ROTATION CURVES OF GALAXIES

A project report submitted in partial fulfilment

for the award of the Masters degree in Physics

by

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CERTIFICATE

This is to certify that the project entitled **Rotation curves of galaxies** submitted by **Parvathy Harikumar**, in partial fulfilment for the award of Masters degree in Physics, is an independent bona fide record of work done by the candidate at University of Madras.

(L. Sriramkumar, Project supervisor)

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Abstract

Understanding the complex working of our universe relies partially on understanding the dynamics of self gravitating systems like galaxies and globular clusters which form an integral part of the universe. Since the components that make up such systems move in their orbits at non relativistic velocities, a Newtonian approach can very well be adopted to study them. In this report, the gravitational potential arising due to the distribution of mass in a galaxy is obtained from the distribution function frozen at a point of time in the phase space. Our aim is to assign a suitable form of density profile to the distribution of dark matter in galaxies such that the flat rotation curves,obtained from the studies of motion of stars and clouds, is reproduced. To get a deeper idea of the concept, we shall also look at how the orbits of stars and their respective orbital periods are determined theoretically.

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CHAPTER 1

Introduction

”The innocent and light minded, who believe that astronomy can be studied by looking at the heavens without knowledge of mathematics, will return in the next life as birds.”

-Plato

The best thing about the dark night sky which blankets half the earth is that half of us can see those beautiful glimmering gems scattered against the black canvas, and if you take a moment to step outside you too can marvel this small glimpse of the universe beyond. The stars that we see do not individually make up the Universe but are all bound by a universal force called gravity, the very reason why the apple fell on Newtons head. He was the first Scientist to understand the role gravity plays in explaining why everything is the way it is like and why Earth and all other planets revolve around the Sun in nearly elliptical orbits. This was a breakthrough in the history of Science as it paved its way to many inventions like rockets, satellites and space stations when we learned what kept us here and we gradually figured out how to escape its effects using sufficient velocity called the escape velocity. A galaxy is formed through many stages of evolution commencing from a huge clump of gas clouds collapsing under its own gravity to form what we see now. In most cases, in the centre of a galaxy dwells a supermassive black hole whose enormous gravity has an inimitable impact on the dynamics of its parent galaxy.

The prodigiousness of gravity lies in the fact that the same force which keeps our feet on the ground can suck in the matter from a star as huge as our Sun to form an accretion disk around a black hole. Most of the dynamics of stars, galaxies and clusters of galaxies can be understood with the Newtonian concept of gravity as the subjects are macroscopic objects.

Centuries ago, when the galaxies were only being discovered one by one, they were thought to be island universes isolated from each other. But only later on did Scientists find that it is not so. Galaxies do interact among their neighbouring galaxies through gravitational forces and just like how the rotation of moon around the Earth cause waves in the ocean, the gravitational field of a galaxy has tidal effects on other galaxies. Many factors influence the formation and evolution of galaxies as a consequence of which we observe different structures in them. Structure simply implies the way stars and other components are distributed in space. Sometimes they are distributed on a disk like structure embedded in a spherical halo whereas in other cases they might be distributed in a spiral or elliptical shape and it goes on. Our focus will be primarily on the former distribution of stars. As we all know dark matter makes up most of the universe. But it interacts very weakly with ordinary matter. Out of the four fundamental forces in nature, dark matter is believed only to exert a gravitational force on matter. Anything non luminous and occupying a significant volume in a galaxy is nothing but dark matter. The distribution of dark matter can be understood from what we call the rotation curves of galaxies. Once we discuss the aspects of the self consistency of the distribution function of stars in a galaxy in the second chapter, we will go on to this simple but interesting idea of how dark matter has an influence of the orbital velocity of the stars in the spherical halo. The basic intention of this report is to find the rotation curves of disk galaxies embedded in a spherical halo consisting of old stars and dark matter starting from the description of such systems by the Collisionless Boltzmann equation(CBE).

CHAPTER 2

Gravitational dynamics

2.1 An overview of self gravitating systems

Gravity is a force we are all familiar with. In fact we take it for granted even when its profound effects permeates through matter and every cell in our body. According to Einstein, mass of an object attributes to its gravity and that bends the space time surrounding it. In a Galaxy, the stars move under the influence of the gravitational field due to the stars in the background. Such systems are generally termed as self gravitating systems as they evolve in a field created by the mass of the entire galaxy to which it contributes its share. Because of self-gravity, stellar systems have to face a natural tendency to collapse. In the absence of dissipation, one may assume equilibrium configurations in which random motions are able to oppose the tendency to collapse. In this chapter, a close analogy is drawn between such systems and a barotropic fluid whose density is a function of pressure of the fluid and vice versa. This assumption will become clear as we go further.

Though everything looks fair enough in this context, unlike in fluids where the molecules (gas or liquid) collide with each other on their path and approach a Maxwellian distribution to achieve a thermal equilibrium, here we assume Galaxies to have reached their steady state configuration. This follows from the fact that all Galaxies that we see have formed billions of years ago. In just the same way as a gas kept in a container comes into equilibrium with the surroundings on leaving it for an ample amount of time, a Galaxy also forms and evolves through different stages to be what they look like now. Moreover Galaxies are assumed to be a collisionless system of stars with a relaxation time much larger than the age of the universe itself. Though close encounters between stars have been reported, we have never witnessed a head on collision among stars.

As we all know, stars are formed from a gas cloud comprised of mostly Hydrogen and Helium, but the temperatures developed as a result of the exothermic nuclear fusion ionises the gas and hence form plasma. Thus inside a star, other than the gravitational forces that keeps the components together there is equally an electrostatic force of attraction or repulsion between the charged ions of the plasma.

2.2 Collisionless boltzmann equation

One of the many crude approximations to study the self gravitating systems is that they are in more or less a steady state. Though evolution is inevitable, stellar systems may be treated as if they are in a thermal equilibrium for the time domain in which they are studied. If we think of the gravitational potential of the galaxy as the sum of a smoothly varying averaged component and a steep potential well near each star,the situation gets more complicated to handle(see **figure.2.1**) [10].

In the beginning of the chapter we had discussed why the constituent particles should evolve in a collisionless manner. In much the same way we can also discuss why the velocity of these particles should approach a Maxwellian distribution in the process of attaining a thermal equilibrium. Rather than thinking of stars as localised point particles in a galaxy,it is more appropriate to treat them as a fluid smoothly distributed from one end to the other. Thus, we can introduce a distribution function f(x,v,t) that gives the number of stars in a phase space volume $d^3\mathbf{x}d^3\mathbf{v}$ at time t assuming stars to move in phase space. So the probability of finding it at any given phase-space location evolves with time. In fluid flow,mass of fluid passing a boundary is conserved as long there are no sources and sinks. So the conservation of fluid mass described by the continuity equation in fluid dynamics for a fluid of density ρ and velocity $\dot{\mathbf{x}}$ is given by,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \mathbf{x}}(\rho \dot{\mathbf{x}}) = 0 \tag{2.1}$$

In our assumption, stars do have a lifetime after which they die and new stars are

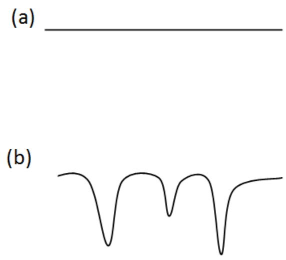

Figure.2.1.(a)Smooth potential,(b)Potential wells near each star

born in the gas clouds of Galaxies. But if the rate of birth of stars, say B and rate of death of stars say D is nearly equal,then B-D $\approx$0. For an evolving system of stars, if C denotes the collision terms $(df/dt) = C$. [1] wherein assuming a collisionless evolution we get the Boltzmann equation.

$$\frac{df}{dt} = 0 \tag{2.2}$$

Expanding out the time derivative and using $\dot{\mathbf{v}} = -\nabla\Phi$, we can express the equations that describe a collisionless gravitating system as

$$\frac{df}{dt} = \frac{\partial f}{\partial \mathbf{x}} + \dot{\mathbf{v}}.\frac{\partial f}{\partial v} + \dot{\mathbf{x}}.\frac{\partial f}{\partial x} = \frac{\partial f}{\partial t} + \mathbf{v}.\frac{\partial f}{\partial \mathbf{x}} - \nabla\Phi.\frac{\partial f}{\partial \mathbf{v}} = 0 \tag{2.3}$$

with the potential determined by

$$\nabla^2\Phi = 4\pi G\rho, \rho(x,t) = m\int f(x,v,t)d^3v \tag{2.4}$$

This is the collisionless Boltzmann equation which is a special case of Liouville's theorem. It states that the flow of stellar phase points through phase space is incompressible, or the phase space density around the phase point of any star remains constant.

