Radiative Processes in Astrophysics

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in
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by

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CERTIFICATE

This is to certify that the project titled **Radiative Processes in Astrophysics** is a bona fide record of work done by **Darshan M Kakkad** towards the partial fulfillment of the requirements of the Master of Science degree in Physics at the Indian Institute of Technology-Madras, Chennai, India.

(L. Sriramkumar, Project supervisor)

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ABSTRACT

Astrophysical objects are distant and most of them have extreme environments such that none of the modern day probes would survive in them. Since we have access to the radiation emitted by them, we should gain an understanding of the properties of the astrophysical source by observing these radiations. In this report we have developed a theoretical framework of synchrotron and bremsstrahlung emission mechanisms. Synchrotron emission is expected to occur in pulsars, active galactic nuclei, even planets such as Jupiter and many other systems where there is an interaction of plasma with high magnetic fields. Bremsstrahlung on the other hand occurs due to an interaction of two unlike charge particles. Our aim is to characterize the properties of these radiations and eventually determine the properties of the astrophysical source from their observation.

Contents

Chapter 1

Introduction

The universe is typically observed through electromagnetic radiation around us which is ubiquitous due to emission from stars, galaxies and other billions of astrophysical systems. We can only access the radiation to get information about the sources such as the emission mechanism, physical properties, their morphology etc.. Hence, it is essential to have a theoretical understanding of the different radiation processes to understand the formation and evolution of these astrophysical objects. A classic example would be that of a pulsar which is a highly magnetized rotating neutron star that emits a beam of electromagnetic radiation. Most of the pulsars emit electromagnetic pulses in the radio frequency band and hence went undetected until the advent of radio telescopes. Similarly, there are radiations from other frequency regimes as well and the obvious question in hand is to understand the reasons for radiation.

There are numerous mechanisms through which electromagnetic radiation is emitted. For example, the Sun radiates energy due to nuclear fusion reactions where protons collide with each other to form helium. A tremendous amount of energy is released in this process in the form of electromagnetic radiation and heat. There are emissions due to electrons moving from excited states to lower energy states in atoms which leads to discrete spectra. Also, there are classical processes where the radiation is emitted due to the interaction of charges with electric and magnetic fields. These interactions cause the charges to accelerate and many interesting phenomenon arise due to this as we will see in the chapters that follow. We shall consider these classical processes in this report. Our essential aim is to understand the mechanism behind radiation to characterize the sources. A theoretical framework is addressed which should be compared with observations to check the credibility of our analysis.

In this report, we shall focus on two classical radiation mechanisms: synchrotron radiation and bremsstrahlung. Generally, we are interested in the spectrum of the radiation from the astrophysical objects. Hence, we shall focus on the spectral properties of the radiation and the factors which change them. Later, we study simple cases specific to these processes and understand how different physical properties such as mass, distribution functions, radius etc.. could be inferred from the spectra.

We shall start with understanding the motion of charge particles in electromagnetic fields, which we will take up in chapter 2. This will be followed by derivation of the equations governing the dynamics of the field itself in chapter 3. A solution to these dynamical equations would then prove that accelerating charges emit radiation which has different characteristics in the relativisitic and non-relativisitic limits.

With the preliminary results in hand, we go on to study synchrotron radiation in detail in chapter 4. We will arrive at the spectrum due to a single electron which will be generalized to a distribution of electrons to obtain an overall spectrum. We will take into account effects such as absorption to complete the picture. In chapter 5, we analyze bremsstrahlung which is due to the interaction of charge particles with a Coulomb field. Again, we focus on spectral properties of the radiation due to a collection of charges moving in a Coulomb field in the relativistic as well as non-relativisitic limits. Later, we study simple cases in chapter 6 to illustrate how different properties of the source could be derived using the spectral characteristics.

Chapter 2

Motion of charges in electromagnetic fields

Before we begin analysis of any radiation mechanism, it is important to understand the effects of interaction between a charge and the electric and magnetic fields. Different field configurations give rise to different particle trajectories. In this chapter, we develop the action of a particle in an electromagnetic field and study the motion of these particles under specific field configurations.

2.1 Action of a particle in an electromagnetic field

One of our first aim is to arrive at the equation of motion of a charge particle in a given field configuration. We shall achieve this using *action* formalism. From the elementary classical field theory, we know that for a mechanical system there exists a certain integral $\mathcal{S}$, called the *action*, which has a minimum value for an actual motion. For a charge particle interacting with an electromagnetic field, we have the action [1] as

$$\mathcal{S}[x^{\mu}] = \int_{a}^{b}(-mcds - \frac{e}{c}A_{\mu}dx^{\mu}), \tag{2.1}$$

where the first term in the integral corresponds to the action of a free particle and the second term is that of the particle interacting with the field. $\int_a^b$ is an integral along the world line of the particle between the two particular events of the arrival of the particle at the initial position and the final position at definite times t_1 and t_2 with a and b as the corresponding two points on the world line of the particle. Note that $A_{\mu} = (\phi, \boldsymbol{A})$ is the four vector potential with ϕ the scalar potential and $\boldsymbol{A}$ the vector potential and $ds = dx_{\mu}dx^{\mu}$ is the line element.

Writing the above action in terms of ϕ and $\boldsymbol{A}$ and noting that $ds = cd\tau$, where $d\tau$ is the proper time, we obtain

$$\mathcal{S} = \int_{t_1}^{t_2} (-mc^2\sqrt{1-\frac{v^2}{c^2}}dt - e\phi dt + \frac{e}{c}\boldsymbol{A}\cdot\frac{d\boldsymbol{r}}{dt}dt) = \int_{t_1}^{t_2} L dt, \tag{2.2}$$

where

$$L = -mc^2\sqrt{1-\frac{v^2}{c^2}} - e\phi + \frac{e}{c}\boldsymbol{A}\cdot\boldsymbol{v}. \tag{2.3}$$

L is the Lagrangian of a particle in an electromagnetic field. The first term is the contribution from the free particle, second and the third term represent the interaction of the charge with the field. With this in hand, we are in a position to deduce the equations of motion in various field configurations as described in the following sections.

2.2 Equations of motion

The equation of motion of a particle is obtained using the principle of least action which states that for an actual motion, the variation in $\mathcal{S}$ with respect to the trajectory of the particle is zero. This gives us the following Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) - \left(\frac{\partial L}{\partial r}\right) = 0, \tag{2.4}$$

for L given by Eq. (2.3). We can also vary this action keeping the trajectory constant and varying the potentials ϕ and $\boldsymbol{A}$. In this case we get the dynamical equations describing the field, which we will take up in a later section. Evaluating this expression term by term

$$\frac{\partial L}{\partial \boldsymbol{r}} = \frac{e}{c}\{(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{A} + \boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{A})\} - e\boldsymbol{\nabla}\phi, \tag{2.5}$$

$$\frac{\partial L}{\partial \boldsymbol{v}} = \frac{\partial}{\partial \boldsymbol{v}}\left(-mc^2\sqrt{1-\frac{v^2}{c^2}} - e\phi + \frac{e}{c}\boldsymbol{A}\cdot\boldsymbol{v}\right) = \boldsymbol{p} + \frac{e}{c}\boldsymbol{A}, \tag{2.6}$$

where $\boldsymbol{p}$ is the generalized momentum. It is now a matter of substituting the expressions (2.5) and (2.6) back into the Euler-Lagrange equation (2.4) to arrive at the equation of motion of a particle in an electromagnetic field. It is found to be

$$\frac{d\boldsymbol{p}}{dt} = -\frac{e}{c}\frac{\partial \boldsymbol{A}}{\partial t} - e\boldsymbol{\nabla}\phi + \frac{e}{c}\boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{A}). \tag{2.7}$$

Let us define electric and magnetic fields in terms of the potentials as

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \boldsymbol{\nabla}\phi, \qquad\qquad \boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}. \tag{2.8}$$

Eq. (2.7) in terms of these fields become

$$\frac{d\boldsymbol{p}}{dt} = e\boldsymbol{E} + \frac{e}{c}\boldsymbol{v}\times\boldsymbol{B}. \tag{2.9}$$

Eq. (2.9) relates the time derivative of the momentum of a particle to the total force acting on it due to a combination of electric and magnetic fields. We call this the Lorentz force equation. It is easy to see that the force due to a magnetic field acts only on charges in motion which have a component of velocity perpendicular to the magnetic field. We can find the rate of doing work by taking the dot product of Eq. (2.9) with velocity to obtain

$$\frac{d\mathcal{E}_{kin}}{dt} = e\boldsymbol{v}\cdot\boldsymbol{E}, \tag{2.10}$$

where $\mathcal{E}_{kin}$ is the kinetic energy of the particle. We see that there is no contribution from the magnetic field term. Consequently, only the electric field does work on a moving charge and not the magnetic field. So, for a particle moving in an electric field free region, its kinetic energy and hence the speed does not change. In the next few subsections, we will derive the particle trajectories corresponding to specific cases.

2.2.1 Motion in a uniform electric field

Assume a uniform electric field of magnitude E to be present along the x-axis and the motion to be confined on the x-y plane. For simplicity, we take the magnetic field to be zero. Under these conditions, the x and y components of Eq. (2.9) can be written as

$$\dot{p}_x = eE, \qquad\qquad \dot{p}_y = 0. \tag{2.11}$$

The momentum along y-axis is a constant, say p_o. For the equation of motion along x direction, we use the relativistic relation between momentum and velocity which is given by

$$\boldsymbol{v} = \frac{c^2}{\mathcal{E}_{kin}}\boldsymbol{p} \tag{2.12}$$

where $\mathcal{E}_{kin}$ is the kinetic energy of the particle. Using the x component of Eqs. (2.11) and (2.12), we obtain

$$v_x = \frac{c^2}{\mathcal{E}_{kin}}p_x = \frac{c^2}{\mathcal{E}_{kin}}eEt, \tag{2.13}$$

$$\mathcal{E}_{kin} = \sqrt{\mathcal{E}_o^2 + (eEct)^2}, \tag{2.14}$$

where $\mathcal{E}_o^2 = m^2c^2 + p_o^2$ is the energy at $t = 0$. Substituting the expression for $\mathcal{E}_{kin}$ into Eq. (2.13), we obtain

$$\frac{dx}{dt} = \frac{c^2eEt}{\sqrt{\mathcal{E}_o^2 + (eEct)^2}}. \tag{2.15}$$

Upon integrating, we obtain the trajectory along the x direction as a function of time as follows

$$x = \frac{1}{eE}\sqrt{\mathcal{E}_o^2 + (eEct)^2}. \tag{2.16}$$

Similarly, we obtain the trajectory along y direction as a function of time to be

$$y = \frac{p_oc}{eE} sinh^{-1}(\frac{ceEt}{\mathcal{E}_o}). \tag{2.17}$$

Eliminating the parameter t from the Eqs. (2.16) and (2.17), we arrive at the relativistic trajectory in the x-y plane to be

$$x = \frac{\mathcal{E}_o}{eE} cosh\left(\frac{eEy}{p_oc}\right). \tag{2.18}$$

In the non-relativistic limit i.e. when v $\ll$ c, $p_o = mv_o$ and $\mathcal{E}_o = mc^2$. Applying these approximations to the relativistic equation and expanding the hyperbolic term yields

$$x = \frac{mc^2}{eE}\left(1 + \frac{e^2E^2y^2}{m^2v_o^2c^2}\right), \tag{2.19}$$

which is the equation of a parabola, as one would expect for a non-relativistic motion under constant electric field.

2.2.2 Motion in a uniform magnetic field

Again for simplicity, we shall assume the magnetic field to be present only along the z direction, $\boldsymbol{B} = B\hat{k}$ and the electric field to be zero. Using the relativistic relation between velocity and momentum given in Eq. (2.12), we get

$$\frac{\mathcal{E}_{kin}}{c^2}\frac{d\boldsymbol{v}}{dt} = \frac{e}{c}\boldsymbol{v} \times \boldsymbol{B}. \tag{2.20}$$

We shall again solve this equation component by component. The z-component of the acceleration is zero as the magnetic field is only along the z-axis. So there is no change in the z component of momentum and hence

$$\dot{v}_z = 0, \qquad \Rightarrow z = kt + z_o, \tag{2.21}$$

where k and z_o are constants to be determined from the initial conditions. Momentum relations along x and y directions yield

$$\frac{\mathcal{E}_{kin}\dot{v}_x}{c^2} = \frac{e}{c}v_y B, \qquad \frac{\mathcal{E}_{kin}\dot{v}_y}{c^2} = -\frac{e}{c}v_x B. \tag{2.22}$$

To solve the two expressions in Eq. (2.22), we shall combine them using complex variables as follows

$$\dot{v}_x + i\dot{v}_y = -i\omega(v_x + iv_y), \tag{2.23}$$

where $\omega \approx eH/mc$. It is easy to integrate the above expression with $v_x + iv_y$ as the variable of integration. The result is given below

$$v_x + iv_y = ae^{-i\omega t}, \tag{2.24}$$

where a is a general complex number and can be written as $a = v_o e^{-i\alpha}$. Here v_o and α are real constants. Substituting the expression for a in Eq. (2.24) and separating the real and imaginary parts, we obtain

$$v_x = v_o cos(\omega t + \alpha), \qquad v_y = -v_o sin(\omega t + \alpha). \tag{2.25}$$

The trajectories along x and y direction follow from integrating Eqs. (2.25) as

$$x = x_o + \frac{v_o}{\omega} sin(\omega t + \alpha), \qquad y = y_o + \frac{v_o}{\omega} cos(\omega t + \alpha), \tag{2.26}$$

where x_o and y_o are constants of integration to be determined from the initial conditions. Along with the z Eq. (2.21) this represents the motion of the particle along a helix as shown in Fig. 2.1. If z-component of velocity is zero, the motion is circular in the x-y plane. We also get a circular trajectory in the non-relativisitic case. However, as we shall see in Sec. 4.1, from an observational point of view we can distinguish these two cases by analyzing the spectrum of the radiation emitted by the charge particles.

