Classical and semi-classical aspects of black holes

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ABSTRACT

Over the last few decades, there has been an exponential growth in the search for a quantum theory of gravity. Gravity, due to it's geometric nature, plays a very important role in a 'theory of everything' and it has not yet been incorporated in the standard model of particle physics. Black holes, one of the most simple objects and one of the earliest known solutions of the Einstein's field equations are at the frontier of this research. They provide a unique setting where quantum field theory meets classical gravity, leading to interesting phenomena. An understanding of such environments is a must for grasping the elusive quantum theory of gravity. The aim of this thesis is to investigate and understand certain aspects of black hole physics which lie at the boundary between these two fundamental theories.

We begin by considering the properties of spacetime around black holes. In particular, we focus on the Schwarzschild and the Kerr solutions so as to develop the core ideas of black holes. The propagation of particles and photons in the Schwarzschild metric is discussed in some detail, and it is followed by discussion of the three, classic experimental tests of general relativity. Then we discuss the behavior of static scalar fields around the Schwarzschild black hole, followed by discussion of Penrose process and superradiance around a Kerr black hole. We then move to the semi-classical or quantum aspects of black holes in the limit that these fields do not contribute to the curvature of spacetime. We discuss the Bogolyubov transformations and their implications. We discuss in detail the phenomena of vacuum polarization and particle production in a time dependent metric (such as the FRW universe). In the end, we discuss the thermal effects that arise due to inequivalent quantization in flat spacetime (viz. the Unruh effect) followed by a discussion of the same phenomenon in a non-rotating black hole spacetime.

CERTIFICATE

This is to certify that the Thesis entitled, **Classical and semi-classical aspects of black holes** and submitted by **Atul Chhotray**, ID No. **2005B5A3453** in partial requirement of BITS C421/422T Thesis embodies the work done by him under my supervision.

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Chapter 1

Black holes

1.1 Introduction

The term **black hole** was coined by the physicist John Archibald Wheeler in 1967 [15]. General relativity defines a black hole as a region of spacetime where due to extremely strong gravity, the curvature of spacetime becomes so great that not even light can escape that region. As per the postulates of relativity, since nothing can travel faster than light in vacuum, no massive object can escape such a region of spacetime either.

The existence of such objects was conceived as far back as the 18th century by John Michel and Pierre Simon Laplace by crude calculations based on Newtonian mechanics [15]. However Newtonian gravity, as we know is an insufficient and approximate way to describe natural phenomena and it fails to describe the physics in regimes of high velocities and strong gravity which persist in the vicinity of these objects. These objects are exact solutions of Einstein's field equations of general relativity. They occupy a special place as only a handful of the exact solutions to these equations are currently known. These equations once solved provide us with the metric which embodies in itself the geometry of spacetime around the black hole.

Black holes are one of the most simplest objects found in nature. They are completely described by just three parameters, mass (M), charge (Q) and angular momentum (J). Black holes are characterized by the metric or the distortion in spacetime they produce which have been named after their discoverers. The various kind of solutions of the field equations describing black hole spacetimes are-

1. Schwarzschild solution - It describes spacetime in the vicinity of a non rotating $(J = 0)$ and uncharged black hole $(Q = 0)$. It is characterized by just one single parameter- mass (M) of the black hole. The corresponding black hole is known as the Schwarzschild black hole.

2. Kerr solution - It describes the spacetime around an uncharged $(Q = 0)$, rotating black hole $(J \neq 0)$. Hence it is characterized by two parameters- mass and angular momentum. The black hole producing this spacetime is known as a Kerr black hole.

3. Reissner-Nordstrom solution - It describes the spacetime geometry surrounding a charged $(Q \neq 0)$ and non rotating black hole $(J = 0)$. The black hole is characterized by two parameters- the mass and the charge on the hole and the black hole is termed Reissner- Nordstrom black hole.

4. Kerr-Newman solution - It is the most general black hole solution of the Einstein's field equations. It describes a black hole which is both charged and rotating.

In this thesis, we shall be focusing on certain classical and semi-classical aspects of the Schwarzschild and the Kerr black holes. In the remainder of this chapter we highlight the essential properties of these black holes.

1.2 The Schwarzschild black hole

The spacetime around a static and spherically symmetric mass, discovered by Karl Schwarzschild, is one of the earliest source free exact solutions of the Einstein's equations. The spacetime metric is hence known as the Schwarzschild metric.

1.2.1 Properties of the metric

The spacetime around a spherically symmetric, non rotating and uncharged object of mass M is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(1-2M/r)dt^2 + (1-2M/r)^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{1.1}$$

Note that the metric depends only on the mass M of the central massive object. The components of this diagonal metric can be written as-

$$g_{tt} = -(1-2M/r) \quad , \quad g_{rr} = (1-2M/r)^{-1} \quad , \quad g_{\theta\theta} = r^2 \quad , \quad g_{\phi\phi} = r^2\sin^2\theta \tag{1.2}$$

Let us observe how the metric behaves as $M \to 0$. The spacetime simplifies to

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{1.3}$$

which is the usual flat spacetime. We obtain the above identical metric as $r \to \infty$. Physically this means that if the mass of the black hole tends to zero or if one is very far away from the black hole, the spacetime behaves identically to Minkowskian spacetime.

Recall that in classical mechanics, if the Lagrangian for a particle is independent of a coordinate q, then that particular coordinate is called a cyclic coordinate. Corresponding to these cyclic coordinates we obtain the integrals of motion for the particle. These integrals of motion occur because of symmetric nature of space and time leading to conservation of quantities like as energy, momentum etc [9]. Similarly if the spacetime metric is independent of certain coordinates then corresponding conserved quantities exist here as well (Section 5.1). As can be clearly seen, the metric is independent of the coordinates t and ϕ. In other words, the metric remains invariant under translations along the t and ϕ directions. Thus such symmetries imply the existence of at least two Killing vector fields $t^\alpha = \partial/\partial t$ and $\phi^\alpha = \partial/\partial\phi$. This symmetry also emphasizes that the spacetime metric is static and spherically symmetric. On carefully observing the metric tensor coefficients one can say that the metric component $g_{tt} = -(1-2M/r)$ is ill behaved at $r = 0$ and the component $g_{rr} = (1-2M/r)^{-1}$ is ill behaved at $r = 2M$. So what does this pathological behaviour imply?

On calculating the curvature one finds it to be

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = 48M^2/r^6 \tag{1.4}$$

which is non-zero and finite quantity at the surface $r = 2M$. So the surface $r = 2M$ represents a coordinate or a removable singularity, an artifact of a bad coordinate system. However, the curvature tensor blows up at the point $r = 0$, which implies that a real curvature singularity exists at that point.

A black hole of mass M has a coordinate singularity at $r = 2M$. This surface is also known as the event horizon or the Schwarzschild radius. Any event happening inside this null surface cannot be viewed by an observer outside this surface. Thus it acts as a one way membrane through which things can enter but nothing, not even light and hence no information can escape. Here onwards we will utilize light cones in analyzing the spacetime around a Schwarzschild black hole.

1.2.2 Behaviour of the null geodesics in the Schwarzschild metric

For radial null geodesics, we have (suppressing the θ and ϕ coordinates)-

$$ds^2 = -(1 - 2M/r)dt^2 + (1 - 2M/r)^{-1}dr^2 = 0 \tag{1.5}$$

$$=> \frac{dt}{dr} = \pm\frac{r}{r - 2M} \tag{1.6}$$

$$=> \frac{dt}{dr} = \pm\left(1 + \frac{2M}{r - 2M}\right) \tag{1.7}$$

$$=> t = \pm[r + 2M\, ln\,|(r/2M) - 1|\,] + C \tag{1.8}$$

where C is an integration constant. For $r > 2M$,

$$\frac{dt}{dr} = \left(1 + \frac{2M}{r - 2M}\right) > 1 \tag{1.9}$$

These represent the outgoing null geodesics. Similarly for $r < 2M$,

$$\frac{dt}{dr} < 1 \tag{1.10}$$

So as r decreases, t increases and these reflect the ingoing null geodesics i.e. $t = -(r + ln|r - 2M| + C)$. As we approach the black hole from infinity, the slope (dt/dr) tends to increase i.e. the light cones start closing up and so massive objects have less freedom to move within the cone. At $r = 2M$, the slope $\frac{dt}{dr} = \infty$ i.e. the null geodesics point vertically upwards or parallel to the time axis. Also the g_{tt} and g_{rr} reverse their character at this surface. Beyond this point the slope is is negative implying that the light cones tip over. Massive objects are only allowed to move within the cone and hence are compelled to fall towards the center of the hole.

1.2.3 Tortoise coordinates

We define a new coordinate r^* such that:

$$r^* = r + 2M\, ln\,|\frac{r - 2M}{2M}|, \tag{1.11}$$

$$=> dr^* = dr + \frac{2Mdr}{r - 2M} \tag{1.12}$$

$$t = \pm r^* + C \tag{1.13}$$

In such coordinates the metric can be expressed as(r is a function of r^*)-

$$ds^2 = \left(1 - \frac{2M}{r}\right)(-dt^2 + dr^{*2}) + r^2 d\Omega^2, \tag{1.14}$$

It should be noted that the surface $r = 2M$ has been pushed to infinity by this transformation.

1.2.4 Kruskal-Szekeres coordinates

As shown in section 5.2, future directed or past directed paths lead us to different regions of spacetime. Let us now use both $\tilde{u}$ and $\tilde{v}$ simultaneously and replace t and r. The metric we obtain is

$$ds^2 = \frac{1}{2}\left(1 - \frac{2M}{r}\right)(d\tilde{u}d\tilde{v} + d\tilde{v}d\tilde{u}) + r^2\Omega^2 \tag{1.15}$$

One can write r in terms of $\tilde{u}$ and $\tilde{v}$ as

$$\frac{1}{2}(\tilde{u} - \tilde{v}) = r + 2M\, ln|\frac{r}{2M} - 1| \tag{1.16}$$

But we can see that via the above change of coordinates, we are again pushing $r = 2M$ to $-\infty$, so we require a new set of coordinates to bring this surface back to a finite value. Such an example is,

$$u' = e^{\frac{\tilde{u}}{4M}} \tag{1.17}$$

$$v' = e^{\frac{\tilde{v}}{4M}} \tag{1.18}$$

Using the above equations we obtain a relation between u′,v′ and the Schwarzschild coordinates-

$$u' = \left(\frac{r}{2M} - 1\right)^{1/2} e^{\frac{r+t}{4M}} \tag{1.19}$$

$$v' = \left(\frac{r}{2M} - 1\right)^{1/2} e^{\frac{r-t}{4M}} \tag{1.20}$$

In the coordinate system (u', v', θ, ϕ) the Schwarzschild metric can be written as-

$$ds^2 = -\frac{16G^3M^3}{r}\, e^{\frac{-r}{2GM}}\,(du'dv' + dv'du') + r^2\Omega^2, \tag{1.21}$$

These coordinates are not pathological as the metric components are completely regular everywhere except at the origin. Now let us redefine coordinates u and v as

$$\begin{aligned} u &= \frac{1}{2}(u' - v') && (1.22)\\ &= (\frac{r}{2M} - 1)^{1/2} e^{\frac{r}{4M}} \cosh(\frac{t}{4M}), && (1.23)\\ v &= \frac{1}{2}(u' + v') && (1.24)\\ &= (\frac{r}{2M} - 1)^{1/2} e^{\frac{r}{4M}} \sinh(\frac{t}{4M}), && (1.25) \end{aligned}$$

The metric can be expressed as-

$$ds^2 = \frac{32G^3M^3}{r} e^{-\frac{r}{2M}}(-dv^2 + du^2) + r^2\Omega^2 \tag{1.26}$$

Using (1.23) and (1.25) we can obtain the expression for r,

$$u^2 - v^2 = (\frac{r}{2M} - 1)e^{\frac{r}{2M}} \tag{1.27}$$

The coordinates (u, v, θ, ϕ) are known as Kruskal-Szekeres Coordinates. To study the properties of this metric, as usual we start with the null geodesics. These are given by-

$$ds^2 = 0 = -dv^2 + du^2 \tag{1.28}$$

$$v = \pm u + \text{ constant} \tag{1.29}$$

So these straight lines describe the null geodesics. Further one can see from (1.27) at $r = 2M$, the event horizon is defined by

$$v = \pm u \tag{1.30}$$

which are straight lines passing through the origin with slope ± 1. This also makes it a null surface. To find surfaces where $t =$ constant we take the ratio of (1.25) and (1.23) i.e.

$$\frac{v}{u} = \tanh(\frac{t}{4M}) \tag{1.31}$$

which are straight lines with an intercept of zero on either axes. It is important to note that as $t \to \pm\infty$, these surfaces coincide with the event horizon. To identify surfaces of constant r, we refer to (1.27)

$$u^2 - v^2 = \text{constant} \tag{1.32}$$

- $r > 2M$: Gives us hyperbole to the right of the horizon.

$$u^2 - v^2 > 0 \tag{1.33}$$

- $r < 2M$: Gives us hyperbole above the horizon.

$$u^2 - v^2 < 0 \tag{1.34}$$

These (u, v) coordinates range over all possible values of u and v i.e. $\infty \leqslant u \leqslant \infty$ with the condition $v^2 < u^2 + 1$. The spacetime diagram with u, v as coordinates is called a Kruskal Diagram.

1.3 The Kerr black hole

The exact solution to the field equations describing spacetime around a massive rotating body was discovered by Roy P. Kerr in 1963. This solution also describes the spacetime in the vicinity of a massive rotating object such as a rotating black hole. This solution unlike the Schwarzschild one describes a stationary and axisymmetric black hole [3, 11, 13]. In the next few subsections, we shall discuss the Kerr metric and it's properties. Then we shall investigate the metric using various classes of observers to gain more insight about the metric.

1.3.1 Properties of the metric

The line element can be written in terms of Boyer Lindquist coordinates as-

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\Sigma}{\rho^2} \sin^2\theta \, d\phi^2 - \frac{4Mar\sin^2\theta}{\rho^2} dt d\phi \tag{1.35}$$

where a is the Kerr parameter and is defined as $a = J/M$ where J and M are the angular momentum and mass of the rotating black hole [13]. So as mentioned in the introduction, rotating black holes are characterized by two parameters- Mass (M) and Angular Momentum (J). The quantities $\rho,\ \Delta$ and Σ appearing in the metric are defined as follows-

$$\rho^2 = r^2 + a^2 cos^2\theta \tag{1.36}$$

$$\Delta = r^2 - 2Mr + a^2 \tag{1.37}$$

$$\Sigma = (r^2 + a^2)^2 - \Delta a^2 \sin^2\theta \tag{1.38}$$

The covariant metric tensor components are evaluated to be-

$$g_{tt} = -(1 - \frac{2Mr}{\rho^2})\ ,\ g_{t\phi} = \frac{-2Mar\sin^2\theta}{\rho^2}\ ,\ g_{rr} = \frac{\rho^2}{\Delta}\ ,\ g_{\theta\theta} = \rho^2 \tag{1.39}$$

$$g_{\phi\phi} = \frac{\Sigma}{\rho^2}\sin^2\theta \tag{1.40}$$

The $g_{\phi\phi}$ can also be expressed as -

$$g_{\phi\phi} = r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2} \tag{1.41}$$

The contravariant metric components for the g^{rr} and $g^{\theta\theta}$ are straight forward to determine. To determine the rest one needs to use the equation-

$$g_{\mu\nu}g^{\mu\nu} = I, \tag{1.42}$$

which provides us with-

$$g^{tt} = \frac{g_{\phi\phi}}{g_{tt}g_{\phi\phi} - (g_{t\phi})^2}\ ,\ g^{t\phi} = \frac{-g_{t\phi}}{g_{tt}g_{\phi\phi} - (g_{t\phi})^2}\ ,\ g^{\phi\phi} = \frac{g_{tt}}{g_{tt}g_{\phi\phi} - (g_{t\phi})^2} \tag{1.43}$$

One can observe from the metric that the Kerr spacetime is stationary and axially symmetric. Therefore we can deduce two killing vectors which are associated with such a spacetime

$$\xi_t^\alpha = (1, 0, 0, 0) = t^\alpha$$

and

$$\xi_\phi^\alpha = (0, 0, 0, 1) = \phi^\alpha \tag{1.44}$$

Note that as $a \to 0$, the line element of Kerr geometry reduces to

$$ds^2 = -(1 - 2M/r)dt^2 + (1 - 2M/r)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{1.45}$$

which is the line element of Schwarzschild spacetime. So the Kerr parameter takes into account the rotation and is related to the angular momentum of the black hole and putting a to zero 'turns off' the hole's rotation.