Depending on the distribution function which is different for different stellar systems a wide variety of solutions are obtained to the coupled equations eqn(2.3) giving rise to many possible models for galaxies which will be discussed in detail in the next section. As of now we limit our discussions to the time independent solutions to the equations where $f(t,\mathbf{x},\mathbf{v}) = f(\mathbf{x},\mathbf{v})$. We know from the definition of an integral of motion that a function in phase space coordinates is an integral only if

$$\frac{d\mathbf{I}(\mathbf{x},\mathbf{v})}{dt} = 0 \tag{2.5}$$

For stars moving in the stationary potential Φ, let $I_i, i = 1, 2..$ be a set of integrals, which is right now unknown. It is obvious that any function $f(I_i)$ of the $I_i's$ will satisfy the steady-state Boltzmann equation;$(df/dt) = (\partial f/\partial I_i)\dot{I}_i = 0$,as $\dot{I}_i$ is identically zero [1]. If we can now determine the Φ from from f self-consistently and populate the orbits of Φ with stars, we have solved the problem.

For the ease of calculations, we will first look at models in which the density and the potential are spherically symmetric. Also it is convenient to shift the origin of Φ by defining a relative potential $\Psi \equiv -\Phi + \Phi_0$, where Φ_0 is a constant. This potential satisfies the equation

$$\nabla^2\Psi = -4\pi G\rho \tag{2.6}$$

and the boundary condition $\Psi \to \Phi_0$ as $|x| \to \infty$.. We also define a shifted energy for particles $\epsilon = -E + \Phi_0$; because $\Phi_0 = \Psi + \Phi, \epsilon = -E + \Psi + \Phi = -\frac{1}{2}v^2 + \Psi$. Sometimes distribution function depends only on the energy ϵ. Such models of galaxies are known as Isotropic models. In other cases it depends on both energy and angular momentum L_z and are known as Anisotropic models. These models are discussed in greater detail in the next section.

2.2.1 Models of Galaxies

Isotropic models

In any steady-state potential $\Phi(x)$, the Hamiltonian H is an integral of motion. Consequently, an equilibrium stellar system is obtained by taking f to be any non-negative function of the Hamiltonian. Distribution functions of this type are called ergodic as every state has a large finite probability to recur. If the potential is constant in an inertial frame, H will be of the form $H = \frac{1}{2}v^2 + \Phi(x)$ and it follows that velocity vanishes everywhere:

$$\bar{\nu(x)} = \frac{1}{\nu(x)} \int d^3\mathbf{v}\mathbf{v} f(\frac{1}{2}v^2 + \Phi(x)) = 0 \tag{2.7}$$

where the second equality follows because the integrand is an odd function of v and the integral is over all velocity space. A similar line of reasoning shows that the velocity-dispersion tensor is isotropic:

$$\sigma_{ij}^2 = \overline{v_i v_j} = \sigma^2 \delta_{ij} \tag{2.8}$$

$$\begin{aligned} \sigma^2(x) &= \frac{1}{\nu(x)} \int dv_z v_z^2 \int dv_y dv_x f(\frac{1}{2}(v_x^2 + v_y^2 + v_z^2) + \Phi(x)) \\ &= \frac{4\pi}{3\nu(x)} \int_0^\infty d\mathbf{v}\mathbf{v}^4 f(\frac{1}{2}v^2 + \Phi). \end{aligned} \tag{2.9}$$

Hence, $f(\mathbf{x}, \mathbf{v}) = f(\epsilon) = f(\Psi - \frac{1}{2}v^2)$. The density $\rho(\mathbf{x})$ corresponding to this distribution is

$$\rho(\mathbf{x}) = \int_0^{\sqrt{2\Psi}} 4\pi v^2 dv f(\Psi - \frac{1}{2}v^2) = \int_0^\Psi 4\pi d\epsilon f(\epsilon)\sqrt{2(\Psi - \epsilon)} \tag{2.10}$$

As we are concerned only about the particles bound in the system's potential, the limits of the integral are the values that the potential takes at the boundaries. Once f is specified the right hand side becomes a known function of Ψ from which density ρ can be determined using the Poisson's equation. For a spherically symmetric potential,

$$\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d\Psi}{dr}) = -4\pi G = -16\pi^2 G \int_0^\Psi d\epsilon f(\epsilon)\sqrt{2(\Psi - \epsilon)} \tag{2.11}$$

From the above equation,$\Psi(r)$ can be obtained with some central value $\Psi(0)$ and the boundary conditions $\Psi'(0) = 0$. Once $\Psi(r)$ is known, all other variables can be computed.

It is quite straightforward to find a relation between a given density $\rho(r)$ and the unique $f(\epsilon)$ generated by it. This is done primarily by determining $\Psi(r)$ from $\rho(r)$ and eliminating r which gives from eqn(2.11) ,

$$\frac{1}{\sqrt{8}\pi}\rho(\Psi) = 2\int_0^\Psi f(\epsilon)\sqrt{\Psi - \epsilon} \tag{2.12}$$

And differentiating both sides with respect to Ψ, we get

$$\frac{1}{\sqrt{8}\pi}\frac{d\rho}{d\Psi} = \int_0^\Psi \frac{f(\epsilon)d\epsilon}{\sqrt{\Psi - \epsilon}}. \tag{2.13}$$

This equation is known as the Abel's integral equation [2] and has the solution,

$$f(\epsilon) = \frac{1}{\sqrt{8}\pi^2}\frac{d}{d\epsilon}\int_0^\epsilon \left(\frac{d\rho}{d\Psi}\right)\frac{d\Psi}{\sqrt{\epsilon - \Psi}}, \tag{2.14}$$

which determines $f(\epsilon)$. Depending upon the choice of $f(\epsilon)$,we can classify spherically symmetric models into three classes of models which are namely the singular isothermal sphere and the Kings model.

The simplest form of the distribution function is a power law with $f(\epsilon) = A\epsilon^{n-\frac{3}{2}}$ for $\epsilon > 0$ and zero otherwise.Using equation (2.12),we get

$$\frac{1}{\sqrt{8}\pi}\rho(\varphi) = 2\int_0^\varphi A\epsilon^{n-\frac{3}{2}}\sqrt{\varphi - \epsilon}d\epsilon \tag{2.15}$$

To solve the above integral, it is appropriate to make use of the Walli's formula given by,

$$2\int_0^1 \sqrt{1 - x^2}dx = \Gamma(\frac{1}{2})\Gamma(\frac{3}{2}) \tag{2.16}$$

On substituting $x^2 = t$ in the formula, we get

$$2\int_0^1 \frac{t^{\frac{1}{2}-1}(1-t)^{\frac{3}{2}-1}}{2}dt = \Gamma(\frac{1}{2})\Gamma(\frac{3}{2}) \tag{2.17}$$

In general for some α and γ, it becomes the famous Euler-Beta integral [9],

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{2.18}$$

The equation(2.15)can also be written as,

$$\frac{1}{\sqrt{8}\pi}\rho(\varphi) \quad = \quad 2\int_0^{\varphi} A\epsilon^{(n-\frac{1}{2})-1}\sqrt{\varphi}(1-\frac{\epsilon}{\varphi})^{\frac{3}{2}-1}d\epsilon$$

On multiplying and dividing by $\varphi^{(n-\frac{1}{2}-1)}$, we get

$$2\int_0^{\varphi} A\epsilon^{(n-\frac{1}{2})-1}\sqrt{\varphi}(1-\frac{\epsilon}{\varphi})^{\frac{3}{2}-1}d\epsilon \quad = \quad 2\int_0^{1} A\frac{\epsilon}{\varphi}^{(n-\frac{1}{2})-1}\varphi^n(1-\frac{\epsilon}{\varphi})^{(}\frac{3}{2}-1)\frac{d\epsilon}{\varphi} \tag{2.19}$$

On substituting $(\epsilon/\varphi) = t$ in eqn(2.19) and comparing it with eqn(2.18) we get $\alpha = n-\frac{1}{2}$and $\beta = \frac{3}{2}$,which gives

$$\frac{1}{\sqrt{8}\pi}\rho(\varphi) = 2A\varphi^n\frac{\Gamma(n-\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(n-\frac{1}{2}+\frac{3}{2})} \tag{2.20}$$

