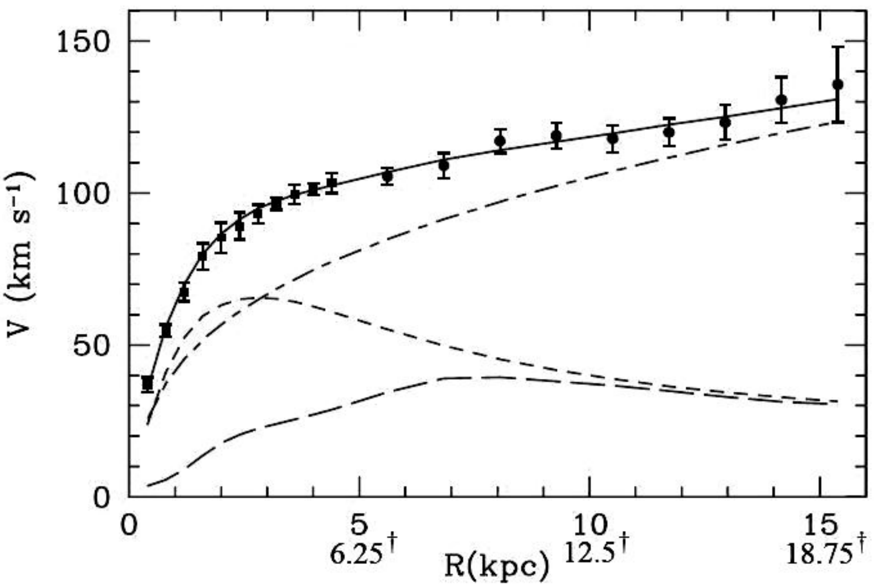

Figure 1.4: M33 rotation curve (points) compared with the best fit model (continuous line). Also shown is the halo contribution (dot-dashed line), the stellar disk (short dashed line) and the gas contribution (long dashed line).[7]

The typical velocities at large distances can be three times higher than that predicted by luminous matter alone, implying ten times more matter than can be directly seen. It is just about possible given present observations that this matter is completely baryonic. However many models based on low mass stars and/or brown dwarfs have been excluded and it is probably difficult to make up all of the halo with them. The popular alternative is to suggest that this additional density is some new form of non-baryonic dark matter that interacts extremely weakly with ordinary matter. This is reinforced by higher estimates for matter density on larger scales.

[7]Source : [2].

Chapter 2

The FLRW Universe

2.1 Arriving at the metric

We construct the simplest model of our universe by imposing homogeneity and isotropy. The geometrical properties of space are determined by the distribution of matter through Einstein equations. Shouldn't that imply that the matter distribution should also be homogeneous and isotropic? It's clearly not the case. But on very large scales (of the order 100 Mpc) matter distribution may be described by a smoothed out average density.

The assumption of homogeneity and isotropy singles out a special class of observers, let's call them the fundamental observers. Any observer moving with respect to them will find the universe to be anisotropic. The most general spacetime interval will have the form

$$ds^2 = g_{ij}dx^i dx^j = g_{00}dt^2 + 2g_{0\alpha}dx^\alpha dt - \sigma_{\alpha\beta}dx^\alpha dx^\beta \tag{2.1}$$

where $\sigma_{\alpha\beta}$ is a positive definite matrix. Isotropy demands that $g_{0\alpha}$ is zero and using the proper time of clocks carried by the observers, we set g_{00} to 1. The spacetime interval thus takes the form

$$ds^2 = dt^2 - \sigma_{\alpha\beta}dx^\alpha dx^\beta = dt^2 - dl^2 \tag{2.2}$$

Isotropy implies spherical symmetry, hence the line element may written as -

$$dl^2 = a(t)^2[\lambda(r)^2 dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)] \tag{2.3}$$

Computing the scalar curvature for this three dimensional space gives us -

$${}^3R = \frac{3}{2a^2r^3}\frac{d}{dr}\left[r^2\left(1 - \frac{1}{\lambda^2}\right)\right] \tag{2.4}$$

Now homogeneity dictates that ${}^3R$ is a constant, hence equating eq. 2.4 to a constant and integrating, we get -

$$r^2\left(1 - \frac{1}{\lambda^2}\right) = c_1 r^4 + c_2 \tag{2.5}$$

where, c_1 and c_2 are constants. To avoid a singularity at $r = 0$ we choose $c_2 = 0$. Thus the full spacetime metric is read off from

$$ds^2 = dt^2 - a(t)^2\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)\right] \tag{2.6}$$

where a(t) is an overall scale known as the expansion factor. This metric know as the Friedmann – Lemaitre – Robertson – Walker (FLRW) metric describes a universe that is spatially homogeneous and isotropic at each instant of time. This particular coordinate system goes by the name - comoving coordinates.

2.2 Analysis of the geometry

The spatial hypersurfaces of the Friedmann universe have positive, negative and zero spatial curvature for k = $+1$, -1 and 0 respectively. The magnitude of the curvature is $\frac{6}{a^2}$ as obtained from eq. 2.4. It is convenient to study the geometry of these spaces by introducing the coordinate -

$$\chi = \int \frac{dr}{\sqrt{1 - kr^2}} \tag{2.7}$$

In terms of (χ, θ, ϕ) the metric becomes -

$$ds^2 = a^2[d\chi^2 + S_k^2(\chi)(d\theta^2 + sin^2\theta d\phi^2)] \tag{2.8}$$

where,

$$S_k(\chi) = \begin{cases} \text{sin } \chi \text{ ,} & \text{if } k = +1 \\ \chi, & \text{if } k = 0 \\ \text{sinh } \chi, & \text{if } k = -1 \end{cases}$$

For $k = 0$ we obtain the familiar flat Euclidean 3-space.

$k = 1$ describes a 3-sphere embedded in a 4-dimensional flat Euclidean space described by -

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = a^2 \tag{2.9}$$

where (x_1, x_2, x_3, x_4) are the Cartesian coordinates of some abstract 4-dimensional space. The angular coordinates (χ, θ, ϕ) on the 3-sphere can be defined as follows -

$$\begin{aligned} x_1 &= a\ cos\chi\ sin\theta\ sin\phi \\ x_2 &= a\ cos\chi\ sin\theta\ cos\phi \\ x_3 &= a\ cos\chi\ cos\theta \\ x_4 &= a\ sin\chi \end{aligned} \tag{2.10}$$

The entire space of the $k = 1$ model is covered by the range of angles - $[0 \le \chi \le \pi;\ 0 \le \theta \le \pi;\ 0 \le \phi \le 2\pi]$ and has a finite volume of $2\pi^2 a^3$.

$k = -1$ represents the geometry of a hyperboloid embedded in a 4-dimensional space with a Lorentzian signature and is described by -

$$x_4^2 - x_1^2 - x_2^2 - x_3^2 = a^2 \tag{2.11}$$

The 3-dimensional hyperboloid may be parametrized by the angular coordinates (χ, θ, ϕ) as follows -

$$\begin{aligned} x_1 &= a\ sinh\chi\ sin\theta\ sin\phi \\ x_2 &= a\ sinh\chi\ sin\theta\ cos\phi \\ x_3 &= a\ snh\chi\ cos\theta \\ x_4 &= a\ cosh\chi \end{aligned} \tag{2.12}$$

The entire space of the $k = -1$ model is covered by the range of angles - $[0 \leq \chi \leq \infty;\ 0 \leq \theta \leq \pi;\ 0 \leq \phi \leq 2\pi]$ and has infinite volume. Friedmann universes with k = -1, 0 and +1 are called open, flat and closed respectively.

2.3 Particle kinematics

To study the geodesics in the Friedmann universe we consider a particle of mass m with 4-velocity u^μ. Parametrizing by the proper length ds, the zeroth component of the geodesic equation becomes -

$$\frac{du^0}{ds} = \Gamma^0{}_{\mu\nu} u^\mu u^\nu \tag{2.13}$$

For the FLRW metric the only non-vanishing component of $\Gamma^0{}_{\mu\nu}$ is $\Gamma^0{}_{ij} = (\dot{a}/a)h_{ij}$, where h_{ij} is the spatial part of the metric. Using the fact that $h_{ij}u^iu^j = |\vec{u}|^2$, the geodesic equation becomes -

$$\frac{du^0}{ds} = \frac{\dot{a}}{a}|\vec{u}|^2 \tag{2.14}$$

Since $(u^0)^2 - |\vec{u}|^2 = 1$, the geodesic equation may be written as -

$$\frac{1}{u^0}\frac{d|\vec{u}|}{ds} + \frac{\dot{a}}{a}|\vec{u}| = 0 \tag{2.15}$$