If we keep a fixed and let $M \to 0$, we obtain the Minkowski metric in ellipsoidal coordinates,

$$ds^2 = -dt^2 + \frac{\rho^2}{r^2 + a^2}dr^2 + \rho^2 d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2 \tag{1.46}$$

where

$$x = \sqrt{r^2 + a^2}\sin\theta cos\phi$$

$$y = \sqrt{r^2 + a^2}\sin\theta\sin\phi$$

$$z = r\cos\theta \tag{1.47}$$

1.3.2 Static and zero angular momentum observers (ZAMOs)

Static observers
Let us begin our discussion with Static observers. By definition, these observers have r, θ and ϕ coordinates fixed i.e. they are fixed in space. The four velocity of such observers will be directly proportional to the timelike Killing vector as in

$$u^\mu = \gamma \xi_t^\mu. \tag{1.48}$$

Using $u^\mu u_\mu = -1$ we can write

$$\gamma^2 \xi_t^\mu . \xi_\mu^t = \gamma^2 g_{tt} \xi^t \xi^t = -1 \tag{1.49}$$

where $\gamma^2 = \frac{-1}{g_{tt}\xi^t\xi^t}$.
So wherever g_{tt} is negative or the Killing vector field ξ_t^μ is timelike, there the four velocity of a static observer will be well defined. However g_{tt} is not timelike for all r. It becomes zero and then positive as r decreases. This can be ascertained from the following expressions

$$g_{tt} = \frac{-(\rho^2 - 2Mr)}{\rho^2} = \frac{-(r^2 + a^2 cos^2\theta - 2Mr)}{\rho^2} = 0, \tag{1.50}$$

$$g_{tt} = 0 \Longrightarrow r^2 + a^2 cos^2\theta - 2Mr = 0. \tag{1.51}$$

The roots of (1.51) are -

$$r_{SL\pm} = M \pm \sqrt{M^2 - a^2 cos^2\theta} \tag{1.52}$$

where the subscript SL denotes the static limit. So basically the four velocity of a static observer is no longer timelike after this value of the radial coordinate, meaning that no observer can remain static inside $r_{SL\pm}$. This surface where $g_{tt} = 0$ is known as the ergosurface (ergosphere is some texts) and also the Killing Horizon as the timelike Killing vector field becomes zero here.
ZAMOs
Zero angular momentum observers are those class of observers whose angular momentum as measured by the rotating black hole is zero. We can utilize the properties of Killing vectors to define conserved quantities such as the energy and angular momentum. These can be calculated as

$$\tilde{E} = -p_\mu \xi_t^\mu \tag{1.53}$$

$$\tilde{L} = p_\mu \xi_\phi^\mu \tag{1.54}$$

where the tilde denotes quantities per unit mass. Using the above equations we obtain,

$$\tilde{L} = p_\mu \xi_\phi^\mu = 0 \tag{1.55}$$

$$p_\mu \xi_\phi^\mu = 0 = g_{\mu\nu} p^\nu \xi_\phi^\mu \tag{1.56}$$

$$g_{\phi\nu} p^\nu = 0 \tag{1.57}$$

$$g_{\phi t} p^t + g_{\phi\phi} p^\phi = 0 \tag{1.58}$$

This gives us-

$$\frac{u^\phi}{u^t} = \frac{-g_{t\phi}}{g_{\phi\phi}} = \omega = \frac{2Mar}{\Sigma} \tag{1.59}$$

The angular velocity of the zero angular momentum observer is given by ω and is in the direction of the black hole's own rotation (it is proportional to a). As this is true for any observer in any frame, so this effect can be attributed to the Kerr spacetime. This is also known as the dragging of inertial frames. Thus an observer who initially had zero angular momentum with respect to an observer far away would gain some as it approaches a rotating massive object because of the fact that the massive object drags the spacetime with it as it rotates.

1.3.3 Stationary observers and the event horizon

Consider observers moving in the Kerr spacetime with arbitrary angular velocity Ω with respect to an observer at infinity. Such an observer is fixed at r, θ coordinates and according to the observer, as he moves the metric is unchanging or constant, hence locally the metric is stationary. With respect to an observer at infinity,

$$\Omega = \frac{d\phi}{dt} = \frac{d\phi/d\tau}{dt/d\tau} = \frac{u^\phi}{u^t} \tag{1.60}$$

The four velocity of such an observer can be expressed as,

$$u^\mu = \gamma(\xi_t^\mu + \Omega\xi_\phi^\mu) \tag{1.61}$$

Using $u^\mu u_\mu = -1$ we can write,

$$g_{\mu\nu}\gamma^2(\xi_t^\mu + \Omega\xi_\phi^\mu)(\xi_t^\nu + \Omega\xi_\phi^\nu) = -1 \tag{1.62}$$

$$\gamma^2(g_{tt} + \Omega^2 g_{\phi\phi} + 2g_{t\phi}\Omega) = -1 \tag{1.63}$$

Thus, we arrive at

$$\gamma^2 = \frac{-1}{g_{\phi\phi}(\Omega^2 - 2\Omega\omega + g_{tt}/g_{\phi\phi})}, \tag{1.64}$$

If the denominator of (1.64) were to change sign then the four velocities of the stationary observers will cease to be timelike. Let us examine the quadratic denominator of (1.64), as it's behaviour is essential to determine the nature of the four velocity ($g_{\phi\phi}$ here is always positive). The roots of the equation are-

$$\Omega_\pm = \omega \pm \sqrt{\omega^2 - \frac{g_{tt}}{g_{\phi\phi}}} \tag{1.65}$$

For Ω lying between Ω_- and Ω_+, γ^2 becomes positive i.e. the four velocity becomes spacelike. At the static limit, $g_{tt} = 0$ and so the values of Ω are-

$$\Omega^2 - 2\Omega\omega = 0. \tag{1.66}$$

$$\Omega_+ = 2\omega = \frac{4Mar}{\Sigma} \tag{1.67}$$

$$\Omega_- = 0 \tag{1.68}$$

$$\tag{1.69}$$

This confirms that an observer cannot remain at rest at the static limit.

One can write the roots $\Omega_\pm$ in an alternate form using the relation $g_{tt}g_{\phi\phi} - g_{t\phi}^2 = \Sigma \sin^2\theta$. Hence the roots are-

$$\Omega_\pm = \omega \pm \frac{\rho^2\sqrt{\Delta}}{\Sigma} \tag{1.70}$$

So $\Omega_\pm = \omega$ for $\Delta^{1/2} = 0$, i.e.

$$\Delta^{1/2} = 0 = r^2 - 2Mr + a^2 \tag{1.71}$$

The roots of the above quadratic are-

$$r_{H\pm} = M \pm \sqrt{M^2 - a^2} \tag{1.72}$$

where the subscript H denotes the horizon. In general, the arbitrary angular velocity of the observer Ω can be expressed as-

$$\Omega_\pm = \frac{2Mar}{\Sigma} \pm \frac{\sqrt{1 - 2M/r + a^2/r^2})}{r[(1 + a^2/r^2)^2 - (1/r^2 - 2Mr/r^4 + a^2/r^4)]} \frac{(1 + \frac{a^2 cos^2\theta}{r^2})}{a^2 \sin\theta}. \tag{1.73}$$

Note that as $r \to \infty$,

$$r\Omega_\pm = \pm 1. \tag{1.74}$$

From (1.64)

$$\Omega_- = \omega \pm \sqrt{w - \frac{g_{tt}}{g_{\phi\phi}}} \tag{1.75}$$

For large r, g_{tt} is timelike and hence negative and so Ω_- is negative. But at r_{SL}, $\Omega_- = 0$. So as r decreases, Ω_- increases from negative to zero and keeps on increasing till the horizon is reached. Ω_+ behaves in the exact opposite fashion i.e it decreases as r decreases (though it never crosses the zero value). The two angular velocities coincide at $r_{H\pm}$ which implies-

$$\Omega_+ = \Omega_- = \omega, \tag{1.76}$$

at

$$r_{H\pm} = M \pm \sqrt{M^2 - a^2}. \tag{1.77}$$

It can be clearly seen that as r decreases the range of values that Ω can assume to make the observer's four velocity timelike decreases. At $r = r_{H\pm}$, there is only one possible value of $\Omega_\pm$ that is allowed by the Kerr geometry i.e.

$$\Omega_\pm = \omega \pm \sqrt{\omega^2 - \frac{g_{tt}}{g_{\phi\phi}}}, \tag{1.78}$$

$$= \omega \pm \frac{\rho^2\sqrt{\Delta}}{\Sigma}\Sigma. \tag{1.79}$$

$\Delta = 0$ at $r = r_{H\pm}$ and so,

$$\Omega_\pm = \omega = \Omega_H, \tag{1.80}$$

$$\Omega_H = -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{2Mar_H}{\Sigma} = \frac{a}{{r_H}^2 + a^2}. \tag{1.81}$$

Hence only those observers whose angular velocity is equal to Ω_H can exist at r_H and all other observers are forbidden. Also, the dot product of the observer's four velocities becomes null at

$r = r_{H\pm}$. So $r_{H\pm}$ denote the event horizons of the Kerr Black hole. Notice that there are two horizons here, an inner and an outer one. Further a horizon should be a null, stationary surface, the normal to any such surface must be directly proportional to $\partial_\alpha r$. This normal would be null if

$$\partial_\alpha r \partial^\alpha r = 0 \tag{1.82}$$

$$g^{\alpha\beta} \partial_\alpha r \partial_\beta r = 0 \tag{1.83}$$

$$g^{rr} = 0 = \frac{\Delta}{\rho^2} \tag{1.84}$$

The above solutions for $\Delta = 0$ are valid only for $M \geq a$. When the equality holds the black holes belong to a special class of black holes called extremal black holes. For $M < a$, the singularity is not cloaked by the event horizons, which leads to a naked singularity. Such singularities are abhorred and existence of such objects are forbidden by the cosmic censorship hypothesis proposed by Roger Penrose [15].
To get a physical picture, let us reconsider all the surfaces of importance. Moving in from infinity towards the hole we first encounter the static limit or the outer ergosurface/ergosphere denoted by r_{SL+}. As we continue to decrease r we shall impinge upon the outer event horizon (r_{H+}). Then we encounter the inner horizon (r_{H-}) and finally the inner static limit (r_{SL-}).

$$r_{SL+} > r_{H+} > r_{H-} > r_{SL-} \tag{1.85}$$

In the upcoming chapters we shall discuss the classical and semi-classical aspects of black hole physics. We will study the propagation of particles and waves in the Schwarzschild and Kerr spacetimes, some tests of the general theory of relativity, quantum field theory in curved spacetime and some of it's implications such as inequivalent quantization and particle production. Now we shall proceed to classical aspects of black holes.

Chapter 2

Classical aspects of black holes

This chapter discusses the classical aspects of black hole physics. It focuses on the behaviour of classical particles and fields in black hole spacetimes. We will investigate the behaviour of particles around the Schwarzschild black hole using the idea of effective potential. In the succeeding sections we will discuss three experimental tests of general relativity and specific observations that set general relativity apart from Newtonian gravity. Towards the end we will discuss two processes namely, Penrose process and superradiance that are unique to the Kerr metric.

2.1 Propagation of particles around the Schwarzschild black hole

In this section we will discuss how test particles respond to a gravitational field produced by a static, spherically symmetric object. The spacetime surrounding such an object is of the Schwarzschild spacetime. Consider a particle propagating in such a spacetime. As the spacetime is spherically symmetric we can always orient the orbit of the particle so that $\theta = \frac{\pi}{2}$ is fixed and thus-

$$d\theta = 0 \tag{2.1}$$

From the Schwarzschild metric one can see that the metric is independent of t and ϕ. Thus t and ϕ are cyclic coordinates and they correspond to two conserved quantities- E and L.

$$p_t = -E \qquad \text{and} \qquad p_\phi = \pm L \tag{2.2}$$

We know that

$$p^\mu p_\mu = -m^2 \tag{2.3}$$

$$g^{00}(p_0)^2 + g^{11}(p_r)^2 + g^{33}(p_\phi)^2 = -m^2 \tag{2.4}$$

The above expression can be rewritten as-

$$-\frac{E^2}{(1-\frac{2M}{r})} + \left(1-\frac{2M}{r}\right)(\frac{dr}{d\lambda})^2 + \frac{L^2}{r^2} + m^2 = 0 \tag{2.5}$$

Let us take $\tilde{E} = E/m$, $\tilde{L} = L/m$ and $\lambda = \tau/m$, we obtain

$$(\frac{dr}{d\tau})^2 = \tilde{E}^2 - \left(1-\frac{2M}{r}\right)(1+\frac{\tilde{L}^2}{r^2}) \tag{2.6}$$

$$= \tilde{E}^2 - \tilde{V}^2(r), . \tag{2.7}$$

Particle orbits

Here we discuss the orbits of particles moving in the vicinity of a Schwarzschild black hole. We will discuss the similarities and differences between orbits of particles in Newtonian gravity and general relativity. We begin with the effective potential obtained in (2.7),

$$\tilde{V}_{eff}^2 = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right) \tag{2.8}$$

Consider a non relativistic particle of unit mass orbiting around an object of mass M producing a static, spherically symmetric gravitational field. The total energy of such a particle is

$$E = \frac{v_r^2}{2} - \frac{M}{r} + \frac{v_\phi^2}{2}. \tag{2.9}$$

The non relativistic version of the effective potential for an orbiting particle can be obtained from the above expression by combining the terms corresponding to the potential energy and the angular kinetic energy. Using $L = v_\phi r$ as the angular momentum,

$$V_{effective} = -\frac{M}{r} + \frac{L^2}{2r^2}. \tag{2.10}$$

If the particle follows a circular orbit then

$$\frac{mv_\phi^2}{r} = \frac{GMm}{r^2}, \tag{2.11}$$

$$=> L^2 = GMm/r \tag{2.12}$$

$$L \propto \sqrt{r} \tag{2.13}$$

This means that for any given angular momentum a particle can have a circular path around the central object. Now, let us continue the analysis of the general relativistic potential. We can also write V_{eff} as-

$$1 - \frac{2M}{r} + \frac{L^2}{r^2} - \frac{2ML^2}{r^3} \tag{2.14}$$

For obtaining stable circular orbits we differentiate with respect to r

$$\frac{dV_{eff}}{dr} = \frac{2M}{r^2} - \frac{2L^2}{R^3} + \frac{6ML^2}{r^4} = 0 \tag{2.15}$$

$$=> 2Mr^2 - 2L^2r + 6ML^2 = 0 \tag{2.16}$$

To solve for real values of r, the discriminant of the above quadratic must be non negative

$$D = 4L^4 - 48M^2L^2 \geq 0 \tag{2.17}$$

$$\Rightarrow L \geq 2\sqrt{3}M \tag{2.18}$$

So stable circular orbits are only possible for angular momenta greater than $2\sqrt{3}M$ which corresponds to a radial coordinate $r = 6M$. This is also known as the radius of the innermost stable circular orbit or r_{ISCO}.