Using the properties of Gamma functions, we obtain the density as a function of the potential to be

$$\begin{aligned}\rho(\varphi) &= 2\sqrt{2}\pi A\varphi^n\frac{\Gamma(n-\frac{1}{2})\sqrt{\pi}}{\Gamma(n+1)}\\ &= B\varphi^n,\end{aligned} \tag{2.21}$$

where

$$B = (2\pi)^{\frac{3}{2}}A\frac{\Gamma(n-\frac{1}{2})}{\Gamma(n+1)} \tag{2.22}$$

$$f(\epsilon) = \frac{\rho_0}{(2\pi\sigma^2)^{\frac{3}{2}}}\exp(\frac{\epsilon}{\sigma^2}) \tag{2.23}$$

where ρ_0 and σ are constants

Isothermal sphere

The Isothermal sphere is a configuration of infinite radius and infinite mass. We know that for barotropic fluids, the pressure P is dependent on the density ρ of the fluid. When $P \propto \rho$, we get an isothermal sphere. Now, it can be shown easily that the following distribution function describes an isothermal sphere,

$$f(\epsilon) = \frac{\rho_0}{(2\pi\sigma^2)^{\frac{3}{2}}} \exp(\frac{\epsilon}{\sigma^2}), \tag{2.24}$$

which is parametrized by two constants ρ_0 and σ. The distribution of velocities at each point in the stellar isothermal sphere is the Maxwellian or Maxwell-Boltzmann distribution. Thus the mean square speed of stars at a point is given by,

$$\langle v^2 \rangle = \frac{\int_0^\infty dvv^4 \exp(\frac{\varphi - \frac{1}{2}v^2}{\sigma^2})}{\int_0^\infty dvv^2 \exp(\frac{\varphi - \frac{1}{2}v^2}{\sigma^2})}$$

By eliminating $\exp(\frac{\varphi}{\sigma^2})$ and substituting $\frac{v^2}{2\sigma^2} = x^2 \Rightarrow v = \sqrt{2}\sigma x$, we get

$$\langle v^2 \rangle \quad = \quad 2\sigma^2 \frac{\int_0^\infty x^4 \exp(-x^2)dx}{\int_0^\infty x^2 \exp(-x^2)dx}$$

Making use of the Gamma integral [9],

$$\int_0^\infty x^m \exp(-ax^2)dx = \frac{\Gamma(\frac{m+1}{2})}{2a^{\frac{m+1}{2}}} \tag{2.25}$$

the equation becomes,

$$\begin{aligned} \langle v^2 \rangle &= \frac{2\sigma^2 \frac{3}{2} \Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2})} \\ &= 3\sigma^2 \end{aligned} \tag{2.26}$$

It is quite straightforward to find the density $\rho(r)$ from $f(\epsilon)$,

$$\begin{aligned} \rho(r) &= \int_0^\varphi 4\pi d\epsilon f(\epsilon)\sqrt{2(\varphi - \epsilon)} \\ &= \frac{2\rho_0}{\sqrt{\pi}\sigma^2} exp(\frac{\varphi}{\sigma^2}) \int_0^\infty \exp(\frac{-(\varphi - \epsilon)}{\sigma^2}) \sqrt{\frac{2(\varphi - \epsilon)}{\sigma^2}} d\epsilon \end{aligned} \tag{2.27}$$

On substituting $(\varphi - \epsilon)/\sigma^2 = x$ and using the Gamma integral,

$$\int_0^\infty x^n \exp(-ax)dx = \frac{\Gamma(n+1)}{a^{n+1}} \tag{2.28}$$

We get,

$$\rho(r) = \frac{2\rho_0}{\sqrt{\pi}} \exp(\frac{\varphi}{\sigma^2})\Gamma(\frac{3}{2}) \tag{2.29}$$

Using $\Gamma(n+1) = n(n+1)$, we get the density of the isothermal sphere to be,

$$\therefore \rho(r) = \frac{2\rho_0}{\sqrt{\pi}} \exp(\frac{\varphi}{\sigma^2})\frac{1}{2}\sqrt{\pi} = \rho_0 \exp(\frac{\varphi}{\sigma^2}) \tag{2.30}$$

Thus v^2 is independent of position. [1] The dispersion in any one component of velocity, for example $(v_r)^{1/2}$, is equal to σ.

Poisson's equation for this system reads

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\varphi}{dr}\right) = -4\pi G\rho \tag{2.31}$$

or with equation(2.30)

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\varphi}{dr}\right) = -4\pi G\rho_0 \exp(\frac{\varphi}{\sigma^2}) \tag{2.32}$$

If we consider a power law density profile $\rho = Cr^{-b}$, we get a singular isothermal sphere who density is infinite at the origin $r = 0$. To obtain a solution that is well behaved at the origin, it is convenient to define the following dimensionless variables.

$$r_0 = \left(\frac{9\sigma^2}{4\pi G\rho_0}\right)^{\frac{1}{2}}; l = \frac{r}{r_0}; \zeta = \frac{\rho}{\rho_0} \tag{2.33}$$

Equation becomes,

$$\frac{1}{r_0^2 l^2}\frac{d}{dl}\left(l^2\frac{d}{dl}\ln\rho/\rho_0\right) = -\frac{4\pi G(\zeta\rho_0)}{\sigma^2} \tag{2.34}$$

Substituting $r_0^2 = 9\sigma^2/4\pi G\rho_0$ and rearranging the terms,

$$\frac{1}{l^2}\frac{d}{dl}\left(l^2\frac{d}{dl}\ln\zeta\right) = -\frac{4\pi G\zeta\rho_0 9\sigma^2}{4\pi G\rho_0\sigma^2} = -9\zeta \tag{2.35}$$

In the above equation,letting $\zeta = \exp(-y)$, we arrive at

$$\frac{1}{l^2}\frac{d}{dl}l^2\frac{dy}{dl} = 9\exp(-y) \tag{2.36}$$

Hence we obtain the equation for an isothermal sphere. Isothermal sphere models are applicable in practice, for example, stellar cores with no nuclear burning, or star clusters etc.

There is another class of galactic models namely **Kings model**, sometimes called lowered isothermal models. They provide a good description of non-rotating globular clusters and open clusters,the distribution function of which is given by,

$$f_k(\epsilon) = \frac{\rho_0}{(2\pi\sigma)^{\frac{3}{2}}}\exp\left\{-\frac{[\Phi(r)+\frac{v^2}{2}]}{\sigma^2} - 1\right\} \tag{2.37}$$

When we integrate this to find $\rho(r)$, and then solve Poisson's equation, the term '-1' acts to reduce the number of stars with high kinetic energy in the outer regions. The average random speeds decreases and the density drops abruptly to zero at some outer truncation radius [10].

Anisotropic Models

As I had mentioned earlier, a distribution function might depend on the angular momentum J as well as on ϵ. This chance howsoever helps us to explore an even wider class of galactic models to fit the same $\rho(r)$. In this case the velocity dispersion of stars remains no longer isotropic. Let us begin with yet another assumption that f depends on ϵ and J^2 only through the combination

$$Q = \epsilon - \frac{J^2}{2R^2}, \tag{2.38}$$

where R is the measure of anisotropy,a free parameter. Such models are called Osipkov-Meritt models.

$$2Q = v_r^2 + (1 + \frac{r^2}{R^2})v_t^2 + 2\Phi(r) \tag{2.39}$$

where v_r, v_t are velocity components parallel and perpendicular to the radius vector **r**.