Which implies that $|\vec{u}| \propto a^{-1}$. In other words the three momentum of the particle red-shifts as a^{-1}. Further, the above analysis also holds for massless particles, since the factor of ds cancels in eq. 2.15. In the quantum mechanical description of light the wavelength is inversely proportional to the momentum. Thus, as the universe expands, the wavelength of a freely moving photon increases in proportion to the scale factor. Astronomers talk in terms of the redshift (z), which is defined as the ratio of the detected wavelength to the emitted wavelength -

$$1 + z = \frac{\lambda_0}{\lambda_1} = \frac{a(t_0)}{a(t_1)} \tag{2.16}$$

2.4 Dynamics of the FLRW universe

The dynamics of the expanding universe only appeared implicitly in the scale factor a(t). To make the time dependence explicit one must solve for the scale factor using the Einstein equations -

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = G_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda\delta_{\mu\nu} \tag{2.17}$$

where $G_{\mu\nu}$ is the Einstein tensor, $T_{\mu\nu}$ is the stress-energy tensor for all the fields present and Λ is the cosmological constant. To be consistent with the symmetries of the metric the stress-energy tensor must be diagonal. Further, isotropy dictates that the spatial components must be equal. The simplest realization of such a stress tensor is that of an ideal fluid, with a time dependent energy density $\rho(t)$ and pressure $p(t)$ -

$$T^{\mu}_{\ \nu} = (\rho, -p, -p, -p) \tag{2.18}$$

The $\mu = 0$ component of the conservation of the stress tensor gives the first law of thermodynamics in the familiar form -

$$d(\rho a^3) = -pd(a^3) \tag{2.19}$$

For the simplest case of an equation of state given by $p = w\rho$, where w is a constant independent of time, $\rho \propto a^{-3(1+w)}$. Examples of interest include -

$$\begin{aligned} \text{Radiation } (p = \frac{\rho}{3}) &\implies \rho \propto a^{-4} \\ \text{Matter } (p = 0) &\implies \rho \propto a^{-3} \\ \text{Vacuum energy } (p = -\rho) &\implies \rho = constant \end{aligned} \tag{2.20}$$

The dynamical equations describing the evolution of the scale factor follows from the Einstein field equations. The non-zero components of the Ricci tensor for the FLRW metric are as follows -

$$\begin{aligned} R_{00} &= -\frac{3\ddot{a}}{a} \\ R_{ij} &= -\left[\frac{\ddot{a}}{a} + \frac{2(\dot{a})^2}{a^2} + \frac{2k}{a^2}\right] g_{ij} \end{aligned} \tag{2.21}$$

The Ricci scalar is given by -

$$R = -6\left[\frac{\ddot{a}}{a} + \frac{(\dot{a})^2}{a^2} + \frac{k}{a^2}\right] \tag{2.22}$$

The 0-0 component of the Einstein equation gives the Friedmann equation -

$$\frac{(\dot{a})^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \tag{2.23}$$

while the i-i component gives -

$$\left[2\frac{\ddot{a}}{a} + \frac{(\dot{a})^2}{a^2} + \frac{k}{a^2}\right] = -8\pi G p \tag{2.24}$$

Of the three field equations, eq. 2.19, eq. 2.23 and eq. 2.24 only two are independent, as they are related by the Bianchi identities. The difference of eq. 2.24 and eq. 2.23 gives the equation for the acceleration $\ddot{a}$ -

$$\frac{\ddot{a}}{a} = \frac{-4\pi G}{3}(\rho + 3p) \tag{2.25}$$

The expansion rate of the universe is determined by the Hubble parameter $H = \dot{a}/a$. H^{-1} sets the time scale of the universe and is known as the Hubble time/radius. The Hubble constant (H_0), defined in chapter 1, is the present day value of the expansion rate. In terms of H the Friedmann equation may be recast as -

$$\frac{k}{H^2 a^2} = \Omega - 1 \tag{2.26}$$

where $\Omega = \rho/\rho_c$, and $\rho_c = 3H^2/8\pi G$ is know as the critical density. Since $H^2 a^2 \geq 0$, there exists a correspondence between the sign of k and $\Omega - 1$ -

$$\begin{aligned} k = +1 &\implies \Omega > 1 \text{ (Closed)} \\ k = 0 &\implies \Omega = 1 \text{ (Flat)} \\ k = -1 &\implies \Omega < 1 \text{ (Open)} \end{aligned} \tag{2.27}$$

At early times when the curvature term was negligible, $H^2 \propto \rho \propto a^{-3}$ for a matter dominated universe. While $H^2 \propto a^{-4}$ for a radiation dominated universe. Since $|\Omega - 1|$ is of the order of unity today, at earlier times -

$$\begin{aligned} &|\Omega - 1| \propto (a/a_0) = (1+z)^{-1} \text{ (Matter Dominated)} \\ &|\Omega - 1| \propto (a_{EQ}/a_0)(a/a_{EQ})^2 = 10^4(1+z)^{-2} \text{ (Radiation Dominated)} \end{aligned} \tag{2.28}$$

where $R_{EQ} = 10^4 R_0$ is the value of R at the transition between matter domination and radiation domination.

If we ignore spatial curvature ($k = 0$) and take the RHS of the Friedmann equation to be dominated by a fluid with pressure $p = w\rho$, then we get -

$$\begin{aligned} &\rho \propto a^{-3(1+w)} \\ &a \propto t^{\frac{2}{3(1+w)}} \end{aligned} \tag{2.29}$$

which can be suitably adjusted for matter dominated, radiation dominated and vacuum dominated universe by suitably choosing the value of w.

Chapter 3

Thermodynamics of the Universe

Before we begin our study of the early universe let us recapitulate some basic thermodynamics.

3.1 Equilibrium thermodynamics

The number density (n), energy density (ρ) and pressure (p) of a dilute weakly interacting gas with g internal degrees of freedom is obtained using the phase space distribution f($\vec{p}$) -

$$n = \frac{g}{(2\pi)^3} \int f(\vec{p}) d^3p \tag{3.1}$$

$$\rho = \frac{g}{(2\pi)^3} \int E(\vec{p}) f(\vec{p}) d^3p \tag{3.2}$$

$$p = \frac{g}{(2\pi)^3} \int \frac{\mid \vec{p} \mid^2}{3E} f(\vec{p}) d^3p \tag{3.3}$$

where $E = |\vec{p}|^2 + m^2$. For a species in kinetic equilibrium f($\vec{p}$) is given by the Fermi-Dirac or Bose-Einstein distribution -

$$f(\vec{p}) = \frac{1}{exp\left(\frac{E-\mu}{T}\right) \pm 1} \tag{3.4}$$

where μ is the chemical potential of the species. One can evaluate equations 3.1, 3.2, 3.3 using the equilibrium distribution 3.4 for different cases. In the relativistic limit ($T >> m$) and $T >> \mu$ -

$$
\begin{aligned}
\rho &= \frac{\pi^2}{30} g T^4 \ \text{(BOSE)} \\
&= \frac{7}{8}\frac{\pi^2}{30} g T^4 \ \text{(FERMI)} \\
\\
n &= \frac{\zeta(3)}{\pi^2} g T^3 \ \text{(BOSE)} \\
&= \frac{3}{4}\frac{\zeta(3)}{\pi^2} g T^3 \ \text{(FERMI)} \\
\\
p &= \frac{\rho}{3}
\end{aligned}
\tag{3.5}
$$

where ζ is the Riemann zeta function. For relativistic bosons or fermions with $\mu < 0$ and $|\mu| < T$ -

$$
\begin{aligned}
\rho &= exp\left(\frac{\mu}{T}\right)\frac{3g}{\pi^2} T^4 \\
\\
n &= exp\left(\frac{\mu}{T}\right)\frac{g}{\pi^2} T^3 \\
\\
p &= exp\left(\frac{\mu}{T}\right)\frac{g}{\pi^2} T^4
\end{aligned}
\tag{3.6}
$$

In the nonrelativistic limit ($m >> T$) p, n and ρ are the same for Bose and Fermi species -

$$\rho = mn$$

$$n = g\left(\frac{mT}{2\pi}\right)^{3/2} exp[-(m-\mu)/T] \tag{3.7}$$

$$p = nT << \rho$$

Since energy density and pressure of nonrelativistic species is exponentially smaller that of the relativistic ones, it is a good approximation to include the contribution of only the relativistic species in calculating the total energy density -

$$\begin{aligned} \rho_R &= g_*\left(\frac{\pi^2}{30}\right)T^4 \\ p_R &= g_*\left(\frac{\pi^2}{90}\right)T^4 \end{aligned} \tag{3.8}$$

where g_* gives the total number of effectively massless degrees of freedom -

$$g_* = \sum_{i=bosons} g_i\left(\frac{T_i}{T}\right)^4 + \frac{7}{8}\sum_{i=fermions} g_i\left(\frac{T_i}{T}\right)^4 \tag{3.9}$$

For $T << MeV$ the only relativistic particles are the three neutrinos and the photon. As will be shown in the next chapter, $T_\nu = (\frac{4}{11})^{1/3}T_\gamma$, thus $g_*(<< MeV) = 3.86$. For $1MeV \lessapprox T \lessapprox 100MeV$, electrons and positrons are additional degrees of freedom and $T_\nu = T_\gamma$, giving $g_* = 10.75$. For $T \gtrapprox 300MeV$, all species in the standard model - 8 gluons, 3 generations of quarks and leptons, $W^\pm, Z^0$ and one complex Higgs doublet should be relativistic, thus yielding $g_* = 106.75$. The temperature dependence of g_* and g_{*S} (defined in the next section) is shown in figure. 3.1.