Let us now discuss the radial motion of a particle in the Schwarzschild spacetime. For such a particle-

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \tag{2.19}$$

As t is a cyclic coordinate, energy is conserved and as obtained as in [4] we arrive at

$$\left(1 - \frac{2M}{r}\right) \dot{t} = k, \tag{2.20}$$

where dot denotes differentiation with respect to an affine parameter. For the case where dot denotes the proper time for the observer, we can write

$$\left(1 - \frac{2M}{r}\right) \dot{t}^2 - (1 - \frac{2M}{r})^{-1}\dot{r}^2 = 1, . \tag{2.21}$$

For a particle dropped with zero initial velocity $(\dot{r} = 0)$ the above equation, as $r \to \infty$ gives

$$\left(1 - \frac{2M}{r}\right) \dot{t}^2 = k = 1 \Rightarrow \dot{t}^2 = 1 \tag{2.22}$$

In other words this means that the energy of the particle equals it's rest mass energy. We arrive at

$$\frac{1}{(1 - \frac{2M}{r})} - \frac{\dot{r}^2}{(1 - \frac{2M}{r})} = 1 \tag{2.23}$$

$$\Rightarrow (\frac{d\tau}{dr})^2 = \frac{r}{2M} \tag{2.24}$$

$$\int \tau d\tau = -\int \frac{r^{1/2}}{(2M)^{1/2}} dr \tag{2.25}$$

where the negative sign has been chosen as the particle falls into the black hole. This gives us-

$$\tau - \tau_0 = \frac{2}{3} \left(\frac{r_0^{3/2} - r^{3/2}}{\sqrt{2M}} \right), \tag{2.26}$$

The above result is identical to the classical result and we observe no singular behaviour as the particle goes into the hole with a finite proper time. Now let calculate the coordinate time taken for such a particle. From (2.21),

$$\frac{dt}{dr} = \frac{\dot{t}}{\dot{r}} = -\sqrt{\frac{r}{2M}} \frac{1}{(1 - \frac{2M}{r})} \tag{2.27}$$

So we have an integral of the type,

$$\int dt = -\frac{1}{\sqrt{2M}} \int \frac{r^{3/2}}{r - 2M} dr \tag{2.28}$$

The integral can be solved by a series of substitutions in the order, $r^3 = s^2$ and then $s^{2/3} = y$. Finally substituting for r we obtain,

$$t - t_0 = -\frac{2}{3\sqrt{2M}}[r^{3/2} - r_0^{3/2} + 6M(r^{1/2} - r_0^{1/2})] + 2M \ln \frac{[r^{1/2} + (2M)^{1/2}][r_0^{1/2} - (2M)^{1/2}]}{[r^{1/2} - (2M)^{1/2}][r_0^{1/2} + (2M)^{1/2}]} \quad (2.29)$$

From the above expression it can be seen that in the frame of an observer located far away from the black hole, as $r \to 2M$ the logarithmic term blows up i.e.

$$t - t_0 \to \infty \quad (2.30)$$

where as the proper time for the observer for that journey will be finite as can be seen from (2.26). Thus it takes forever for a particle to fall into the black hole with respect to an observer watching from a large distance but the proper time taken by the particle itself is finite. From these results we can conclude that the Schwarzschild coordinates are pathological in nature and it is prudent to work with other coordinate systems.

2.2 Experimental tests of general relativity

Any physical theory, irrespective of it's elegant and mathematical beauty must be subjected to the verification via experiments before it is accepted as a theory which explains the workings of the universe. General relativity has been put through various experimental tests over the past century and has come out with flying colours. We will discuss a few of these experiments which have confirmed that the theory provides the correct description of our universe.

2.2.1 Precession of the perihelion of Mercury

Mercury, the closest planet to the sun displays strange behaviour. An anomaly was found in it's orbit i.e. the perihelion of the orbit was observed to precess, an observation that Newtonian gravity failed to predict. The solution to this puzzle was provided by general relativity. First we would briefly touch upon the Kepler problem in celestial mechanics. Considering a test particle in a gravitational field, we are aware that the energy and angular momentum of such a particle will be conserved. In the planar polar coordinates R, ϕ using these quantities we can obtain the orbital equation for the particle as-

$$\frac{d^2u}{d\phi^2} + u = \frac{\mu}{h^2}, \quad (2.31)$$

where $u = \frac{1}{r}$, $h = \frac{L}{m} = r^2\dot{\phi}$ and m is the mass of the test particle. Here L denotes the angular momentum and μ is a constant denoting the product of mass of the central object and the gravitational constant. This equation is also known as Binet's equation and the solution to this are-

$$u = \frac{\mu}{h^2} + A\cos(\phi - \phi_0) \quad (2.32)$$

$$\frac{l}{R} = 1 + e\cos(\phi - \phi_0) \quad (2.33)$$

where A, ϕ_0 are constants and $l = h^2/\mu$ and $e = Ah^2/\mu$. Binet's equation describes the planar motion as a function of ϕ in terms of conic section. Now let us begin with the general relativistic treatment of the Kepler problem. Consider test particles in the Schwarzschild spacetime i.e a spherically symmetric gravitational field. Using the variational methods as given in [4] we can write-

$$(\frac{ds}{d\tau})^2 = -(1-2M/r)\dot{t}^2 + (1-2M/r)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2 = 2K = 1, \tag{2.34}$$

$$\frac{d}{d\tau}[-(1-2M/r)\dot{t}] = 0 \tag{2.35}$$

$$\frac{d}{d\tau}(r^2\dot{\theta}) - r^2\sin\theta cos\theta\dot{\phi}^2 = 0 \tag{2.36}$$

$$\frac{d}{d\tau}(r^2\sin^2\theta\dot{\phi}) = 0 \tag{2.37}$$

where dot denotes differentiation with respect to proper time. For planar motion we take $\theta = \pi/2$ and so $\dot{\theta} = 0$ giving us

$$r^2\dot{\phi} = h, \tag{2.38}$$

We also have,

$$\left(1 - \frac{2M}{r}\right)\dot{t} = k = \text{constant} \tag{2.39}$$

Setting $u = 1/r$, and using (2.34) and (2.38) we can obtain the following-

$$(\frac{du}{d\phi})^2 + (u)^2 = \frac{k^2-1}{h^2} + \frac{2M}{h^2}u + 2Mu^3. \tag{2.40}$$

Differentiating with respect to ϕ we get

$$\frac{d^2u}{d\phi^2} + u = \frac{M}{h^2} + 3Mu^2,. \tag{2.41}$$

This is the general relativistic Binet's equation. On comparing this with the classical one (2.31) we note the presence of an extra term $3Mu^2$. The equation can be solved approximately by the use of the perturbation method as in [4] by introduction of parameter

$$\varepsilon = \frac{3M^2}{h^2}. \tag{2.42}$$

Now (2.41) can be rewritten as

$$\frac{d^2u}{d\phi^2} + u = \frac{M}{h^2} + \frac{u^2h^2\varepsilon}{M} \tag{2.43}$$

The general solution to the above equation (till first order) is-

$$u = u_0 + \frac{\varepsilon M}{h^2}[1 + e\phi\sin\phi + e^2(\frac{1}{2} - \frac{\cos 2\phi}{6})] \tag{2.44}$$

where $u_0 = \frac{M}{h^2}(1 + e\cos\phi)$. The $e\phi\sin\phi$ term is an important correction to the u_0 term as it increases with ϕ i.e. after each revolution. Neglecting other corrections we obtain -

$$u = \frac{1}{r} \simeq \frac{M}{h^2}[1 + e\cos\phi(1-\varepsilon)] \tag{2.45}$$

We can clearly see that the above function is periodic with the period

$$\frac{2\pi}{1-\varepsilon} = 2\pi(1-\varepsilon)^{-1} \approx 2\pi + 2\pi\varepsilon \tag{2.46}$$

In general relativity the orbit of the test particle is remains an ellipse but only approximately. The test particle will return to the same value of u (and so r) after an angular displacement of $2\pi(1+\varepsilon)$. The perihelion is the point on the orbit where the orbiting body is closest to the central object which occurs at a specific value of u i.e. u_{Peri}. This value shifts with the angular coordinate and thus though the orbit remains elliptical but the ellipse itself rotates in the ϕ direction between the points of closest approach, thus the name precession of perihelion. The magnitude of this precession can be computed as

$$2\pi\varepsilon = 2\pi\frac{3M^2}{h^2} \simeq \frac{24\pi^3 a^2}{c^2T^2(1-e^2)}. \tag{2.47}$$

2.2.2 Bending of light in a gravitational field

General relativity says that the presence of mass curves spacetime and hence any entity traveling across such a curved region of spacetime must deviate from it's straight line path. The bending of light due to a gravitational field was one of the predictions of general relativity which was qualitatively confirmed by Sir Arthur Eddington during a total solar eclipse in 1919 [4].

Here we consider the trajectory of a photon or light in Schwarzschild geometry. Light in general relativity travels on null geodesics

$$(\frac{ds}{d\lambda})^2 = -(1-2M/r)\dot{t}^2 + (1-2M/r)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2 = 0. \tag{2.48}$$

where the dots denote differentiation with respect to an affine parameter say λ. Following a similar process as the last section we obtain the null geodesics in polar coordinates as-

$$\frac{d^2u}{d\phi^2} + u = 3Mu^2, \tag{2.49}$$

Far away from the central object the above equation reduces to-

$$\frac{d^2u}{d\phi^2} + u = 0 \tag{2.50}$$

The above can be solved as-

$$u = \frac{\sin[\phi-\phi_0]}{D} = u_0 \tag{2.51}$$

$$r = \frac{D}{\sin[\phi-\phi_0]} \tag{2.52}$$

where D is a constant. The above is an equation for straight line in polar coordinates and this conforms with the predictions of Newtonian gravity. The solution of the general relativistic version can be treated as a perturbation of the classical version where we treat Mu as small. We are looking for an approximate solution to (2.49) having the form,

$$u = u_0 + 3Mu_1 \tag{2.53}$$

The equation for u_1 is

$$\frac{d^2u_0}{d\phi^2} + u_0 - 3Mu_0{}^2 + 3M\frac{d^2u_1}{d\phi^2} + 3Mu_1 - 27M(Mu_1)^2 = 0 \tag{2.54}$$

Using (2.49) and neglecting terms of the order $(Mu_1)^2$ we obtain-

$$\frac{d^2u_1}{d\phi^2} + u_1 = u_0{}^2 = \frac{\sin^2\phi}{D^2} \tag{2.55}$$

Taking the solution of u_1 of the form-

$$u_1 = \frac{1 + C\cos\phi + \cos^2\phi}{3D^2} \tag{2.56}$$

and substituting in (2.23) we obtain the general solution approximately,

$$u \simeq \frac{\sin\phi}{D} + \frac{M(1 + C\cos\phi + \cos^2\phi)}{D^2} \tag{2.57}$$

For the case where M/D the second term in the last equation seems like a perturbation from the straight line solution. Let us study the case of light passing close to the sun. For the case under study, light approaching the sun gets deflected by some angle say δ and then again resumes it's straight line path. When $r \to \infty$, $u \to 0$ and so the right hand side of the above equation must become zero. The values of ϕ for which light travels in straight lines must give zero. Let ϕ for the ray after deflection be $-\varepsilon_1$ and for the ray approaching the sun be $\pi + \varepsilon_2$. Using small angle approximation, for the approaching ray-

$$-\frac{\varepsilon_2}{D} + \frac{M(2+C)}{D^2} = 0 \tag{2.58}$$

For the receding ray,

$$-\frac{\varepsilon_1}{D} + \frac{M(2-C)}{D^2} = 0 \tag{2.59}$$

The total deflection is-

$$\delta = \varepsilon_1 + \varepsilon_2 = \frac{4M}{D} = \frac{4GM}{c^2 D} \tag{2.60}$$

where we have displayed G and c explicitly.

2.2.3 Gravitational redshift

The gravitational redshift employs the idea that light must lose energy as it escapes from the gravitational field of a massive object such as the sun and the Earth. This loss of energy shows up as the decrease in the frequency of light. An experiment based on the Mossbauer effect performed in 1960 by Pound and Rebeka verified this result of general relativity [4].

Let us imagine an observer located at an arbitrary radial coordinate r in the Schwarzschild geometry. In the frame of that observer if a particle passes by, the energy of the particle is given by-

$$E_{local} = p^{\mu}.\hat{x}_0,, \tag{2.61}$$

where p^μ is the particle's four momentum in the observer's frame and $\hat{x}_0$ is the observer's timelike vector (in a tetrad frame). The above gives-

$$p^\mu \sqrt{g_{00}} x_0 = \sqrt{1 - \frac{2M}{r}} p^0 \tag{2.62}$$

We know that

$$p^0 = g^{00} p_0 = \left(1 - \frac{2M}{r}\right)^{-1} p_0, \tag{2.63}$$

Using (2.61), (2.63) we obtain

$$E_{local} \sqrt{1 - \frac{2M}{r}} = E \tag{2.64}$$

As $r \to \infty$ the quantity E on the right hand side of the previous equation denotes the energy of the particle as seen at infinity.

$$E_{local} \sqrt{g_{00}} = E_\infty \tag{2.65}$$

where E_{local} is the energy of a particle measured by an observer when the particle crosses him. E_∞ is the energy measured by an observer located far away from the massive object producing the spacetime curvature. The above equation can be written as-

$$E_{local} \sqrt{1 - \frac{2M}{r}} = E, \tag{2.66}$$

The r in the above equation is the value of the coordinate at which the observer locally measures the energy of the particle. The previous equation can also be interpreted (for all r) as-

$$E_{local} \geq E \tag{2.67}$$

Let us consider an in falling photon. At infinity,

$$E = E_\infty = \frac{hc}{\lambda_\infty} \tag{2.68}$$

where $lambda_\infty$ denotes the wavelength seen by an observer who is located far away from the central object/black hole. The energy as measured by an observer located at a radial coordinate r is

$$E_{local} = E_r = \frac{hc}{\lambda_r} \tag{2.69}$$

where λ_r denotes the wavelength measured by an observer at radial coordinate r. Using the expression (2.66) we obtain the relations among the wavelengths as

$$\lambda_\infty > \lambda_r. \tag{2.70}$$

From the above equations it is clear that an in falling photon or an outgoing photon will gain or lose energy while interacting with a gravitational field of an object. As can be seen from the equations obtained, the wavelength of the photon will increase or the frequency will decrease towards to the red end of the visible electromagnetic spectrum and hence the name gravitational redshift.