The density ρ is the integral over all velocities of f:

$$\rho(r) = 2\pi \int\int v_t dv_t dv_r f(\epsilon, J) \tag{2.40}$$

which can be written

$$\rho(r) = \frac{2\pi}{r^2} \int_{\Phi}^{0} dQ f(Q) \int_{0}^{2r^2(Q-\Phi)/(1+\frac{r^2}{R^2})} dJ^2 [2(Q-\Phi) - \frac{J^2}{r^2}] \tag{2.41}$$

or equivalently

$$\rho(r) = \frac{2\pi\sqrt{8}}{(1+\frac{r^2}{R^2})} \int_0^{\Psi} f(Q)\sqrt{\Psi - Q}dQ \equiv \frac{2\pi\sqrt{8}}{(1+\frac{r^2}{R^2})}\mu(\Psi). \tag{2.42}$$

where the second equality defines $\mu(\Psi)$.Comparing eqn(2.14)and eqn(2.42),we get

$$f(Q) = \frac{1}{\pi^2\sqrt{8}} \frac{d}{dQ} \int_0^{Q} \left(\frac{d\mu}{d\Psi}\right) \frac{d\Psi}{\sqrt{Q-\Psi}}. \tag{2.43}$$

Mestel disks

Sometimes it also happens that the distribution function depends only on the z component of angular momentum vector,J_z. Such systems are highly flattened that there is little or no vertical distribution of stars and Mestel disks are one among them. For flattened systems, very few stars have orbits taking them close to the z axis, with $J_z \approx 0$. [10]But relatively many will follow near-circular orbits in the equatorial plane with large J_z. Galaxies are flattened by an excess of kinetic energy in the equatorial plane relative to the meridional plane.The distribution function for a Mestel disk is,

$$f(\epsilon, J_z) = AJ_z^n \exp(\epsilon/\sigma^2), J_z \geq 0. \tag{2.44}$$

$$= 0, J_z \leq 0$$

For a disk with surface density,

$$\Sigma = \Sigma_0 \frac{R_0}{R} \tag{2.45}$$

the circular velocity [1],

$$v_c^2 = -R\frac{\partial \Psi}{\partial R} = 2\pi G \Sigma_0 R_0 \tag{2.46}$$

We set the arbitrary constant involved in the definition of the relative potential such that $\Psi(R_0) = 0$ and integrate the above equation with respect to R, to find

$$\Psi = -v_c^2 \ln(\frac{R}{R_0}) \tag{2.47}$$

Now inserting eq.(2.47) into eq.(2.44) and integrating over all velocities,we get

$$\begin{aligned} \Sigma'(R) &= AR^n \int_0^\infty dv_\phi v_\phi^n \int_{-\infty}^\infty dv_R \exp\left[\frac{-v_c^2}{\sigma^2}\ln(\frac{R}{R_0}) - \frac{(v_R^2+v_\phi^2)}{2\sigma^2}\right] \\ &= AR^n(\frac{R}{R_0})^{-v_c^2/\sigma^2} \int_0^\infty dv_\phi v_\phi^n \exp(-\frac{v_\phi^2}{2\sigma^2}) \int_{-\infty}^\infty dv_R \exp(-\frac{v_c^2}{2\sigma^2}) \end{aligned} \tag{2.48}$$

Evaluating the integrals taking $x^2 = v_\phi^2/2\sigma^2$ and $y^2 = v_R^2/2\sigma^2$,we get

$$
\begin{aligned}
\int_0^\infty dv_\phi v_\phi^n \exp(-\frac{v_\phi^2}{2\sigma^2}) &= (\sqrt{2})^{n-1} \sigma^{n+1} \left(\frac{n-1}{2}\right)! \\
\int_{-\infty}^\infty dv_R \exp(-\frac{v_c^2}{2\sigma^2}) &= \sqrt{\frac{\pi}{2}}\sigma
\end{aligned}
\tag{2.49}
$$

Substituting these expressions back into eq.,

$$
\Sigma'(R) = AR^n (\frac{R}{R_0})^{-v_c^2/\sigma^2} 2^{\frac{n}{2}} \sigma^{n+2} \left(\frac{n-1}{2}\right)! \sqrt{\pi} \tag{2.50}
$$

Comparing equations (2.45) and (2.50), we see that the DF of equation (2.44) will self-consistently generate the Mestel disk if we set

$$
n = \frac{v_c^2}{\sigma^2} - 1, \qquad\qquad A = \frac{\Sigma_0}{2^{\frac{n}{2}} \sqrt{\pi} \left(\frac{n-1}{2}\right)! \sigma^{n+2}} \tag{2.51}
$$

The parameter n that appears in the distribution function (2.44) of the Mestel disk is a measure of the degree to which the disk is centrifugally supported.The mean azimuthal velocity,

$$
\begin{aligned}
\overline{v_\phi} &= \frac{\int d^2\mathbf{v} v_\phi f(\varepsilon, L_z)}{\int d^2\mathbf{v} f(\varepsilon, L_z)} \\
&= \frac{\int dv_\phi v_\phi v_\phi^n \exp(\frac{-v_\phi^2}{2\sigma^2})}{\int dv_\phi v_\phi^n \exp(\frac{-v_\phi^2}{2\sigma^2})}
\end{aligned}
$$

On taking $x = \frac{v_\phi}{\sqrt{2}\sigma}$ we get,

$$
\begin{aligned}
\overline{v_\phi} &= \sqrt{2}\sigma \frac{\int dx x^{n+1} \exp(-x^2)}{\int dx x^n \exp(-x^2)} \\
&= \sqrt{2}\sigma \frac{(\frac{n}{2})!}{(\frac{n-1}{2})!}
\end{aligned}
\tag{2.52}
$$

For large n, $v_\phi/\sigma = \sqrt{n}[1 + O(n-1)]$, all stars are on circular orbits, and $v_\phi = v_c$.

Kalnajs disks

A more complicated set of disk models,called Kalnajs disks can be obtained from the distribution function that has the form

$$f(\epsilon, J_z) = A[(\Omega_0^2 - \Omega^2)a^2 + 2(\epsilon + \Omega J_z)]^{-\frac{1}{2}} \tag{2.53}$$

when the term in brackets is positive and zero otherwise. We already know that the energy and the component of angular momentum in the z direction are $\epsilon = \Psi - \frac{1}{2}v^2 = \Psi - \frac{1}{2}(v_\phi^2 + v_R^2)$ and $J_z = Rv_\phi$ respectively. The mean systematic or rotational velocity of the system,

$$\langle v_\phi \rangle = \Omega R \tag{2.54}$$

where Ω is a free parameter. The Relative Potential is defined as

$$\Psi(R) = -\phi(R) + C \tag{2.55}$$

The constant is chosen such that $\Psi(R) = -\frac{1}{2}\Omega_0^2 R^2$. On Subsituting eqn(2.54) , eqn(2.55), the denominator of the distribution function becomes,

$$\begin{aligned} (\Omega_0^2 - \Omega^2)a^2 + 2(\epsilon + \Omega J_z) &= (\Omega_0^2 - \Omega^2)a^2 + -\Omega_0^2 R^2 + \Omega^2 R^2 - (v_\phi - \Omega R)^2 - v_R^2 \\ &= (\Omega_0^2 - \Omega^2)(a^2 - R^2) - (v_\phi - \Omega R)^2 - v_R^2 \end{aligned} \tag{2.56}$$

Sub in eqn(2.53) and integrating over all velocities, we find the surface density $\Sigma(R)$

generated by this DF in the potential of our disk to be

$$\begin{aligned}
\Sigma(R) &= \int f(\epsilon, J_z) d^2v = \int f(\epsilon, J_z) dv_\phi dv_R \\
&= A \int_{v_{\phi_1}}^{v_{\phi_2}} dv_\phi \int_{v_{R_1}}^{v_{R_2}} \frac{1}{\sqrt{\underbrace{(\Omega_0^2 - \Omega^2)(a^2 - R^2) - (v_\phi - \Omega R)^2} - v_R^2}} dv_R \\
\Sigma(R) &= A \int_{v_{\phi_1}}^{v_{\phi_2}} dv_\phi \int_{-b}^{+b} \frac{1}{\sqrt{b^2 - R^2}} dv_R
\end{aligned} \tag{2.57}$$

The limits v_{R_1}, v_{R_2} of the inner integral in equation (2.57) are just the values of v_R for which the integrands denominator vanishes. Hence $b^2 = (\Omega_0^2 - \Omega^2)(a^2 - R^2) - (v_\phi - \Omega R)^2$. Using the standard integral,

$$\int \frac{1}{\sqrt{a^2 - x^2}} = \sin^{-1}(\frac{x}{a}) \tag{2.58}$$

the integral becomes,

$$\int_{-b}^{+b} \frac{1}{\sqrt{b^2 - R^2}} = \sin^{-1} \frac{v_R}{b}|_{-b}^{+b} = \frac{\pi}{2} - \sin^{-1}(-\sin\frac{\pi}{2}) = \pi \tag{2.59}$$

