Classical and semi-classical aspects of black holes

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by

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CERTIFICATE

This is to certify that the project titled **Classical and semi-classical aspects of black holes** is a bona fide record of work done by **Ramit Kumar Dey** towards the partial fulfillment of the requirements of the Master of Science degree in Physics at the Indian Institute of Technology, Madras, Chennai 600036, India.

(L. Sriramkumar, Project supervisor)

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ABSTRACT

The aim of the project is to understand various classical and semi-classical aspects of black holes. After arriving at the metric describing static and rotating black holes, we go on to study various classical phenomena involving dynamics of particles and photons such as the precession of the perihelion of Mercury, gravitational bending of light and the Penrose process. We then go on to analyze the behavior of classical fields around a black hole. In particular we study the phenomena of super radiance around rotating black hole. We also study the behavior of quantum fields near a black hole. In this context we begin by studying the behavior of quantum fields in the presence of horizons in spacetime, such as it occurs in a Rindler frame (i.e. the frame of a uniformly accelerated observer). We then go on to analyze the origin of Hawking radiation and its implication for the thermodynamics of black holes.

Contents

Chapter 1

Introduction

Einstein's theory of general relativity provides a new view point of looking at gravity in terms of curvature of the spacetime. General theory of relativity is a classical theory and it has predicted some remarkable results about the structure of the universe and other large scale objects present in the universe. This theory also predicts the presence of spacetime singularities and spacetime horizons in the form of black holes. In this work, we concentrate on the study of black holes and look at various classical and semi-classical aspects of black holes. When the curvature of spacetime becomes infinite, as in the case of a black hole, we cannot account for the structure of the spacetime using purely classical physics. This motivates one to look beyond a classical theory of gravity (general relativity) and adapt some new approach. Among various new methods which intends to solve the problems present in general theory of relativity we look into, the semi-classical way of treating gravity. In this approach, the gravitational field is inherently classical but the fields present in the background are assumed to be quantum in nature.

This report is broadly divided into two parts. The first part of the report deals with the study of black holes in the classical regime. Trajectories of particles are studied around a static black hole and few of the first experimentally verified predictions of general relativity, such as precession of perihelion of mercury and bending of light around a massive gravitating object are reviewed. General relativity also predicts the presence of rotating black holes, formed from collapsing stars having some angular momentum. We look at how particles and fields behave around such rotating black holes and study the phenomenon of Penrose process and super-radiance. In the case of rotating black hole, we can have an analogy between thermodynamics and these classical processes (Penrose process, super-radiance)

[1, 2, 3]. In the case of super-radiance we have stimulated emission when some scalar field is incident on the black hole. From this fact one can guess that there might be spontaneous emission from a black hole when quantum fields are studied around it and this motivated the study of quantum fields in the background of a black hole.

In the second part of the thesis, we deal with the study of quantum fields in flat and curved spacetimes. Study of quantum fields in Minkowski and Rindler spacetime led to Unruh effect [4, 5, 6, 7]. We also survey how Unruh effect emerges due to inequivalent quantization in Minkowski and Rindler spacetime. Thermal Green's function are periodic in imaginary time and when Green's function defined in Minkowski space is projected on Rindler space it behaves like one defined at a finite temperature. Using this, we can easily obtain the translation probability rate of an accelerated detector in the Minkowski vacuum and verify that the thermality condition holds [8, 9, 10].

Study of quantum fields in the background of a collapsing star, which eventually settles down as a static black hole, reveals the fact that static black holes can spontaneously emit particles and this phenomenon is known as Hawking radiation [2, 6, 11, 12, 13] . This result has some remarkable implications as it shows that we can associate a temperature (Hawking temperature) with the black hole horizon and thus a black hole truly behaves as a thermodynamic system. In the purely classical regime, it is not possible to associate a physical temperature with the black hole, and this illustrates the importance of semi-classical studies of black holes. It can be also shown that eternal black holes exhibit Hawking radiation by the study of thermal Greens function. The particles created by black holes are observed at $\mathcal{I}^+$ and the concept of particle is ill-defined near the black hole horizon. Due to Hawking radiation the mass of the black hole must deplete and this can be understood in a better way by studying the expectation value of the stress-energy tensor defined near the black hole horizon [7, 14]. Since the stress-energy tensor is a local object we can study Hawking radiation in the vicinity of a black hole using it. Next we study black hole thermodynamics in this context and survey the laws of black hole thermodynamics defining an entropy for the black hole event horizon [15, 16, 17, 18] .

In this report all Latin indices, $a, b, c, ...$ runs over $0, 1, 2, 3$where 0-index denotes the time dimension and the other three index denotes spatial dimension. Throught the report we used the metric signature $(+ - --)$ except in chapter two we have used the signature $(- +$

$++$). We also used " geometrized units", in which the speed of light, c, and the Newtonian gravitation constant, G, is set to unity.

Chapter 2

Static black holes

2.1 Solving the Einstein equation

General theory of relativity tells us how spacetime is influenced in the presence of sources and the way this source is constrained due to geometry of the spacetime. This can be better understood by looking at the field equation, known as Einstein equation, which is given as:

$$G_{ab} = \kappa T_{ab},$$

Here G_{ab} contains second derivative of the metric and is known as the Einstein tensor, κ is $8\pi G$, where G is the Newton's gravitational constant and T_{ab} is the stress energy tensor. The Einstein equations are a set of second order, non-linear, coupled differential equations that relates the energy source (T_{ab}) with the geometry (G_{ab}) of the spacetime. In four dimensions there are ten field equations in ten different variables, viz. the ten independent components of the metric tensor g_{ab}.