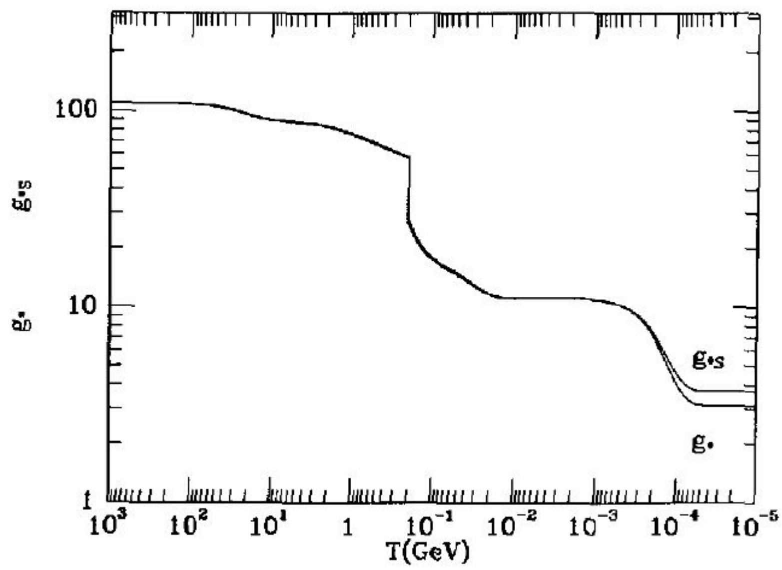

Figure 3.1: Evolution of relativistic degrees of freedom assuming standard model.[1]

3.2 A brief discussion on entropy

During most of the history of the universe , the particle interaction rates were much larger than the expansion factor, implying that local thermal equilibrium must have existed. This in turn implies that the entropy per unit comoving volume must have remained constant.

In an expanding universe, the second law of thermodynamics applied to a unit comoving volume, with physical volume $V = a^3$ gives

$$TdS = d(\rho V) + pdV \tag{3.10}$$

where p and ρ are the equilibrium pressure and energy density. Now the integrability con-

[1]Source [3]

dition reads -

$$\frac{\partial^2 S}{\partial T \partial V} = \frac{\partial^2 S}{\partial V \partial T} \tag{3.11}$$

appplying to eq. 3.10, we get -

$$dp = \frac{\rho + p}{T} dT \tag{3.12}$$

Substituting eq. 3.12 in eq. 3.10, it follows that -

$$dS = d\left[\frac{(p+\rho)V}{T} + const.\right] \tag{3.13}$$

i.e, upto an additive constant the entropy $S = a^3(\rho + p)/T$. Now the first law can be written as -

$$d(\rho + p)V = V dp \tag{3.14}$$

Substituting eq. 3.14 in eq. 3.12 , we get -

$$d\left[\frac{(\rho + p)V}{T}\right] = 0 \tag{3.15}$$

which implies that entropy per comoving volume is conserved. It is useful to define the entropy density s as -

$$s = \frac{(\rho + p)}{T} \tag{3.16}$$

The entropy density is dominated by relativistic species, so that to a very good approximation -

$$s = \frac{2\pi^2}{45} g_{*s} T^3 \tag{3.17}$$

where,

$$g_{*s} = \sum_{i=bosons} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{i=fermions} g_i \left(\frac{T_i}{T}\right)^3 \tag{3.18}$$

Conservation of S implies s $\propto a^{-3}$ and $g_{*s}a^3T^3$ is constant. Since $N = na^3$ -

$$N \propto \frac{n}{s} \tag{3.19}$$

And hence for a species in thermal equilibrium -

$$\begin{aligned} N &= \frac{45\zeta(3)g}{2\pi^4 g_{*s}} \quad (\text{T} >> \text{m}, \mu) \\ &= \frac{45g}{4\sqrt{2}\pi^5 g_{*s}}(m/T)^{3/2}exp(-m/T + \mu/T) \quad (\text{T} << \text{m}) \end{aligned} \tag{3.20}$$

Although the baryon to photon ratio $\eta = n_B/n_\gamma = 1.8g_{*S}(n_B/s)$ does not remain constant as g_{*S} is a function of time, after the era of electron positron annihilation, it is a constant. So $\eta \approx 7(n_B/s)$ and (n_B/s) can be used interchangeably.

The second fact , that $S = g_{*s}a^3T^3 =$ const. implies that -

$$T \propto g_{*s}^{-1/3}a^{-1} \tag{3.21}$$

When g_{*S} is a constant, the familiar result $T \propto a^{-1}$ follows.

3.3 The idea of decoupling

Consider a massless species initially in local thermal equilibrium that decouples at $t = t_D$ when the temperature was T_D and the expansion factor a_D. After decoupling the energy of each of the masssless species is redshifted because of the expansion of the universe : $E(t) = E(t_D)a(t_D)/a(t)$. In addition the number density of the species falls off as : $n \propto a^{-3}$.

As a result the phase space distribution function will be precisely that of the species in local thermal equilibrium with a temperature $T(t) = T(t_D)a_D/a(t)$ -

$$f(\vec{p}, t) = f\left(\vec{p}\frac{a}{a_D}, t_D\right) = \frac{1}{exp\left(\frac{E\ a}{a_D T_D}\right) \pm 1} \tag{3.22}$$

Thus the distribution function for decoupled massless particle remains self similar while the temperature redshifts as a^{-1}.

A similar argument for a massive nonrelativistic species ($T << m$), for which the momentum redshifts as : $|\vec{p}(t)| = |\vec{p}(t_D)|(a(t_D)/a(t))$, the kinetic energy as :$E_K(t) = E_K(t_D)a^2(t_D)/a^2(t)$ and the number density falls off as $n \propto a^{-3}$, would imply that that the decoupled species would have an equilibrium distribution described by -

$$\begin{aligned} T &= T_D\frac{a_D^2}{a^2} \\ \mu &= m + (\mu_D - m)\frac{T(t)}{T_D} \end{aligned} \tag{3.23}$$

Ignoring the temperature variation of g_* , $T \propto a^{-1}$, and thus the expansion rate $H = -\dot{T}/T$. So long as the interactions needed for the distribution function to adjust to the changing temperature are rapid compared to the expansion, the universe will evolve through a series of near thermal equilibrium states. The usual rule of thumb is that the reaction is occurring rapidly enough to maintain thermal equilibrium when $\Gamma \gtrapprox H$, where $\Gamma = n\sigma|\vec{v}|$ is the interaction rate per particle. Here n is the number density of the target particles, σ is the interaction cross-section and $|\vec{v}|$ the relative velocity appropriately averaged.

$\Gamma < H$ is not a sufficient condition for departure from equilibrium. The rate of some reaction that is essential for maintaining thermal equilibrium must be less than H. The correct way to evolve particle distributions is to integrate the Boltzmann equation as will be discussed in the next section. For the moment $\Gamma < H$ ($\Gamma > H$) is used as the condition for the species to be decoupled (coupled) from (to) the thermal plasma of the universe.

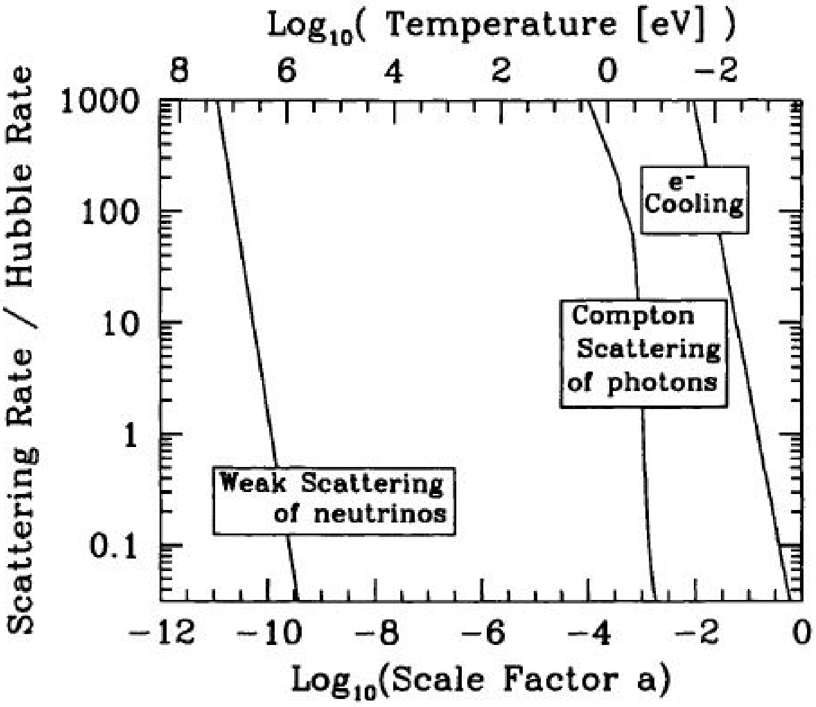

Figure 3.2: Rates as a function of the scale factor.[2]