Hence the equation for surface density reduces to

$$\Sigma(R) = \pi A \int_{v_{\phi_1}}^{v_{\phi_2}} dv_\phi = \pi A(v_{\phi_2} - v_{\phi_1}) \tag{2.60}$$

But v_{ϕ_1} and v_{ϕ_2} are the roots of the quadratic equation, $b^2 = (\Omega_0^2 - \Omega^2)(a^2 - R^2) - (v_\phi - \Omega R)^2 = 0$ Thus we obtain,

$$\begin{aligned}
v_\phi^2 + \Omega^2 R^2 - 2v_\phi \Omega R - (\Omega_0^2 - \Omega^2)(a^2 - R^2) &= 0 \\
v_\phi^2 - (2\Omega R)v_\phi - (a^2 - R^2)\Omega_0^2 + \Omega^2 a^2 &= 0
\end{aligned} \tag{2.61}$$

On solving the above quadratic equation,we get

$$\begin{aligned}
v_{\phi_2} &= \Omega R + \sqrt{(a^2 - R^2)(\Omega_0^2 - \Omega^2)} \\
v_{\phi_1} &= \Omega R - \sqrt{(a^2 - R^2)(\Omega_0^2 - \Omega^2)}
\end{aligned} \tag{2.62}$$

By substituting these values back into eq.(2.60),

$$\begin{aligned}
\Sigma(R) &= 2\pi A \sqrt{a^2 - R^2}\sqrt{\Omega_0^2 - \Omega^2} \\
&= \Sigma_0 \sqrt{(1 - \frac{R^2}{a^2})}
\end{aligned} \tag{2.63}$$

where,$\Sigma_0 = 2\pi A a \sqrt{(\Omega_0^2 - \Omega^2)}$

It is straightforward to verify that the mean angular speed Ω of the stars in a Kalnajs disk is independent of position, and relative to this mean speed the stars have isotropic velocity dispersion in the disk plane,

$$\overline{v_x^2} = \overline{v_y^2} = \frac{1}{3}a^2(\Omega_0^2 - \Omega^2)(1 - \frac{R^2}{a^2}) \tag{2.64}$$

Thus Kalnajs disks range from hot systems with $\Omega << \Omega_0$, in which the support against self-gravity comes from random motions, to cold systems with $\Omega \approx \Omega_0$, in which all stars move on nearly circular orbits and the random velocities are small.

The coldness of the disk is also a criterion to understand the roles of gases and interstellar dust in the formation of a galaxy. A highly flattened disk is formed by the dissipation of energy yet conserving angular momentum. This happens when the gas clouds dissipate energy and condense to form a disk like structure to minimize the energy. Thus a galaxy disk can be thought of as basically consisting of two components,Population I and Population II. Population I is dominated by cold gas,in atomic or molecular form, but contains significant amounts of stars recently born in the interstellar medium. This component is in a thin layer (at least within the bright optical disk) and is characterized by very low-velocity dispersion. Whereas Population II is dominated by old stars in a thicker layer and is characterized by higher-velocity dispersions; that is, it is warmer from the dynamical point of view. The population II stars form a spherical halo which has little or no mean rotation. [4] This interesting aspect leads to a differential rotation of the galaxy. So the orbits of the stars in a stellar system,like a galaxy can act as a guide to understanding its structure and composition. This will form the basis of our discussion in the next chapter.

CHAPTER 3

Orbits of stars

So far we have only been looking at the self consistent gravitational potential arising due to a distribution of stars in a galaxy and we could see that different distribution function results in different forms of such potentials.But we are yet to throw light upon the orbits of stars in this smooth potential. Will they remain the same if we vary the distribution function? In this chapter, our focus is on finding the nature of the orbits of stars analytically and their circular velocities as a function of radius from the galactic centre.

Within a galaxy, the distribution of stars may not be even everywhere.It may vary for huge radial distances. Our own Milky way galaxy is a disk galaxy surrounded by a spherical halo of globular clusters. For the disk, it is convenient to use an axisymmetric potential whereas for the halo a spherically symmetric potential would be ideal. In such cases, it is interesting to see how the orbits of stars get modified with the radial distance. Although galaxies are composed of stars, we shall neglect the forces from individual stars and consider only the large-scale forces from the overall mass distribution, which is made up of thousands of millions of stars. This fundamental approximation is inevitable for our discussion in the following section. Also, since we are dealing only with gravitational forces, the trajectory of a star in a given field does not depend on its mass. Hence, we examine the dynamics of a particle of unit mass, and quantities such as momentum, angular momentum, and energy, and functions such as the Lagrangian are normally written per unit mass.

3.1 Orbits in spherically symmetric potentials

We first consider orbits in a static, spherically symmetric gravitational field. Such fields are appropriate for globular clusters, which are usually nearly spherical. The motion of a star in a centrally directed gravitational field is greatly simplified by the familiar law of conservation of angular momentum which says $\mathbf{r} \times \dot{\mathbf{r}}$ is a constant vector, $\mathbf{L}$, the angular momentum per unit mass. For a star orbiting in a potential $\Phi(r)$, the Lagrangian can be written as,

$$L = \frac{1}{2}(\dot{r}^2 + (r\dot{\psi})^2) - \Phi(r) \tag{3.1}$$

From the Lagrangian equations of motion of the star,we obtain $r^2\dot{\psi}$ =constant= $\mathbf{L}$.According to Kepler who studied the orbits of planets in the Solar System,$\mathbf{L}$ is equal to twice the rate at which the radius vector sweeps out the area.

$$\frac{d}{dt} = \frac{L}{r^2}\frac{d}{d\psi} \tag{3.2}$$

Lagrangian equations of motion then becomes,

$$\frac{L^2}{r^2}\frac{d}{d\psi}\left(\frac{1}{r^2}\frac{dr}{d\psi}\right) - \frac{L^2}{r^2} = -\frac{d\Phi}{dr} \tag{3.3}$$

On substituting $(1/r) = u$ in the above equation,we get

$$\frac{d^2u}{d\psi^2} + u = \frac{1}{L^2u^2}\frac{d\Phi}{dr}(\frac{1}{u}) \tag{3.4}$$

The solutions of this equation are of two types: along unbound orbits $r \to \infty$ and hence $u \to 0$, while on bound orbits r and u oscillate between finite limits. Thus each bound orbit is associated with a periodic solution of this equation. Multiplying by $\frac{du}{d\psi}$ and integrating over ψ, we obtain an equation for the radial energy of stars in their orbits,

$$(\frac{du}{d\psi})^2 + \frac{2\Phi}{L^2} + u^2 = constant = \frac{2E}{L^2} \tag{3.5}$$

For bound orbits,$du/d\psi = 0$.Like how a simple pendulum oscillates between the two extreme limits, a star in its bound orbit will have two turning points,the values of which are the two roots u_1 and u_2 of the equation,

$$u^2 + 2\frac{[\Phi(\frac{1}{u}) - E]}{L^2} = 0 \tag{3.6}$$

Thus the orbit is confined between an inner radius $r_1 = u_1^{-1}$ known as the pericenter distance, and an outer radius $r_2 = u_2^{-1}$ called the apocenter distance. Both are equal for a circular orbit. The radial period T_r is the time required for the star to travel from apocenter to pericenter and back [1]. It is obtained by eliminating $\dot{\psi}$ from equation(3.5) using $r^2\dot{\psi} = \mathbf{L}$.We find,

$$(\frac{dr}{dt})^2 = 2(E - \Phi) - \frac{L^2}{r^2} \tag{3.7}$$

which may be rewritten as,

$$\frac{dr}{dt} = \pm\sqrt{2(E - \Phi(r)) - \frac{L^2}{r^2}} \tag{3.8}$$

The two possible signs arise because the star moves alternately in and out. Thus it follows from the above equation that the radial period is,

$$T_r = 2\int_{r_1}^{r_2} \frac{dr}{\sqrt{2(E - \Phi(r)) - \frac{L^2}{r^2}}} \tag{3.9}$$

During this period, the azimuthal angle changes by the amount,

$$\triangle\psi = 2\int_{r_1}^{r_2} \frac{d\psi}{dr}dr = 2\int_{r_1}^{r_2} \frac{L}{r^2}\frac{dt}{dr}dr = 2L\int_{r_1}^{r_2} \frac{dr}{r^2\sqrt{2[E - \Phi(r)] - (\frac{L^2}{r^2})}} \tag{3.10}$$