Until now we have studied the behaviour of particles around a Schwarzschild black hole. Now we shall discuss some interesting features of the rotating black hole which are unique to the Kerr geometry.

2.3 Penrose Process

The Penrose process allows one to extract energy from a rotating black hole. It is based on the nature of the time like killing vector field ξ_t^α, which becomes null at the ergosphere and takes on positive values (becomes spacelike) inside the static limit. As a result the energy of the particle can be negative within the ergosphere (with respect to an observer at infinity). Now suppose that the in falling object in the ergosphere splits into two parts such that, one part falls into the hole and the other has gains the necessary velocity to escape the ergosphere and hence come out of the hole. Because of the Penrose process, the outgoing object can have sufficient velocity cum angular momentum such that the energy of the object emitted from the hole can be greater that the energy of the object which initially fell into the hole. As a result due to the absorption of a negative energy object, the mass of the hole decreases and it can also be shown that the angular momentum of hole decreases as well. This introduces the prospect of rotational energy extraction from the hole, which is transported by the outgoing particle. It should be emphasized that this process is still classical and not quantum mechanical in nature.

Let us allow the above particle to fall into the Kerr black hole. As we are already aware, the particle will first encounter the ergosphere's boundary i.e. the static limit. Here the timelike Killing vector field becomes null and it behaves in a spacelike fashion afterwards ($r < r_{SL}$). The static limit occurs when

$$g_{tt} = -(1 - 2Mr/\Sigma) = 0 \tag{2.71}$$

at

$$r = M + \sqrt{M^2 - a^2cos^2\theta} \tag{2.72}$$

Consider a particle having and energy E with respect to an observer at infinity. The energy is defined as-

$$E = p_\mu \xi_t^\alpha \tag{2.73}$$

$$= g_{\mu\nu} p^\nu \xi_t^\mu, xi_t^\mu = d/dt \tag{2.74}$$

For $r \to \infty$ we have $E = p^t = -p_t$. With respect to an observer at infinity, for a particle outside the ergoregion where ξ_t^μ is timelike, the energy is always positive.

Now let us call the particle that enters the black hole ergoregion as particle A. Inside the ergoregion, A disintegrates to give particles B and C. The disintegration happens in such a manner that particle B escapes to infinity whereas C falls into the hole. Applying conservation of momentum immediately before and just after the disintegration process we obtain

$$p_A^\mu = p_B^\mu + p_C^\mu, \tag{2.75}$$

The dot product of (2.75) with the Killing vector field ξ_t^α gives

$$P_A.\xi_t^\alpha = P_B.\xi_t^\alpha + P_C.\xi_t^\alpha \tag{2.76}$$

$$E_A = E_B + E_C \tag{2.77}$$

$$=> E_B = E_A - E_C \tag{2.78}$$

As particle C falls into the hole and the disintegration happens inside the ergosphere, E_C can be negative giving us

$$E_B > E_A \tag{2.79}$$

Therefore, particle B can carry more energy out of the black hole that particle A had brought in. Consider a more generalized Killing vector field, being a linear combination of the two fields discussed in the previous chapter.

$$\xi_K^\nu = \xi_t^\nu + \Omega_H \xi_\phi^\mu \tag{2.80}$$

The above vector field is timelike outside the Event Horizon. So for a particle located just outside the horizon we have,

$$\mathbf{P}.\xi_K^\nu = \mathbf{P}.\xi_t^\nu + \Omega_H \mathbf{P}.\xi_\phi^\mu < 0 \tag{2.81}$$

$$-E + \Omega_H L < 0 \tag{2.82}$$

$$\Omega_H L < E, \tag{2.83}$$

As E is a negative quantity just outside the horizon and Ω_H is positive by definition therefore, L must be negative for the previous equation to be satisfied. So, a particle having negative energy must also possess negative angular momentum. By swallowing such a particle the black hole loses mass as well as angular momentum. Hence,

$$E \propto \delta M \qquad \text{and} \qquad L \propto \delta J \tag{2.84}$$

Using (2.83)

$$\delta M = \Omega_H \delta J \tag{2.85}$$

Using definition of Ω_H from (1.81)

$$\delta M = \frac{a}{(r_H^2 + a^2)} \delta J \tag{2.86}$$

where r_H denotes the radial coordinate of the event horizon.

The existence of entities with negative energy is not against the fundamentals of physics. The principle that all existing particles have positive energy is based upon the assumption that for all r and t, we have a timelike Killing vector field. The Kerr geometry is special case because here unlike in the Schwarzschild black hole, the timelike Killing vector field becomes spacelike outside the horizon making the process of energy extraction viable.

2.4 Behaviour of scalar fields in black hole spacetimes

2.4.1 Static, scalar field in Schwarzschild spacetime

Consider a scalar field produced by a star which collapses to form a Schwarzschild black hole. We can ask what would happen to scalar field after the black hole is formed? Let us try to answer it.

A massless, static scalar field in Schwarzschild spacetime satisfies:

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \psi) = 0 \tag{2.87}$$

The scalar field can be expanded in terms of the spherical harmonics as the spacetime is spherically symmetric [11].

$$\psi = \sum_l \varphi(t, r) Y_{lm}(\theta, \phi) \tag{2.88}$$

For a static field, we arrive at the following radial equation

$$\left(1-\frac{2M}{r}\right)^{-1}\frac{d^2\varphi}{dr^{*2}}+\frac{2}{r}\frac{d\varphi}{dr^*}-\ell(\ell+1)\frac{\varphi}{r^2}=0, \tag{2.89}$$

where r^* are the Tortoise coordinates as defined in (1.11). The acceptable asymptotic solutions to this equation are

$$\varphi \sim r^{*-(\ell+1)} \tag{2.90}$$

as $r^* \to \infty$ and

$$\varphi \sim \text{ constant} \tag{2.91}$$

as $r^* \to -\infty$. More details can be found in [14]. Now for the scalar field to exist we must be able to connect these two asymptotic solutions together. It can be seen that to connect a constant solution at one end with a decreasing solution at the other, the function must have an inflection point $(d^2\varphi/dr^{*2}=0)$ in between. It is clear graphically that in such a case, the function and its first derivative would have opposite signs. But it can be seen from (2.89) that the wave equation does not satisfy this condition.

Hence, from this we conclude that a static, scalar field cannot exist outside a Schwarzschild black hole.

2.4.2 The phenomenon of superradiance

We have previously discussed the Penrose process, a classical process of energy extraction from a rotating black hole. Here we discuss the wave analog of the same, i.e. a wave having a specified energy is incident upon a rotating black hole and the outgoing wave is found to have a greater amplitude than the incident wave. It is thus another way to ferry energy out of a rotating black hole.
Let us attempt to solve the wave equation for a scalar field in Kerr spacetime. We know that the equation satisfied by massless scalar field is-

$$\Box\psi = 0 \tag{2.92}$$

The wave equation in Kerr metric is of the form-

$$\frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial x^\mu}\sqrt{g}g^{\mu\nu}\frac{\partial}{\partial x^\nu}\psi\right]=0 \tag{2.93}$$

Using the fact that $\sqrt{g}=\Sigma\sin\theta$. This can be reduced to the form-

$$\sin\theta\frac{\partial}{\partial r}\left(\Delta\frac{\partial\psi}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\Sigma\sin\theta\left(g^{tt}\frac{\partial^2\psi}{\partial t^2}+g^{\phi\phi}\frac{\partial^2\psi}{\partial\phi^2}+2g^{t\phi}\frac{\partial^2\psi}{\partial t\partial\phi}\right), \tag{2.94}$$

Due to the stationary and axially symmetric nature of Kerr spacetime, the modes describing the classical field can be expanded in terms of Spheroidal harmonics [5].

$$\psi = R(r)S(\theta)e^{-i\omega t+im\phi} \tag{2.95}$$

Substituting the above in (2.94) and after a few lines of algebra one can obtain

$$\frac{\sin\theta}{R(r)}\frac{\partial}{\partial r}\left(\Delta\frac{\partial R(r)}{\partial r}\right)+\frac{1}{S(\theta)}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S(\theta)}{\partial\theta}\right)-\Sigma\sin\theta\left[\omega^2g^{tt}+m^2g^{\phi\phi}-2g^{t\phi}\omega m\right]=0 \tag{2.96}$$

Using $g_{tt}g_{\phi\phi} - (g_{t\phi})^2 = \Delta \sin^2\theta$ and separating the functions $R(r)$ and $S(\theta)$ we obtain-

$$\Delta\frac{\partial}{\partial r}\left[\Delta\frac{\partial R(r)}{\partial r}\right] - [\omega^2(r^2+a^2)^2 - 4Ma\omega mr + m^2a^2 - \Delta(w^2a^2+\lambda)]R = 0, \tag{2.97}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial S(\theta)}{\partial\theta}\right] + [w^2a^2cos^2\theta - \frac{m^2}{\sin^2\theta} + \lambda]S(\theta) = 0 \tag{2.98}$$

The mode functions can also be expanded as given in [6].

$$u(l,m,p) = N(p)(r^2+a^2)^{-1/2}R_{lm}(p,a)S_{lm}(a\epsilon|cos\theta)e^{(im\phi - i\epsilon t)} \tag{2.99}$$

where S_{lm} are spheroidal harmonics. Let us introduce a new coordinate,

$$r^* = r + \frac{M}{\sqrt{M^2-a^2}}\left(r_+\ln\frac{r-r_+}{r_+} - r_-\ln\frac{r-r_-}{r_-}\right) \tag{2.100}$$

$$\Rightarrow \frac{dr^*}{dr} = \frac{r^2+a^2}{\Delta} \tag{2.101}$$

As in the Schwarzschild case this new coordinate system pushes the horizon to minus infinity. We can rewrite the radial equation as obtained in (2.97) by replacing the function $R(r)$ by $R_{lm}(p,a)(r^2+a^2)-1/2$. As a result the radial equation simplifies to

$$\left[\frac{d^2}{dr^{*2}} - V_{lm}\right]R_{lm}(p,a) = 0 \tag{2.102}$$

The exact form of V_{lm} can be found in [6]. When $r* \to \infty$ the potential reduces to-

$$V_{lm} \to -(\epsilon)^2 \tag{2.103}$$

whereas in the region $r^* \to -\infty$,

$$V_{lm} \to -(\epsilon - m\Omega_H)^2 \tag{2.104}$$

For a radial wave originating close to the horizon we can asymptotically write,

$$\overrightarrow{R}_{lm}(p,a) \;\to\; e^{ipr^*} + \overrightarrow{A}_{lm}(p,a)e^{-ipr^*}\,,\; r^*\to-\infty,\; \epsilon = p + m\Omega_H \tag{2.105}$$

$$\overrightarrow{R}_{lm}(p,a) \;\to\; \overrightarrow{B}_{lm}\,e^{i(p+m\Omega_H)r^*}\,,\; r^*\to\infty,\; \epsilon = p + m\Omega_H \tag{2.106}$$

Similarly for waves generated at infinity,

$$\overleftarrow{R}_{lm}(p,a) \;\to\; \overleftarrow{B}_{lm}\,e^{i(p-m\Omega_H)r^*}\,,\; r^*\to-\infty,\; \epsilon = p \tag{2.107}$$

$$\overleftarrow{R}_{lm}(p,a) \;\to\; e^{-ipr^*} + \overleftarrow{A}_{lm}e^{ipr^*}\,,\; r^*\to\infty,\; \epsilon = p \tag{2.108}$$

As we do not have a timelike Killing vector beyond the static limit, so it is here that the wave can have negative energy. The Wronskian is constant for all r^* i.e.

$$R\frac{\partial R^*}{\partial r^*} - R^*\frac{\partial R}{\partial r^*} \tag{2.109}$$

for each set of solutions i.e. waves originating at infinity or near the horizon. Matching the Wronskian obtained for the solutions, we can write-

$$1 - |\overrightarrow{A}_{lm}(p,a)|^2 = \frac{p + m\Omega_H}{p} |\overrightarrow{B}_{lm}|^2 \tag{2.110}$$

$$1 - |\overleftarrow{A}_{lm}(p,a)|^2 = \frac{p - m\Omega_H}{p} |\overleftarrow{B}_{lm}|^2 \tag{2.111}$$

The most important thing here is that for the last equation if $m\Omega_H > p$ then the waves produced at infinity are reflected with a greater amplitude than they started with.Similarly for outgoing waves if $m\Omega_H < -p$ then the waves produced at the horizon are reflected back with a larger amplitude as compared amplitude at which they were produced. This phenomena is called superradiance and by this process a wave can gain energy from a rotating black hole.

Chapter 3

Quantum aspects of black holes

In this chapter we shall focus exclusively on aspects of quantum field theory in a classical general relativistic background i.e. the quantum field's energy does not contribute to the curvature of spacetime. We will follow the approach of canonical quantization of scalar fields in curved spacetime. We will discuss topics such as Bogolyubov transformations, inequivalent quantization and vacuum polarizations before moving onto the subject of quantum fields around black holes.

3.1 Canonical quantization in curved spacetime

The basic formalism of quantum field theory in flat spacetime can be generalized to a curved spacetime in a straightforward manner [2]. A field can be described in a curved spacetime by the generally covariant version of flat spacetime Lagrangian, variation of which leads us to the generally covariant field equations. We invoke quantization by postulating a set of commutation relations for the field or for the creation and annihilation operators. Depending upon the spacetime we then write the field in terms of it's normal modes i.e. the positive and negative frequency modes. After obtaining the normalized form of these positive frequency modes we define the vacuum state for the quantum field in the given spacetime. Here we shall discuss some important results which will form the basis for the later sections. We will concern ourselves only with quantization of scalar fields. The action for a real massless scalar field ϕ is given by

$$S[\psi] = \int d^4x \sqrt{-g}\; L(\psi) = \frac{1}{2} \int d^4x \sqrt{-g}\; g_{\mu\nu}\partial^{\mu}\psi\, \partial^{\nu}\psi \tag{3.1}$$

Subtle differences arise when one tries to quantize fields in curved spacetime. Actually the departures from flat spacetime QFT arises even when quantum fields are quantized in a non-inertial coordinate system in flat spacetime. Further as physics is coordinate independent there is no reason why field quantization should be carried out in the Minkowski coordinates alone. As an example an accelerating observer will find it more natural to carry out the field quantization in a coordinate system obtained by a suitable transformation of the Minkowski coordinates. It then turns out, that the vacuum state defined in an inertial coordinate system and the vacuum state defined in a non-inertial coordinate system can,in general, be different. As a result, the definition of a particle in the two different systems can also be different.

3.2 Bogolyubov transformations

Physics as we know should be independent of the coordinate system employed. Hence in a given spacetime one should in principle be able to carry out the field quantization any arbitrary coordinates. Let us suppose that we have carried out such a quantization of a field Φ in two different coordinate systems say U and V and so we obtain a set of complete, orthonormal modes for the same namely $u_k(t,x)$ and $v_l(\tau,\xi)$ respectively. These two decompositions lead to two vacuum states $|0_u\rangle$ and $|0_v\rangle$ and their associated Fock space. Now, are these two quantization equivalent?