To demonstrate the above discussion we consider the following two cases : (i) interactions mediated by a massless gauge boson, e.g. the photon and (ii) those mediated by massive gauge bosons, e.g. $W^{\pm}$ and Z^0, below the scale of electroweak symmetry breaking $(T \lessapprox 300\ GeV)$. The rates of these interactions as a function of the scale factor is shown in figure. 3.2. In the first case, for a $2 \leftrightarrow 2$ scattering of relativistic particles with significant momentum transfer, $\sigma \sim \alpha^2/T^2$ ($g = \sqrt{4\pi\alpha}$ is the gauge coupling strength). In the second case for $T \lessapprox m_X$, $\sigma \sim G_X^2 T^2$, where m_X is the mass of the gauge boson and $G_X \sim \alpha/m_X^2$. For $T >> m_X$ the cross section is same as that for massless gauge bosons. Thus for the first case $\Gamma \sim \alpha^2 T$ and during the radiation dominated epoch $H \sim T^2/m_{pl}^2$, so that $\Gamma/H \sim \alpha^2 m_{pl}/T$. Therefore for $T \lessapprox \alpha^2 m_{pl} \sim 10^{16} GeV$ such reactions are occurring rapidly while for $T \gtrapprox \alpha^2 m_{pl} \sim 10^{16} GeV$ such reactions have essentially frozen out. For in-

[2]Source [4]

teractions mediated by the massive gauge bosons $\Gamma \sim G_X^2 T^5$ and $\Gamma/H \sim G_X^2 m_{pl} T^3$. Thus for $m_X >> T >> G_X^{2/3} m_{pl}^{-1/3} \sim (m_X/100\ GeV)^{4/3}$ MeV such reactions are occurring rapidly, while for $T \lessapprox (m_X/100\ GeV)^{4/3}$ MeV such reactions have effectively frozen out. For $T \gtrapprox \alpha^2 m_{pl} \sim 10^{16} GeV$, which corresponds to times earlier than $10^{-38} s$ all perturbative interactions are frozen out and thus ineffective in maintaining or establishing thermal equilibrium. Perhaps there are other unknown interactions that thermalize the universe at such early epochs or maybe the universe was not in thermal equilibrium during it's earliest epoch.

3.4 The Boltzmann equation

For most of the history of the universe, it's constituents have been in thermal equilibrium, making the equilibrium description a good approximation. However there have been a number of notable departures from equilibrium, resulting in some important relics. We have already given a simplistic description of decoupling in the previous section, however the evolution of particle distributions near the epoch of decoupling is challenging.

In order to properly treat decoupling one must follow the particles microscopic phase space distribution $f(x^\mu\ , p^\nu)$, which is governed by the Boltzmann equation -

$$\hat{\mathbf{L}}[f] = \hat{\mathbf{C}}[f] \tag{3.24}$$

where **C** is the collision operator and $\hat{\mathbf{L}}$ the liouville operator, whose covariant relativistic generalization is given by -

$$\hat{\mathbf{L}} = p^\alpha \frac{\partial}{\partial x^\alpha} - \Gamma^\alpha{}_{\beta\gamma}\, p^\beta p^\gamma \frac{\partial}{\partial p^\alpha} \tag{3.25}$$

For the FRW model, space is homogeneous and isotropic which implies $f = f(E,t)$, so the Liouville operator becomes -

$$\hat{\mathbf{L}}[f(E,t)] = E\frac{\partial f}{\partial t} - \frac{\dot{a}}{a}|\vec{p}|^2 \frac{\partial f}{\partial E} \tag{3.26}$$

Using the definition of number density eq. 3.1 and integrating by parts, the Boltzmann equation may be written as -

$$\frac{dn}{dt} + 3\frac{\dot{a}}{a}n = \frac{g}{2\pi^3}\int \mathbf{C}[f]\frac{d^3p}{E} \tag{3.27}$$

The collision term for the process $\psi + a + b + ... \leftrightarrow i + j + ...$ is given by -

$$\begin{aligned} \frac{g}{2\pi^3}\int \mathbf{C}[f_\psi]\frac{d^3p_\psi}{E_\psi} = -\int d\prod_\psi d\prod_a d\prod_b ... d\prod_i d\prod_j ... \times (2\pi)^4\delta^4(p_\psi + p_a + ... - p_i - p_j...) \\ \times [|M|^2_{\psi+a+b+...\to i+j+...} f_\psi f_a f_b ... (1 \pm f_i)(1 \pm f_j)... \\ - \quad |M|^2_{i+j+...\to\psi+a+b+...i+j+...} f_i f_j f_k ... (1 \pm f_\psi)(1 \pm f_a)...] \end{aligned} \tag{3.28}$$

where f's are the phase space distributions for the different species. Here "+" applies to bosons and "$-$" to fermions; and -

$$d\prod = \frac{g}{(2\pi)^3}\frac{d^3p}{E} \tag{3.29}$$

where g counts the internal degrees of freedom, the delta function imposes energy momentum conservation and the matrix element squared $|M|^2$ is averaged over all initial and final spins.

The Boltzmann equation gives a set of integral partial differential equations. There are two well motivated approximations that greatly simplify eq. 3.28. The first is the assumption of T or CP invariance which implies -

$$|M|^2_{\psi+a+b+...\to i+j+...} = |M|^2_{i+j+...\to\psi+a+b+...i+j+...} \tag{3.30}$$

The second assumption is the use of Maxwell-Boltzmann statistics instead of FD or BE statistics. In the absence of Bose condensate and Fermi degeneracy, the blocking and stimulated emission factors may be ignored $(1 \pm f \approx 1)$. And $f(E_i) = exp(-(E_i - \mu)/T)$. With these two assumptions the Boltzmann equation can be recast in the form -

$$\dot{n_\psi} + 3\ H\ n_\psi = -\int d\prod_\psi d\prod_a d\prod_b ...\ d\prod_i d\prod_j ... \times (2\pi)^4 \delta^4(p_\psi + p_a + ... - p_i - p_j ...) \times |M|^2 [f_\psi f_a f_b ... - f_i f_j ...] \tag{3.31}$$

It is useful to scale out the expansion of the universe by using the entropy density s and defining the dependent variable $Y = n_\psi/s$. Using the conservation of entropy ($sa^3 = const.$), we get -

$$\dot{n_\psi} + 3\ H\ n_\psi = s\dot{Y} \tag{3.32}$$

Furthermore, since the interaction term will depend explicitly on the temperature rather than time, it is useful to define as the independent variable $(x = m/T)$. During the radiation dominated epoch x and t are related by -

$$t = .301 g_*^{-1/2} \frac{m_{pl}}{T^2} = .301 g_*^{-1/2} \frac{m_{pl}}{m^2} x^2 \tag{3.33}$$

So the Boltzmann equation may be written as -

$$\frac{dY}{dx} = -\frac{x}{H(m)\ s} \int d\prod_\psi d\prod_a d\prod_b ...\ d\prod_i d\prod_j ... \times (2\pi)^4 \delta^4(p_\psi + p_a + ... - p_i - p_j ...) \times |M|^2 [f_\psi f_a f_b ... - f_i f_j ...] \tag{3.34}$$

where $H(m) = 1.67 g_*^{1/2} \frac{m^2}{m_{pl}}$ and $H(x) = H(m) x^{-2}$.

Chapter 4

Thermal History and Decoupling

4.1 Summary of the thermal history of the universe

The thermal history of the universe is based upon extrapolating our present knowledge of the universe and particle physics back to the planck epoch ($t \approx 10^{-43}$ or $T \approx 10^{19} GeV$), the point at which quantum corrections to general relativity sets in. At the earliest times the universe was a plasma of relativistic particles, including quarks, leptons, gauge bosons and Higgs bosons. A number of spontaneous symmetry breaking (SSB) phase transitions must have taken place in the early universe. Theses include grand unification phase transitions at a temperature of $10^{14}\ GeV$ to $10^{16}\ GeV$ and electroweak SSB phase transition at about 300 GeV. During these transitions some particles and gauge bosons acquire mass via the Higgs mechanism and the full symmetry of the theory is broken down to a lower symmetry. Subsequent to these phase transitions, interactions mediated by the X bosons which acquire mass, are now characterized by a coupling strength G_X as discussed earlier. Particles interacting only via these X bosons will thus decouple from the thermal plasma at $T \approx G_X^{-2/3} m_{pl}^{-1/3}$. At a temperature of about 100 MeV to 300 Mev the universe should undergo a transition associated with chiral symmetry breaking and colour confinement, after which strongly interacting particles form colour - singlet - quark - triplet states (baryons) and colour - singlet - quark - antiquark states (mesons). The epoch of nucleosynthesis follows when $t \approx 10^{-2} s$ to $10^{2} s$ and $T \approx 10 MeV$ to $0.1 MeV$. At a time of about $t \approx 10^{11} s$ the matter density becomes equal to that of radiation, which marks the beginning of the current matter dominated epoch and the start of structure formation. Finally at a time of about $10^{13} s$, the ions and electrons combine to form atoms, matter and radiation decouples, ending the epoch of near thermal equilibrium that existed in the early universe. The surface of last scattering of the cosmic microwave background radiation is the universe itself at decoupling. We will now discuss some of these events in more details.