The corresponding azimuthal period is,

$$T_\psi = \frac{2\pi}{\triangle\ \psi}T_r. \tag{3.11}$$

Apart from the knowledge that stars move in smoothed potentials, we are yet to figure out the form of the potential. The deserving candidates for a realistic potential are the Kepler's potential,where the whole mass is assumed to be concentrated at a single point like planets moving under the gravitational pull of the Sun at the centre and the harmonic potential where mass is believed to be distributed over a sphere of radius equal to that of the system under consideration. A star on a Kepler orbit completes a radial oscillation in the time required for ψ to increase by $\triangle\psi = 2\pi$, whereas a star that orbits in a harmonic-oscillator potential has already completed a radial oscillation by the time

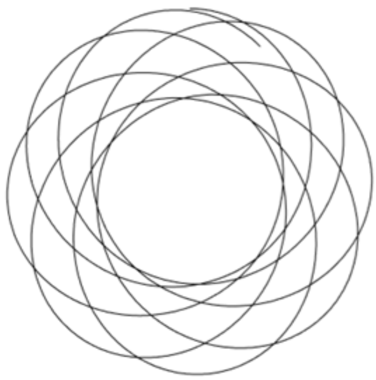

Figure.3.1.Orbits of stars in an Isochrone potential forms a rosette

has increased by $\triangle\psi = \pi$. Since galaxies are more extended than point masses, and less extended than homogeneous spheres, a typical star in a spherical galaxy completes a radial oscillation after its angular coordinate has increased by an amount that lies somewhere in between these two extremes; $\pi < \triangle\psi < 2\pi$. Thus, we expect a star to oscillate from its apocenter through its pericenter and back in a shorter time than is required for one complete azimuthal cycle about the galactic center. To meet these expectations we can think of a potential of the form, [1]

$$\Phi = -\frac{GM}{b + \sqrt{(r^2 + b^2)}} \tag{3.12}$$

called the Isochrone potential.**Figure.3.1** [1] shows a typical orbit in this potential.When $b \to 0$,this leads to the Kepler potential whereas for $b \to \infty$ the potential takes the form a harmonic potential. This potential is generated by a density distribution of the form

$$\rho(r) = \frac{1}{4\pi G}\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d\Phi}{dr}) = M[\frac{3(b+a)a^2 - r^2(b+3a)}{4\pi(b+a)^3 a^3}] \tag{3.13}$$

where $a = \sqrt{b^2 + r^2}$,which has the limiting forms

$$\rho(0) = \frac{3M}{16\pi G b^3}, \rho(r) \simeq \frac{bM}{2\pi r^4}, \qquad (r >> b). \tag{3.14}$$

It is convenient to define an auxiliary variable s by

$$s \equiv -\frac{GM}{b\Phi} = 1 + \sqrt{\frac{r^2}{b^2} + 1}, \qquad \frac{r^2}{b^2} = s^2(1 - \frac{2}{s}), \qquad (s \geq 0) \tag{3.15}$$

Given this one-to-one relationship between s and r, we may employ s as a radial coordinate in place of r. The integrals (3.9) and (3.10) for T_r and $\triangle\psi$ both involve the infinitesimal quantity,

$$dI = \frac{dr}{\sqrt{2(E - \Phi) - \frac{L^2}{r^2}}} \tag{3.16}$$

Eliminating r from this expression,we get

$$dI = \frac{b(s-1)ds}{\sqrt{2Es^2 - 2(2E - GM/b)s - 4GM/b - L^2/b^2}} \tag{3.17}$$

As the star moves from pericenter r_1 to apocenter r_2, s varies from the smaller root s_1 of the quadratic expression in the denominator of equation (3.17) to the larger root s_2. Thus, combining equations (3.9) and (3.17), the radial period is

$$\begin{aligned} T_r &= \frac{2b}{\sqrt{-2E}} \int_{s_1}^{s_2} \frac{(s-1)ds}{\sqrt{(s_2 - s)(s - s_1)}} \\ &= \frac{2b}{\sqrt{-2E}} \int_{s_1}^{s_2} \frac{ds}{\sqrt{-s^2 + (s_1 + s_2)s - s_1 s_2}} \left(\frac{-(s_1 + s_2)}{2} + 1 \right) \\ &= \frac{2b}{\sqrt{-2E}} \left[-\sin^{-1} \left(\frac{-2s + b}{\sqrt{(s_1 - s_2)^2}} \right) \right]_{s_1}^{s_2} = \frac{2\pi b}{\sqrt{-2E}} \left[\frac{1}{2}(s_1 + s_2) - 1 \right] \end{aligned} \tag{3.18}$$

But from the denominator of equation (3.17) it follows that the roots s_1 and s_2 obey

$$s_1 + s_2 = 2 - GM/Eb \tag{3.19}$$

Hence the radial period is,

$$T_r = \frac{2\pi GM}{(-2E)^{\frac{3}{2}}}. \tag{3.20}$$

Note that T_r depends on the energy E but not on the angular momentum L; it is this unique property that gives the isochrone its name.Equation (3.10), for the increment $\triangle\psi$ in azimuthal angle per cycle in the radial direction, yields

$$\begin{aligned} \triangle\,\psi &= 2L\int_{s_1}^{s_2}\frac{dt}{r^2} = \frac{2L}{b\sqrt{-2E}}\int_{s_1}^{s_2}\frac{(s-1)}{s(s-2)\sqrt{(s_2-s)(s-s_1)}}ds \\ &= \pi(1+\frac{L}{\sqrt{L^2+4GMb}}). \end{aligned} \tag{3.21}$$

From this expression we see that $\pi \leq \triangle\psi \leq 2\pi$. This proves the fact that isochrone potential neither behaves like Kepler nor like the harmonic potential,which we already discussed. Because galaxies are more extended objects unlike point masses, a typical star takes less time to complete one radial oscillation that what it takes to complete once azimuthal cycle.

3.2 Rotation curves of galaxies

All stars do not rotate around the galactic centre with the same velocity. If we know how mass is distributed, we can find the resulting gravitational force and from this we can calculate how the positions and velocities of stars and galaxies will change over time. As the radial distance of the star from the centre increases their velocity decreases according to the conservation of angular momentum. This results in a differential rotation of the galaxy contributing to a circular velocity given by $v = \Omega r$. The inner parts of the galaxy rotates faster than the outer parts.as was mentioned in the last chapter gravity can be supported either by the centrifugal force or the random velocities of the components of a galaxy. A galaxy forms from a huge clump of gas and dust which gradually in the process of attaining the thermal equilibrium condenses to form a disk which is the highly flattened component in which newly formed stars rotate almost in circular orbits lying along the same plane. In this case gravity is supported against the centrifugal force of rotation, whereas the stars in the spherically distributed halo surrounding the disk forms in the early stages of the evolution of a galaxy where the gravity is supported against the random velocities of motion of stars. They are called globular clusters. Thus the ratio of Mass to Luminosity ceases to be equal to unity in the halo as the observational data suggests.The rotation curve, a plot of rotational speed V versus distance from the galactic center, is an indicator of the mass distribution within the galaxy.The rotation curves of most galaxies,including our own,indicate that large quantities of dark matter are associated with the individual galaxies. On equating the centrifugal force and gravitational force, we get

$$\frac{v_c^2}{R} = \frac{G}{R^2} = \frac{\partial \Phi}{\partial R}$$

It is obvious from the relation that velocity should increase linearly with the gravitational potential gradient upto the extent of the disk and decrease in a Keplerian fashion as most of the matter in the halo is dark. It could be non luminous remnants of a star or other fragments of matter which never became luminous(Jupiter like planets, black holes, neutrinos or monopoles) [6]. We can also use the stellar motions to tell us where the mass is. Only luminous matter were thought to contribute to the potential. But when the rotation curves of such galaxies were studied it was found that the circular velocity does not decrease instead remains almost flat. The reason behind this flat curve is the dark matter present in the interstellar medium of globular clusters of galaxies. This was a breakthrough for those people who were searching for observational evidence of dark matter other than gravitational lensing where gravity was found to bend the light rays coming from distant galaxies or stars.