2.2.3 Motion in a uniform electric and magnetic field

Up till now, we examined the motion of a charge particle in the presence of electric and magnetic fields separately. In practice, we come across a combination of the fields which we

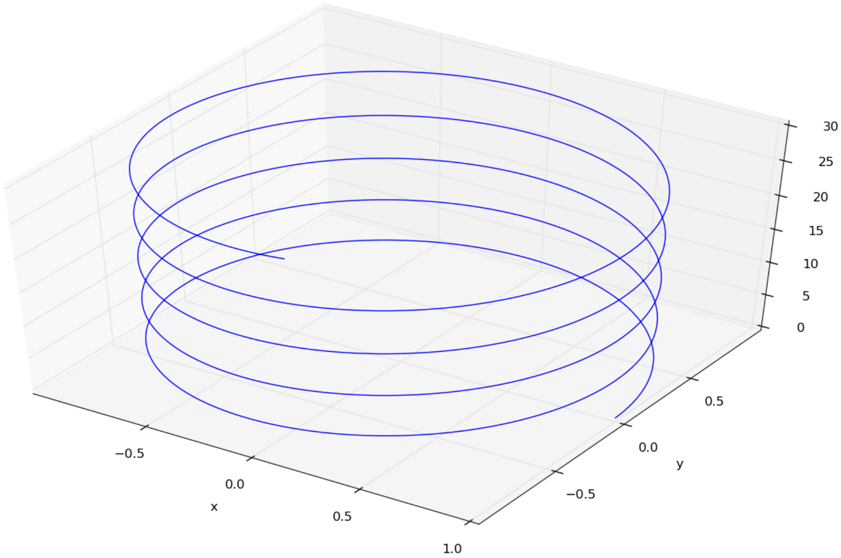


Figure 2.1: Helical motion in a uniform magnetic field

will now take up for discussion. We shall assume the non-relativistic limit of Eq. (2.9) as follows

$$m\frac{d\boldsymbol{v}}{dt} = e\boldsymbol{E} + \frac{e}{c}\boldsymbol{v}\times\boldsymbol{B}. \tag{2.27}$$

We again follow the heuristic approach to solutions under the following assumptions. The magnetic intensity, $\boldsymbol{B}$ is only along the z direction and the electric field $\boldsymbol{E}$ along x direction is assumed to be zero to make calculations simpler. The equation corresponding to the z direction can be easily obtained as given below

$$m\ddot{z} = eE_z, \qquad \Rightarrow mz = \frac{eE_z t^2}{2} + v_{oz}t, \tag{2.28}$$

where the second equation is obtained on integrating the first equation twice, v_{oz} being a constant of integration. The x and y equations can be coupled as we did in Sec. 2.2.2 and it takes the following form here:

$$m(\ddot{x} + i\ddot{y}) = ieE_y + \frac{e}{c}yB - \frac{iexB}{c}, \tag{2.29}$$

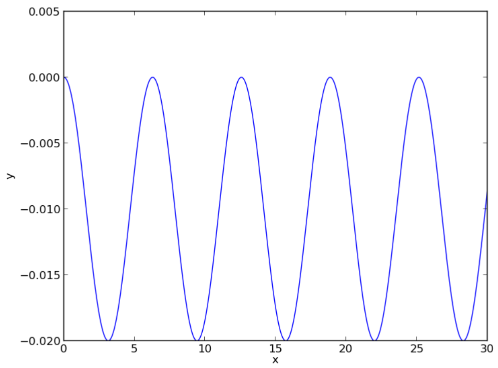
Integrating once using $\dot{x} + i\dot{y}$ as the variable gives us

$$\dot{x} + i\dot{y} = ae^{-i\omega t} + \frac{cE_y}{B}, \tag{2.30}$$

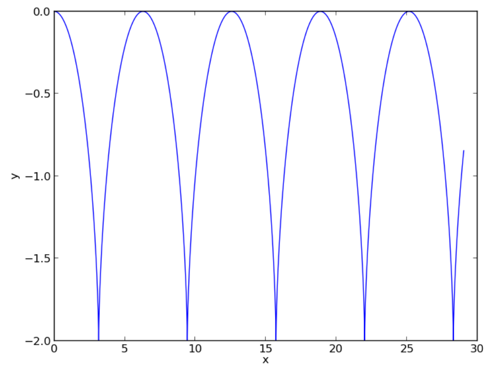
where a is a generic complex number which could be written as $a = be^{i\alpha}$. Constants b and α are real and determined by initial conditions. Real and imaginary parts of Eq. (2.30) give the trajectories along x and y directions respectively after integration as follows

$$x = \frac{a}{\omega} sin(\omega t) + \frac{cE_y t}{B}, \qquad y = \frac{a}{\omega}(cos(\omega t) - 1). \tag{2.31}$$

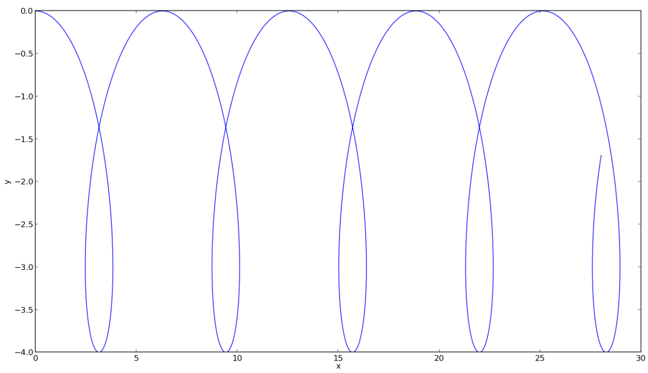
Together with Eq. (2.28), these determine the motion of a particle in a combination of electric and magnetic fields which is dependent on the nature of a. The different cases are depicted in Fig. 2.2 [1].



(a) $|a| < cE_y/B$



(b) $|a| = cE_y/B$



(c) $|a| > cE_y/B$

Figure 2.2: Motion in electric and magnetic field depends on the nature of a or in other words, the ratio of the electric and magnetic field. Relative magnitude of the fields change the trajectory of the particle under different circumstances [1].

2.3 Electromagnetic field tensor

In Sec. 2.2, we derived the equations of motion in the conventional three dimensional form. Now we shall derive the equation of motion of a particle in a covariant form [1]. As mentioned in Sec. 2.1, the action describing a particle in an electromagnetic field can be written as

$$\mathcal{S}[x^\mu] = \int_a^b (-mcds - \frac{e}{c}A_\mu dx^\mu). \tag{2.32}$$

To obtain the equations of motion, we vary the action with respect to the trajectory and expect $\delta\mathcal{S} = 0$ for the actual motion. Using $ds = \sqrt{dx_\mu dx^\mu}$ and $v_\mu = dx_\mu/ds$, we obtain

$$\int (mcv_\mu d\delta x^\mu + \frac{e}{c}A_\mu d\delta x^\mu + \frac{e}{c}\delta A_\mu dx^\mu) = 0. \tag{2.33}$$

v_μ is the four velocity of the particle. Integrating the first term by parts and noting that the end points of the trajectory are kept constant, i.e. $\delta x^\mu = 0$, we get the following result

$$\int (mcdv_\mu \delta x^\mu + \frac{e}{c}\delta x^\mu dA_\mu - \frac{e}{c}\delta A_\mu dx^\mu) = 0. \tag{2.34}$$

Now $\delta A_\mu = (\partial A_\mu/\partial x^\nu)\delta x^\nu$ and $\mathrm{d}A_\mu = (\partial A_\mu/\partial x^\nu)dx^\nu$. Substituting these expressions back into Eq. (2.33) and after a simple algebra we arrive at

$$mc\frac{dv_\mu}{ds} = \frac{e}{c}\left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}\right)v^\nu. \tag{2.35}$$

Let us call the term inside the brackets as $F_{\mu\nu}$ which is called the *electromagnetic field tensor*, a 4×4 matrix, the elements of which constitute different components of the electric and magnetic fields. Rewritting Eq. (2.35) in terms of the field tensor, $F_{\mu\nu}$, we get

$$mc\frac{dv_\mu}{ds} = \frac{e}{c}F_{\mu\nu}v^\nu, \tag{2.36}$$

where

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{2.37}$$

Eq. (2.36) is the equation of motion of a charge particle in electric and magnetic fields in covariant form. It is easy to see that Eq. (2.36) gives back Eq. (2.9) when the field tensor is expanded. The electromagnetic field tensor characterizes the fields and has many utilities in classical and quantum field theory. It remains invariant under a gauge transformation i.e. under $A_\mu \to A_\mu + \partial_\mu \Lambda$ where Λ is a scalar function. From the definition, it is easy to see that it is an antisymmetric tensor with 6 independent components: 3 electric field components and 3 magnetic field components. As we shall see in the next chapter, this tensor reduces and simplifies Maxwell's equations as four non-covariant equations to two covariant ones and it is highly useful in simplification of algebra involving field derivations.

Chapter 3

Radiation from an accelerating charge

A charge moving with constant velocity creates a Coulomb field and magnetic field due to its motion. However, a charge under acceleration creates an additional radiation field which has many interesting features which will be later useful for the analysis of different radiation mechanisms such as synchrotron emission. In this chapter, our aim is to arrive at the energy radiated by an accelerating charge in the relativistic as well as non-relativistic limits. A generalized expression for the emission spectrum will be obtained towards the end of this chapter.

3.1 First pair of Maxwell's equations

We derived the properties of a charge particle under motion in an electromagnetic field in chapter 2. In this section and Sec. 3.3, we will arrive at equations which describe the properties of the field itself. From the electromagnetic field tensor, $F_{\mu\nu}$ derived in Sec. 2.3, we can easily show the following identity [1]

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0. \tag{3.1}$$

Expanding the indices μ,ν and λ and using the components of the field tensor derived in Sec. 2.3, it is easy to derive the non-covariant forms as given below

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{3.2}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0. \tag{3.3}$$

These form the first pair of Maxwell's equations which are source free equations. These define the constraints on the dynamical fields $\boldsymbol{E}$ and $\boldsymbol{B}$. Eq. (3.2) is nothing but the Ampere's law which states that an electric field is induced on changing magnetic flux while Eq. (3.3) states that the divergence of $\boldsymbol{B}$ is zero which means that the magnetic field lines exist in closed loops and that magnetic monopoles do not exist.

3.2 Action governing the electromagnetic fields

Up till now, we have been concentrating on the action function of a particle in an electromagnetic field. To complete the picture, we shall now arrive at the form of the action describing the electromagnetic field itself using the following basic arguments [1]:

- The electromagnetic fields follow the principle of superposition. Hence the action function should contain a quadratic term which upon variation leads to a linear equation of motion consistent with our requirement for superposition.
- Potentials could not enter the expression for the action as they are not uniquely determined. The action should be gauge invariant and the best candidate for this would be to make use of the field tensor $F_{\mu\nu}$.
- Finally action has to be a scalar quantity which could be described by, say $F_{\mu\nu}F^{\mu\nu}$.

Based on the points above, we can write the action describing the electromagnetic fields as

$$\mathcal{S}_f = -\frac{1}{16\pi c}\int F_{\mu\nu}F^{\mu\nu}d\Omega, \tag{3.4}$$

where $d\Omega = cdtd^3\boldsymbol{x}$ and the constant $-1/(16\pi c)$ is determined by experiments. Together with Eq. (2.1), the overall action of a particle in an electromagnetic field can now be written as

$$\mathcal{S}[x^\mu] = -\sum\int mcds - \sum\int \frac{e}{c}A_k dx^k - \frac{1}{16\pi c}\int F_{\mu\nu}F^{\mu\nu}d\Omega, \tag{3.5}$$

where the first term represents the action of a collection of free particles, the second term describes the interaction of the charge particle with the electromagnetic field and the last term the action of electromagnetic field itself. The summation in the first two terms of Eq. (3.5) has been done over all the charge particles interating with the electromagnetic field. Note that the action described by Eq. (3.5) satisfies all the requirements described in the points above such as gauge invariance, requirement of scalar function etc..

3.3 Second pair of Maxwell's equations

Using the action proposed in the previous section, we shall now derive the dynamical field equations or the second pair of Maxwell's equations involving the sources [1]. We use the principle of least action again, however this time we will vary the potentials instead of the trajectory. Varying the action in Eq. (3.5), we get

$$\delta\mathcal{S} = -\int \frac{1}{c}\left(\frac{1}{c} j^{\mu}\delta A_{\mu} + \frac{1}{8\pi} F_{\mu\nu}\delta F^{\mu\nu}\right) d\Omega = 0. \tag{3.6}$$

Expand $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ and subsequent evaluation gives

$$\delta\mathcal{S} = -\int \frac{1}{c}\left(\frac{1}{c} j^{\mu}\delta A_{\mu} - \frac{1}{4\pi} F^{\mu\nu}\frac{\partial}{\partial x^{\nu}}\delta A_{\mu}\right) d\Omega. \tag{3.7}$$

Integrating the second part of Eq. (3.7) by parts, we obtain

$$\delta\mathcal{S} = -\frac{1}{c}\int \left(\frac{1}{c} j^{\mu} + \frac{1}{4\pi}\frac{\partial}{\partial x^{\nu}} F^{\mu\nu}\right)\delta A_{\mu} d\Omega - \frac{1}{4\pi c}\int F^{\mu\nu}\delta A_{\mu} dx_{\nu}, \tag{3.8}$$

where $j^{\mu} = (\rho c, \rho \boldsymbol{v})$ is the four current. The second term of Eq. (3.8) vanishes at the limits $(-\infty \to \infty)$ as it is a surface term. Since the factors outside the first term in Eq. (3.8) are arbitrary, we can set the factor inside the brackets to be zero

$$\frac{\partial}{\partial x^{\nu}} F^{\mu\nu} = -\frac{4\pi}{c} j^{\mu}. \tag{3.9}$$

It is easy to see by expanding $F_{\mu\nu}$ that this covariant equation results in the following two equations in non-covariant form

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = 4\pi\rho, \tag{3.10}$$

$$\nabla\times\boldsymbol{B} = \frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t} + \frac{4\pi}{c}\boldsymbol{j}. \tag{3.11}$$

This is the second pair of Maxwell's equations involving the sources. Eqs. (3.1) and (3.9) are the two pairs of Maxwell's equation in covariant form. While the first pair determine the constraints on the field, the second pair describes the dynamical equations. Maxwell's equations form the foundation of classical electrodynamics. They describe some of the fundamental laws of electromagnetic theory such as the continuity equation [1] for charge currents given by

$$\partial_{\mu} j^{\mu} = 0, \tag{3.12}$$

which follows from Eq. (3.9). Here again j^μ is the four current. In non-covariant form, this could be expressed as

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0, \tag{3.13}$$

which is the conventional three dimensional equation describing conservation of charge. In the following section, we will obtain the scalar and vector potentials due to a moving charge using the Green's function for the Maxwell's equations which will allow us to calculate radiation fields.

3.4 Field of moving charges

A charge at rest creates a static electric field around it while a charge in motion produces both electric as well as magnetic fields which will be apparent from the solutions of the Maxwell's equations obtained in the following discussion. Using an appropriate choice of gauge, Maxwell's equations reduce to wave equations whose solutions can be obtained using Green's functions. It is important to emphasize that these waves travel with a finite velocity. Hence, the effects due to the fields are not instantaneous but takes a finite time.

3.4.1 Retarded Green's functions

As mentioned above, a moving charge creates an electromagnetic field around it. This field propagates with a finite velocity equal to the speed of light in vacuum. Hence, the effect of this emission at any point happens after a finite amount of time given by R/c where R is the distance from the source where the effect is measured. In this section, we shall develop the relevant equations describing this process [1, 2]. We will modify the Maxwell's equations derived in the previous sections in a convenient form which will allow us to calculate the fields easily. Eqs. (3.1) and (3.9) could be written in terms of the four potentials using the relations in Eq. (2.8) as

$$\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{\nabla}\cdot\boldsymbol{A} = -4\pi\rho, \tag{3.14}$$

$$\nabla^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} - \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\boldsymbol{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right) = -\frac{4\pi}{c}\boldsymbol{j}. \tag{3.15}$$

If we choose the Lorentz gauge i.e. $\boldsymbol{\nabla}\cdot\boldsymbol{A}+(\partial\phi)/(c\partial t)=0$, then the above two equations get uncoupled leading to a general form as follows

$$\nabla^2\psi-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}=-4\pi f(\boldsymbol{x},t), \tag{3.16}$$

where $f(\boldsymbol{x},t)=\boldsymbol{j}(\boldsymbol{x},t)/c$ if $\psi=\boldsymbol{A}$ and $f(\boldsymbol{x},t)=\rho(\boldsymbol{x},t)$ if $\psi=\phi$. Notice that we have now included the time dependence of the potentials as well. Eq. (3.16) can be solved using Green's functions which is a conventional tool for solving inhomogenous differential equations. First we find the solutions to the following equation

$$\left(\nabla_x^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\boldsymbol{x},t;\boldsymbol{x'},t')=-\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x'})\delta^{(1)}(t-t'), \tag{3.17}$$

where $G(\boldsymbol{x},t;\boldsymbol{x}',t')$ is the appropriate Green's function for the given problem. The function ψ can then be found using

$$\psi(\boldsymbol{x})=\int G(\boldsymbol{x},t;\boldsymbol{x}',t')f(\boldsymbol{x}',t')d^3x'dt'. \tag{3.18}$$

To find the solution of Eq. (3.17), let us write the delta function as

$$\delta^{(3)}(\boldsymbol{x}-x')\delta^{(1)}(t-t')=\frac{1}{(2\pi)^4}\int d^3k\int d\omega e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}e^{-i\omega(t-t')}. \tag{3.19}$$

Using inverse Fourier transform, we can write $G(\boldsymbol{x},t;\boldsymbol{x}',t')$ as

$$G(\boldsymbol{x},t;\boldsymbol{x'},t')=\int d^3k\int d\omega g(\boldsymbol{k},\omega)e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}e^{-i\omega(t-t')}. \tag{3.20}$$