Since both set of modes are complete then we can express one set of modes in terms of the other set and vice versa. This gives us-

$$v_l\left[\tau(t,x),\xi(t,x)\right] = \int_{-\infty}^{\infty} dk \left(\alpha(l,k)\, u_k(t,x) + \beta(l,k)\, u_k^*(t,x)\right). \tag{3.2}$$

and

$$u_k[t(\tau,\xi),x(\tau,\xi)] = \int_{-\infty}^{\infty} dl \left(\alpha^*(l,k)\, v_l(\tau,\xi) - \beta(l,k)\, v_l^*(\tau,\xi)\right) \tag{3.3}$$

These relations are known as the Bogolyubov transformations. The quantities $\alpha(l,k)$ and $\beta(l,k)$ are called the Bogolyubov coefficients. Using equation (3.2) and the orthogonality relations for the mode functions, the Bogolyubov coefficients can be expressed as

$$\alpha(l,k) = (v_l, u_k) \qquad \text{and} \qquad \beta(l,k) = -\left(v_l, u_k^*\right). \tag{3.4}$$

Using the orthogonality relations for the normal modes u_k and v_l it can be shown that the annihilation operators for U and V can be expressed as

$$\hat{a}_k = \int_{-\infty}^{\infty} dl\ \left(\alpha(l,k)\,\hat{b}_l + \beta^*(l,k)\,\hat{b}_l^\dagger\right) \tag{3.5}$$

and

$$\hat{b}_l = \int_{-\infty}^{\infty} dk\ \left(\alpha^*(l,k)\,\hat{a}_k - \beta^*(l,k)\,\hat{a}_k^\dagger\right). \tag{3.6}$$

the above expressions. These transformations also possess the following properties

$$\int_{-\infty}^{\infty} dk \left(\alpha(l,k))\,\alpha^*(l',k) - \beta(l,k)\,\beta^*(l',k)\right) \quad = \quad \delta_D(l-l'), \tag{3.7}$$

$$\int_{-\infty}^{\infty} dk \left(\alpha(l,k))\,\beta(l',k) - \beta(l,k)\,\alpha(l',k)\right) \quad = \quad 0. \tag{3.8}$$

It can be easily seen from equations (3.5) and (3.6) that the two Fock spaces constructed out of the modes u_k and v_l will be different if the Bogolyubov coefficient β is nonzero. For example, if β proves to be nonzero then it can be easily seen from equation (3.6) that the u_k vacuum $|0_u\rangle$ will not be annihilated by the annihilation operator $\hat{b}_l$ for the mode v_l.

3.3 Inequivalent quantization in flat spacetime

The phenomena of inequivalent quantization arises primarily because field quantization involves expanding the field in terms of mode functions which are characterized by a certain proper time. For spacetimes in general, there is no global timelike fields thus it is difficult to determine positive frequency modes and sometimes quantization is only possible in a certain specific regions of spacetime.

3.3.1 Unruh effect

The Unruh effect (or the Fulling Davies Unruh Effect) was theoretically discovered and described in the middle 1970's [2, 12]. It states that a uniformly accelerating observer in Minkowski spacetime observes particles having a thermal spectrum although inertial observers find none. The temperature of these observed particles is directly proportional to the acceleration of these observers.

Before beginning with the description of this effect we should discuss some aspects of the Rindler coordinates. These are used to describes the motion of a uniformly accelerating frame in flat spacetime. These uniformly accelerating observers are called Rindler observers. Let us begin by studying accelerated motion in flat spacetime. For simplicity, we shall work in a (1+1)D spacetime. The flat space metric is given by

$$ds^2 = dt^2 - dx^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta \tag{3.9}$$

Using the proper time as a parameter,

$$u^\alpha(\tau) = \frac{dx^\alpha}{d\tau} = (\dot{t}(\tau), \dot{x}(\tau)), \tag{3.10}$$

which satisfies

$$\eta_{\alpha\beta} du^\alpha du^\beta = 1, \tag{3.11}$$

Differentiating the above expression we get

$$\eta_{\alpha\beta} da^\alpha da^\beta = 0 \tag{3.12}$$

In the co-moving frame with respect to the accelerating observer, he will be at rest. So $\dot{x}(\tau)$=0 and so $u^\alpha(\tau) = (1, 0)$. This implies that the acceleration of the observer is constant i.e. $a^\alpha(\tau) = (0, \text{constant } = a)$. We can obtain an expression for the covariant acceleration as well,

$$\eta_{\alpha\beta} a^\alpha a^\beta = \eta_{\alpha\beta} \ddot{x}^\alpha(\tau) \ddot{x}^\beta(\tau) = -a^2, \tag{3.13}$$

We shall also discuss inertial lightcone coordinates defined as-

$$u = t - x \tag{3.14}$$

$$v = t + x \tag{3.15}$$

The metric in lightcone coordinates becomes

$$ds^2 = du dv = g^M_{\alpha\beta} dx^\alpha dx^\beta, \tag{3.16}$$

where $x^0 = u, x^1 = v$ and the metric is $g^M_{\alpha\beta}$ where M stands for Minkowski. Let us see the trajectory of an accelerated observer in the Minkowski spacetime. In the lightcone coordinates,

$$x^\alpha(\tau) = (u(\tau), v(\tau)) \tag{3.17}$$

From (3.10), (3.11) and (3.13) we obtain

$$\dot{u}(\tau)\dot{v}(\tau) = 1 \tag{3.18}$$

$$\dot{u}(\tau)\dot{v}(\tau) = -a^2 \tag{3.19}$$

Using the above equations,

$$\ddot{u} = -\frac{\ddot{v}}{\dot{v}^2} \tag{3.20}$$

which gives us

$$(\frac{\ddot{v}}{\dot{v}})^2 = a^2 \tag{3.21}$$

On integrating for $v(\tau)$, we obtain as solution

$$v(\tau) = \frac{A}{a}e^{at} + B \tag{3.22}$$

Similarly we obtain,

$$u(\tau) = -\frac{1}{Aa}e^{-at} + C \tag{3.23}$$

where B and C are integration constants which can be set to one by shifting the origin of our inertial frame. We can set $A = 1$ by a suitable Lorentz transformation. So in the lightcone coordinates the trajectory of a uniformly accelerated observer is,

$$u(\tau) = -\frac{1}{a}e^{-at}, v(\tau) = \frac{1}{a}e^{at} \tag{3.24}$$

Expressing the same result in terms of Minkowski coordinates(t, x)

$$t(\tau) = \frac{u+v}{2} = \frac{1}{a}\sinh a\tau, x(\tau) = \frac{v-u}{2} = \frac{1}{a}\cosh a\tau, \tag{3.25}$$

This is the trajectory of an observer who decelerates while approaching from $x \to -\infty$, makes a stop at $x = a^{-1}$ and then accelerates back to infinity. Notice from (3.25) that the trajectory is a hyperbola given by

$$x^2 - t^2 = \frac{1}{a^2}. \tag{3.26}$$

For large values of τ, the worldline of the accelerated observer seems to asymptotically approach the lightcone.

The co-moving frame

We can also find a co-moving frame with coordinates (ξ^0, ξ^1) for a uniformly accelerated observer. In such a coordinate system, the observer will be at rest in the frame at $\xi^1 = 0$ and $\xi = 0$ should coincide with the observer's proper time. Since any metric in (1+1) dimensions is con formally flat, we should try to express the metric of the co-moving coordinates in the same. So we desire the metric to take the form of-

$$ds^2 = \Omega^2(\xi^0, \xi^1)[(d\xi^0)^2 - (d\xi^1)^2] \tag{3.27}$$

We can shift to the lightcone coordinates of the co-moving frame where,

$$\tilde{u} = \xi^0 - \xi^1, \tilde{v} = \xi^0 + \xi^1 \tag{3.28}$$

The metric here can be written as

$$ds^2 = \Omega^2(\tilde{u}, \tilde{v}) d\tilde{u} d\tilde{v} \tag{3.29}$$

The worldline of the observer in lightcone coordinates can be expressed as-

$$\xi^0(\tau) = \tau \,,\; \xi^1(\tau) = 0 \tag{3.30}$$

Therefore, from the definition of lightcone coordinates, As ξ^0 is the proper time at the observer's location the conformal factor becomes-

$$\Omega^2(\tilde{u} = \tau, \tilde{v} = \tau) = 1 \tag{3.31}$$

As we are describing Minkowskian spacetime in different coordinates so using (3.16)we have

$$ds^2 = du\, dv = \Omega^2(\tilde{u}, \tilde{v}) d\tilde{u}\, d\tilde{v} \tag{3.32}$$

It can be seen that the u and v can depend on either $\tilde{u}$ or $\tilde{v}$. Let us select the following,

$$u = u(\tilde{u}) => v = v(\tilde{v}) \tag{3.33}$$

To obtain the functions explicitly we use the velocities in these coordinate systems.

$$\frac{du(\tau)}{d\tau} = \frac{du(\tilde{u})}{d\tilde{u}} \frac{d\tilde{u}(\tau)}{d\tau} \tag{3.34}$$

$$e^{-a\tau} = 1.\frac{du(\tilde{u})}{d\tilde{u}} \tag{3.35}$$

which on solving gives us

$$u = De^{-a\tilde{u}} \tag{3.36}$$

where D is a constant. Similarly we find that

$$v = Ee^{a\tilde{v}} \tag{3.37}$$

At proper time τ in the frame of the accelerated observer, we use the definition of the metric to obtain $-a^2DE = 1$. This gives us $E = 1/a$ and $D = -1/a$ as possible solutions. So we obtain the functions as

$$u = -\frac{1}{a}e^{-a\tilde{u}} \,,\; v = \frac{1}{a}e^{a\tilde{v}}, \tag{3.38}$$

Using the definitions of our lightcone coordinates we can obtain the relations between the inertial coordinates t, x and the coordinates of the co-moving frame and they are as follows-

$$t(\xi^0, \xi^1) = \frac{e^{a\xi^1} \sinh a\xi^0}{a} \,,\; x(\xi^0, \xi^1) = \frac{e^{a\xi^1} \cosh a\xi^0}{a}, \tag{3.39}$$

Writing the metric in terms of the co-moving coordinates gives us the Rindler spacetime, which is con formally related to the Minkowski spacetime.

$$ds^2 = e^{2a\xi^1}[(d\xi^0)^2 - (d\xi^1)^2] \tag{3.40}$$

Using (3.39) and as $\xi^0 \to \pm\infty$ we can see that the Rindler spacetime covers only a part of Minkowski spacetime. So there are regions of the Minkowski spacetime which the Rindler coordinates cannot access. To investigate this property of Rindler spacetime that we set $\xi^0 = \text{constant}$. Thus the metric becomes

$$ds^2 = e^{2a\xi^1} - (d\xi^1)^2 \tag{3.41}$$

On integrating between $-\infty$ to 0, we get the distance s as

$$s = 1/a \tag{3.42}$$

i.e. an infinite length of one coordinate spans a finite physical distance. From (3.39), we can see that the as $\xi^0 \to \infty$, $x = t$ i.e. the worldline of the observer merges with the light cone itself. This means that the observer will not be able to receive information about certain events in the upper part of spacetime. So the acceleration of the observer shields from him the information about events occurring above the horizon which is analogous to the event horizon of a black hole.

Inequivalent quantization

In this section we will quantize quantum fields in inertial and accelerated frames. Canonical quantization of fields requires us to define the mode functions in particular the positive frequency modes which are defined using the respective time coordinates. We can write the action for a massless scalar field in 1+1- dimensions as

$$S[\psi] = \frac{1}{2}\int g^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta}\sqrt{-g}d^2x \tag{3.43}$$

It is here that we can use the 1+1-dimensional spacetime to our advantage. In such a spacetime the above action is conformally invariant. Also the equation of motion for the quantum field is identical to that in flat spacetime. So action in the lightcone coordinates of the two frames i.e. inertial and Rindler frames can be written as-

$$S = 2\int \partial_u\psi\partial_v\psi du dv = 2\int \partial_{\tilde{u}}\psi\partial_{\tilde{v}}\psi d\tilde{u}d\tilde{v} \tag{3.44}$$

Thus we arrive at

$$\partial_u\partial_v\psi = 0\,,\; \partial_{\tilde{u}}\partial_{\tilde{v}}\psi = 0 \tag{3.45}$$

The solutions to these are -

$$\psi(u,v) = A(u) + B(v)\,,\; \psi(\tilde{u},\tilde{v}) = \tilde{A}(\tilde{u}) + \tilde{B}(\tilde{v}) \tag{3.46}$$

In flat space we know that the solutions to the wave equation are of the form $e^{-i(t-x)\omega} = e^{-iu\omega}$. We take them to be the right moving modes. Similarly the modes in the Rindler spacetime will be of the form $e^{-i\Omega\tilde{u}} = e^{-i\Omega(\xi^0-\xi^1)}$. The left moving modes behave in a similar fashion. As we are carrying out canonical quantization we can expand the field in terms of it's mode functions. Considering only modes propagating to the right, for the inertial frame we have-

$$\hat{\psi} = \int_0^\infty \frac{d\omega}{\sqrt{2\pi}}\frac{1}{\sqrt{2\omega}}[e^{(-i\omega u)}\hat{a_\omega^-} + e^{(i\omega u)}\hat{a_\omega^+}] \tag{3.47}$$

For the Rindler frame,

$$\hat{\psi} = \int_0^\infty \frac{d\Omega}{\sqrt{2\pi}}\frac{1}{\sqrt{2\Omega}}[e^{(-i\Omega\tilde{u})}\hat{b_\Omega^-} + e^{(i\Omega\tilde{u})}\hat{b_\Omega^+}] \tag{3.48}$$

where the operators $\hat{a}^{\pm}_{\omega}$ and $\hat{b}^{\pm}_{\Omega}$ satisfy the following relations-

$$[\hat{a}^{-}_{\omega}, \hat{a}^{+}_{\omega'}] = \delta(\omega - \omega'), [\hat{b}^{-}_{\Omega}, \hat{b}^{+}_{\Omega'}] = \delta(\Omega - \Omega') \tag{3.49}$$

We have seen the mode functions for the two frames are different and the coordinates of the two related by non trivial transformation and thus the vacuums defined by the annihilation operators will be different in these frames. We define the Minkowski vacuum and Rindler vacuum respectively as follows-

$$\hat{a}^{-}_{\omega}|0_M\rangle = 0\ ,\ \hat{b}^{-}_{\Omega}|0_R\rangle = 0 \tag{3.50}$$

If we calculate the zero point energies of the two vacuum states, they will be different and this difference leads to observation of particles. The Rindler observer will observe particles of frequency Ω in the Minkowski spacetime as with respect to him only vacuum state $|0_R\rangle$ is the state of zero particles.