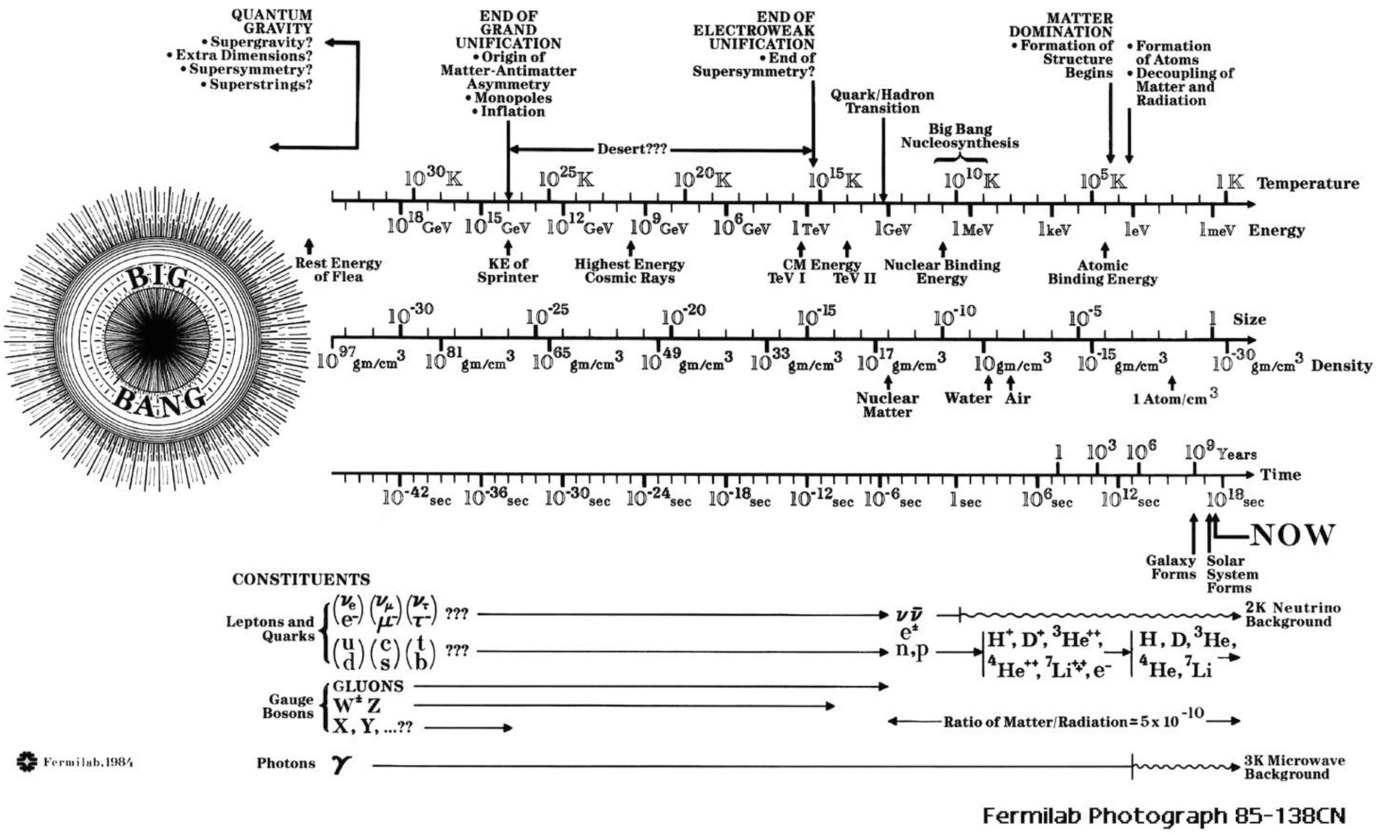

Figure 4.1: The complete history of the universe.[1]

4.2 Neutrino decoupling

In the early universe three species of left handed neutrinos and their CP conjugated states are excited in the primeval plasma. They are maintained in kinetic and chemical equilibrium by leptons, baryons and photons via weak interactions. In this regime the neutrino distribution is the Fermi-Dirac type, with negligible contribution of mass to the energy. The temperature is that of the photons which falls off as a^{-1}, because of entropy conservation, except near the

[1]Source : [3]

times when particles disappear from the thermal bath releasing their energy to the lighter particles. For the time being we will neglect the chemical potential for the neutrinos.

From our discussion in section 3.3, we can get a quick estimate of the neutrino decoupling temperature. The leading process responsible for keeping the neutrinos in thermal equilibrium with the plasma is $e^+e^- \leftrightarrow \nu\bar{\nu}$. Recalling the case when the temperature T is smaller than the masses of the $Z^{\pm}$, W bosons and that $H \approx \sqrt{g_*}T^2/m_{pl}$ for the radiation dominated epoch, the decoupling temperature is given by -

$$G_X^2 T_{\nu D}^5 = \sqrt{g_*}T_{\nu D}^2/m_{pl} \implies T_{\nu D} \approx g_*^{1/6} MeV \tag{4.1}$$

We see that this temperature is of the order of MeV. Only photons, electrons / positrons and the neutrinos themselves contribute to g_* at this temperature, and thus -

$$g_* = 2 + \frac{7}{8} \times 4 + \frac{7}{8} \times 6 = \frac{43}{4} = 10.75 \tag{4.2}$$

A more refined calculation of the decoupling temperature can be done by soving the Boltzmann equations in terms of the $x = ma$ and $y = pa$, with m some suitable mass scale -

$$\begin{aligned} Hx\frac{\partial f_{\nu_e}}{\partial x} &= -\frac{80G_X^2(\tilde{g}_R^{l2} + g_R^{l2})m^9}{3\pi^3 x^5} y f_e \\ Hx\frac{\partial f_{\nu_{\mu,\tau}}}{\partial x} &= -\frac{80G_X^2(g_R^{l2} + g_R^{l2})m^9}{3\pi^3 x^5} y f_{\mu,\tau} \end{aligned} \tag{4.3}$$

To obtain these expressions one uses neutrino interaction rate amplitudes [5], and approximates the particle distributions as Boltzmann functions. Because during radiation domination $H \propto a^{-2}$, the solutions are functions of the combination x/y^3 or yT^3. In particular the decoupling temperatures are found to be $T_{\nu_D} = 2.7y^{-1/3}$ MeV and $T_{\mu_D,\tau_D} = 4.5y^{-1/3}$ MeV. Taking the average value of momenta y $\approx$ 3, we get $T_{\nu_D} = 1.87$ MeV and $T_{\mu_D} = 3.12$ MeV. The slightly lower value for ν_e is because they can interact with the electrons and positrons via charged current processes in addition to neutral currents, so they remain in equilibrium slightly later. The limit in which we consider the decoupling as an instantaneous event happening at T_{ν_D}, is known as the instantaneous decoupling limit. However, decoupling actually happen over a finite time interval, as the interaction cross-sections are energy dependent and not all neutrinos have the same momentum. Implying that high energy neutrinos will be kept at thermal equilibrium longer than the ones with lower energy.

After decoupling, they propagate freely, their distribution remains unchanged except for the redshift of momentum. They are an example of hot relics (decouples when relativistic). In the instantaneous decoupling limit the, the distribution of the neutrinos after decoupling is given by eq. 3.22 with a $+$ sign (F-D statistics).