Substituting expressions (3.19) and (3.20) back into Eq. (3.17) and comparing the two sides of the equation, we arrive at

$$g(\boldsymbol{k},\omega)=\frac{1}{4\pi^3}\frac{1}{k^2-\frac{\omega^2}{c^2}}. \tag{3.21}$$

Substitute the expression for $g(\boldsymbol{k},\omega)$ in Eq. (3.20), we obtain

$$G(\boldsymbol{x},t;\boldsymbol{x'},t')=\int d^3k\int d\omega\left(\frac{1}{4\pi^3}\frac{1}{k^2-\frac{\omega^2}{c^2}}\right)e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}e^{-i\omega(t-t')}. \tag{3.22}$$

Since this integration is performed over the entire space of k and ω, there is a singularity at $k^2=\omega^2/c^2$. Also, the wave disturbance is created by a source at point $\boldsymbol{x}'$ at time t'. So, to satisfy causality i.e. the effects due to a field could not be observed before its emission, $G = 0$ for $t' < t$ and G must represent outgoing waves for $t > t'$. Let us rewrite the Eq. (3.22) as

$$G(\boldsymbol{x}, t; \boldsymbol{x}', t') = \frac{1}{4\pi^3} \int d^3k \int d\omega \frac{e^{i(\boldsymbol{k}.\boldsymbol{R} - \omega\tau)}}{k^2 - \frac{1}{c^2(\omega + i\epsilon)^2}}, \tag{3.23}$$

where $\boldsymbol{R} = \boldsymbol{x} - \boldsymbol{x}'$, $\tau = (t - t')$ and ϵ is very small. Using basic contour integration over ω, we obtain the expression for the Green's function as follows

$$G(\boldsymbol{R}, \tau) = \frac{c}{2\pi^2} \int d^3k e^{i\boldsymbol{k}.\boldsymbol{R}} \frac{sin(c\tau k)}{k}. \tag{3.24}$$

Integration over k using $d^3k = k^2 sin\theta d\theta d\phi$ and writing $\boldsymbol{k} \cdot \boldsymbol{R} = kRcos\theta$ gives the retarded Green's function as

$$G_r(\boldsymbol{x}, t; \boldsymbol{x}', t') = \frac{\delta^{(4)}(ct' + |\boldsymbol{x} - \boldsymbol{x}'| - ct)}{|\boldsymbol{x} - \boldsymbol{x}'|} \Theta(ct - ct'), \tag{3.25}$$

where $\Theta(ct - ct')$ imposes the causality condition. The above equation clearly expresses the idea that the effect observed at point $\boldsymbol{x}$ at time t is due to a disturbance which originated at an earlier or retarded time $ct' = ct - |\boldsymbol{x} - \boldsymbol{x}'|$ at point $\boldsymbol{x}'$ (hence the name *retarded* Green's function). In the integral in Eq. (3.22), we chose our contour in such a way so as to arrive at the retarded Green's function. However, another choice of contour can lead to advanced Green's function, which violates causality but is also a solution of Eq. (3.17). Advanced Green's function is given by

$$G_a(\boldsymbol{x}, t; \boldsymbol{x}', t') = \frac{\delta^{(4)}(ct' - |\boldsymbol{x} - \boldsymbol{x}'| - ct)}{|\boldsymbol{x} - \boldsymbol{x}'|} \Theta(ct' - ct). \tag{3.26}$$

We can put both forms of Green's functions in covariant form using the following identity

$$\begin{aligned} \delta^{(4)}[(x - x')^2] &= \delta^{(4)}[(ct - ct')^2 - |\boldsymbol{x} - \boldsymbol{x}'|^2], & (3.27) \\ &= \frac{1}{2R}[\delta^{(4)}(ct - ct' - \boldsymbol{R}) + \delta^{(4)}(ct - ct' + \boldsymbol{R})], & (3.28) \end{aligned}$$

where $\boldsymbol{R} = |\boldsymbol{x} - \boldsymbol{x}'|$. The delta functions select any one of the Green's functions. Thus we can write the two forms of Green's function in terms of the expression (3.28) as

$$G_r = \frac{1}{2\pi} \Theta(ct - ct') \delta^{(4)}[(x - x')^2], \qquad G_a = \frac{1}{2\pi} \Theta(ct' - ct) \delta^{(4)}[(x - x')^2]. \tag{3.29}$$

Due to causality, we prefer to use the form of retarded Green's function.

3.4.2 Liénard-Wiechert potentials

With the appropriate Green's function in hand, we can now arrive at the potentials describing the fields. Using Eq. (3.18), the four potentials can be derived as

$$A^{\mu}(x) = \frac{4\pi}{c} \int d^4x' G(\boldsymbol{x}, t; \boldsymbol{x}', t') j^{\mu}(x'), \tag{3.30}$$

where j^{μ} is the four current which in this case can be written as

$$j^{\mu}(x') = ce \int d\tau v^{\mu}(\tau) \delta^{(4)}[x' - r^{\mu}(\tau)], \tag{3.31}$$

where v^{μ} is the four velocity. Substituting the expressions for $j^{\mu}(x')$ and $G(\boldsymbol{x}, t; \boldsymbol{x}', t')$ obtained in Sec. 3.4.1 in Eq. (3.30), we get

$$\begin{aligned} A^{\mu}(x) &= \frac{4\pi}{c} \int d^4x' \frac{1}{2\pi} \Theta(ct - ct') \delta[(x - x')^2] ce \int d\tau v^{\mu}(\tau) \delta^{(4)}(x' - r(\tau)), && (3.32) \\ &= 2e \int d\tau v^{\mu}(\tau) \Theta(ct - r_o(\tau)) \delta^{(4)}[(x - r(\tau))^2]. && (3.33) \end{aligned}$$

This equation is non-zero only when $[x - r(\tau_o)]^2 = 0$ for some $\tau = \tau_o$ and $ct > r_o(\tau)$, which follows from causality. The above integral can be solved using the following identities

$$\delta^{(n)}[f(x)] = \sum_i \frac{\delta^{(n)}(x - x_i)}{|df/dx|\Big|_{x_i}}, \tag{3.34}$$

where x_i are the zeroes of $f(x)$. In our case $f(x) = [x - r(\tau)]^2$ and hence

$$\frac{d}{d\tau}[x - r(\tau)]^2 = -2(x - r(\tau))_{\beta} v^{\beta}(\tau). \tag{3.35}$$

Using the expressions obtained in Eqs. (3.34) and (3.35) in Eq. (3.33), we obtain

$$\begin{aligned} A^{\mu}(x) &= \frac{a v^{\mu}(\tau)}{v \cdot [x - r(\tau)]}\Big|_{\tau_o}, && (3.36) \\ &= \frac{a v^{\mu}(\tau_o)}{\gamma c \boldsymbol{R} - \gamma \boldsymbol{v} \cdot \hat{n} R}. && (3.37) \end{aligned}$$

Expressing this result in the non-covariant form, we get the scalar and the vector potentials as

$$\phi(\boldsymbol{x}, t) = \left(\frac{e}{(1 - \boldsymbol{\beta} \cdot \hat{n}) R} \right)_{ret}, \qquad \boldsymbol{A}(\boldsymbol{x}, t) = \left(\frac{e \boldsymbol{\beta}}{1 - \boldsymbol{\beta} \cdot \hat{n}) R} \right)_{ret}, \tag{3.38}$$

where $\boldsymbol{\beta} = \boldsymbol{v}/c$ and the subscript "ret" implies that the expressions have been evaluated at retarded time. These equations clearly express the fact that the effect of a field is due to emission at an earlier or retarded time. As we have mentioned before, this is a consequence of the finite speed of electromagnetic waves.

3.4.3 Electromagnetic field due to accelerating charges

Recall that we are ultimately interested in determining the form of the field emitted by an accelerating charge. The fields can be determined using Eq. (2.8) where we can just substitute the expression for potentials obtained in Eq. (3.38). However, for making the algebra simple, we shall follow a different method as described below. Let us write the expression for electric field in covariant form

$$\boldsymbol{E} = -F^{0i} = -\partial^0 A^i + \partial^i A^0. \tag{3.39}$$

The second equality follows from the definition of field tensor. For simplicity, we shall begin with Eq. (3.33) and differentiate it as follows

$$\partial^\alpha A^\beta(x) = 2e\int d\tau v^\beta(\tau)\Big[\partial^\alpha \Theta(ct - r_o(\tau))\delta^{(4)}[(x-r(\tau))^2]\Big] + \Theta(ct - r_o(\tau))\partial^\alpha\delta^{(4)}[(x-r(\tau))^2]. \tag{3.40}$$

Clearly, the first term of Eq. (3.40) is non-zero only when $ct = r(\tau_o)$ or $[x - r(\tau)]^2 = -R^2 = 0$ due to the condition imposed by the delta function. So the contribution from the first term is only when $R = 0$. For the second term, we use the following identity

$$\partial^\alpha \delta[f] = \partial^\alpha f \frac{d\tau}{df}\frac{d}{d\tau}\delta[f], \tag{3.41}$$

with $f = [x - r(\tau)]^2$ in our case. Substituting this in Eq. (3.40) and further simplification leads to

$$\partial^\alpha A^\beta(x) = 2e\int d\tau \frac{d}{d\tau}\left(\frac{v^\beta (x-r(\tau))^\alpha}{v\cdot(x-r(\tau))}\right)\Theta(ct - r_o(\tau))\delta^{(4)}[(x-r(\tau))^2]. \tag{3.42}$$

The expression for the field tensor now takes the form

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha = 2e\int d\tau \frac{d}{d\tau}\left(\frac{(x-r)^\alpha v^\beta - (x-r)^\beta v^\alpha}{v\cdot(x-r)}\right)\Theta(ct - r_o(\tau))\delta^{(4)}[(x-r(\tau))^2]. \tag{3.43}$$

Integrating to eliminate the theta and delta function gives

$$F^{\alpha\beta} = \frac{e}{v\cdot(x-r)}\frac{d}{d\tau}\left(\frac{(x-r)^\alpha v^\beta - (x-r)^\beta v^\alpha}{v\cdot(x-r)}\right). \tag{3.44}$$

The differentiation can be carried out using elementary calculus. The intermediate results are mentioned below. We have used $(x-r)^\alpha = (R, R\hat{n})$ and $v^\alpha = (\gamma c, \gamma c\boldsymbol{\beta})$.

$$\frac{dv^o}{d\tau} = c\frac{d\gamma}{d\tau} = c\gamma^4 \boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}, \tag{3.45}$$

$$\frac{d\boldsymbol{v}}{d\tau} = c\gamma^2\dot{\boldsymbol{\beta}} + c\gamma^4\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}), \tag{3.46}$$

$$\frac{d}{d\tau}[v\cdot(x-r)] = -c^2 + (x-r)_\alpha \frac{dv^\alpha}{d\tau}, \tag{3.47}$$

$$= -c^2 + Rc\gamma^4\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}} - R\hat{n}c\gamma^2\dot{\boldsymbol{\beta}} - Rc\gamma^4(\hat{n}\cdot\boldsymbol{\beta})(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}). \tag{3.48}$$

Substituting the results of Eqs. (3.45) - (3.48) back into the expression for F^{0i}, hence $\boldsymbol{E}$, we arrive at the fields as given below. The magnetic field is calculated using $\boldsymbol{B} = \hat{n}\times\boldsymbol{E}$

$$\boldsymbol{E} = e\left[\frac{\hat{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\hat{n})^3R^2}\right] + \frac{e}{c}\left[\frac{\hat{n}\times\left\{(\hat{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right\}}{(1-\boldsymbol{\beta}\cdot\hat{n})^3R}\right], \tag{3.49}$$

$$\boldsymbol{B} = -\frac{e(\hat{n}\times\boldsymbol{\beta})}{\gamma^2(1-\boldsymbol{\beta}\cdot\hat{n})^3R^2} + \frac{e}{c}\frac{\left\{(\hat{n}\times\dot{\boldsymbol{\beta}})(1-\boldsymbol{\beta}\cdot\hat{n}) + (\hat{n}\times\boldsymbol{\beta})(\hat{n}\cdot\boldsymbol{\beta})\right\}}{(1-\boldsymbol{\beta}\cdot\hat{n})^3R}. \tag{3.50}$$

As we can see, both the electric and magnetic fields contain two terms - acceleration dependent and acceleration independent parts. The acceleration independent part for electric field gives us the Coulomb field and it is easy to see that under non-relativistic conditions or for a particle at rest, it reduces to the case of electrostatics where field is proportional to the inverse square of the distance R. Similarly for the magnetic field, it is easy to see that the accelerated independent part reduces to Biot-Savart law for non-relativistic particles. Due to acceleration, an additional term is introduced and the radiation emitted by this accelerated charge has interesting characteristics which we study below. We call this additional term as the radiation field. Unlike Coulomb field, radiation field is inversely proportional to the distance and hence it is the dominating term at large distances. Hence, it is important to study various aspects of this term as it finds many applications in astrophysical situations.

3.5 Energy radiated by an accelerating charge

As we have seen above, the field generated by an accelerated charge has an additional term which depends on the acceleration of the particle. Our aim is to find the power emitted due to this radiation field in the relativistic and non-relativistic limits. Let us concentrate on this second term in Eq. (3.49) and write it as $\boldsymbol{E}_a$.

$$\boldsymbol{E}_a = \frac{e}{c}\left[\frac{\hat{n}\times\left\{(\hat{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right\}}{(1-\boldsymbol{\beta}\cdot\hat{n})^3R}\right]. \tag{3.51}$$

For $v \ll c$ i.e. in the non-relativistic case, this reduces to

$$\boldsymbol{E}_a = \frac{e}{c}\left[\frac{\hat{n}\times(\hat{n}\times\dot{\boldsymbol{\beta}})}{R}\right]. \tag{3.52}$$

We can infer from the above expression that the electric field is confined in the plane of $\dot{\boldsymbol{\beta}}$ and $\hat{n}$. The instantaneous energy flux is given by the Poynting vector $\boldsymbol{S} = (c/4\pi)\boldsymbol{E}\times\boldsymbol{B}$ whose direction points to the direction of the flow of energy flux density. Using expression for $\boldsymbol{E}_a$ from Eq. (3.51), we obtain

$$\boldsymbol{S} = \frac{c}{4\pi}|\boldsymbol{E}_a|^2\hat{n}. \tag{3.53}$$

The magnitude of Poynting vector $|\boldsymbol{S}| = dP/dA$ gives the power emitted per unit area. Writing dA as $R^2 d\Omega$, $d\Omega$ being the solid angle, we get the power emitted per unit solid angle as

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{c}{4\pi}|R\boldsymbol{E}_a|^2, &(3.54)\\ &= \frac{e^2}{4\pi c}|\hat{n}\times(\hat{n}\times\dot{\boldsymbol{\beta}})|^2, &(3.55)\\ &= \frac{e^2}{4\pi c^3}|\dot{\boldsymbol{v}}|^2 sin^2\theta, &(3.56)\end{aligned}$$

where θ is the angle between $\hat{n}$ and $\boldsymbol{\beta}$. Hence, the angular distribution of the power emitted in case of a non-relativistic particle goes as the square of a sinusoidal pattern. A plot for this angular distribution is given in Fig. 3.1a. It is just a matter of integrating over the solid angle $d\Omega = sin\theta d\theta d\phi$ which will finally give us the power emitted by a charge particle moving with non-relativistic speed

$$P = \frac{2e^2}{3c^3}|\dot{\boldsymbol{v}}|^2. \tag{3.57}$$

Thus, the emitted power depends only on the acceleration of the particle and not its velocity.