The Rindler observer 'observes' the modes in a region which is a part of a larger Minkowskian spacetime. Considering the common region among the two, we can write the relation between the operators $\hat{a}^{\pm}_{\omega}$ and $\hat{b}^{\pm}_{\Omega}$ via the Bogolyubov transformations, follows as

$$\hat{b}^{-}_{\Omega} = \int_0^{\infty} d\omega[\alpha_{\Omega\omega}\hat{a}^{-}_{\omega} - \beta_{\Omega\omega}\hat{a}^{+}_{\omega}] \tag{3.51}$$

where $\alpha_{\Omega\omega}$ and $\beta_{\Omega\omega}$ are Bogolyubov coefficients. Using the commutation relations we can obtain,

$$\int_0^{\infty} d\omega[\alpha_{\Omega\omega}\alpha^{*}_{\Omega\omega} - \beta_{\Omega\omega}\beta^{*}_{\Omega\omega}] = \delta(\Omega - \Omega') \tag{3.52}$$

We should note that the current transformations are of a more general character as both the inertial positive and negative frequency modes are involved. However, we cannot define an inverse transformation as Rindler coordinates do not cover the Minkowski spacetime completely. Comparing the coefficients of the annihilation operator we get,

$$\frac{1}{\sqrt{\omega}}e^{-i\omega u} = \int_0^{\infty} \frac{d\Omega'}{\sqrt{\Omega'}}(\alpha_{\Omega'\omega}e^{-i\Omega'\tilde{u}} - \beta^{*}_{\Omega'\omega}e^{i\Omega'\tilde{u}}) \tag{3.53}$$

The above can be used to obtain the Bogolyubov coefficients as

$$\alpha_{\Omega\omega} = +\frac{e^{+\frac{\pi\Omega}{2a}}}{2\pi a}\sqrt{\frac{\Omega}{\omega}}\, exp\frac{i\Omega\ln(\omega/a)}{a}\Gamma\left(-\frac{i\Omega}{a}\right) \tag{3.54}$$

$$\beta_{\Omega\omega} = -\frac{e^{-\frac{\pi\Omega}{2a}}}{2\pi a}\sqrt{\frac{\Omega}{\omega}}\, exp\frac{i\Omega\ln(\omega/a)}{a}\Gamma\left(-\frac{i\Omega}{a}\right) \tag{3.55}$$

For details refer to [12]. It can be clearly seen that the coefficients are related as-

$$|\alpha_{\Omega\omega}|^2 = e^{\frac{2\pi\Omega}{a}}|\beta_{\Omega\omega}|^2, \tag{3.56}$$

Unruh temperature

Let us see how the Rindler observer views the Minkowski vacuum as a bath of particles having a thermal distribution. We need to calculate the number of Rindler particles(b particles) as observed by a Rindler observer in Minkowski spacetime. Using the b particle number operator we have-

$$\langle \hat{N}_\Omega \rangle = \langle 0_M | \hat{b}^+_\Omega \hat{b}^-_\Omega | 0_M \rangle \tag{3.57}$$

Using (3.56) we can reduce the above relation to-

$$\langle \hat{N}_\Omega \rangle = \int d\omega |\beta_{\Omega\omega}|^2 \tag{3.58}$$

Now, using the above equations we obtain

$$\int d\omega |\beta_{\Omega\omega}|^2 = \left(\frac{1}{e^{\frac{2\pi\Omega}{a}} - 1} \right) \delta(0) \tag{3.59}$$

This $\delta(0)$ term in arises as we are dealing with continuous momenta. If we used a finite volume to normalize the mode functions we would have obtained discrete energies and momenta and $\delta(0)$ would have been replaced by V(the finite volume). Thus particle density becomes-

$$\hat{n}_\Omega = \frac{1}{e^{\frac{2\pi\Omega}{a}} - 1}, \tag{3.60}$$

The above distribution function resembles the Bose Einstein distribution function having a specific temperature. So the accelerated observer observes the Minkowski vacuum to be populated with a thermal bath of particles, which corresponds to a temperature

$$T_{Unruh} = \frac{a}{2\pi} \tag{3.61}$$

This temperature is known as the Unruh temperature.

At the end we must mention the fact that the above effect is yet to be theoretically observed. In flat spacetime QFT is invariant under the PoincarÃl' group which consists of linear coordinate transformations. This section has demonstrated that under a set of non-linear coordinate transformations concepts such as particles, vacuum etc become coordinate dependent. Further in curved spacetime the PoincarÃl' group is not a symmetry group. Hence in curved spacetime also quantization in different coordinates can be inequivalent. We shall show this in the section on quantum field theory around black holes.

3.4 Vacuum polarization and particle production

Casimir Effect: An example of vacuum polarization

Casimir effect was first proposed by Hendrik B. Casimir. The effect is that in a vacuum two uncharged conducting plates attract each other due to an aspect of quantum field theory known as vacuum polarization. The presence of these plates causes polarization of the vacuum in between the plates. This effect manifests itself as a force on the plates, pushing them together.

Here as a simple example, we shall discuss the vacuum polarization that arises due to the non-trivial topology.

Consider a spacetime having a two dimensional Minkowski line element

$$ds^2 = dt^2 - dx^2 \tag{3.62}$$

The spacetime can be imagined as a cylinder with a periodic length L. This length can be thought of as the circumference of the universe. In such a spacetime the normal modes for the massless, scalar fields are of the form

$$u_k = \frac{1}{\sqrt{2\omega L}} e^{i\omega(x-t)}, \tag{3.63}$$

where $k = \frac{2\pi n}{L}$ and n is an integer. Let these modes represent the waves that travel from the left towards right. Due to the imposition of the periodic boundary conditions the mode functions form a discrete set. The energy momentum tensor for the field is

$$T_{\mu\nu} \;=\; \psi_{,\mu}\psi_{,\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta} \tag{3.64}$$

The Cartesian components of this tensor are

$$T_{tt} = T_{xx} = \frac{1}{2}\left(\frac{\partial\psi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial\psi}{\partial x}\right)^2 \tag{3.65}$$

and

$$T_{tx} \;=\; T_{xt} \;=\; \frac{\partial\psi}{\partial t}\frac{\partial\psi}{\partial x} \tag{3.66}$$

Let the vacuum state for the modes defined in (3.63) be denoted by $|0_L\rangle$ where L denotes the periodicity length. It is instructive to note that as $L \to \infty$, this vacuum state reduces to the Minkowskian vacuum state $|0\rangle$. Using the equation

$$\langle 0|T_{\mu\nu}|0\rangle \;=\sum_k \hat{T}_{\mu\nu}\left[u_k, u_k^*\right] \tag{3.67}$$

obtained in [2], for this particular spacetime we can write

$$\langle 0_L|T_{tt}|0_L\rangle \;=\; \frac{2\pi}{L^2}\sum_0^\infty n. \tag{3.68}$$

It is evident that the Hamiltonian density diverges. It's behaviour is similar to the behaviour one obtains in Minkowskian spacetime. Using the concept of normal ordering of operators we can write

$$\begin{aligned} \langle 0_L| : T_{\mu\nu} : |0_L\rangle \;&=\; \langle 0_L|T_{\mu\nu}|0_L\rangle \;-\langle 0|T_{\mu\nu}|0\rangle, \\ &=\; \langle 0_L|T_{\mu\nu}|0_L\rangle \;-\lim_{L'\to\infty}\langle 0_{L'}|T_{\mu\nu}|0_{L'}\rangle., \end{aligned} \tag{3.69}$$

Using the process of regularization we introduce the cut-off factor into (3.68) we obtain

$$\langle 0_L|T_{tt}|0_L\rangle \;=\; \frac{2\pi}{L^2}\lim_{\alpha\to 0}\sum_0^\infty n e^{-\alpha 2\pi n/L}, \tag{3.70}$$

$$=> \lim_{\alpha \to 0} \frac{2\pi}{L^2} e^{2\pi\alpha/L} (e^{2\pi\alpha/L} - 1)^{-2} \tag{3.71}$$

Expanding about $\alpha = 0$,

$$\langle 0_L | T_{tt} | 0_L \rangle = \frac{1}{2\pi\alpha^2} - \frac{\pi}{6L^2} + O(\alpha^3) \tag{3.72}$$

Also,

$$\lim_{L' \to \infty} \langle 0_{L'} | T_{tt} | 0_{L'} \rangle = \frac{1}{2\pi\alpha^2} \tag{3.73}$$

Substituting in (3.69) we obtain as $\alpha \to 0$

$$\langle 0_L | : T_{tt} : | 0_L \rangle = -\frac{\pi}{6L^2} \tag{3.74}$$

We have shown that the expectation value of the stress energy tensor diverges for the states $|0_M\rangle$ and $|0_L\rangle$ but their difference is finite as we have calculated. If $\langle 0| : T_{\mu\nu} : |0\rangle = 0$, then the state $|0_L\rangle$ has a finite energy density associated with it which is negative. It is given as

$$\rho = p = \langle 0_L | : T_{tt} : | 0_L \rangle = \langle 0_L | : T_{xx} : | 0_L \rangle = -\frac{\pi}{6L^2} \tag{3.75}$$

So the total energy in this cylindrical universe is

$$E = -\frac{\pi}{6L^2} L = -\frac{\pi}{6L}. \tag{3.76}$$

Particle production in curved spacetime

The phenomena of particle production arises because of the multiple ways possible to define the positive frequency modes in a given spacetime. In this section we will illustrate by an example how a time dependent metric can naturally give rise to difference in positive frequency modes at different times leading to creation of particles.
Consider a two dimensional Friedman-Robertson-Walker universe with the line element

$$ds^2 = dt^2 - a^2(t)\, dx^2. \tag{3.77}$$

where the scale factor, $a(t)$ describes how the spatial sections of the spacetime behave as a function of time(t). We introduce a new time parameter called the conformal time parameter which is defined as

$$\eta = \int \frac{dt}{a(t)} \tag{3.78}$$

the metric (3.77) can be expressed in terms of the conformal time η as

$$\begin{aligned} ds^2 &= a^2(\eta) \left(d\eta^2 - dx^2 \right) \\ &= C(\eta) \left(d\eta^2 - dx^2 \right), \end{aligned} \tag{3.79}$$

where the conformal scale factor is defined as $C(\eta) = a^2(\eta)$. This form of the line element is manifestly conformal to the flat spacetime line element in Minkowski coordinates. Let us define $C(\eta)$ as-

$$C(\eta) = A + B \tanh(\rho\eta). \tag{3.80}$$

where A, B and ρ are constants. Hence in the asymptotic past and the asymptotic future the spacetime becomes Minkowskian i.e.

$$C(\eta) \to A \pm B, \quad \text{as} \quad \eta \to \pm\infty. \tag{3.81}$$

Consider a real, massive scalar field described by the action

$$S[\psi] = \int d^2x \sqrt{-g}\; L(\psi) = \frac{1}{2}\int d^2x \sqrt{-g}\; \left(g_{\mu\nu}\,\partial^\mu\psi\,\partial^\nu\psi - m^2\,\psi^2\right). \tag{3.82}$$

Varying this action with respect to the scalar field ψ, we obtain the equation of motion satisfied by the scalar field to be

$$\left(\Box + m^2\right)\psi \equiv \left(\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right) + m^2\right)\psi = 0. \tag{3.83}$$

Substituting the metric (3.79) here provides us with

$$\left(\partial_\eta^2 - \partial_x^2 + m^2 C(\eta)\right)\psi(\eta, x) = 0. \tag{3.84}$$

The field can be decomposed in terms of the normal modes which are

$$u_k(\eta, x) \propto \chi_k(\eta)\, e^{ikx}, \tag{3.85}$$

where the functions $\chi_k(\eta)$ obey the following differential equation

$$\frac{d^2\chi_k}{d\eta^2} + \left(k^2 + m^2 C(\eta)\right)\chi_k = 0. \tag{3.86}$$

When $C(\eta)$ given by (3.80), this differential equation can be solved in terms of hyper-geometric functions [2]. The normalized positive frequency modes in the asymptotic past (*i.e.* as η, $t \to -\infty$) are

$$\begin{aligned} u_k^{in}(\eta, x) \quad &= \quad \frac{1}{\sqrt{4\pi\omega_{in}}}\ \exp\, i\Big\{kx - \omega_+\eta - (\omega_-/\rho)\ln\left[2\cosh(\rho\eta)\right]\Big\} \\ &\qquad \times F\Big(1 + (i\omega_-/\rho), i\omega_-/\rho, 1 - (i\omega_{in}/\rho), \left[1 + \tanh(\rho\eta)\right]/2\Big) \\ &\xrightarrow{\eta\to-\infty} \quad \frac{1}{\sqrt{4\pi\omega_{in}}}\ \exp -i\left(\omega_{in}\eta - kx\right), \end{aligned} \tag{3.87}$$

where

$$\left.\begin{aligned} \omega_{in} \ &= \left(k^2 + m^2(A - B)\right)^{1/2} \\ \omega_{out} \ &= \left(k^2 + m^2(A + B)\right)^{1/2} \\ \omega_\pm \ &= \tfrac{1}{2}\left(\omega_{out} \pm \omega_{in}\right). \end{aligned}\right\} \tag{3.88}$$

The positive frequency modes in the asymptotic future can be expressed (*i.e.* as $\eta,\, t \to \infty$) are as follows

$$\begin{aligned} u_k^{out}(\eta, x) &= \frac{1}{\sqrt{4\pi\omega_{out}}} \exp i\Big\{kx - \omega_+\eta - (\omega_-/\rho)\ln\left[2\cosh(\rho\eta)\right]\Big\} \\ &\quad \times F\Big(1 + (i\omega_-/\rho), i\omega_-/\rho, 1 + (i\omega_{out}/\rho), \left[1 - \tanh(\rho\eta)\right]/2\Big) \\ &\xrightarrow{\eta\to\infty} \frac{1}{\sqrt{4\pi\omega_{out}}} \exp -i\left(\omega_{out}\eta - kx\right). \end{aligned} \tag{3.89}$$

It is evident that u_k^{in} and u_k^{out} are not identical which implies that the Bogolyubov coefficient β relating these two modes is be non-zero. To see we can use the linear transformation properties of hyper-geometric functions (see [1]) to write u_k^{in} in terms of u_k^{out} as

$$u_k^{in}(\eta, x) = \alpha(k)\, u_k^{out}(\eta, x) + \beta(k)\, u_{-k}^{out*}(\eta, x), \tag{3.90}$$

where

$$\alpha(k) = \left(\frac{\omega_{out}}{\omega_{in}}\right)^{1/2} \left(\frac{\Gamma\left[1 - (i\omega_{in}/\rho)\right]\, \Gamma(-i\omega_{out}/\rho)}{\Gamma(-i\omega_+/\rho)\, \Gamma\left[1 - (i\omega_+/\rho)\right]}\right), \tag{3.91}$$

$$\beta(k) = \left(\frac{\omega_{out}}{\omega_{in}}\right)^{1/2} \left(\frac{\Gamma\left[1 - (i\omega_{in}/\rho)\right]\, \Gamma(i\omega_{out}/\rho)}{\Gamma(i\omega_-/\rho)\, \Gamma\left[1 + (i\omega_-/\rho)\right]}\right) \tag{3.92}$$

and $\Gamma(z)$ represents the Gamma function. On comparing the equations for the modes we can obtain the Bogolyubov coefficients as

$$\alpha(k, k') = \alpha(k)\, \delta_D(k - k') \qquad \text{and} \qquad \beta(k, k') = \beta(k)\, \delta_D(k + k'). \tag{3.93}$$

From the above expression we can write

$$|\alpha(k)|^2 = \left(\frac{\sinh^2(\pi\omega_+/\rho)}{\sinh(\pi\omega_{in}/\rho)\, \sinh(\pi\omega_{out}/\rho)}\right), \tag{3.94}$$

$$|\beta(k)|^2 = \left(\frac{\sinh^2(\pi\omega_-/\rho)}{\sinh(\pi\omega_{in}/\rho)\, \sinh(\pi\omega_{out}/\rho)}\right), \tag{3.95}$$

from which the normalization condition

$$|\alpha(k)|^2 - |\beta(k)|^2 = 1, \tag{3.96}$$

is obtained. While working in the Heisenberg picture, let us consider the situation when the field is initially in the *in*-vacuum state $|0_{in}\rangle$ as defined by a co-moving observer in the limit $\eta,\, t \to -\infty$. With time the spacetime expands and reaches the asymptotic future in the limit $\eta,\, t \to \infty$ the quantum field still exists in the vacuum $|0_{in}\rangle$. But now the c0-moving observer describes the spacetime differently and thus defines a different vacuum state at that time as $|0_{out}\rangle$, which seems to him to be populated with $|\beta(k)|^2$ (given by equation (3.95)) particles. With respect to that the observer particle production has occurred. The concept of particles being produced during the expansion is not meaningful as it is difficult to define them at intermediate times such as during the expansion.