The neutrino to photon temperature ratio after neutrino decoupling can be determined from entropy conservation of the electromagnetic plasma. When the temperature drops below a T_{ann} comparable to the electron mass, $e^{\pm}$ pair annihilation's cannot be sufficiently compensated for by inverse pair productions. Subsequently all electrons and positrons disappear except for a tiny relic electron density constrained by electric neutrality. As a result photons are heated, and their temperature in this phase is not decreasing as a^{-1}. In the instantaneous decoupling limit, neutrinos are left undisturbed by $e^{\pm}$ pair annihilation. For $T >> m_e$, electrons and positrons are still relativistic, implying -

$$g_{*s} = 2 + \frac{7}{8}(2+2) = \frac{11}{2} \tag{4.4}$$

In the opposite limit, electrons and positrons are no longer relativistic, hence -

$$g_{*s} = 2 \tag{4.5}$$

Since entropy per comoving volume is conserved and $T_\nu \propto a^{-1}$ -

$$g_{*s}(T)\frac{T^3}{T_\nu^3} = constant \tag{4.6}$$

At neutrino decoupling, $T = T_\nu$. And after the $e^\pm$ annihilation phase ($T << m_e$), the value of g_{*s} drops, giving -

$$\frac{11}{2} = 2\frac{T^3}{T_\nu^3} \implies \frac{T_\nu}{T} = \left(\frac{4}{11}\right)^{1/3} \tag{4.7}$$

Using the know value of CMB today, $T_0 = 2.725$ K, we see that the neutrinos are as cold as $T_{\nu,0} = 1.945$ K.

4.3 Recombination

When the temperature drops to about 1 eV, photons are still tightly coupled to electrons and electrons to protons. Even though energetics favor otherwise, there is very little Hydrogen, due to the large photon to baryon ratio. As long as the reaction $e^- + p \leftrightarrow H + \gamma$ is in equilibrium, it follows from the Boltzmann equation that (Saha approximation)

$$\frac{n_p n_e}{n_H} = \frac{n_p^0 n_e^0}{n_H^0} \tag{4.8}$$

where n is the number density as discussed earlier, the subscripts e, p and H, corresponds to free electrons, protons and Hydrogen atom respectively. n_i^0 is given by -

$$\begin{aligned} n_i^0 = g_i \int \frac{d^3p}{2\pi^3} exp\left(\frac{-E_i}{T}\right) &= g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} exp\left(\frac{-m_i}{T}\right) \quad (m_i >> T) \\ &= g_i \frac{T^3}{\pi^2} \quad (m_i << T) \end{aligned} \tag{4.9}$$

We define the free electron fraction

$$X_e = \frac{n_e}{n_e + n_H} \tag{4.10}$$

further, electric neutrality demands $n_e = n_p$. Evaluating the integrals on the RHS of eq. 4.8, we get -

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_e + n_H}\left[\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-[m_e + m_p - m_H]/T}\right] \tag{4.11}$$

where we have neglected the mass difference of H and p in the prefactor. The exponential can be written as $(-\epsilon_0/T)$, where $\epsilon_0 = m_e + m_p - m_H$. Neglecting the relatively small number of Helium atoms, the denominator is approximately equal to the baryon number density ($\approx 10^{-9}T^3$). So when the temperature is of the order of ϵ_0, the RHS is of the order $\approx 10^{15}$. In that case the above equation is satisfied only when $X_e \approx 1$: all Hydrogen is ionized. In order to follow the free electron fraction accurately, we need to solve the Boltzmann equation explicitly. After simplification we get -

$$\frac{dX_e}{dt} = [(1 - X_e)\beta - X_e^2 n_b \alpha^{(2)}] \tag{4.12}$$

where the ionization rate is given by -

$$\beta = \langle \sigma v \rangle \left(\frac{m_e T}{2\pi} \right)^{3/2} e^{\epsilon_0/T} \tag{4.13}$$

and the recombination rate -

$$\alpha^{(2)} = \langle \sigma v \rangle \tag{4.14}$$

The only way for recombination to proceed is via the capture of an electron to some excited state of Hydrogen, then to a good approximation -

$$\alpha^{(2)} = 9.78 \frac{\alpha^2}{m_e^2} (\epsilon_0/T)^{1/2} ln(\epsilon_0/T) \tag{4.15}$$

The Saha approximation does a good job at predicting the redshift of recombination, however for a detailed evolution of X_e, one needs to use eq. 4.12, the numerical integration results of which are shown in figure. 4.2. Recombination at z $\approx$ 1000 is directly tied to decoupling of photons from matter, it occurs roughly when the rate of photons to Compton scatter off electrons becomes smaller than the expansion rate. The scattering rate is given by -

$$n_e \sigma_T = X_e n_b \sigma_T \tag{4.16}$$

where $\sigma_T = 0.665 \times 10^{-24} cm^2$ is the Thomson scattering cross - section. n_b can be eliminated for $\Omega_b h^2$, where h is a parameter that sets the uncertainty in the measured value of the expansion rate. So we get -

$$n_e \sigma_T = 7.477 \times 10^{-30} cm^{-1} X_e \Omega_b h^2 a^{-3} \tag{4.17}$$

Dividing by the expansion rate, gives us -

$$\frac{n_e \sigma_T}{H} = 0.0692 \; X_e \Omega_b h a^{-3} \frac{H_0}{H} \tag{4.18}$$

Further analysis of the above equation indicates that photons decouple when the free electron fraction drops below 10^{-2}. From figure. 4.2, we see that X_e drops very quickly from unity to 10^{-3} around $z \approx 1000$. Thus decoupling takes place during recombination.

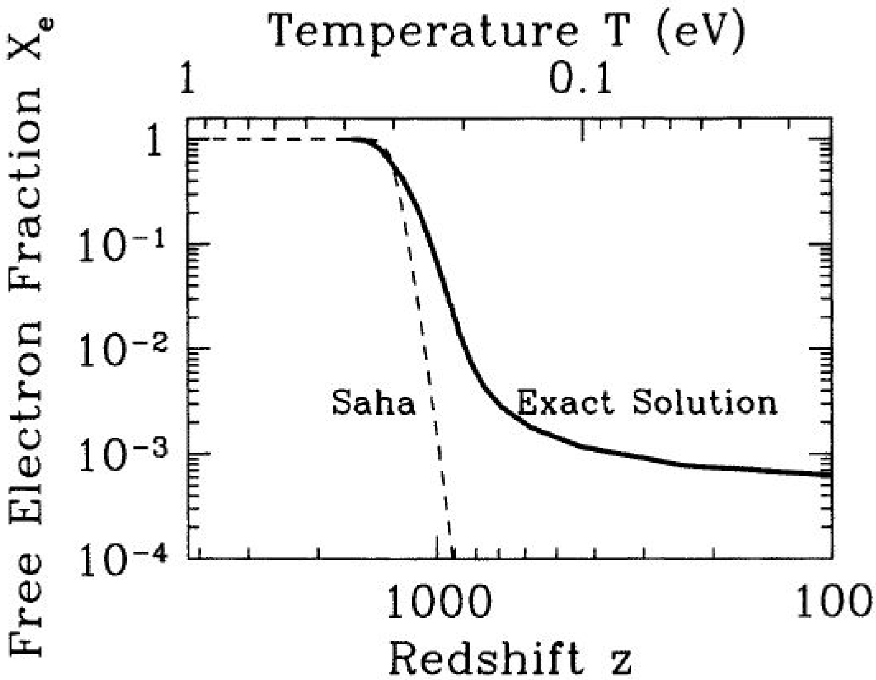

Figure 4.2: Free electron fraction as a function of redshift.[2]

[2]Source : [4]

4.4 WIMP decoupling and dark matter

There is strong evidence for non-baryonic dark matter in the universe with $\Omega_{dm} \approx 0.3$, the most accurate value comes from the Planck collaboration. Standard candles that can be seen at largest distance are type Ia supernova. Figure. 4.3 shows a Hubble diagram for something known as the luminosity distance as a function of redshift. The three curves depicts three different possibilities : flat matter dominated, open and flat with cosmological constant. The current best fit is a universe with about 70% of the energy in the form of cosmological constant or some other form of dark energy.

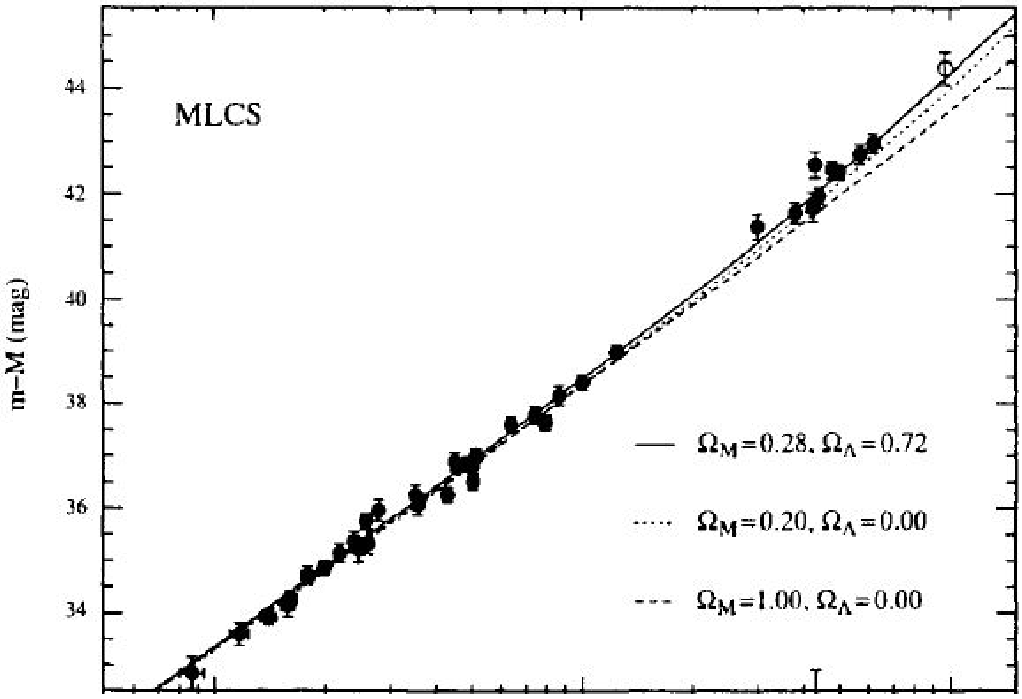

Figure 4.3: Hubble diagram for distant type Ia supernova - apparent magnitude (indicator of distance) as a function of redshift.[4]