Moving on to the case of a relativistic particle, the emitted power depends both on the velocity and acceleration as we see in the following discussion. Using Eqs. (3.51) and (3.53), the radial component of the Poynting vector can be written as

$$(\boldsymbol{S}\cdot\hat{n})_{ret} = \frac{c}{4\pi}\frac{e^2}{c^3}\left|\frac{\hat{n}\times\left\{(\hat{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right\}}{(1-\boldsymbol{\beta}\cdot\hat{n})^3 R}\right|^2. \tag{3.58}$$

There are two kinds of relativistic effects: first due to spatial relation between $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$ which determines detailed angular distribution and second due to the term $(1-\boldsymbol{\beta}\cdot\hat{n})$ which arises as a result of a transformation from the rest frame of the particle to the observer's frame. The expression of $\boldsymbol{S}\cdot\hat{n}$ above is the energy per unit area per unit time detected at an observation point at time t of radiation emitted by a charge at an earlier time $t' = t - r(t')/c$. Energy, $\mathcal{E}$ emitted from $t' = T_1$ to $t' = T_2$ is given by

$$\mathcal{E} = \int_{T_1+\frac{R(T_1)}{c}}^{T_2+\frac{R(T_2)}{c}} \boldsymbol{S}\cdot\hat{n}\Big|_{ret} dt = \int_{t'=T_1}^{t'=T_2} (\boldsymbol{S}\cdot\hat{n})\frac{dt}{dt'}dt', \tag{3.59}$$

where $(\boldsymbol{S}\cdot\hat{n})dt/dt'$ represents power radiated per unit area in terms of charge's own time. Hence, power radiated per unit solid angle can now be written as

$$\frac{dP(t')}{d\Omega} = R^2(\boldsymbol{S}\cdot\hat{n})\frac{dt}{dt'} = R^2(\boldsymbol{S}\cdot\hat{n})(1-\boldsymbol{\beta}\cdot\hat{n}). \tag{3.60}$$

We shall assume that the charge is accelerated only for a short time during which $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$ are essentially constant in magnitude and direction. If we observe this radiation far away such that $\hat{n}$ and R change negligibly during this time interval, then the above equation is proportional to angular distribution of energy radiated and thus we can write

$$\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c}\frac{|\hat{n}\times\left\{(\hat{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right\}|^2}{(1-\boldsymbol{\beta}\cdot\hat{n})^5}. \tag{3.61}$$

It is apparent from this expression that the power emitted in a relativistic case depends both on the particle velocity as well as its acceleration as stated before. There are two cases which are of particular interest to us which are described below.

3.5.1 When velocity is parallel to acceleration

In this case $\boldsymbol{\beta} \parallel \dot{\boldsymbol{\beta}}$ which leads to $(\hat{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} = \hat{n} \times \dot{\boldsymbol{\beta}}$. Thus, Eq. (3.61) takes the form

$$\frac{dP(t')}{d\Omega} = \frac{e^2\dot{v}^2}{4\pi c^3}\frac{sin^2\theta}{(1-\beta cos\theta)^5}. \tag{3.62}$$

An angular distribution of the power emitted per unit solid angle given by expression above is shown in Fig. 3.1b. If $\beta \ll 1$, we again arrive at the non-relativistic expression (3.56) for power emitted per unit solid angle which was proportional to $sin^2\theta$. But as $\beta \to 1$, the angular distribution is tipped more towards the direction of motion and increases in magnitude. Maximum intensity occurs at an angle θ_{max} which could be obtained by the extrema of the power emitted

$$\frac{d}{d\theta}\left(\frac{dP(t')}{d\Omega}\right)\Bigg|_{\theta_{max}} = 0. \tag{3.63}$$

Using the expression for $dP/d\Omega$ from Eq. (3.62), we get

$$\theta_{max} = cos^{-1}\left(\frac{1}{3\beta}(\sqrt{1+15\beta^2}-1)\right). \tag{3.64}$$

As $\beta \to 1$, we can write $\beta^2 = 1 - 1/\gamma^2$. On expanding β as a series in γ, we see that $\theta \to 1/2\gamma$ under ultra-relativisitic conditions [3]. The radiation thus seems to be confined in a cone along the direction of motion. The cone angle becomes smaller with increase in the speed of the particle. For such angles

$$\frac{dP(t')}{d\Omega} \simeq \frac{e^2\dot{v}^2}{c^3}\frac{1}{4\pi}\frac{\theta^2}{\left\{1-\beta\left(1-\frac{\theta^2}{2}\right)\right\}^5}. \tag{3.65}$$

Again, expanding β in terms of a series in γ and further simplification leads to

$$\frac{dP(t')}{d\Omega} = \frac{8}{\pi}\frac{e^2\dot{v}^2}{c^3}\frac{(\gamma\theta)^2}{(1+\gamma^2\theta^2)^5}. \tag{3.66}$$

Total power radiated is obtained by integrating the expression obtained in Eq. (3.62) with respect to the angle variables.

$$P_{total} = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{e^2\dot{v}^2}{4\pi c^3}\frac{sin^3\theta}{(1-\beta cos\theta)^5}d\theta d\phi, \tag{3.67}$$

$$= \frac{2}{3}\frac{e^2}{c^3}\dot{v}^2\gamma^6. \tag{3.68}$$

Comparing this expression to that of the non-relativistic one, we see that an additional factor of γ^6. In the non-relativisitic limit, this reduces to the expression obtained in Eq. (3.57).

3.5.2 When velocity perpendicular to acceleration

Next let us consider the case when $\boldsymbol{\beta} \perp \dot{\boldsymbol{\beta}}$. We shall choose a coordinate system such that instantaneously $\boldsymbol{\beta}$ is in the z-direction and $\dot{\boldsymbol{\beta}}$ in the x direction. Let the angle between the velocity and $\hat{n}$ be θ, so $\boldsymbol{\beta}\cdot\hat{n} = \beta cos\theta$. So under the given conditions

$$\hat{n}\times\left\{(\hat{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right\} = \hat{n}\dot{\beta}sin\theta cos\phi - \dot{\boldsymbol{\beta}}(1-\beta cos\theta) - \boldsymbol{\beta}\dot{\beta}sin\theta cos\phi. \tag{3.69}$$

Substituting this expression into Eq. (3.61) for the power emitted per unit solid angle, we obtain

$$\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c^3}\frac{|\dot{\boldsymbol{v}}|^2}{(1-\beta cos\theta)^3}\left[1-\frac{sin^2\theta cos^2\phi}{\gamma^2(1-\beta cos\theta)^2}\right]. \tag{3.70}$$

The angular distribution of power emitted in this case is shown in Fig. 3.1c. Again, under relativistic limit we shall expand β as a series in γ [3] which gives the following result

$$\frac{dP(t')}{d\Omega} \approx \frac{2}{\pi}\frac{e^2}{c^3}\gamma^6 \frac{|\dot{\boldsymbol{v}}|^2}{1+\gamma^2\theta^2}\left[1-\frac{4\gamma^2\theta^2 cos^2\phi}{(1+\gamma^2\theta^2)^2}\right]. \tag{3.71}$$

Total power radiated can be obtained by integrating the expression in Eq. (3.70) over the solid angle

$$P_{total} = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{e^2}{4\pi c^3}\frac{|\dot{\boldsymbol{v}}|^2}{1+\gamma^2\theta^2}\left[1-\frac{4\gamma^2\theta^2 cos^2\phi}{(1+\gamma^2\theta^2)^2}\right] sin\theta d\theta d\phi \tag{3.72}$$

$$= \frac{2}{3}\frac{e^2}{c^3}|\dot{\boldsymbol{v}}|^2\gamma^4. \tag{3.73}$$

Here again there is an additional factor of γ^4 due to relativistic effects and the expression reduces to that of a non-relativistic case when $\gamma \to 1$. The angular distribution of the radiation for the three cases- non relativisitic and the two relativistic are given in Fig. 3.1.

Notice from Fig. 3.1 that for a non-relativisitic particle, the emission is dipolar in nature. But as we keep increasing the velocity either perpendicular or parallel to the acceleration, there is a tendency for the radiation to be emitted along the direction of motion. This is called *relativistic beaming* [4] and its effect gets stronger with increasing velocities. We will see in the next chapter that this has a lot of implications in the spectrum of synchrotron radiation which spans across a wide range of frequencies.

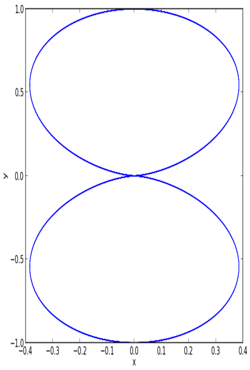
3.5.3 Power spectrum of an accelerating charge

Generally we are also interested to know the frequency components of the radiation which may give further insights into the radiative mechanism taking place. On taking the Fourier transform of Eq. (3.55), we obtain

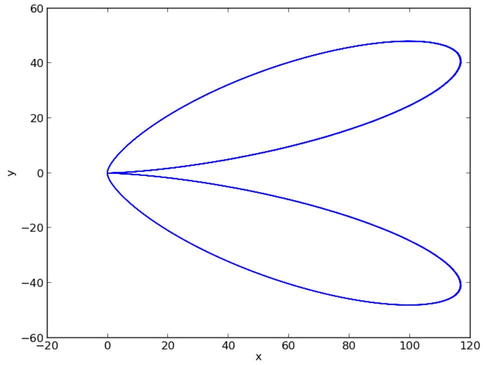
$$\frac{dW}{d\omega d\Omega} = \frac{c}{4\pi^2}\left|\int [R\boldsymbol{E}(t)]e^{i\omega t}dt\right|^2 \tag{3.74}$$

$$= \frac{q^2}{4\pi^2 c}\left|[\hat{n}\times\{(\hat{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\kappa^{-3}]e^{i\omega t}dt\right|^2, \tag{3.75}$$

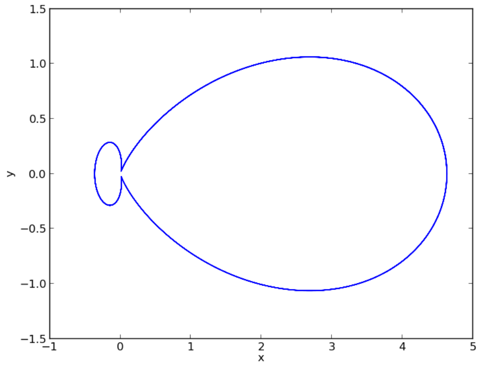
where $E(t)$ is given by the Eq. (3.51) and we have defined $(1-\boldsymbol{\beta}\cdot\hat{n}) = \kappa$. The expression inside the brackets is evaluated at retarded time, $t' = t - R(t')/c$. Using this relation and changing



(a) Non-relativistic particle



(b) Relativistic particle with velocity parallel to acceleration



(c) Relativistic particle with velocity perpendicular to acceleration

Figure 3.1: Angular distribution of radiation in relativistic and non-relativisitic regimes. The motion is along the positive x direction. Under relativisitic speeds, there is a tendency for the radiation to be emitted along the direction of motion. Greater the speed, lesser the cone angle about which the radiation is confined. This is called *relativistic beaming* [4].

the variable of integration from $t \to t'$, we can write $dt = \kappa t'$. Expand $R(t') \approx |r| - \hat{n} \cdot \boldsymbol{r}_o$, valid for $|\boldsymbol{r}_o| \ll |\boldsymbol{r}|$ and substitute the preceding relations in the above integral, we get

$$\frac{dW}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\int \hat{n} \times \{(\hat{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\}\kappa^{-2} exp[i\omega(t' - \hat{n} \cdot \boldsymbol{r}_o(t')/c)]dt'\right|^2. \tag{3.76}$$

Finally Eq. (3.76) can be integrated by parts to obtain an expression involving only $\boldsymbol{\beta}$ as follows

$$\frac{dW}{d\omega d\Omega} = (q^2\omega^2/4\pi^2 c)\left|\int \hat{n} \times (\hat{n} \times \boldsymbol{\beta}) exp[i\omega(t' - \hat{n} \cdot \boldsymbol{r}_o(t')/c)]dt'\right|^2. \tag{3.77}$$

This is a very useful result and gives the power spectrum of a charge under acceleration given its velocity and trajectory. The power spectrum gives the different frequency components present in the radiation which might also point to various periodicities present in the system. For example, the spectrum of a non-relativisitic charge particle in a circular trajectory is a single line which peaks at the frequency of the circular motion. Hence, from the spectrum itself, one gets to know many properties of the astrophysical system which could not be inferred from the time dependence otherwise. Hence, it is essential to understand the spectrum due to various processes which will be discussed in the succeeding chapters. We shall use the result obtained in Eq. (3.77) in the analysis of synchrotron and bremsstrahlung mechanisms.

Chapter 4

Synchrotron radiation

As we have seen from the previous chapter that a charge particle under acceleration emits radiation. For a relativistic particle, this radiation is confined to a cone along the direction of motion. This phenomenon has important application in the analysis of synchrotron radiation as we shall see in this chapter. So, we consider charges accelerated by the magnetic fields in our discussions. Our aim would be to arrive at a spectrum of synchrotron emitting electrons and understand its characteristics.

4.1 Cyclotron and synchrotron

A charge particle moving in a uniform magnetic field undergoes a circular motion as we derived in Sec. 2.2.2. If the velocity is non-relativistic, the radiation emitted by the particle will have an angular distribution as given in Fig. 3.1a and we call it as the cyclotron radiation. For an observer in the plane of motion of the particle, the amplitude of the electric field varies sinusoidally as shown in Fig. 4.1a [4]. The frequency spectrum of this radiation (Fig. 4.1b) peaks only at one frequency which is the gyration frequency given by $\omega_g = qB/m$.

For a particle moving with relativistic speed in a uniform magnetic field, the emitted radiation shows beaming effect and its angular distribution is as given in Fig. 3.1c. In such a case, the particle is said to be emitting synchrotron radiation. Because of the beaming effect, the radiation is observed only when the line of sight is within the cone angle of $2/\gamma$. Since the motion is still circular, the amplitude of the electric field varies in the form of pulses as shown in Fig. 4.1c [4]. Due to the finite width of the pulses, the frequency spectrum would be spread around a range of frequencies (Fig. 4.1d [4]) as we shall derive in the next sec-

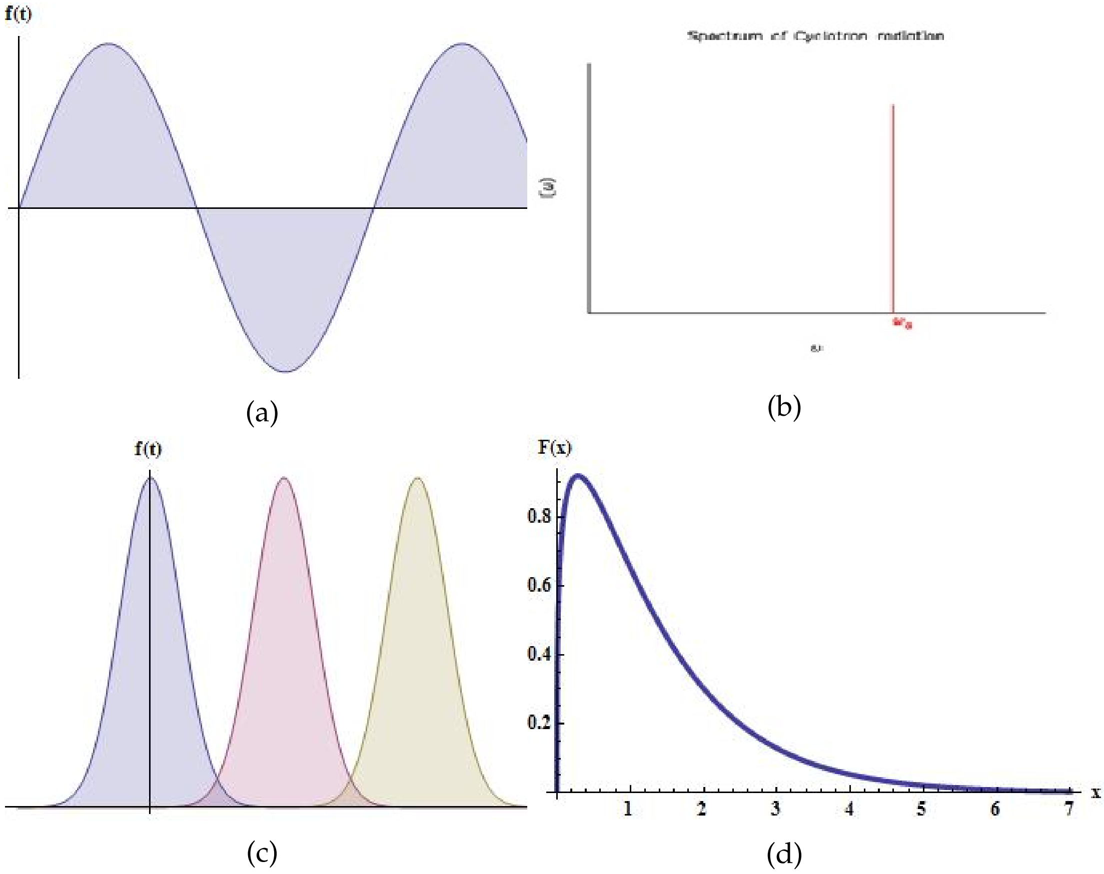


Figure 4.1: Spectral properties of cyclotron and synchrotron radiation. (a) The sinusoidal variation of the amplitude of the electric field as a function of time for a cyclotron when observed in the plane of circular motion, (b) Spectrum of (a) obtained by a Fourier transform which peaks at the gyration frequency, (c) Amplitude variation as a function of time in case of an electron emitting synchrotron radiation when observed in the plane of circular motion. The pulses are due to the beaming effect, (d) Spectrum of an electron emitting synchrotron radiation which will be derived in this chapter [4].

tion. We can show by elementary calculation that the width of the observed pulses for the uniform circular motion as

$$\Delta t = \frac{1}{\gamma^3 \omega_B sin\alpha}, \tag{4.1}$$

where $\omega_B = (qB/\gamma mc)$ is the gyration frequency and α is the angle between velocity and the magnetic field. We see that the width of the observed pulses is smaller than the gyration period by a factor of γ^3. In this context, it is useful to define a cut off frequency which is conventionally written as

$$\omega_c = \frac{3}{2}\gamma^3 \omega_B sin\alpha. \tag{4.2}$$

The definition of cut off frequency is based on the assumption that maximum power is emitted around ω_c. As we shall see in the sections that follow, the power spectrum could be written in terms of this cut-off frequency and that it peaks around ω_c.