3.2.1 Orbits in axisymmetric systems

When it comes to axisymmetric systems such as disks embedded in a halo, we encounter interesting features in the nature of orbits of the stars as was mentioned in the beginning of the section. Thus our goal is to frame a theoretical basis to support the observational data collected so far on different galactic rotation curves. Disk galaxies may or may not have a bulge component at the centre. Either way the disk is believed to have a surface density profile that decreases exponentially with the radial distance from the galactic centre. It gives us the liberty to choose the following form of surface density to account for the mass distribution in a disk.

$$\Sigma(R) = \Sigma_0(-R/R_d) \tag{3.22}$$

We can now determine the gravitational potential that is due to such a disk and the properties of stellar orbits. For the ease of calculations, it is convenient to use the fact that in axially symmetric cases, the functions [3]

$$\Phi_{\pm}(R, z) = \exp(\pm kz) J_0(kR) \tag{3.23}$$

(where J_0 is the Bessel function) are the solutions to the Laplace equation. Consider now the function given by,

$$\Phi_k(R, z) = \exp(-k|z|) J_0(kR) \tag{3.24}$$

which has the discontinuity in the derivative at $z = 0$ given by,

$$\lim_{z \to 0^+} \left(\frac{\partial \Phi_k}{\partial z} \right) = -kJ_0(kR), \qquad \lim_{z \to 0^-} (-\frac{\partial \Phi}{\partial z}) = +kJ_0(kR) \tag{3.25}$$

We know from Gauss theorem that a surface density distribution will result in a gravitational potential Φ_k. Hence it follows that,

$$\Sigma_k(R) = -\frac{k}{2\pi G} J_0(kR), \tag{3.26}$$

located at $z = 0$. Superposing the density distribution by using a weightage function S(k),we find that the surface density profile,

$$\Sigma(R) = \int_0^\infty S(k)\Sigma_k(R)dk = -\frac{1}{2\pi G}\int_0^\infty S(k)J_0(kR)kdk, \tag{3.27}$$

will generate the gravitational potential

$$\Phi(R,z) = \int_0^\infty S(k)\Phi_k(R,z)dk = \int_0^\infty S(k)J_0(kR)\exp(-k|z|)dk \tag{3.28}$$

Though this analysis is very general and could be done with a wide class of radial functions,Bessel functions make it a lot easier due to the fact that it is invertible [3].i.e.

$$S(k) = -2\pi G\int_0^\infty J_0(kR)\Sigma(R)RdR \tag{3.29}$$

Using this in eqn(3.28),we can express the potential directly in terms of the density $\Sigma(R)$, thereby providing the complete solution to the problem. Given the form of the potential, we can also determine the rotation velocity in terms of $S(k)$;

$$v_c^2(R) = R(\frac{\partial\Phi}{\partial R})_{z=0} = -R\int_0^\infty S(k)J_1(k)kdk \tag{3.30}$$

Similarly we can obtain surface density in terms of rotation curves in the following manner;we know that for axisymmetric disks,z=0.Hence the surface density becomes,

$$\Sigma(R) = -\frac{1}{2\pi G}\int_0^\infty S_0(k)J_0(kR)kdk \tag{3.31}$$

and the potential is expressed in terms of Bessel functions as,

$$\begin{aligned}\Phi(R,0) &= \int_0^\infty dkS_0(k)J_0(kR)\\ \frac{\partial\Phi(R,0)}{\partial R} &= \frac{\partial}{\partial R}\int_0^\infty dkS_0(k)J_0(kR)\end{aligned} \tag{3.32}$$

Using the property of Bessel functions,

$$\frac{dJ_0(x)}{dx} = -J_1(x)$$

we obtain the rotation velocity as,

$$v_c^2 = -R\int_0^\infty dkkS_0(k)J_1(kR) = 2\pi GR\int_0^\infty dkkJ_1(kR)\int_0^\infty J_0(kR')\Sigma(R')R'dR \tag{3.33}$$

Applying to eqn(3.33), the inversion formula for Hankel transform,we get

$$S_0(k) = -\int_0^\infty dR'v_c^2(R')J_1(kR') \tag{3.34}$$

Substituting the above equation in eqn(3.31)

$$\Sigma(R) \quad = \quad \frac{1}{2\pi G}\int_0^\infty J_0(kR)kdk\int_0^\infty dR\prime v_c^2(R')J_1(kR') \tag{3.35}$$

For an exponential disk, we can evaluate the integral in eq. to obtain $S(k)$,

$$S(k) = -2\pi G\Sigma_0\int_0^\infty J_0(kR)\exp(\frac{-R}{R_d})RdR \tag{3.36}$$

On substituting $kR = x$,

$$S(k) = -\frac{2\pi G\Sigma_0}{k^2}\int_0^\infty J_0(x)\exp(-\frac{x}{kR_d})xdx \tag{3.37}$$

We know from the properties of Bessel functions [9] for $a = \pm 1, \int x\exp(ax)J_0(x)dx$

$$= \exp(ax)\left[\frac{ax}{a^2+1}J_0(x) + \frac{x}{a^2+1}J_1(x)\right] - \frac{a}{a^2+1}\int \exp(-ax)J_(x)dx \tag{3.38}$$

Using the above property, equation(3.37) becomes

$$S(k) = -\frac{2\pi G\Sigma_0}{k^2}\left[-\frac{a}{a^2+1}\int_0^\infty \exp(ax)J_0(x)dx\right] \tag{3.39}$$

where $a = -1/kR_d$.We also know that,

$$\int_0^\infty \exp(ax)J_0(x)dx = \frac{1}{\sqrt{a^2+b^2}} \tag{3.40}$$

Substituting the value of 'a' back into eqn(3.39) and using the above property,we get

$$S(k) \;=\; -\frac{2\pi G\Sigma_0 R_d^2}{[1+(kR_d)^2]^{\frac{3}{2}}} \tag{3.41}$$

The potential is now given by,

$$\Phi(R,z) = -2\pi G\Sigma_0 R_d^2 \int_0^\infty \frac{J_0(kR)\exp(-k|z|)}{[1+(kR_d)^2]^{\frac{3}{2}}}dk \tag{3.42}$$

which unfortunately cannot be expressed in simple closed form for arbitrary z.In the $z = 0$ plane,however, the result can be expressed in terms of modified Bessel functions by

$$\Phi(R,0) = -\pi G\Sigma_0 R[I_0(y)K_1(y) - I_1(y)K_0(y)], \qquad y \equiv \frac{R}{2R_d} \tag{3.43}$$

with the corresponding circular speed

$$v_c^2(R) = R\left(\frac{\partial\Phi}{\partial R}\right) = 4\pi G\Sigma_0 R_d y^2[I_0(y)K_0(y) - I_1(y)K_1(y)]. \tag{3.44}$$

Figure 3.2. shows $v_c^2(R)/4\pi G\Sigma_0 R_d$ as a function of (R/R_d). It is evident from the figure that the rotation curves have to do with the luminous matter distribution in the plane of a disk. As is expected from the constant M/L ratio for a disk where the only components contributing to the gravitational potential are the stars, gas and the interstellar dust, the rotational velocity reaches a peak almost at a distance twice as large the

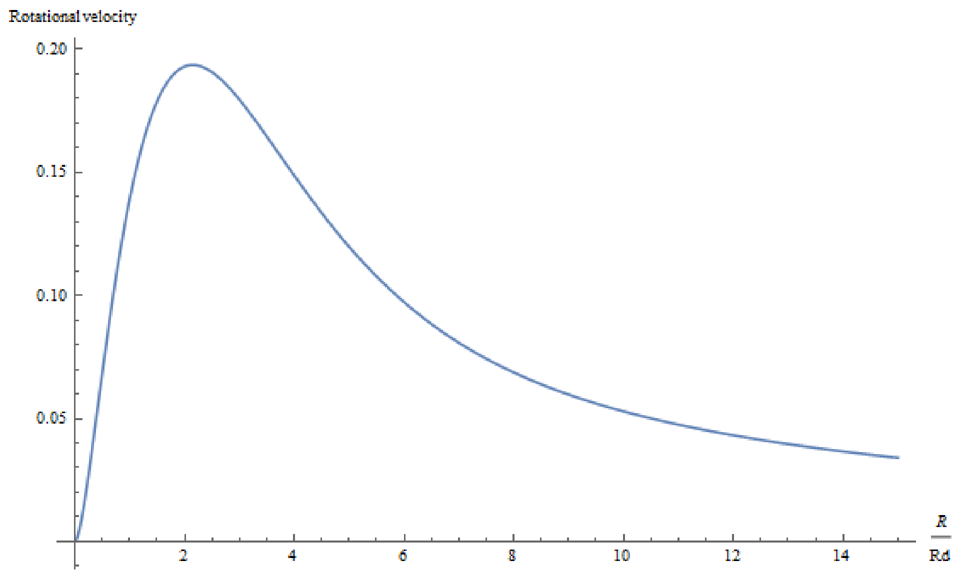

Figure.3.2.Rotation curve of disk

disk radius R_d and falls off as $V(r) \propto r^{-1/2}$ in the outer parts. But the curves obtained theoretically by assuming an exponentially varying surface density profile failed to be consistent with the observed rotation curves as it was found that the rotation curve flattens out or falls only slowly with the radius beyond a certain distance from the galactic centre. This led us to think of another component which can effectively account for this major discrepancy in the nature of the curves.