Now we move to quantum field theory in black hole spacetimes. We shall discuss inequivalent quantization in Schwarzschild spacetime.

3.5 Quantum field theory around black holes

3.5.1 Inequivalent quantization in black hole spacetime

In this section we will discuss the aspects of field quantization in the spacetime around a black hole. We shall give example of a 1+1 dimensional black hole spacetime where quantization proves to be inequivalent and just as in the flat spacetime case, the observer finds the spacetime to be flooded with particles having a thermal distribution.

Let us consider a static, non rotating black hole of mass M in a 1+1 dimensional spacetime. The metric is given by

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} \tag{3.97}$$

We can rewrite metric in the tortoise coordinates as obtained in (1.14) in 1+1 dimensions-

$$ds^2 = \left(1 - \frac{2M}{r}\right)(dt^2 - dr^{*2}), \tag{3.98}$$

Let us define the lightcone tortoise coordinates as-

$$\tilde{u} \equiv t - r^* \qquad \text{and} \qquad \tilde{v} \equiv t + r^* \tag{3.99}$$

The metric can be written in terms of these coordinates as-

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tilde{u}\, d\tilde{v} \tag{3.100}$$

It is clear that the Tortoise coordinates are singular and cover only a part of the entire available spacetime. Thus we now transform to Kruskal-Szekeres coordinates as follows

$$1 - \frac{2M}{r} = \frac{2M}{r}\, e^{\left(1 - \frac{r}{2M}\right)}\, e^{\left(\frac{\tilde{v} - \tilde{u}}{4M}\right)}. \tag{3.101}$$

As a result the metric in (3.98) can be written as

$$ds^2 = \frac{2M}{r}\, e^{\left(1 - \frac{r}{2M}\right)}\, e^{\frac{\tilde{u}}{4M}}\, e^{\frac{\tilde{v}}{4M}}\, d\tilde{u}\, d\tilde{v} \tag{3.102}$$

Defining the Kruskal-Szekeres lightcone coordinates as

$$u = -4M\, e^{-\frac{\tilde{u}}{4M}},\; v = 4M\, e^{\frac{\tilde{v}}{4M}},, \tag{3.103}$$

the metric now can be expressed as

$$ds^2 = \frac{2M}{r(u,v)}\, e^{\left(1 - \frac{r(u,v)}{2M}\right)}\, du\, dv \tag{3.104}$$

In terms of timelike and spacelike coordinates (T, R) defined as,

$$u = T - R\,,\; v = T + R\,, \tag{3.105}$$

the spacetime metric becomes

$$ds^2 = \frac{2M}{r(T,R)} e^{\left(1 - \frac{r(T,R)}{2M}\right)} (dT^2 - dR^2). \tag{3.106}$$

Field quantization

We can follow the scheme for calculating the modes as we have previously done for the case of inequivalent quantization in flat spacetime. Expanding the field operator in terms of the right moving mode functions we obtain

$$\hat{\psi} = \int_0^\infty \frac{d\Omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\Omega}} [e^{-i\Omega\tilde{u}}\, \hat{b}^-_\Omega + e^{i\Omega\tilde{u}}\, \hat{b}^+_\Omega] \tag{3.107}$$

The vacuum state $|0_B\rangle$ is defined by the expression

$$\hat{b}^-_\Omega |0_B\rangle = 0. \tag{3.108}$$

This vacuum state is referred to as the Boulware vacuum and hence the subscript B and by definition is devoid of particles for an observer far away. This vacuum is defined only outside the black hole as the tortoise coordinates do not embody the spacetime within the hole. The Kruskal-Szekeres coordinates however, are valid throughout the entire Schwarzschild spacetime. Close the the event horizon the metric becomes

$$ds^2 \rightarrow \, du\, dv = \, dT^2 - dR^2 \tag{3.109}$$

The field in the Kruskal-Szekeres lightcone coordinates is of the form

$$\psi \propto e^{-i\omega(T-R)} = e^{-i\omega u} \tag{3.110}$$

for the right-moving positive frequency modes. Thus we can expand the field as

$$\hat{\psi} = \int_0^\infty \frac{d\omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} [e^{-i\omega u}\, \hat{a}^-_\omega + e^{i\omega u}\, \hat{a}^+_\omega] \tag{3.111}$$

The Kruskal vacuum state $|0_K\rangle$is defined as

$$\hat{a}^-_\omega |0_K\rangle = 0. \tag{3.112}$$

As these coordinates are nonsingular everywhere, the vacuum state is nonsingular on the horizon. Now the analogy between the Schwarzschild spacetime and flat spacetime is clear. The Rindler spacetime covers only a part of the flat spacetime as the tortoise coordinates covers only a portion of the Schwarzschild spacetime. The Rindler observer finds the Minkowski vacuum to be populated with particles. Using this connection we can say, that the observer associated with the tortoise coordinates will find the Kruskal vacuum to be full of particles. We can find the factor in the black hole spacetime analogous to acceleration in flat spacetime by comparing the equations which express the relation among lightcone coordinates in these coordinate systems. Comparing (3.38) and (3.103)

$$a = \frac{1}{4M} \tag{3.113}$$

An interesting fact to note is that for a non rotating black hole the surface gravity K defined by comes out to be $1/4M$. Replacing a by K in (3.60) we get the number operator as

$$\langle \hat{n}_\Omega \rangle = \left(\frac{1}{e^{\frac{2\pi\Omega}{K}} - 1} \right) \tag{3.114}$$

Hence the observer associated with the tortoise coordinates finds the Kruskal vacuum to be populated with a thermal distribution of particles having a temperature

$$T = \frac{K}{2\pi} \tag{3.115}$$

Thus inequivalent quantization leads to different vacuum states and which is interpreted by an observer as flux of particles. However it is a remarkable fact that the observed number density is obtained to have a thermal distribution with a temperature proportional to the surface gravity of the black hole. Another remarkable fact that we should emphasize is the role of the horizon in discussed cases. In both the spacetimes one of the coordinates becomes singular at the event horizon.

3.5.2 Hawking radiation

In section 3.4, we discussed the behaviour of scalar fields in a spacetime described by a time dependent metric and the phenomenon of particle production. In the last section, we have shown how thermal effects arise in the spacetime surrounding a Schwarzschild black hole. Now, consider an imploding star which is about to form a non-rotating black hole. Due to the collapse the metric is time dependent, hence for a quantum field the vacuum states $|0_{in}\rangle$ and $|0_{out}\rangle$ before and after the collapse will differ, leading to particle production. As the final state is that of a black hole, thermal effects arise hence the discussions in the previous sections are a stepping stone for studying Hawking radiation [8].

Chapter 4

Summary

Black holes are purely general relativistic in nature. They can be completely described by just three parameters- mass, charge and angular momentum, making them one of the simplest known entities. Anything crossing the the event horizon of a black hole can never return back. A rotating black hole is more queer as it has two horizons along with two ergosurfaces where no observers can be static. In this thesis, we began by studying the classical aspects of black holes, focusing on the discussion of the propagation of particles in the Schwarzschild spacetime, followed by a discussion of the three, classic experimental tests of the general theory of relativity. Then had moved onto the Penrose process around a rotating black hole where we had demonstrated the extraction of energy from the hole using the idea of negative energy particles in the ergosphere. This was followed by a section on the behaviour of scalar fields around black holes where we had found that a static, scalar field is impossible in the Schwarzschild spacetime. At the end, we had considered the wave analog of the Penrose process called superradiance.

Before introducing the semi-classical aspects of black holes we had reviewed the basic formalism of field quantization in flat spacetime and its generalization to curved spacetime. We had then discussed using the Bogolyubov transformation and the implication of the non-zero β Bogolyubov coefficient. Using these relations we had shown that quantization of massless, scalar fields is inequivalent in the 1+1 dimensional Minkowski and Rindler spacetimes. The consequence of this is the Unruh effect i.e. an accelerating observer finds a thermal distribution of particles with the temperature proportional to the acceleration. We had then discussed the vacuum polarization in non-trivial topology, where the difference between the infinite vacuum energy of the Minkowski and a flat cylindrical spacetime was shown to be finite. Next, we had discussed particle production in a time dependent spacetime which we had illustrated with an example of particle production in the FRW universe. We had shown that the vacuum states in the asymptotic past and the future are different and hence, an observer in the asymptotic future finds particles. We had then moved onto the topic of quantum field theory around black holes where we had discussed thermal effects due to inequivalent quantization in the Schwarzschild spacetime. Using the scheme employed to study the Unruh effect, we had shown that the inequivalent quantization in the Kruskal and tortoise coordinates leads to a thermal distribution of particles as well. The temperature was found to be proportional to the surface gravity of the black hole which is analogous to the acceleration of a Rindler observer in the flat spacetime case. Another fact to note is the presence of a horizon in both the spacetimes where the thermal distribution is observed.

Black holes thus provide a unique environment where both general relativity and quantum field theory merge leading to interesting phenomena. Some of the semi-classical results obtained around black holes are thought to depict approximately the quantum mechanical behaviour of black holes. Hence the study of such areas is essential as these offer clues which may prove to be vital in the understanding of the still elusive, quantum theory of gravity.

Chapter 5

Appendix

5.1 Killing vectors

In classical mechanics it is well known that symmetries of the Lagrangian lead to conservation laws and hence we get quantities which are conserved throughout the motion of the object.

In general relativity Killing vectors are a standard tool for description of symmetry and provides an alternate procedure for obtaining the constants of motion. Let the metric $g_{\mu\nu}$ be independent of certain coordinate x^K in the basis dx^α. This gives us

$$\frac{\partial g_{\mu\nu}}{dx^\alpha} = 0 \qquad \text{for} \qquad \alpha = K. \tag{5.1}$$

Geometrically this means that any curve or function translated in the x^K direction will give us an equivalent curve. To show this fact let a curve $x^\alpha = c^\alpha(\lambda)$ be translated in the x^K direction by an amount $\Delta x^K = \varepsilon$.

$$x^\alpha(\lambda) = x^\beta(\lambda) + x^K(\lambda) \qquad \text{where} \qquad \beta \neq K \tag{5.2}$$

After translation

$$x'^\alpha(\lambda) = c^\beta(\lambda) + c^K(\lambda) + \varepsilon. \tag{5.3}$$

The length of the curve is given by

$$\int_{s_2}^{s_1} ds = \int_{s_2}^{s_1} \sqrt{g_{\mu\nu}\, dx^\mu dx^\nu} = L \tag{5.4}$$

which in terms of parameter λ becomes

$$\int_{\lambda_2}^{\lambda_1} d\lambda \sqrt{\left(g_{\mu\nu}[x(\lambda)]\,\frac{dx^\mu}{d\lambda}\,\frac{dx^\nu}{d\lambda}\right)} = L \tag{5.5}$$

The length of the translated curve is

$$\int_{\lambda_2}^{\lambda_1} d\lambda \sqrt{[g_{\mu\nu}(x(\lambda) + \Delta x^K)]\,\frac{dx^\mu}{d\lambda}\,\frac{dx^\nu}{d\lambda}} \tag{5.6}$$

This can be expressed using (5.1) as

$$\int_{\lambda_2}^{\lambda_1} d\lambda \sqrt{\left[g_{\mu\nu}[x(\lambda)] + \frac{\partial g_{\mu\nu}}{dx^K}\varepsilon\right] \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} = L \tag{5.7}$$

Thus under such a transformation the length of the curve remains constant.

$$\frac{\partial L}{\partial \varepsilon} = \frac{\partial L}{\partial x^K} = 0 \tag{5.8}$$

Hence the vector

$$\xi = \frac{\partial}{\partial \varepsilon} = \frac{\partial}{\partial x^K} \tag{5.9}$$

denotes a Killing vector that describes an infinitesimal translation that preserves the length of the curve. We will now show that the Killing vectors satisfy a Killing equation, which is

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0, \tag{5.10}$$

Consider the geodesic coordinate system [4]. If we establish that the Killing vectors satisfy the Killing equation in the geodesic coordinate system, then they shall satisfy the same in every other coordinate system as well. In the geodesic coordinate system,

$$\xi^\mu = \delta^\mu_K = \frac{\partial}{\partial x^K} \tag{5.11}$$

The first term of (5.10) can be written as

$$\xi_{\mu;\nu} = \frac{1}{2}\left[\frac{\partial g_{\gamma\mu}}{\partial x^\nu} + \frac{\partial g_{\nu\mu}}{\partial x^\gamma} - \frac{\partial g_{\gamma\nu}}{\partial x^\mu}\right]\xi^\gamma \tag{5.12}$$

$$=> \xi_{\mu;\nu} = \frac{1}{2}\left[g_{K\mu,\nu} - g_{K\nu,\mu}\right] \tag{5.13}$$

Similarly

$$\xi_{\nu;\mu} = \frac{1}{2}\left[g_{K\nu,\mu} - g_{K\mu,\nu}\right] = -\xi_{\mu,\nu}. \tag{5.14}$$

which then leads to (5.10).

Now we state a theorem known as the Constancy theorem, followed by a proof of the same. The theorem states that in a geometry which is endowed with a symmetry described by a Killing vector field ξ, the scalar product of the tangent vector along any geodesic with the Killing vector is a constant

$$\mathbf{p}.\xi = \text{constant} = p_\mu \xi^\mu = p_\mu \delta^\mu_K = p_K \tag{5.15}$$

As there is no change in the value of the scalar product along a geodesic, so

$$\frac{dp_K}{d\lambda} = 0. \tag{5.16}$$

The left-hand side of above expression can be rewritten as

$$\frac{dp^\mu \xi_\mu}{d\lambda} \tag{5.17}$$

which in turn can be expressed as

$$(p^\mu \xi_\mu)_{;\nu}\, p^\nu = p^\mu_{;\nu}\, \xi_\mu p^\nu + p^\mu\, p^\nu\, \xi_{\mu;\nu} \tag{5.18}$$

$$= p^\mu_{;\nu}\, \xi_\mu p^\nu + \frac{1}{2} p^\mu\, p^\nu\, \xi_{\mu;\nu} + \frac{1}{2} p^\nu\, p^\mu\, (-\xi_{\mu;\nu})$$

$$= 0. \tag{5.19}$$

Thus by the virtue of the geodesic equation and the Killing equation the first term and the second terms vanish. Thus

$$\frac{dp_K}{d\lambda} = \frac{dp^\mu \xi_\mu}{d\lambda} = 0. \tag{5.20}$$

If $\mathbf{p}$ denotes the four momentum of a particle moving along a geodesic then the constancy theorem implies the constancy of K-th component of momentum i.e.

$$\mathbf{p}.\xi = \quad \text{constant} \tag{5.21}$$

Therefore whenever $g_{\mu\nu}$ is independent of a coordinate x^K, that coordinate is called cyclic and the corresponding conserved quantity p_K is termed conjugate momentum.