Figure. 4.4 shows the predictions of big bang nucleosynthesis for light element abundances. The boxes and arrows show the current estimates, which happen to be consistent with the predicted values. Since we know how densities scale as the universe evolves, we can turn the measurement of light element abundance to a measure of baryon density to-

[4]Source : [4]

day. In particular the abundance of primordial deuterium pins down the baryon density extremely accurately to only a few percent of the critical density. Baryons contribute atmost 5% of the critical density. Since the total matter density today is almost certainly larger than this, nucleosynthesis provides a compelling evidence for non-baryonic dark matter.

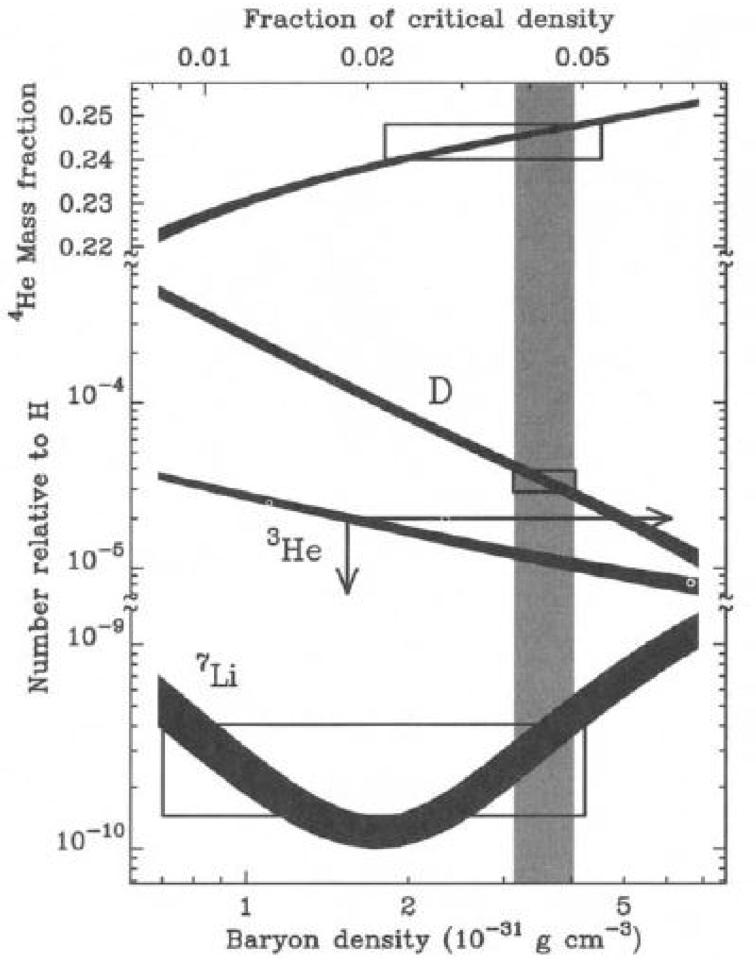

Figure 4.4: Constraints on baryon density from big bang nucleosynthesis[5]

The most plausible candidate for dark matter is a weakly interacting massive particle (WIMP), which was in close contact with the rest of the cosmic plasma at high temperatures, but they experienced freeze-out as the temperature dropped below their mass, and annihilations were no longer capable of maintaining equilibrium. We will try to solve the

[5]Source : [4]

Boltzmann equation for such a particle and determine the epoch of freeze - out and their relic abundance. By fixing the relic abundance to Ω_{dm}, we can learn about their mass and cross-section. We can then use this knowledge to detect such particles in the laboratory.

In the generic scenario two WIMP's (X) can annihilate into two light particles l. The light particles are assumed to be tightly coupled to the cosmic plasma, thus, $n_l = n_l^0$. Using the Boltzmann equation, one can thus write -

$$a^{-3}\frac{dn_X}{dt} = \langle\sigma v\rangle\left[(n_X^0)^2 - n_X^2\right] \tag{4.19}$$

Using the fact that the temperature scales as a^{-1} and defining -

$$Y = \frac{n_X}{T^3} \tag{4.20}$$

eq. 4.19 can be written as -

$$\frac{dY}{dt} = T^3\langle\sigma v\rangle\left[Y_{EQ}^2 - Y^2\right] \tag{4.21}$$

with $Y_{EQ} = n_X^0/T^3$. As before, we introduce the new time variable -

$$x = \frac{m}{T} \tag{4.22}$$

where m is the mass of the WIMP. Dark matter production typically occurs deep in the radiation era, thus using the results of section 3.4, eq. 4.21 can be written as -

$$\frac{dY}{dx} = -\frac{\lambda}{x^2}\left[Y_{EQ}^2 - Y^2\right] \tag{4.23}$$

where the ratio of annihilation rate to expansion rate is parametrized by -

$$\lambda = \frac{m^3\langle\sigma v\rangle}{H(m)} \tag{4.24}$$

which we will assume to be a constant.

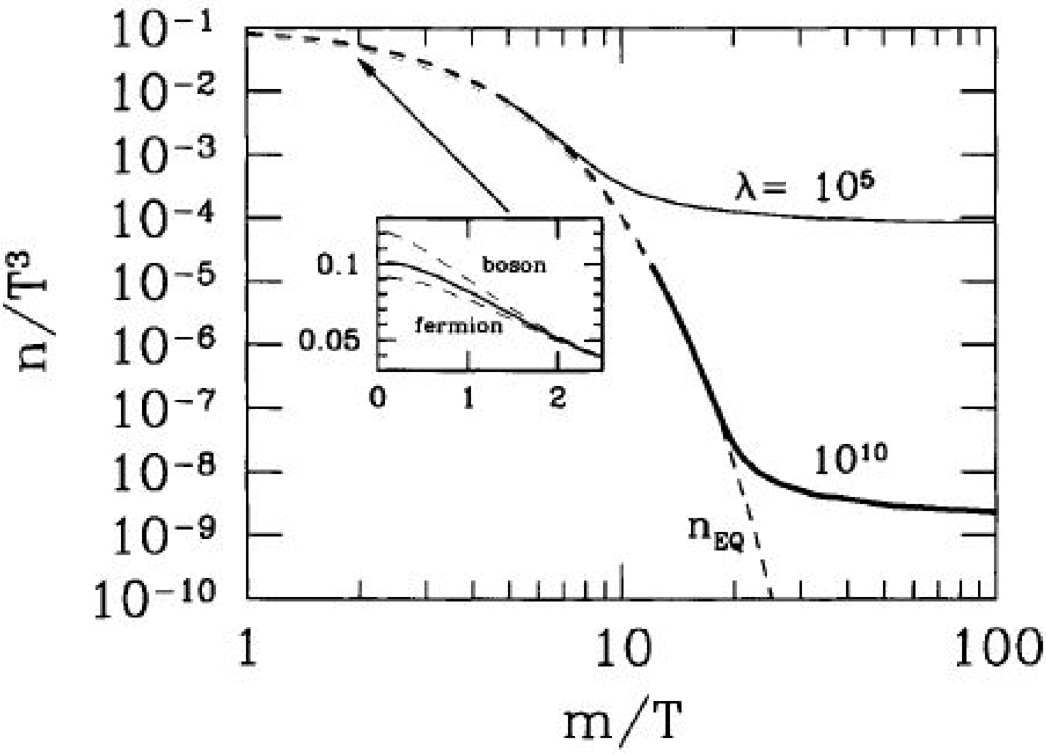

Figure 4.5: Abundance of heavy stable particle, as temperature drops below it's mass.[6]

Well after freeze-out, Y will be much larger than Y_{EQ} : the X particles will not be able to annihilate fast enough to maintain equilibrium. Thus at late times -

$$\frac{dY}{dx} \approx -\frac{\lambda}{x^2} Y^2 \quad (x >> 1) \tag{4.25}$$

Integrating from the epoch of freeze-out (x_f) until very late times ($x = \infty$), we get -

$$\frac{1}{Y_\infty} - \frac{1}{Y_f} = \frac{\lambda}{x_f} \tag{4.26}$$

[6]Source : [4]