3.2.2 Evidence for the existence of dark matter

Neptune was discovered in the year of 1846, when it struck the intelligent minds on Earth that there was something unusual about the kinematics of Uranus. Such anomalies that we see often reveals the hidden. The dark matter is mostly hidden in the darkness of the universe that even a slight evidence of its existence can be a guide to understanding our universe better. As far as the search for this most dominant component in the galaxies are concerned, the rotation curves act as the primary tool to locate and study their distribution. The spectroscopic studies based on the Doppler shifts in the emitted or absorbed lines can tell us a great deal about the rotation curves of galaxies. Of all

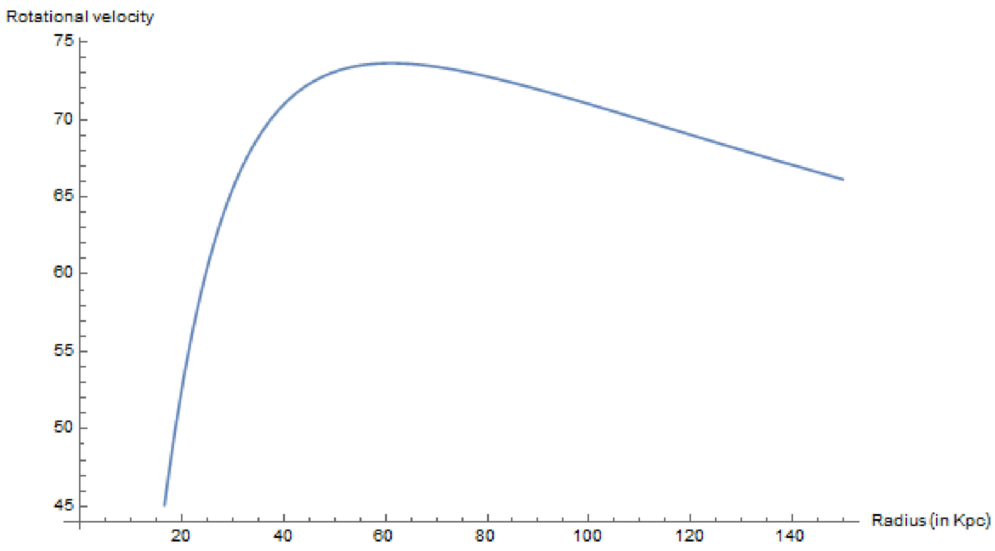

Figure.3.3.Rotation curve of a disk embedded in a halo

frequencies in the EM spectrum, the radio frequencies are best suited to probe rotation curves as the most decisive observations come from the studies of cold atomic hydrogen which do not confine just to the disk alone. The density profile of the spherically distributed halo needs to be obtained by a comparison of the observed rotation curve with the one that is due to the exponential disk. When this comparison was made, the density profile of the dark halo,responsible for the flat rotation curves, resembled that of an isothermal sphere which was already discussed in section 2.3 of the last chapter. Thus a density function given below would be appropriate to account for a spherically symmetric dark halo. [3]

$$\rho_h(r) = \frac{\rho_h(0)}{1 + (r/a)^\gamma} \tag{3.45}$$

where a is the radius of the halo which being uncertain usually varies from 7kpc to 12kpc. γ can take values in the range $1.9 < \gamma < 2.9$. This halo produces a rotational velocity given by,

$$v_h^2(r) = \frac{GM_h(r_h)}{r} = \frac{4\pi G}{r} \int_0^r dx x^2 \rho_h(x) \tag{3.46}$$

The observed rotation curves in the plane of the galaxy is the sum of the contributions in the quadrature:$v_c^2 = v_{disk}^2 + v_h^2$. **Figure 3.3.** illustrates this degeneracy in a simple situation. The y axis gives v_c^2 in units of $4\pi G\Sigma_0 R_d$ as a function of (R/a),where R is the radius of the disk [3]. Hence we conclude that though the major contribution to luminosity comes from the visible matter, the support of the rotation curve is shared by both the visible and dark components. A flat rotation curve directly implies an increase in mass to luminosity ratio with radius extending beyond the optical disk. This is a strong indication that dark matter is present in significant amounts both in the outer regions of the galaxy.

One of the earliest indications for the existence of relatively flat rotation curves in external galaxies arises from the study of M31 in Andromeda. The density profile of the dark halo provides little or no information about its ingredients. As long as the shape of the halo remains uncertain, it is difficult to obtain a possible explanation as to why it does not interact through other forces in nature. Nevertheless, we can convince ourselves that oblate and fairly flat halos cannot be adapted with non-baryonic matter as it would lack the dissipation that can make a system collapse to a disk. [4]

CHAPTER 4

Conclusion

In this report, starting with the description of galaxies, assuming them to have attained their equilibrium configuration through years of evolution, we assigned a distribution function constant in time to them in order to study the different structures they evolve into. We saw in chapter 2 that, the dynamics of a galaxy can be explained with the help of the collisionless Boltzmann equation which immediately follows from the fact that stars in a galaxy are not discrete infinite dips in the potential but are rather distributed smoothly in space. With this idea in mind, we could obtain the gravitational potentials of some isotropic and anisotropic models of galaxies classified based on the form of their distribution function.

In chapter 3, we found that apart from the stars, there were other components that tend to influence the velocity with which they rotate about the galactic centre. This realization followed from the flat rotation curves that were inferred from the spectroscopic studies of the motion of stars, gas and dust in a galaxy. We obtained the same flat rotation curves of galaxies theoretically assigning a density function similar to that of an isothermal sphere discussed in chapter 2 simply to account for the spherical symmetry of the dark halo surrounding the highly luminous disks of the galaxies. Thus it was found that the theoretical prediction of rotation curves taking a Keplerian form (i.e.$\propto r^{-1/2}$) in the outer parts was wrong as the dark matter residing in the halo provides a fairly good amount of potential to maintain a flat rotation curve even when there is little or no luminous matter. Despite the fact that we are nothing but a frivolous dust in this universe, there lies so much beyond our naked eyes expecting to be revealed sometime in the near future through our constant efforts to understand the universe better. Dark matter is what tops the list. Though we were able to study how dark matter is distributed it is still a matter of dedicated research as to what makes up dark matter.

REFERENCES

[1] James Binney, Scott Tremaine. Galactic dynamics, Second edition, Princeton Series in Astrophysics, Princeton University Press, 1987

[2] T.Padmanabhan, Theoretical Astrophysics.Vol.I:Astrophysical Processes, Cambridge University Press, Cambridge,2000.

[3] T.Padmanabhan, Theoretical Astrophysics.Vol.III:Galaxies and Cosmology, Cambridge University Press, Cambridge,2000.

[4] Bertin, G. (Giuseppe), Dynamics of galaxies, Second edition, Cambridge University Press, Cambridge, 2000.

[5] Hale Bradt. Astrophysics Processes. Cambridge University Press, Cambridge, 2008.

[6] Rubin, Vera(1997). Bright galaxies, dark matter. Woodbury, NY: AIP Press.

[7] Peter Schneider. Extragalactic Astronomy, Springer-Verlag, Berlin, Heidelberg 2006.

[8] Houjun Mo, Frank van den Bosch, Simon White. Galaxy evolution and formation, Cambridge University Press, Cambridge, 2010.

[9] Spiegel,.Schaum's. Mathematical Handbook of Formulas and Tables.(0070382034)(McGraw-Hill,.1998).

[10] Linda S. Sparke, Gallagher. Galaxies in the Universe, Cambridge University Press, Cambridge, 2007.

[11] T.S. van Albada, J.N. Bahcall, K. Begeman und R. Sancisi: Astrophys.J.,295(1985)305