Note that synchrotron radiation is observable only in case of charges with low mass such as electrons as the power emitted depends inversely on the square of the mass of the particle. The exact expression for power can be derived using Eqs. (2.20) and (3.57). The result is as follows

$$P = \frac{4}{3}\sigma_T c\beta^2\gamma^2 U_B, \tag{4.3}$$

where $\sigma_T = 8/3\pi r_o^2$ is the Thomson cross-section with $r_o = e^2/mc^2$, $\beta = v/c$ (v being the speed of the particle), $U_B = B^2/8\pi$. So, for massive particles like protons, the emitted power is negligible as the Thomson cross-section is small on account of higher mass of proton.

4.2 Spectrum of synchrotron radiation

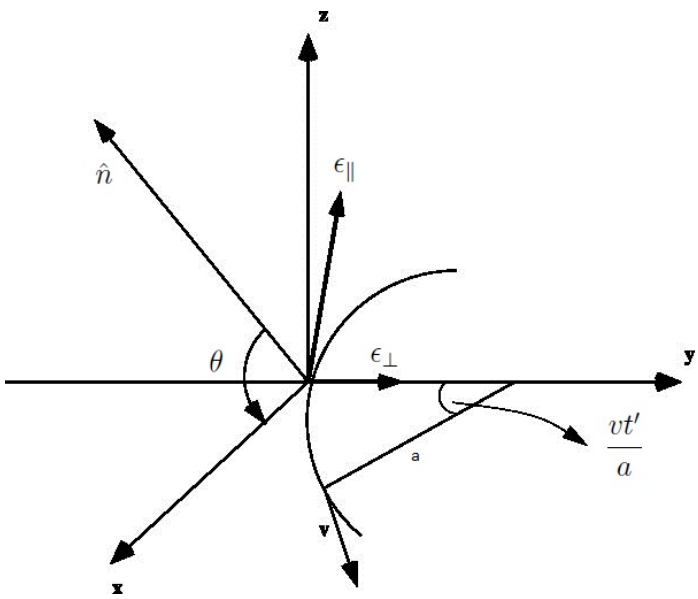


Figure 4.2: Geometry of synchrotron radiation [4].

The spectrum of synchrotron radiation extends across broad range of frequencies due to a combination of relativistic beaming and Doppler effect. Now we shall formally derive the power spectrum due to a single electron emitting synchrotron radiation. Consider the orbital trajectory $\boldsymbol{r}(t')$ as given in Fig. 4.2 [4]. The origin of the coordinates is the location of the particle at retarded time $t' = 0$ when the velocity vector is along the x-axis. The center of the circular motion with radius a is assumed to be on the y-axis. We also define $\epsilon_\perp$ as the unit vector along y-axis and $\epsilon_\parallel = \hat{n} \times \epsilon_\perp$ where $\hat{n}$ is a unit vector from the origin to the point of observation. From Fig. 4.2, we can write

$$\hat{n} \times (\hat{n} \times \boldsymbol{\beta}) = -\epsilon_\perp sin\left(\frac{vt'}{a}\right) + \epsilon_\parallel cos\left(\frac{vt'}{a}\right) sin\theta, \tag{4.4}$$

$$\begin{aligned} t' - \frac{\hat{n} \cdot \boldsymbol{r}(t')}{c} &= t' - \frac{a}{c} cos\theta sin\left(\frac{vt'}{a}\right) & (4.5) \\ &\approx \frac{1}{2\gamma^2}\left[(1+\gamma^2\theta^2)t' + \frac{c^2\gamma^2 t'^3}{3a^2}\right]. & (4.6) \end{aligned}$$

We use both these results in the power law expression obtained in Eq. (3.77). Note that here the expression for power consists of two terms - one due to $\epsilon_\perp$ and the other due to $\epsilon_\parallel$. Hence the net power can be written as the sum of the powers emitted along the two differential directions $\epsilon_\perp$ and $\epsilon_\parallel$

$$\frac{dW}{d\omega d\Omega} = \frac{dW_\perp}{d\omega d\Omega} + \frac{dW_\parallel}{d\omega d\Omega}, \tag{4.7}$$

$$\frac{dW_\perp}{d\omega d\Omega} = (e^2\omega^2/4\pi^2 c)\left|\int \frac{ct'}{a} exp\left[i\frac{\omega}{2c\gamma^2}\left((1+\gamma^2\theta^2)t' + \frac{c^2\gamma^2 t'^3}{3a^2}\right)\right]dt'\right|^2, \tag{4.8}$$

$$\frac{dW_\parallel}{d\omega d\Omega} = (e^2\omega^2\theta^2/4\pi^2 c)\left|\int exp\left[i\frac{\omega}{2c\gamma^2}\left((1+\gamma^2\theta^2)t' + \frac{c^2\gamma^2 t'^3}{3a^2}\right)\right]dt'\right|^2. \tag{4.9}$$

To simplify the algebra, we use the following change of variables

$$y = \frac{\gamma c t'}{a\theta_\gamma}, \qquad \eta = \frac{\omega a \theta_\gamma^3}{3c\gamma^3}, \tag{4.10}$$

where $\theta_\gamma = 1 + \gamma^2\theta^2$. Applying these changes to Eqs. (4.8) and (4.9), we get

$$\frac{dW_\perp}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left(\frac{a\theta_\gamma^2}{\gamma^2 c}\right)\left|\int_{-\infty}^{\infty} y exp\left[\frac{3}{2}i\eta(y + \frac{1}{3}y^3)\right]dy\right|^2, \tag{4.11}$$

$$\frac{dW_\parallel}{d\omega d\Omega} = \frac{e^2\omega^2\theta^2}{4\pi^2 c}\left(\frac{a\theta_\gamma^2}{\gamma^2 c}\right)\left|\int_{-\infty}^{\infty} exp\left[\frac{3}{2}i\eta(y + \frac{1}{3}y^3)\right]dy\right|^2. \tag{4.12}$$

Both of these integrals are functions of the parameter η. Since the cone angle, θ through which the radiation is emitted is nearly zero for ultra-relativistic particles we can write

$$\eta \approx \frac{\omega a}{3c\gamma^3} = \frac{\omega}{2\omega_c}, \tag{4.13}$$

where in the second equality, we have used the expression for the critical frequency given in Eq. (4.2). Hence, the dependence of power in ω is only through η or (ω/ω_c). The integrals in Eqs. (4.11) and (4.12) can be expressed in terms of the Macdonald's function $K_n(x)$ (or modified Bessel function) using the following relation [5]

$$Ai(x) = \frac{1}{\pi}\int_0^{\infty} cos(xu + \frac{u^3}{3})du = \frac{1}{\sqrt{3}\pi}\sqrt{x}K_{1/3}(\frac{2}{3}x^{3/2}). \tag{4.14}$$

Here $Ai(x)$ is the Airy's function and the order n of the Macdonald's function being 1/3. Using this result in the integrals (4.11) and (4.12), we obtain

$$\frac{dW_{\perp}}{d\omega d\Omega} = \frac{e^2\omega^2}{3\pi^2 c}\left(\frac{a\theta_\gamma^2}{\gamma^2 c}\right)^2 K_{2/3}^2(\eta), \tag{4.15}$$

$$\frac{dW_{\parallel}}{d\omega d\Omega} = \frac{e^2\omega^2\theta^2}{3\pi^2 c}\left(\frac{a\theta_\gamma}{\gamma c}\right)^2 K_{1/3}^2(\eta). \tag{4.16}$$

It is just the matter of integration over the solid angle to give the energy per unit frequency range radiated by the particle per complete orbit in the projected normal plane. Due to the helical motion of the particle, the radiation is confined along the shaded region in Fig. 4.3 which lies within an angle of $1/\gamma$ of a cone of half-angle α. Hence, we can write the solid angle as $d\Omega = 2\pi sin\alpha d\theta$ and we can make a little error by extending the limit of integration over θ from $0 \to \pi$ to $-\infty \to \infty$. Eqs. (4.15) and (4.16) can now be written as

$$\frac{dW_{\perp}}{d\omega} = \frac{2e^2\omega^2 a^2 sin\alpha}{3\pi c^3\gamma^4}\int_{-\infty}^{\infty}\theta_\gamma^4 K_{2/3}^2(\eta)d\theta, \tag{4.17}$$

$$\frac{dW_{\parallel}}{d\omega} = \frac{2e^2\omega^2 a^2 sin\alpha}{3\pi c^3\gamma^2}\int_{-\infty}^{\infty}\theta_\gamma^2\theta^2 K_{1/3}^2(\eta)d\theta. \tag{4.18}$$

After an algebraic simplification, we can further reduce the above integrals as follows [6, 7]

$$\frac{dW_{\perp}}{d\omega} = \frac{\sqrt{3}e^2\gamma sin\alpha}{2c}(F(x) + G(x)), \tag{4.19}$$

$$\frac{dW_{\parallel}}{d\omega} = \frac{\sqrt{3}e^2\gamma sin\alpha}{2c}(F(x) - G(x)), \tag{4.20}$$

where $x = (\omega/\omega_c)$ and

$$F(x) = x\int_x^{\infty} K_{5/3}(\zeta)d\zeta, \qquad G(x) = xK_{2/3}(x). \tag{4.21}$$

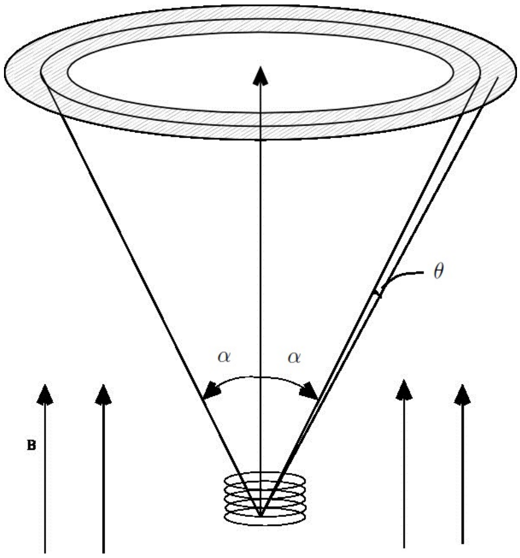


Figure 4.3: Synchrotron emission from a particle in a helical motion with pitch angle α. The emission is confined in the shaded region due to this motion [4].

The emitted power per unit frequency is obtained by dividing the above equation by the time period, $T = (2\pi/\omega_B)$

$$P_{\parallel}^{\perp} = \frac{\sqrt{3}e^3 B sin\alpha}{4\pi mc^2}[F(x) \pm G(x)]. \tag{4.22}$$

The total power emitted per unit frequency can be obtained using Eq. (4.7) as follows

$$P_{total} = P_{\perp}(\omega) + P_{\parallel}(\omega) = \frac{\sqrt{3}e^3 B sin\alpha}{2\pi mc^2} F(\omega/\omega_c). \tag{4.23}$$

Different ways to plot the power emitted per unit frequency range due to synchrotron emission of a single electron is given in Fig. 4.4.

In astrophysical situations, we come across emission from a distribution of electrons rather than a single electron. Often, the number density of particles with energies between E and $E + dE$ can be approximately expressed in the form of a power law as give below

$$N(E)dE = CE^{-p}dE, \qquad E_1 < E < E_2. \tag{4.24}$$

The index p is called the power law index. The total power radiated per unit volume per unit frequency for a power law distribution is given by

$$P_{total} = \int \{N(E)dE\}P(\omega) \propto \int_{E_1}^{E_2} F(\frac{\omega}{\omega_c})E^{-p}dE. \tag{4.25}$$

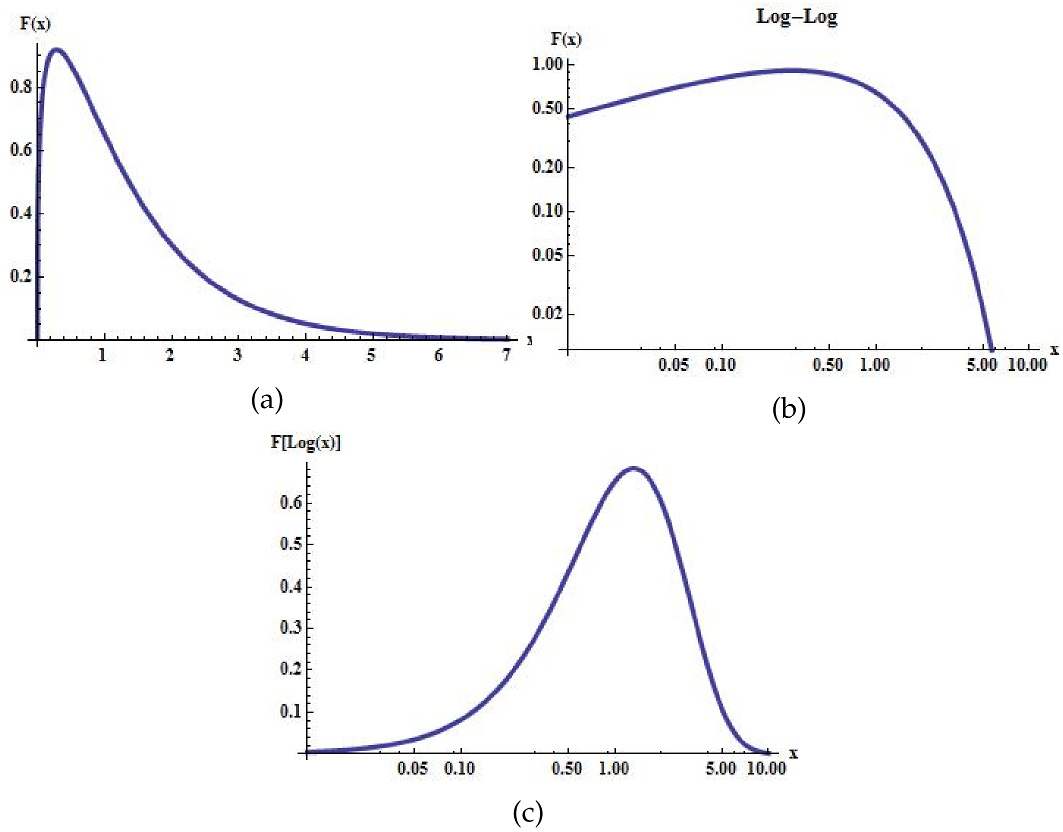


Figure 4.4: This figure shows three ways to plot synchrotron spectrum. Although, all of them plot the same spectrum, they convey or suppress information in different ways. (a) simply plots $F(x)$ vs x on linear axes. It peaks at x = 0.29 and completely obscures the spectrum below this peak. (b) plots $F(x)$ on a logarithmic axes which shows that lower frequency spectrum has a slope of $1/3$. However, it obscures the fact that most power is emitted around the critical frequency i.e. at $x \approx 1$. Remember $F(x)$ is proportional to power emitted per unit frequency. (c) plots $F(log(x))$ with a linear ordinate but a logarithmic abscissa. This curve is clearly consistent with the fact that most of the emission occurs around $x \sim 1$ [8].