5.2 Eddington-Finkelstein coordinates for Schwarzschild black hole

Consider changing to the null coordinates such that the null geodesics become straight lines throughout the Schwarzschild spacetime. So we change to a new coordinates $\tilde{u}$ and$\tilde{v}$,

$$\tilde{u} = t + r^*, \tag{5.22}$$

$$\tilde{v} = t - r^*, \tag{5.23}$$

where r^* is defined in (1.14). Let us analyze the lightcone structure here. In the Tortoise coordinates, null geodesics lie along $t = \pm r^* + constant$.This gives us-

$$\tilde{u} = constant \tag{5.24}$$

for in falling photons and

$$\tilde{u} = 2r^* + \text{constant} \tag{5.25}$$

for outgoing ones. We can write their slopes as-

$$\frac{d\tilde{u}}{dr} = 0 \tag{5.26}$$

$$\frac{d\tilde{u}}{dr} = 2\frac{dt}{dr} = \frac{2r}{r - 2M} \tag{5.27}$$

The above equations give the slopes for the in falling and outgoing photons respectively.

The metric can be written as

$$ds^2 = -\left(1 - \frac{2M}{r}\right) d\tilde{u}^2 + (d\tilde{u}dr^* + dr^* d\tilde{u}) + r^2 d\Omega^2 \tag{5.28}$$

For an observer at rest the metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) d\tilde{u}^2 \tag{5.29}$$

$$ds^2 = -(1 - \frac{r}{2M})\, dt^2 \tag{5.30}$$

As $r \to \infty$,

$$ds^2 = -dt^2 \tag{5.31}$$

we obtain the metric in flat spacetime. Via this coordinate system we have extended the spacetime to include regions where the Schwarzschild coordinates broke down. Let us analyze this coordinate system a little more. From (5.22) we can say that as t always increases, r^* must decrease for the sum to be a constant. Thus the horizon be only crossed along future directed paths only. Another thing to notice is that as the horizon approaches $r* \to -\infty$ and so $t \to \infty$. (5.23) tells us that the horizon in this case can only be crossed along past directed paths or as $r \to 2M, t \to -\infty$. If we follow the above different paths we arrive at different points in spacetime. This is what it means to extend spacetime analytically.

5.3 Penrose-Carter diagrams

Penrose-Carter diagrams or Penrose diagrams (hereafter) belong to a family of spacetime diagrams which represent the infinite spacetime as a finite region. It is an excellent tool to study the large scale causal structure of spacetime. It involves a conformal transformation or conformal compactification of the metric i.e.

$$\bar{g}_{\alpha\beta} = \Omega^2(x) g_{\alpha\beta}, \tag{5.32}$$

where Ω is known as the conformal factor. Proper values of this conformal factor enables us to compactify infinity to an accessible point i.e. a finite coordinate value. Another important observation from (5.32) and [4] is that the null geodesics will be the identical for both the metrics. This is important as this implies that the light cone structure would remain identical in both the metrics in question which is an excellent opportunity to study the causal structure of spacetime at infinity. We shall first obtain the Penrose diagram for Minkowski spacetime and then we shall do the same in Schwarzschild spacetime. In the end we will discuss the same for the case of gravitational collapse.

5.3.1 Minkowski spacetime

The metric of flat spacetime is given as

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2. \tag{5.33}$$

As we have to compactify an infinite spacetime it is advisable to keep a track of all coordinates ranges. Currently, the ranges are

$$-\infty < t < \infty \qquad \text{and} \qquad 0 \leq r < \infty. \tag{5.34}$$

Let us define the null coordinates in this spacetime as

$$u = \frac{1}{2}(t + r)\,,\; v = \frac{1}{2}(t - r), \tag{5.35}$$

with the ranges as

$$\begin{aligned} -\infty < \quad & u \quad < \infty \\ -\infty < \quad & v \quad < \infty \\ v \quad \leq \quad & u. \end{aligned} \tag{5.36}$$

In these new coordinates the metric can be expressed as

$$ds^2 \;=\; -2(dudv + dvdu) + (u - v)^2 d\Omega^2 \tag{5.37}$$

To bring in infinity to a finite value we can transform to the following coordinates

$$\begin{aligned} U \;&=\; tan^{-1}u \\ V \;&=\; tan^{-1}v \end{aligned} \tag{5.38}$$

The ranges now become

$$\begin{aligned} \frac{-\pi}{2} \leq \quad & U \quad \leq \frac{\pi}{2} \\ \frac{-\pi}{2} \leq \quad & V \quad \leq \frac{\pi}{2} \\ V \quad \leq \quad & U \end{aligned} \tag{5.39}$$

The metric in this new coordinate system is

$$ds^2 = \frac{1}{\cos^2 U \cos^2 V}[-2(dUdV + dVdU) + sin^2(U - V)d\Omega^2], \tag{5.40}$$

where

$$\begin{aligned} dU \quad &= \quad \frac{du}{1+u^2}, \\ cos(tan^{-1}u) \quad &= \quad \frac{1}{\sqrt{1+u^2}}, \\ (u\ - v\,)^2 \quad &= \quad \frac{sin^2(U-V)}{\cos^2 U \cos^2 V}. \end{aligned} \tag{5.41}$$

Again introducing the timelike and the spacelike coordinates (η, χ) respectively as

$$\begin{aligned} \eta \quad &= \quad U + V \\ \chi \quad &= \quad U - V, \end{aligned} \tag{5.42}$$

with their corresponding ranges are

$$\begin{aligned} -\pi < \quad & \eta \quad < \pi \\ 0 \leq \quad & \chi \quad < \pi \end{aligned} \tag{5.43}$$

Rewriting the metric in terms of these new coordinates we obtain

$$ds^2 \;=\; \frac{1}{\omega^2}(-d\eta^2 + d\chi^2 + sin^2\chi\, d\Omega^2), \tag{5.44}$$

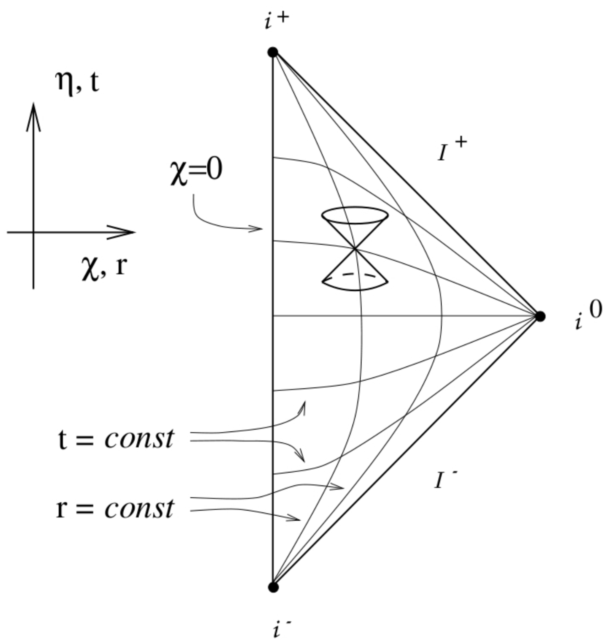

Figure 1: Penrose diagram for the Minkowski spacetime (figure from ([3]))

$$=> d\bar{s}^2 \; = \; -d\eta^2 + d\chi^2 + sin^2\chi \, d\Omega^2, \tag{5.45}$$

where

$$\omega^2 = \cos^2 U \cos^2 V \tag{5.46}$$

We can clearly see that the unphysical metric we have obtained via these seemingly arbitrary transformations is related via a conformal factor to the Minkowski metric. This is the metric of Einstein's static universe [3]. Unrolling the shaded region we obtain the Penrose diagram for the Minkowski spacetime.

As can be seen from the Figure 1, few features are marked in the diagram. Let us analyze these features so as to obtain information regarding the large scale causal structure of spacetime. First let us focus on the three vertices of the triangle. The point $\imath^+(\eta = \pi, \chi = 0)$ corresponds in the Minkowski spacetime to future timelike infinity. Similarly the point $\imath^-(\eta = -\pi, \chi = 0)$ gives us the past timelike infinity and the point $\imath^0(\eta = 0, \chi = \pi)$ denotes spatial infinity. Now let us see what the sides of the triangle represent. The segment connecting $\imath^+$ with $\imath^0$ is described by the equation $\eta = \pi - \chi, 0 < \chi < \pi$. Firstly and most importantly, this line segment represents a null surface and this segment denotes the future null infinity $\Im^+$. Similarly the segment connecting $\imath^0$ and $\imath^-$ would denote the past null infinity $\Im^-$. It is important to note that these features would not be visible on a flat Minkowski spacetime as these are at infinity. The lightcone structure is identical to the structure flat spacetime with the angle made by the radial null rays is $\pm$ 45 °with respect the axis. $\chi = 0$ in this diagram for a particle implies that it is at rest, and moving from past timelike infinity to the future timelike infinity along the vertical line. Particle having some constant velocity will start from $\imath^-$, follow a path within their lightcone so as to end at $\imath^+$. Similarly purely spacelike geodesics end at $\imath^0$.

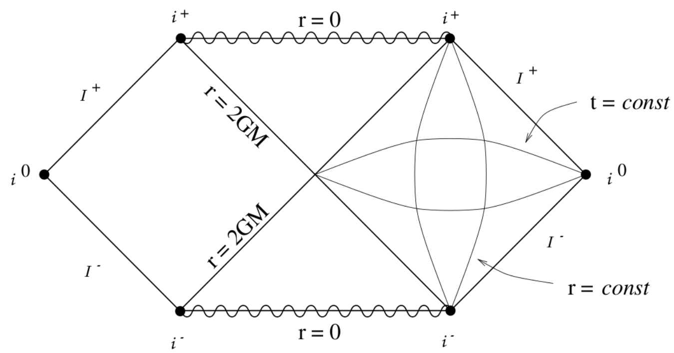

Figure 2: Penrose diagram for Schwarzschild spacetime (figure from ([3]))

5.3.2 Kruskal extension for Schwarzschild spacetime

We shall start off from the metric obtained in terms of the null Kruskal-Szekeres coordinates as given in (1.21). Using the following transformations

$$U' = \tan^{-1}(\frac{u'}{\sqrt{2M}}),$$
$$V' = \tan^{-1}(\frac{v'}{\sqrt{2M}}), \tag{5.47}$$

and the ranges are-

$$-\frac{\pi}{2} < U' < \frac{\pi}{2},$$
$$-\frac{\pi}{2} < V' < \frac{\pi}{2},$$
$$-\pi < U' + V' < \pi. \tag{5.48}$$

Transforming again to spacelike(χ') and timelike (η') coordinates

$$\eta' = U' + V',$$
$$\chi' = U' - V', \tag{5.49}$$

with the ranges

$$-\pi < \eta < \pi$$
$$0 \leq \chi < \pi \tag{5.50}$$

It can be shown that the surface $r = 2M$, is represented by $U' = V' = 0.$ which are straight lines at ± 45 °in the (η', χ') coordinate system. Let us discuss some features of the diagram qualitatively. As $r \to \infty$, the metric must reduce to the flat space metric and thus the Penrose diagram at spatial infinity for the two cases would be identical. The Penrose diagram (Kruskal extension) for the Schwarzschild spacetime is shown in Figure 2.

As can be seen from the diagram there are four regions in the diagram just as there were in the Kruskal diagram. Features such as $\imath^0, \Im^+, \Im^-$ that we had defined in the Penrose diagram for flat

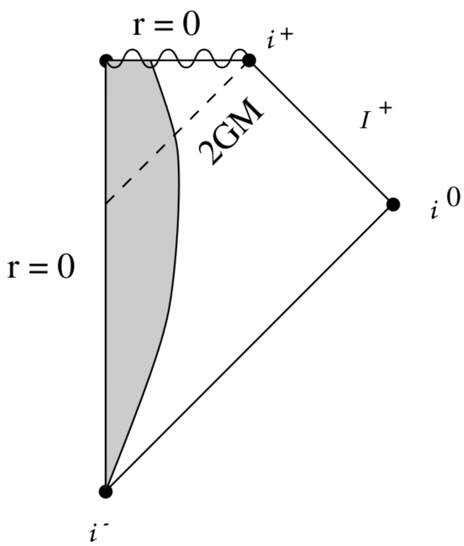

Figure 3: Penrose diagram for gravitational collapse to a black hole (figure from ([3]))

spacetime will remain identical here as well as this spacetime becomes flat asymptotically. The singularity $r = 0$ is shown as a wavy line in the past and the future. We must keep in mind that the spacetime we are describing has an eternal black hole. It is also clear from the structure of the lightcones inside the horizon that any massive object crossing into the event horizon has no choice but to proceed to it's inevitable demise at the singularity. The disconnected spacetime on the left can be thought of as a parallel universe and spacetime at the bottom as a white hole.

Penrose diagram for gravitational collapse to a black hole

In this section we shall focus on the Penrose diagram for gravitational collapse of an object such as a massive, spherically symmetric star leading to the formation of a black hole.

Consider the star to be in existence for a very long time. The spacetime at large values of r will be Minkowskian. It is instructive to compare this diagram with the Penrose diagrams for flat and Schwarzschild spacetime. The shaded region in diagram represents the star's interior. Initially the radius of the star is assumed to be fixed and so follows the it's worldline along the constant r surface. Let the collapse start at $\eta = 0$ after which the worldline of the star's surface keeps moving towards the left in the diagram (the radius decreases) until it hits the future singularity. It is essential to note the fact that as the worldline of the surface crosses the event horizon there is no turning back. There is no singularity in the past though we get one in the future at $r = 0$. Note the difference between the future timelike infinity $\imath^+$ and the singularity.

Bibliography

[1] Abramowitz, M. and Stegun, I.A. (eds.) 1965, *Handbook of Mathematical Functions* (Dover, New York).

[2] Birrell, N.D. and Davies, P.C.W. 1989, *Quantum fields in curved space* (Cambridge University Press, Cambridge).

[3] Carroll, S. 1997, *Lecture Notes in General Relativity*. arXiv:gr-qc/9712019v1.

[4] d' Inverno, R. 1992, *Introducing Einstein's Relativity* (Oxford University Press, Oxford)

[5] Detweiler, D. 1980, Phys. Rev D **22**, 10

[6] Dewitt, B.S. 1975, *Quantum Field Theory in Curved Spacetime* (Physics Reports, North Holland Publishing Company)

[7] Greiner, W. and Reinhardt, J. 1996, *Field Quantization* (Springler-Verlag, Berlin)

[8] Hawking, S.W. 1975, Commun. Math. Phys. **43**, 199

[9] Landau, L.D. and Lifshitz, E.M. 1976, *Course on Theoretical Physics Volume 1:Mechanics* 3^{rd} ed. (Butterworth Heinemann)

[10] Landau, L.D. and Lifshitz, E.M. 1976, *Course on Theoretical Physics Volume 2:Classical Theory of fields* 4^{th} revised ed. (Pergamom Press)

[11] Misner, C.W., Thorne, K.S. and Wheeler, J.A. 1973, *Gravitation* (Freeman, San Francisco)

[12] Mukhanov, V.F. and Winitzki, S., 2007, *Introduction to Quantum Effects in Gravity* (Cambridge University Press, Cambridge)

[13] Poisson, E. 2004, *A RELATIVIST'S TOOLKIT The mathematics of Black-Hole Mechanics* (Cambridge University Press, Cambridge)

[14] Price, R.H. 1972, Phys. Rev. D **5**,10

[15] Thorne, K.S. 1994, *Black Holes and Time Warps Einstein's Outrageous Legacy* (W.W. Norton and Company, New York)