Typically Y_f is much greater than Y_∞, so a simple analytic approximation is -

$$Y_\infty \approx \frac{x_f}{\lambda} \tag{4.27}$$

A simple order of magnitude estimate for the dark matter problem is $x_f \approx 10$. Figure. 4.5 shows the numerical solution to eq. 4.23, for different values of λ. The rough estimate $Y_\infty \approx 10/\lambda$, is seen to be a good approximation for the relic abundance. One should also note from figure. 4.5, that the distinction between Bose-Einstein, Fermi-Dirac and Boltzmann statistics is relevant only for temperatures above the mass of the WIMP. For temperatures relevant to freeze-out, Boltzmann distribution is a good approximation. After freeze-out the heavy particle density simply falls off as a^{-3}. So it's energy density today is $m(a_1^3/a_0^3)$ times the number density. Here a_1 corresponds to times when Y has reached it's asymptotic value Y_∞. The number density at that time is $Y_\infty T_1^3$. So -

$$\rho_X = mY_\infty T_0^3 \left(\frac{a_1 T_1}{a_0 T_0}\right)^3 \approx \frac{mY_\infty T_0^3}{30} \tag{4.28}$$

where $(a_1T_1/a_0T_0)^3 \approx 1/30$, follows from an analysis for g_{*s} , similar to that of section. 4.2. To find the fraction of the critical density contributed by X today, we insert the expressions for Y_∞ and ρ_c -

$$\Omega_X = \frac{x_f}{\lambda}\frac{mT_0^3}{30\rho_c} = \frac{H(m)x_f T_0^3}{30m^2\rho_c\langle\sigma v\rangle} \tag{4.29}$$

So to find the present density of these heavy particles, we need to know the Hubble rate when the temperature was equal to X mass (H(m)), for which we need the energy density when the temperature was equal to m. The energy density in the radiation era is given by eq. 3.8. Therefore -

$$\Omega_X = \left[\frac{4\pi^2 G g_*(m)}{45}\right]^{1/2} \frac{x_f T_0^3}{30m^2\rho_c\langle\sigma v\rangle} \tag{4.30}$$

which implies that it is the cross-section that determines the relic abundance rather than the mass, which is still there but implicitly. At the temperatures of interest for dark matter production ($T \approx 100GeV$), $g_*(m)$ includes all particles of the standard model and is of the order 100. Normalizing g_* and x_f by their nominal values, we get -

$$\Omega_X = 0.3h^{-1}\left(\frac{x_f}{10}\right)\left(\frac{g_{*s}(m)}{100}\right)^{1/2}\frac{10^{-39}cm^2}{\langle\sigma v\rangle} \tag{4.31}$$

The fact that this estimate is of order unity for cross-section of order $10^{-39}cm^2$ is a good sign : there are several theories that predict particles with cross-sections this small. It is easy to see that the standard model of particles contains no suitable candidates. The Z^0 and H^0 are neutral and massive, but have lifetimes of only a fraction of a second and hence are not viable candidates for structure formation. The only known stable baryon is the proton which is not invisible. The neutron if free decays within 10 minutes and if bound to the nucleus it counts as visible matter. The three known neutrinos have masses below 1 eV, which makes them hot at the time of structure formation, something that is disfavored by observations

The most notable of the theories for dark matter is supersymmetry, the theory predicts that every particle has a partner with opposite statistics. Also, supersymmetry is broken and these partners are massive with masses greater than 10 - 100 GeV. Now these particles must be neutral and stable. The first condition requires it to be a partner of a neutral particle like the Higgs or photon. The second condition requires them to be the lightest (LSP for Lightest Supersymmetric Particle).

Among alternate models can be mentioned those with universal extra dimensions Kaluza-Klein (KK) models. The lightest electrically neutral particle in this model is a U(1) boson. An analysis of the present LHC limit gives a limit > 600 -700 GeV , for the mass scale of these models. One may even construct a minimilastic model of dark matter by extending the Standard Model with one extra singlet or doublet Higgs. Although this model is still viable, the currently allowed parameter space is on the verge of being greatly reduced with the next generation of experiments. All the discussed models fall under the category of WIMP. Of course, there are candidates that have other motivations from particle physics, we will not be discussing them here.

Experimentally there are atleast three ways to decipher the mystery of dark matter. These are direct detection, indirect detection and production at colliders. Direct detection's are commonly carried out in underground laboratories, they rely on signals from interaction of dark matter with ordinary matter. Indirect detection is often carried out in outer space where one searches for relic signatures from dark matter annihilations during thermal freeze-out in the early universe. Collider searches such as that at LHC, looks for production of dark matter from standard model particles. Figure. 4.6 shows the constraints on the available parameter space of existing WIMP dark matter models, from various experiments. For a detailed discussion see [6].

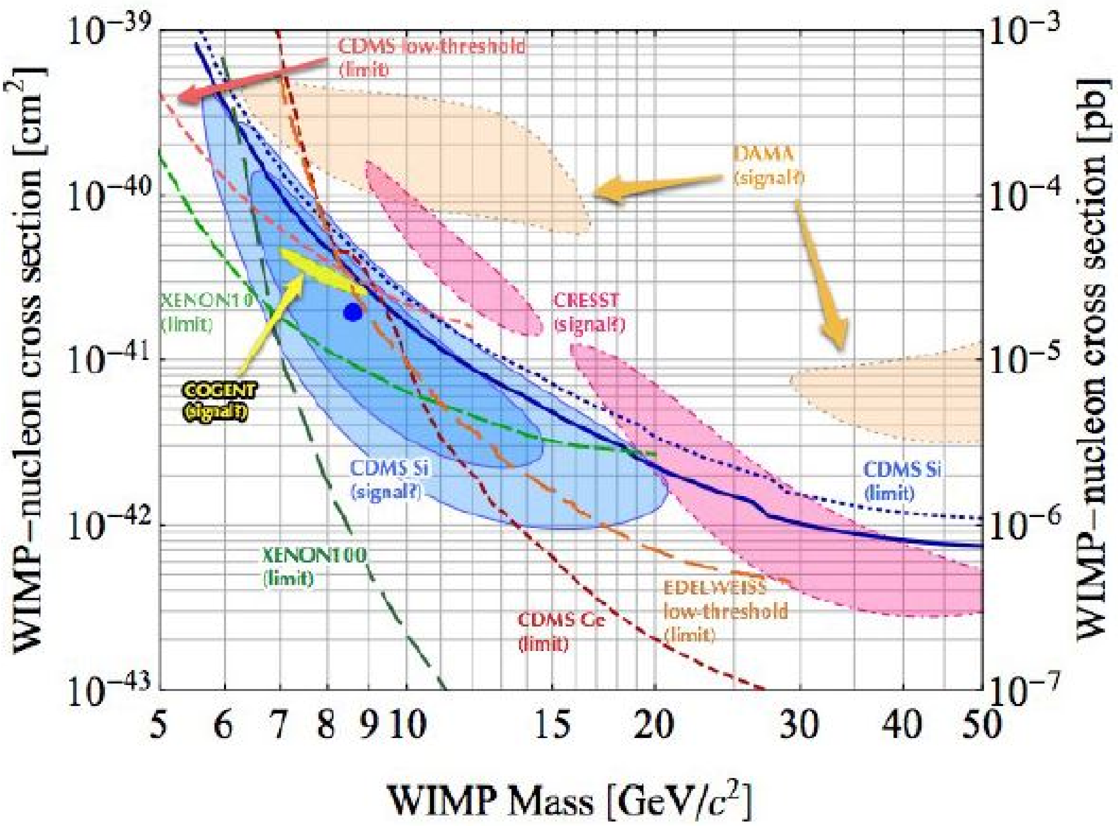

Figure 4.6: Experimental upper limits for the WIMP nucleon cross-section as a function of WIMP mass.[8]

[8]Source : [6]

Chapter 5

Summary

So we are at the end of the thesis, lets take a final look back and ask, how far have we come from where we began? We started out by highlighting some of the cornerstones of modern cosmology, thus motivating what lay ahead. We then tried to describe the simplest model of the universe in an even simpler way. Our cherished concepts from statistical mechanics were adopted to apply to our universe at large. We stood witness to the marriage between particle physics and the ideas we developed in cosmology. This in turn led to the idea of decoupling and thermal relics, which stands as one of the strongest evidence in support of big bang cosmology. Decouplings are perhaps the most important events in the thermal history of our universe. To develop a better understanding of these events, we explored some of them from first principles. Along the way, we motivated the need for non-baryonic dark matter in the universe, and discussed the decoupling of one of the most plausible dark matter candidate called WIMP. We have definitely come a long way $\sim$ 379,000 years upto the time of recombination. But we have certainly not done justice to it, trying to capture it in some 40 pages. There is still a long road in spacetime to cover and a lot more to explore, but let's leave that for another thesis.

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