Again changing the variables to $x = (\omega/\omega_c)$ and assuming the energy limits to be sufficiently wide, we get

$$P_{total} \propto \omega^{-(p-1)/2} \int_0^\infty F(x) x^{\frac{p-3}{2}} dx \propto \omega^{-s}, \tag{4.26}$$

where $s = (p-1)/2$ is the spectral index and is a very useful quantity which could be observed to give the power law index, p. Thus spectral index could be used to determine the energy distribution of particles. From the above expression, it is clear that the synchrotron spectrum for a power law distribution is a straight line with slope $-s$. We can think of this as a superposition of the individual electron spectra to give an overall power spectrum as

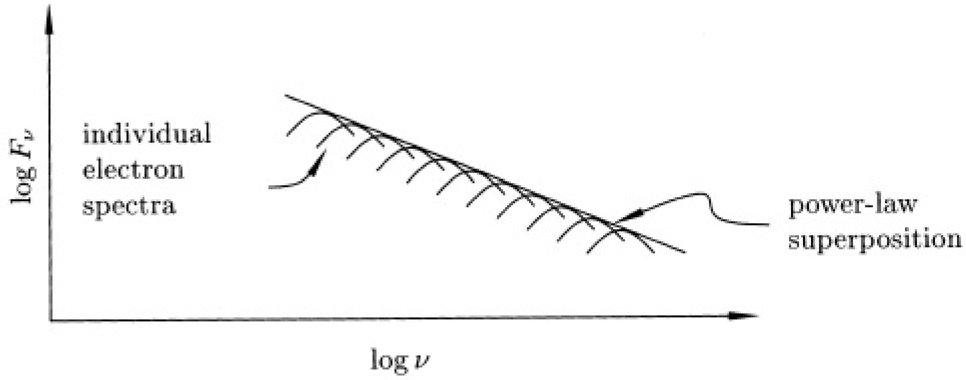


Figure 4.5: Superposition of individual electron spectrum for a power law distribution of electrons (image from Ref. [9]).

shown in Fig. 4.5. Using the following identity for the integration of $F(x)$

$$\int_0^\infty x^\mu F(x)dx = \frac{2^{\mu+1}}{\mu+2}\Gamma\left(\frac{\mu}{2}+\frac{7}{3}\right)\Gamma\left(\frac{\mu}{2}+\frac{2}{3}\right) \tag{4.27}$$

and along with Eqs. (4.23) and (4.25) we can arrive at the exact expression for the power spectrum as follows

$$P_{total}(\omega) = \frac{\sqrt{3}q^3 CBsin\alpha}{2\pi mc^2(p+1)}\Gamma\left(\frac{p}{4}+\frac{19}{12}\right)\Gamma\left(\frac{p}{4}-\frac{1}{12}\right)\left(\frac{mc\omega}{3qBsin\alpha}\right)^{-(p-1)/2}. \tag{4.28}$$

which has the frequency dependence as predicted by Eq. (4.26).

4.3 Polarization of synchrotron radiation

As we have seen in the previous section, differential power is emitted in the two directions $\epsilon_\perp$ and $\epsilon_{||}$ for an electron or a distribution of electrons emitting synchrotron radiation. So, we can infer that in general the radiation is elliptically polarized. Whether the polarization is right handed or left handed depends on whether the observation is made inside or outside the cone in Fig. 4.3. For a reasonable distribution of particles that vary smoothly with pitch angle, the elliptical component will cancel out as the emission cones contribute equally from both the sides of the line of sight. So the radiation will be partially linearly polarized, the degree of linear polarization given by

$$\Pi(\omega) = \frac{P_\perp(\omega) - P_{||}(\omega)}{P_\perp(\omega) + P_{||}(\omega)} = \frac{G(x)}{F(x)}. \tag{4.29}$$

The second equality is obtained using Eq. (4.22). Note that this is the expression for the case of a single electron. For a power law distribution of electrons, we find the degree of linear polarization to be

$$\Pi = \frac{\int \{P_\perp(\omega) - P_{||}(\omega)\} N(E) dE}{\int \{P_\perp(\omega) + P_{||}(\omega)\} N(E) dE} \tag{4.30}$$

$$= \frac{p+1}{p+7/3}. \tag{4.31}$$

Thus the degree of linear polarization can go as high as 70% for a power law index of 2. So the presence of linear polarization is a characteristic feature of synchrotron radiation.

4.4 Synchrotron self-absorption

The synchrotron model described above works well for high frequency regimes. However, if the synchrotron radiation intensity within a source becomes sufficiently high, then re-absorption of the radiation by synchrotron electrons themselves become important. This phenomenon is called self-absorption. There is a possibility of stimulated emission as well, just as in laser theory. The three processes: spontaneous emission, stimulated emission and self-absorption are related by the Einstein coefficients A and B[1].

The states of the emitting particle are nothing but the free particle states. According to statistical mechanics, there is one quantum state associated with translational degree of freedom of a particle within a volume of phase space of magnitude h^3. So we break the entire phase space into elements of size h^3 and consider the transition between these discrete states. In terms of the Einstein coefficients, the absorption coefficient for a two level system is given by

$$\alpha = \frac{h\nu}{4\pi} n_1 B_{12} \phi(\nu), \tag{4.32}$$

where B_{12} is the Einstein B coefficient for absorption and $\phi(\nu)$ is the line profile function which is introduced because the energy difference between any two levels is not infinitely sharp. The profile function is often approximated by a delta function with its peak at the transition frequency $\nu_o = (E_2 - E_1)/h$. However, here we are dealing with a large number of

[1]For a two level system: A_{21} is the Einstein A coefficient which defines the transition probability per unit time for spontaneous emission, B_{12} is the Einstein B coefficient for absorption and B_{21} is the Einstein B coefficient for stimulated emission. A_{21} and B_{21} are related by $A_{21} = (2h\nu^3/c^2)B_{21}$.

states instead of two. So in our case, the formula for absorption coefficient should contain sum over all upper states 2 and lower states 1

$$\alpha = \frac{h\nu}{4\pi}\sum_{E_1}\sum_{E_2}[n(E_1)B_{12} - n(E_2)B_{21}]\phi_{21}(\nu). \tag{4.33}$$

The summations can be restricted to the states differing by an energy $h\nu = E_2 - E_1$, by assuming the profile function to be a delta function. In doing so, we have assumed the emission and absorption to be isotropic which requires the magnetic field $\boldsymbol{B}$ to be tangled and have no net direction and the particle distribution to be isotropic.

We can now derive the power spectrum due to the self absorption process. In terms of the Einstein coefficients, the power spectrum is given by

$$\begin{aligned} P(\nu, E_2) &= h\nu\sum_{E_1} A_{21}\phi_{21}(\nu) && (4.34)\\ &= \left(\frac{2h\nu^3}{c^2}\right)h\nu\sum_{E_1} B_{21}\phi_{21}(\nu). && (4.35)\end{aligned}$$

The absorption coefficient in Eq. (4.33) can now be written in terms of power emitted as

$$\alpha = \frac{c^2}{8\pi h\nu^3}\sum_{E_2}[n(E_2 - h\nu) - n(E_2)]P(\nu, E_2), \tag{4.36}$$

where we have used $B_{12} = B_{21}$ and $E_1 = E_2 - h\nu$. Introducing the isotropic distribution of electrons by the $f(p)$ such that $f(p)d^3p$ represent the number of electrons per unit volume with momenta in d^3p about p. According to statistical mechanics, the number of quantum states per unit volume in the range d^3p is simply $2h^{-3}d^3p$ which gives the electron density per quantum state as $2h^{-3}f(p)$. Thus, we can make the following replacements in the expression (4.36):

$$\sum_{E_2} \rightarrow \frac{2}{h^3}\int d^3p_2 \qquad\qquad n(E_2) \rightarrow \frac{h^3}{2}f(p_2), \tag{4.37}$$

Substituting these expressions in Eq. (4.36), we get the general expression for the absorption coefficient as

$$\alpha = \frac{c^2}{9\pi h\nu^3}\int d^3p_2[f(p_2^*) - f(p_2)]P(\nu, E_2), \tag{4.38}$$

where p_2^* is the momentum corresponding to energy $E_2 - h\nu$. Since the electron distribution is isotropic, it is convenient to use energy rather than momentum to describe the distribution function as follows

$$N(E)dE = f(p)d^3p = f(p)4\pi p^2 dp. \tag{4.39}$$

We shall substitute this in the expression for absorption coefficient in Eq. (4.38) and generalize our expression for any energy E. Since we are dealing with classical electrodynamics, we can assume $h\nu \ll E$

$$\begin{aligned} \alpha &= \frac{c^2}{8\pi h\nu^3}\int dEP(\nu,E)E^2\Big[\frac{N(E-h\nu)}{(E-h\nu)^2}-\frac{N(e)}{E^2}\Big] & (4.40)\\ &= -\frac{c^2}{8\pi\nu^2}\int dEP(\nu,E)E^2\frac{\partial}{\partial E}\Big(\frac{N(E)}{E^2}\Big), & (4.41)\end{aligned}$$

where in the second step, we have used a taylor expansion of $N(E-h\nu)$. For a power law distribution of particles, Eq. (4.41) reduces to

$$\alpha = \frac{(p+2)c^2}{8\pi\nu^2}\int dEP(\nu,E)\frac{N(E)}{E}. \tag{4.42}$$

We can now substitute the expression for power emitted from Eq. (4.23) and integrate the results to obtain the self-absorption coefficient using the identity given by Eq. (4.27). A relation between x and E can be obtained by noting that $x = (\omega/\omega_c)$ and using the definition of ω_c from Eq. (4.2), we can infer $x \propto E^{-2}$. The final result is

$$\alpha = \frac{\sqrt{3}q^3}{8\pi m}\left(\frac{3q}{2\pi m^3c^5}\right)^{p/2} C(B sin\alpha)^{(p+2)/2}\Gamma\left(\frac{3p+2}{12}\right)\Gamma\left(\frac{3p+22}{12}\right)\nu^{-(p+4)/2}. \tag{4.43}$$

The source function can be found from

$$S_\nu \propto \frac{P(\nu)}{\alpha_\nu} \propto \nu^{5/2}. \tag{4.44}$$

Thus, the source function is independent of the power law index. For an optically thin synchrotron emission, the observed intensity is proportional to the emission function (4.27) while for optically thick synchrotron emission the observed intensity if proportional to the source function given by Eq. (4.44). The overall spectrum taking into account the absorption effects is given in Fig. 4.6.

Radio galaxies, Active Galactic Nuclei (AGN) and pulsars are some examples of astrophysical sources which are believed to emit radiation via synchrotron mechanism [10]. Although the mechanism is particularly important for radio astronomy, depending on the energy of

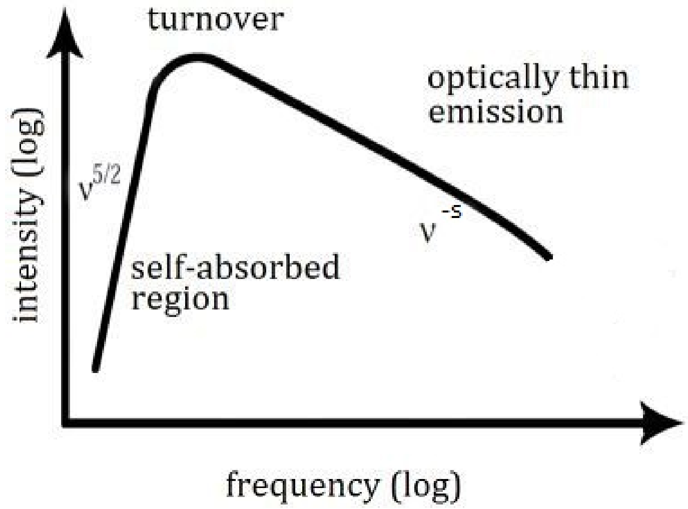


Figure 4.6: Overall spectrum of synchrotron emission due to a power law distribution of electrons. Note that low frequency spectrum is dominated by the source function which does not depend on the power law index [11].

the electron and strength of the magnetic field, the emission can occur at visible, ultraviolet and X-ray wavelengths. We can calculate the peak frequency using Eq. (4.2) if the various factors become known to us. Also, from the overall spectrum given in Fig. 4.6, it is possible to calculate the size of the astrophysical source and the constraint on the distribution of the electrons to be optically thin or optically thick. We will study a simple case in chapter 6 which will help us to infer such physical properties of the system from an observed spectrum.

Chapter 5

Bremsstrahlung

In the previous chapter, we had considered the radiation emitted by ultra-relativistic electrons accelerating in a magnetic field. Its power spectrum has features which could help in determination of the properties of the astrophysical sources such as the distribution of charge particles in the system, its radius etc.. In this chapter, we will consider the radiation due to the acceleration of a charge in the Coulomb field of another charge. This is called *bremsstrahlung* or free-free emission. Again, we shall follow a classical treatment and we shall include the quantum corrections in the form of a Gaunt factor. There is no radiation due to collision between like particles in the dipole approximation as the dipole moment is zero for the system. Hence, we shall start with a system containing an electron moving in the Coulomb field of a massive ion.

5.1 Bremsstrahlung due to a single electron

Consider the situation depicted in Fig. 5.1. To make calculations simpler, we shall assume electron moves rapidly enough to ignore any deviations from the straight path. If $\boldsymbol{d} = -e\boldsymbol{r}$ is the dipole moment of the system

$$\ddot{\boldsymbol{d}} = -e\ddot{R} = -e\dot{\boldsymbol{v}}. \tag{5.1}$$

Taking Fourier transform on both sides, we obtain

$$-\omega^2 \hat{d}(\omega) = -\frac{e}{2\pi}\int_{-\infty}^{\infty} \dot{\boldsymbol{v}} e^{i\omega t} dt. \tag{5.2}$$

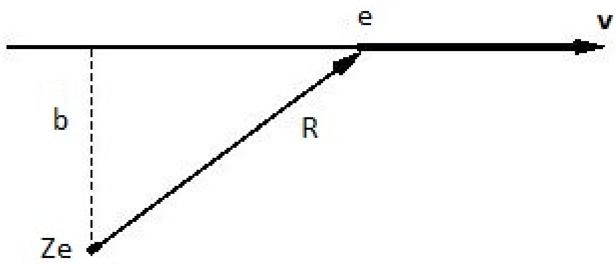


Figure 5.1: An electron of charge e moving in the Coulomb field of an ion of charge Ze [4].

The electron is in a close interaction with the ion over a time interval, called as the collision time, which is of the order

$$\tau \approx \frac{b}{v}. \tag{5.3}$$

For $\omega\tau \gg 1$, the exponential in the integral in Eq. (5.2) oscillates very rapidly and hence the integral is small. On the other hand, when $\omega\tau \ll 1$, the exponential will be of the order 1. Thus, we have two cases

$$\hat{d}(\omega) \sim \begin{cases} \frac{e}{2\pi\omega^2}\Delta\boldsymbol{v} & ,\omega\tau \ll 1 \\ 0 & ,\omega\tau \gg 1 \end{cases} \tag{5.4}$$

For a dipole system, the radiation field (3.49) takes the form

$$|E_a(t)| = \left|\frac{\hat{n}\times(\hat{n}\times\ddot{\boldsymbol{d}}(t))}{c^2R}\right| = \frac{\ddot{d}(t)sin\theta}{c^2R}, \tag{5.5}$$

where θ is the angle between $\hat{n}$ and $\ddot{\boldsymbol{d}}$. Taking Fourier transform on both sides of Eq. (5.5) and using the relation (3.55) for power emitted per unit solid angle for a non-relativistic case, we get

$$\frac{dP}{d\omega} = \frac{8\pi\omega^4}{3c^3}|\hat{d}(\omega)|^2. \tag{5.6}$$

Substituting the expressions for $\hat{d}(\omega)$ from Eq. (5.4) into (5.6), we obtain

$$\frac{dP}{d\omega} = \begin{cases} \frac{2e^2}{3\pi c^3}|\Delta\boldsymbol{v}|^2 & ,\omega\tau \ll 1 \\ 0 & ,\omega\tau \gg 1 \end{cases} \tag{5.7}$$

From Fig. 5.1, it is easy to see that the acceleration normal to the path of the particle is

$$\frac{\Delta v}{\Delta t} = \frac{Ze^2}{m}\frac{b}{(b^2+v^2t^2)^{3/2}}. \tag{5.8}$$

Integrating over time gives us the change in velocity of the particle as follows

$$\Delta v = \frac{2Ze^2}{mbv}. \tag{5.9}$$

Substituting the expression for Δv obtained in Eq. (5.9) back into Eq. (5.7) gives the energy spectrum of an electron undergoing bremsstrahlung radiation

$$\frac{dW(b)}{d\omega dt} = \begin{cases} \frac{8Z^2e^6}{3\pi c^3 m^2 v^2 b^2} & ,b \ll v/\omega \\ 0 & ,b \gg v/\omega \end{cases} \tag{5.10}$$

where $W(b)$ is the energy emitted which is a function of the impact parameter b. Again, in an astrophysical system, we encounter many electrons instead of a single electron considered above. Suppose we have a medium with ion density n_i and electron density n_e for a fixed electron speed v. So the flux of electrons incident on one ion is simply $n_e v$. About an ion, the element of area is $2\pi b db$. Therefore, total emission per unit time per unit volume per unit frequency range is

$$\frac{dW}{d\omega dv dt} = n_e n_i 2\pi v \int_{b_{min}}^{\infty} \frac{dW(b)}{d\omega} b db, \tag{5.11}$$

where b_{min} is some minimum value of the impact parameter whose choice is discussed below. Now, if we substitute the expression obtained for $dW(b)/d\omega dt$ (5.10), we get a divergent value for the upper limit at infinity. So, we approximate the value of the upper limit at B_{max} which is some value of b beyond which the $b \ll v/\omega$ asymptotic result in invalid and the contribution to the integral becomes negligible. From Eqns. (5.10) and (5.11), we can write

$$\frac{dW}{d\omega dv dt} = \frac{16e^6}{3c^3m^2v} n_e n_i Z^2 \int_{b_{min}}^{b_{max}} \frac{db}{b}, \tag{5.12}$$

$$= \frac{16e^6}{3c^3m^2v} n_e n_i Z^2 ln\left(\frac{b_{max}}{b_{min}}\right). \tag{5.13}$$

We are now left with determination of the b values. The value of b_{max} is uncertain but it is of the order v/ω. However, since it lies inside the logarithm function, its precise value is not important and we can simply take

$$b_{max} \approx \frac{v}{\omega}. \tag{5.14}$$

In doing so, we make a small but not a significant error. The value of b_{min} can be estimated in two ways. First, the straight line approximation ceases to be valid when $\Delta v \sim v$ and hence

$$b_{min}^{(1)} = \frac{4Ze^2}{\pi m v^2}. \tag{5.15}$$

Second way is to treat the problem quantum mechanically. Using uncertainity principle $\Delta x \Delta p \geq \hbar$, take $\Delta x \sim b$ and $\Delta p \sim mv$, we obtain

$$b_{min}^{(2)} \simeq \frac{\hbar}{mv}. \tag{5.16}$$

Since $b^{(1)}_{min} \gg b^{(2)}_{min}$, the classical description is valid and we use $b_{min} = b^{(1)}_{min}$. It is important to understand the two cases: the classical treatment is valid whenever the kinetic energy of the electron $(1/2)mv^2$ is much smaller than Z^2R_y, where R_y is the Rydberg energy for the hydrogen atom. On the other hand, when $(1/2)mv^2$ is much greater than Z^2R_y, the uncertainity principle plays an important role.

As stated in the beginning, we have followed a classical treatment of the problem. Quantum corrections are obtained by expressing the results in terms of a gaunt factor $g_{ff}(v,\omega)$

$$\frac{dW}{d\omega dvdt} = \frac{16\pi e^6}{3\sqrt{3}c^3m^2v} n_e n_i Z^2 g_{ff}(v,\omega). \tag{5.17}$$

Comparing Eqs. (5.13) and (5.17), we see that

$$g_{ff}(v,\omega) = \frac{\sqrt{3}}{\pi} ln\left(\frac{b_{max}}{b_{min}}\right). \tag{5.18}$$

5.2 Thermal bremsstrahlung

In the previous section we obtained the expression for bremsstrahlung in case of single-speed electrons. Now, we average this expression obtained over thermal distribution of speeds. The phenomenon then is called as thermal bremsstrahlung. We know from Maxwell-Boltzmann distribution of velocities that the probability dP that a particle has velocity in a range d^3v about v is

$$dP \propto e^{-E/k_BT} d^3v \propto exp\left[-\frac{mv^2}{2k_BT}\right] d^3v, \tag{5.19}$$

where k_B is the Boltzmann constant. For an isotropic distribution of velocities, we can write $d^3v = 4\pi v^2 dv$ and for emission to occur, the electron velocity should be such that

$$h\nu \leq \frac{1}{2}mv^2, \tag{5.20}$$

otherwise the photon of energy $h\nu$ would not be emitted. This cut-off in the lower limit of the integration over electron velocities is called photon discreteness effect. So, for a thermal distribution of velocities we have the following expression

$$\frac{dW'(T,\omega)}{dvdtd\omega} = \int_{v_{min}}^{\infty} (dW(T,\omega)/dvdtd\omega) v^2 exp\left(-\frac{mv^2}{2k_BT}\right) dv \int_0^{\infty} exp\left(-\frac{mv^2}{2k_BT}\right) dv, \tag{5.21}$$

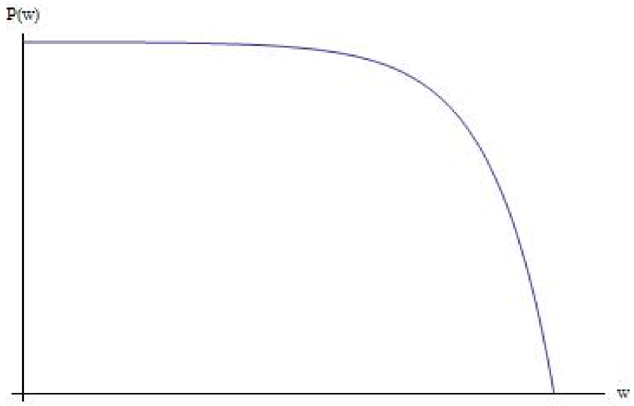


Figure 5.2: Spectrum (Log-log) of electrons exhibiting thermal bremsstrahlung which is rather flat upto the cut-off frequency given by $\nu \approx k_B T/h$ after which the intensity of the spectrum drops sharply.

where $(dW(T,\omega)/dvdtd\omega)$ is the single-speed expression. Substituting Eq. (5.17) into (5.21) and integrating

$$\frac{dW(T,\omega)}{dVdtd\nu} = \frac{2^5 \pi e^6}{3mc^3} \left(\frac{2\pi}{3k_B m} \right)^{1/2} T^{-1/2} Z^2 n_e n_i e^{-h\nu/k_B T} \tilde{g}_{ff}(T,\nu), \tag{5.22}$$

where $\tilde{g}_{ff}$ is the velocity averaged gaunt factor. The expression above is called the emissivity function for thermal bremsstrahlung. In the integration above, we have used the fact that $< v > \propto T^{1/2}$ and $dW/d\omega dvdt \propto v^{-1}$ from Eq. (5.17). The analytic formulas for $\tilde{g}_{ff}$ differ for different orders of $u \equiv (h\nu/k_B T)$ or for the cases when uncertainity principle becomes important or not etc.. For $u \gg (h\nu/k_B T)$, the values of $\tilde{g}_{ff}$ are not important because the spectrum cuts off at these values. $\tilde{g}_{ff}$ is of order unity for $u \sim 1$ and is in the range 1 to 5 for $10^{-4} < u < 1$. So, a good order of magnitude estimates can be made by setting $\tilde{g}_{ff}$ to unity.

We see that bremsstrahlung has a rather flat spectrum in the log-log scale (Eq. (5.22) with $\tilde{g}_{ff}$ set to unity) upto its cut-off at about $h\nu \sim k_B T$ as shown in Fig. 5.2. So far we looked at the thermal distribution of velocities. For a non-thermal distribution, we need to have the actual distribution of velocities and the formula for single speed electron must be averaged over that distribution as we have done here in case of thermal distribution. In such cases, the requirement of appropriate Gaunt factors become necessary. Integrating Eq. (5.22) over frequency gives the total power per unit volume emitted via thermal bremsstrahlung mechanism

$$\frac{dW}{dVdt} = \left(\frac{2\pi k_B T}{3m} \right)^{1/2} \frac{2^5 \pi e^6}{3hmc^3} Z^2 n_e n_i \bar{g}_B, \tag{5.23}$$

where $\bar{g}_B$ is the frequency and velocity averaged Gaunt factor which is in the range 1.1 to 1.5. Choosing 1.2 gives an accuracy to within 20% [4].

5.3 Thermal bremsstrahlung absorption

Just like in the case of synchrotron self-absorption, we have to take into account the absorption effects in case of bremsstrahlung radiation as well. We will relate the absorption of radiation to the preceding bremsstrahlung emission process. Let us start with the Kirchoff's law of thermal radiation which is defined as

$$j_\nu^{ff} = \alpha_\nu^{ff} B_\nu(T), \tag{5.24}$$

where j_ν^{ff} is the emission coefficient for free-free absorption, α_ν^{ff} is the free-free absorption coefficient and $B_\nu(T)$ is the Planck function given by

$$B_\nu(T) = \frac{2h\nu^3/c^2}{exp\left(\frac{h\nu}{k_BT}\right) - 1}. \tag{5.25}$$

We can also define the power emitted per unit volume per unit frequency in terms of j_ν^{ff} as

$$\frac{dW}{dtdVd\nu} = 4\pi j_\nu^{ff}. \tag{5.26}$$

Using Eqns. (5.22), (5.24) and (5.26), we get the free-free absorption coefficient as

$$\alpha_\nu^{ff} = \frac{4e^6}{3mhc}\left(\frac{2\pi}{3k_Bm}\right)^{1/2} T^{-1/2} Z^2 n_e n_i \nu^{-3}(1 - e^{-h\nu/k_BT})\bar{g}_{ff}. \tag{5.27}$$

There are two cases of particular interest to us. When $h\nu \gg k_BT$, the exponential is negligible and we can write $\alpha_\nu^{ff} \propto \nu^{-3}$. For $h\nu \ll k_BT$, we can expand the exponential in powers of $(h\nu/k_BT)$ to arrive at

$$\alpha_\nu^{ff} = \frac{4e^6}{3mk_Bc}\left(\frac{2\pi}{3k_Bm}\right)^{1/2} T^{-3/2} Z^2 n_e n_i \nu^{-2} \bar{g}_{ff}. \tag{5.28}$$

Note that this is the Rayleigh-Jeans regime where $\alpha_\nu^{ff} \propto \nu^{-2}$.

5.4 Relativistic bremsstrahlung

Up till now, we have considered bremsstrahlung radiation due to a non-relativistic charge particle. In this section, we shall consider classical relativisitic bremsstrahlung. From Sec.

3.4.3, we know that the electric field due to a charge moving with a constant velocity is given by

$$\boldsymbol{E} = q\left[\frac{(\boldsymbol{n}-\boldsymbol{\beta})(1-\beta^2)}{(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^3 R^2}\right]. \tag{5.29}$$

Let us consider a situation as shown in Fig. 5.3 where a charge particle moves with a uniform velocity. In the given configuration, we have the various field components as follows

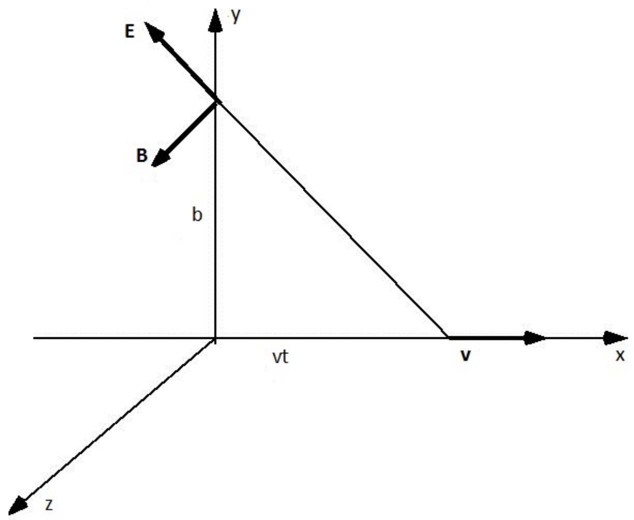


Figure 5.3: Electric and magnetic field due to a charge moving with a uniform velocity [4].

$$E_z = 0, \qquad E_y = \frac{q\gamma b}{(\gamma^2 v^2 t^2 + b^2)^{3/2}}, \qquad E_x = -\frac{q v \gamma b}{(\gamma^2 v^2 t^2 + b^2)^{3/2}}, \tag{5.30}$$

where we have considered the limit $\gamma \gg 1$. A plot of E_x and E_y is shown in Fig. 5.4. As it is clear from the graphs, the fields are strong only when t is of the same order as $b/\gamma v$. Hence the field of moving charge is confined only in the plane transverse to its motion within an angle $1/\gamma$ (cf. Sec. 3.5). We can find the equivalent spectrum of the pulse as follows

$$\hat{E}(\omega) = \frac{1}{2\pi}\int E_y(t) e^{i\omega t} dt, \tag{5.31}$$

where $\hat{E}(\omega)$ is the Fourier transform of $|\boldsymbol{E}|$. We have only taken E_y in this formula as its magnitude is much larger than the other components. Substituting E_y from Eq. (5.30) above and then integrating, we get the result

$$\hat{E}(\omega) = \frac{q}{\pi b v}\frac{b\omega}{\gamma v} K_1\left(\frac{b\omega}{\gamma v}\right), \tag{5.32}$$

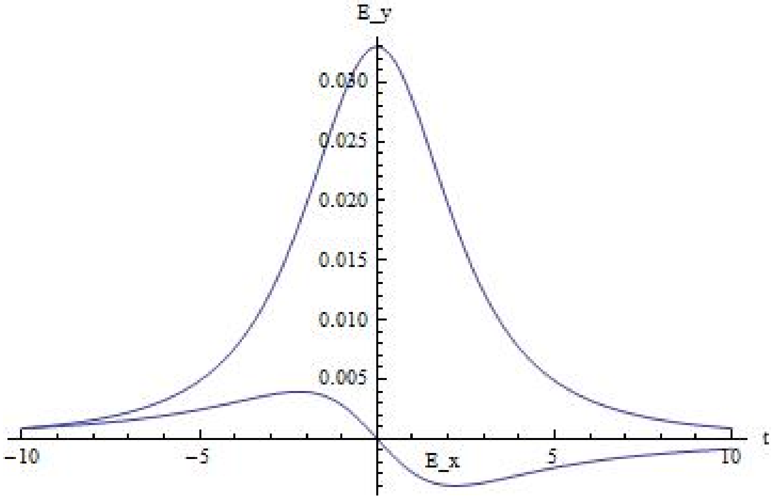


Figure 5.4: The amplitude of the x and y components of the electric field as a function of time due to a charge in a uniform motion as shown in Fig. 5.3. The field is essentially confined in a direction perpendicular to the motion i.e. $E_y \gg E_x$ [4].

where $K_n(x)$ is the Macdonald function or modified Bessel function of order n. The power spectrum can be obtained as follows

$$\frac{dP}{dAd\omega} = c|\hat{E}(\omega)|^2 = \frac{q^2 c}{\pi^2 b^2 v^2}\left(\frac{b\omega}{\gamma v}\right)^2 K_1^2\left(\frac{b\omega}{\gamma v}\right). \tag{5.33}$$

Now, let us consider a collision between an electron and a heavy ion of charge Ze. In the rest frame of the electron, the ion seems to move rapidly towards the electrons. We can assume the ion moving towards the electron along x-axis with electron placed on the y-axis at distance b from the origin. As we have seen above, for an electron the field due to the ion seems to be like a pulse (or a photon). So the electron compton scatters off the ion to produce the emitted radiation. However, if we look from the rest frame of the ion, we obtain bremsstrahlung radiation.

Thus, relativisitic bremsstrahlung can be regarded as compton scattering of virtual quanta of the ion's electrostatic field as seen in electron's frame. In the electron's rest frame, the spectrum of the pulse of virtual quanta has the form

$$\frac{dP'}{dA'd\omega'} = \frac{Z^2 e^2}{\pi^2 b^2 c}\left(\frac{b'\omega'}{\gamma v}\right)^2 K_1^2\left(\frac{b'\omega'}{\gamma v}\right), \tag{5.34}$$

where we have used Eq. (5.33) and took the ultra-relativistic limit $v \sim c$. The prime indicates that we are working in the rest frame of the electron. For low frequencies, the power per unit frequency can be written in terms of Eq. (5.34) as

$$\frac{dW'}{d\omega'} = \sigma_T \frac{dW'}{dA' d\omega'}, \tag{5.35}$$

where σ_T is the Thomson cross-section. We want to look the spectrum from the lab frame. Since the energy and frequency transform identically under Lorentz transformation

$$\frac{dW'}{d\omega'} = \frac{dW}{d\omega}. \tag{5.36}$$

Moving to the lab frame, we make the following changes: $b' = b$ and $\omega = \gamma\omega'(1 + \beta cos\theta')$ which is the doppler effect as one goes from one frame to the other. Since scattering is forward-backward symmetric, we can average over the angle θ' to get $\omega = \gamma\omega'$. Making these substitutions in Eq. (5.34), we get the emission in the lab frame as

$$\frac{dP}{d\omega} = \frac{8Z^2 e^6}{3\pi b^2 c^5 m^2} \left(\frac{b\omega}{\gamma^2 c}\right)^2 K_1^2 \left(\frac{b\omega}{\gamma^2 c}\right). \tag{5.37}$$

For a plasma with electron and ion densities n_e and n_i respectively, we can derive the low frequency limit of the above spectrum which comes out to be

$$\frac{dW}{dt dV d\omega} \sim \frac{16 Z^2 e^6 n_e n_i}{3c^4 m^2} ln\left(\frac{0.68\gamma^2 c}{\omega b_{min}}\right), \tag{5.38}$$

where in this case $b_{min} \sim h/mc$.

Examples of astrophysical sources which emit radiation via bremsstrahlung mechanism include regions containing ionised gases such as gaseous nebulae and in hot intracluster gas of clusters of galaxies. Thermal bremsstrahlung is believed to occur in giant elliptical galaxy such as M87 [10]. As we have seen in Sec. 5.2, the spectrum due to such emission process is rather flat upto a cut off frequency. Just like in the case of synchrotron spectrum, this could be used to determine various properties of the astrophysical system for which an example will be taken up in the next chapter.

Chapter 6

Case studies

In this chapter, we consider two examples one each of synchrotron and bremsstrahlung emission mechanisms to illustrate how to decipher the physical properties of the astrophysical source such as its size, mass, density etc.. from the observed spectrum. We shall make use of the theoretical concepts developed in chapters 4 and 5 in arriving at the conclusions (in this context, see Ref. [4]).

6.1 Synchrotron radiation

Let us consider a spectrum shown in Fig. 6.1 which is observed from a point source at an unknown distance d. A model for this source is a spherical mass of radius R emitting synchrotron radiation in a magnetic field of strength $\boldsymbol{B}$. Let us assume that the space between us and the source is uniformly filled with a thermal bath of hydrogen which emits and absorbs mainly by bound-free transitions, and that the hydrogen bath is unimportant compared to the synchrotron source at frequencies where the former is optically thin. The synchrotron source function is given by

$$S_{\nu} = A(ergcm^{-2}s^{-1}Hz^{-1})\left(\frac{B}{B_o}\right)^{-1/2}\left(\frac{\nu}{\nu_o}\right)^{5/2}. \tag{6.1}$$

The absorption coefficient for synchrotron radiation can be written as

$$\alpha_{\nu}^{s} = C(cm^{-1})\left(\frac{B}{B_o}\right)^{(p+2)/2}\left(\frac{\nu}{\nu_o}\right)^{-(p+4)/2}, \tag{6.2}$$

and for bound-free transitions

$$\alpha_{\nu}^{bf} = D(cm^{-1})\left(\frac{\nu}{\nu_o}\right)^{-3}, \tag{6.3}$$

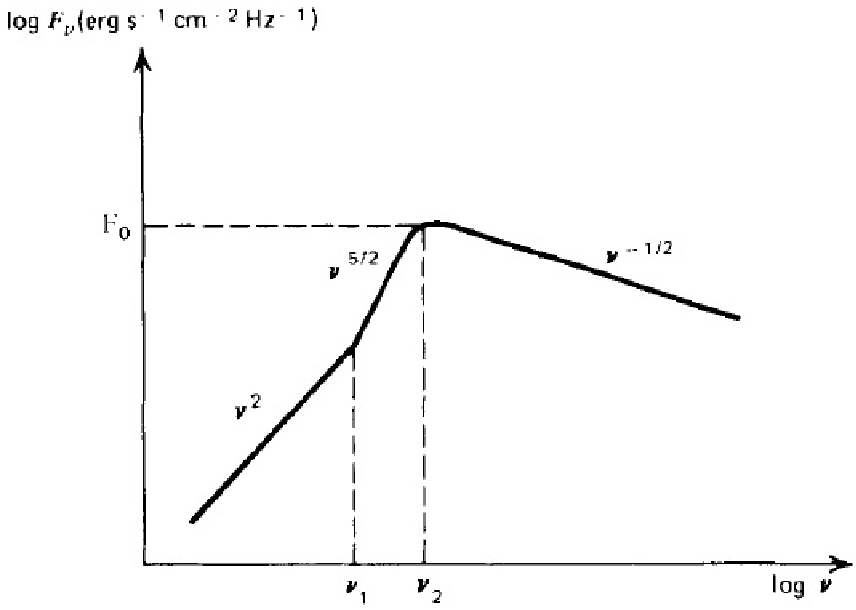


Figure 6.1: Observed spectrum from a point source (image from Ref. [4])

where A, B_o, ν_o, C and D are constants and p is the power law index for the assumed power law distribution of relativistic electrons in the synchrotron source.

The problem at hand is to find the size of the source R and the magnetic field strength B in terms of the solid angle $\Omega = \pi(R^2/d^2)$ subtended by the source and the constants A, B_o, ν_o, C and D. Secondly, we would also like to know the solid angle of the source and its distance from us.

- **Power law index:** Recall from Sec. 4.2 that for an optically thin source, the power spectrum has a frequency dependence proportional to ν^{-s} where $s = (p-1)/2$ is the spectral index and p the power law index. From the spectrum given in Fig. 6.1, we see that when the source is optically thin, $P(\nu) \propto \nu^{-1/2}$. Thus, $s = (p-1)/2 = 1/2$ which gives $p = 2$. Hence the power law index for the distribution of electrons is 2.

- **Magnetic field:** From Fig. 6.1, it is apparent that the spectrum makes a transition at two frequencies ν_1 and ν_2. The flux of a source can be written as

$$F_\nu = S_\nu \times \Omega, \tag{6.4}$$

where F is the flux of the source at frequency ν, S_ν is the source function which is given by Eq. (6.1) and Ω, the solid angle. From Fig. 6.1, the flux at ν_2 is F_o. Substituting the relevant values in Eq. (6.4), we get

$$F_o = A\left(\frac{B}{B_o}\right)^{-1/2}\left(\frac{\nu_2}{\nu_o}^{5/2}\right)\Omega, \tag{6.5}$$

where the expression for source function (6.1) is evaluated at the transition frequency ν_2. Re-arranging the above equation, we get the magnetic field in terms of the known constants as

$$B = B_o\left[\frac{A\Omega}{F_o}\left(\frac{\nu_2}{\nu_o}\right)^{5/2}\right]^2. \tag{6.6}$$

- **Size of the source, R:** The absorption coefficient for synchrotron radiation at frequency ν_2 is just sufficient for the electron to travel a distance R within the source. If dl is the mean free path of a particle within the source, we can write

$$\int \alpha_\nu^s dl = 1. \tag{6.7}$$

  If we assume the absorption coefficient given by Eq. (6.2) to be constant while the electron moves a distance R, we obtain

$$C\left(\frac{B}{B_o}\right)^{(p+2)/2}\left(\frac{\nu_2}{\nu_o}\right)^{-3} R \sim 1, \tag{6.8}$$

  where we have evaluated α_ν^s at the frequency ν_2 and set $p = 2$. Rearranging the expression above and using Eq. (6.6), we get the size of the source in terms of the required constants

$$R = C^{-1}\left(\frac{\nu_2}{\nu_o}\right)^{-7}\left(\frac{A\Omega}{F_o}\right)^{-4}. \tag{6.9}$$

- **Solid angle, Ω** and **distance**, d: We know that the solid angle subtended by a source of area d^2, d being the size of the source which is at a distance R is given by $\Omega = \pi R^2/d^2$. Using the mean free path concept mentioned in the previous part, we can write

$$\alpha_{\nu_1}^{bf} d \sim 1, \tag{6.10}$$

  where this time ν_1 is the frequency at which hydrogen becomes optically thick. It is now easy to obtain d using Eqs. (6.3) and (6.10). We can also arrive at the expression for solid angle using its definition mentioned above and Eqs. (6.9) and (6.11). The final results are as follows

$$d = D^{-1}\left(\frac{\nu_1}{\nu_o}\right)^3, \tag{6.11}$$

$$\Omega = \pi A^{-8} C^{-2} D^2 \left(\frac{\nu_1}{\nu_o}\right)^{-6} \left(\frac{\nu_2}{\nu_o}\right) F_o^8 \tag{6.12}$$

Hence, from the information given in the spectra, we were able to arrive at the properties of the astrophysical source such as the magnetic field (6.6), size of the source (6.9), its distance (6.11) and the solid angle subtended by it (6.12).

6.2 Bremsstrahlung

Suppose X-rays are received from a source of known distance L with a flux $F(ergcm^{-2}s^{-1})$. Let the spectrum be of the form as shown in Fig. 6.2. It is conjectured that these X-rays are due to bremsstrahlung from an optically thin, hot, plasma cloud, which is in dynamic equilibrium around a central mass M. Assuming that the cloud thickness ΔR is roughly its radius R, $\Delta R \sim R$, we have to find R and the density of the cloud, ρ, in terms of the known observations and conjectured mass M [4].

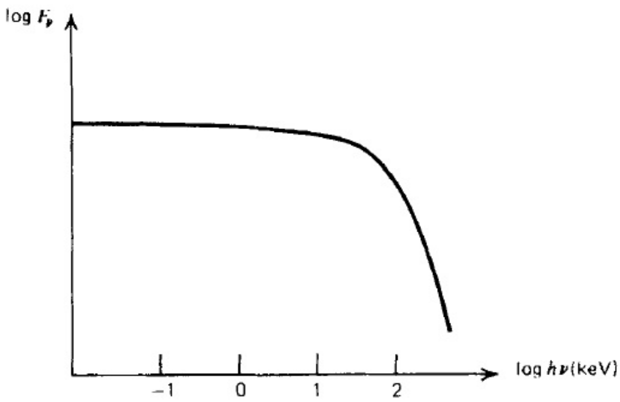


Figure 6.2: Detected spectrum from an X-ray source (image from Ref. [4]).

Since the cloud is in a hydrostatic equilibrium around the mass M, the gravitational pull is supported by the kinetic energy of the particles in the plasma. According to virial theorem for gravitational potential $U = -GMm/R$, $2K = -U$ where K is the kinetic energy and for a system at temperature T, this is equal to $(3/2)k_BT$ and m is the mass of the particles constituting the plasma. From Fig. 6.2, we see that the spectrum has a cut off at about $h\nu \simeq 10^2 keV = k_BT$, where the second equality follows from the discussion in Sec. 5.2. We obtain the temperature of the system as $10^9 K$. Substituting the relevant values in the virial theorem, we arrive at the size of the source in terms of the known constants to be

$$R = \frac{GMm}{3k_BT} \tag{6.13}$$

To find the density ρ, we make use of Eq. (5.38) in CGS units where we shall approximate g_B to 1.2. This is the emissivity function of the source. Note that the gas cloud we have considered is optically thin, so this formula is valid. Assuming the number densities of the electrons and the ions to be equal, we can write $n_e = n_i = \rho/m$. Making these changes in Eq. (5.38), we obtain that

$$\epsilon^{ff} = 1.68 \times 10^{-27} \sqrt{T}\frac{\rho^2}{m^2} Z^2 ergs^{-1}cm^{-3} \tag{6.14}$$

In a volume $V = (4/3)\pi R^3$, the total energy of the emitted radiation is give by $\epsilon^{ff} \times V$. Flux observed at a distance L is then given by

$$F = \frac{\epsilon^{ff}V}{4\pi L^2} \tag{6.15}$$

Using Eqns. (6.13), (6.14) and (6.15), we obtain the density as given below

$$\rho = \left(48.2 \times 10^{27} \frac{L^2 k_B^3 F T^{5/2}}{G^3 M^2 m Z^2}\right)^{1/2} \tag{6.16}$$

Hence we have arrived at the expressions for the size and the density of the hot plasma cloud in terms of the known constants assuming that the cloud is optically thin.

6.3 Discussion

These two examples clearly reflect the importance of spectral analysis in astrophysics. Various physical properties of the sources such as mass, size, density, its distance etc.. could be analyzed using them. The theoretical concepts addressed in chapters 5 and 6 make the foundation for such analysis.

Chapter 7

Summary

To summarize the contents of this report, we started with writing the action of charge particles interacting with electromagnetic fields. A variation in this action led to various equations of motion in different field configurations and the Maxwell's equations decribing the dynamics of the field itself. The solutions to these Maxwell's equations was obtained rigorously using Green's function which led us to the conclusion that accelerating charges emit electromagnetic radiation. This radiation has many interesting features such as relativistic beaming due to which the emission is confined in a narrow cone along the direction of motion.

We saw that relativistic beaming leads to a broad spectrum in synchrotron radiation. Using the radiation field due to an accelerating charge in a magnetic field, we were able to obtain the power spectrum for a distribution of synchrotron emitting electrons. Synchrotron radiation has characteristic features such as presence of linear polarization and a transition in the power spectrum dependence from a source function to an emission function. We also arrived at the spectrum from a charge accelerating in a Coulomb field which is called bremsstrahlung. A classical treatment was followed in all the cases above with quantum corrections applied wherever necessary. Various physical properties of an astrophysical source were also obtained using an observed spectrum as an example to emphasize the importance of the power spectrum to characterize the sources.

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