2.1.1 Static vacuum solution

At present there exist a huge number of exact and approximate solutions to the Einstein equation. The first and the simplest solution of the Einstein equations was given by Karl Schwarzschild in 1916 for a static and spherically symmetric spacetime having no external energy source.

A spacetime is said to be static when it possesses a timelike Killing vector field which is hypersurface orthogonal [2, 21]. The coordinate basis which is used for defining the metric has the Killing parameter t as one of the coordinate. This implies that the metric must be independent of the t coordinate. As this Killing vector field is orthogonal to the $t = constant$ hypersurface there cannot be any cross term such as $dtdx^{\mu}$ in the metric. This implies static spacetime not only has time translation symmetry but it also remains invariant under the diffeomorphism, $t \to -t$, i.e. it has time reflection symmetry. Spherical symmetry implies that the spacetime must be invariant under rotation about an axis and this requires a two dimensinal part within the $t = constant$ hypersurface to be a two dimensional sphere. Choosing t, r, θ, ϕ as our coordinate basis we can write the metric of the full spacetime as

$$ds^2 = f(r)dt^2 - g(r)dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{2.1}$$

Using vacuum Einstein equation the two variables of the above metric can be determined and we can write the solution as

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2), \tag{2.2}$$

where M is the mass of the gravitating test body. This solution of the vacuum Einstein equation is known as the Schwarzschild metric and this metric makes some remarkable predictions, few of them will be reviewed in this report.

It can be seen directly from the solution (2.2) that this metric becomes singular at two points corresponding to $r = 0$ and $r = 2M$. The singularity related to the $r = 0$ point is a true singularity of the spacetime while the singularity at $r = 2M$ is just a pathological singularity corresponding to the particular choice of the coordinate system we are working in. The strength of this solution lies in the fact that this solution is unique for the given symmetries of the spacetime and the predictions made by the Schwarzschild metric are the some of the experimentally verified tests of general relativity.

2.2 Trajectories of particles and photons in Schwarzschild spacetime

As mentioned earlier, there are two singularities in the Schwarzschild metic which add some peculiarity to the orbit of particles and photons around the black hole [1] [22]. One way to understand that the singularity at the $r = 2M$ surface is due to the choice of coordinates is by considering radially infalling particles in different time frames.

2.2.1 Trajectory of a radially infalling particle

For a radially infalling particle, we have

$$\dot{\theta} = \dot{\phi} = 0. \tag{2.3}$$

Considering ξ^μ to be an arbitrary Killing vector, we know that for a particle moving along a geodesic the quantity $p_\mu \xi^\mu$ is a constant. As Schwarzschild spacetime is static and spherically symmetric it has four Killing vector fields. ξ_t is the only timelike Killing vector field exhibited by this spacetime. For ξ_t, the conserved quantity is

$$\left(1 - \frac{2M}{r}\right) p^t = \tilde{E}, \tag{2.4}$$

where $\tilde{E}$ is a constant. Now considering the metric given in (2.2) and differentiating with respect to proper time τ, we get for a radially infalling particle

$$\left(1 - \frac{2M}{r}\right) \dot{t}^2 - \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 = 1, \tag{2.5}$$

where the overdots represent differentiation with respect to τ. Using (2.4), we can write this equation as

$$\left(1 - \frac{2M}{r}\right) = -E^2 + \dot{r}^2. \tag{2.6}$$

The constant E is defined as $\tilde{E}/m$ and it can have different values corresponding to different initial conditions. If we consider a particle dropped in the black hole from infinity with zero initial velocity then, $E = 1$

Using this particular choice for the constant and integrating (2.6) we get

$$\tau - \tau_0 = \frac{2}{3\sqrt{2M}} \left(r^{\frac{3}{2}} - r_0^{\frac{3}{2}}\right) \tag{2.7}$$

Now, if we want the motion of the particle in terms of the coordinate time 't' we get from (2.5) and (2.6), taking $E = 1$

$$\frac{dt}{dr} = -\left(\frac{r}{2M}\right)^{\frac{1}{2}} \left(1 - \frac{2M}{r}\right)^{-1}. \tag{2.8}$$

Integrating this equation gives

$$t - t_0 = \frac{2}{3\sqrt{2}M}\left(r^{\frac{2}{3}} - r_0^{\frac{2}{3}} + 6Mr^{\frac{1}{2}} - 6Mr_0^{\frac{1}{2}}\right) + 2Mln\frac{(r^{\frac{1}{2}} + (2M)^{\frac{1}{2}})(r_0^{\frac{1}{2}} - (2M)^{\frac{1}{2}})}{(r^{\frac{1}{2}} - (2M)^{\frac{1}{2}})(r_0^{\frac{1}{2}} + (2M)^{\frac{1}{2}})} \tag{2.9}$$

From Eq. (2.7) we can see that the motion of the particle is smooth across the $r = 2M$ surface. Thus the singularity at $r = 2M$ is not a physical singularity and it does not affect the motion of the particle. When we consider the motion of the particle with respect to its coordinate time we can see from (2.9) that at $r = 2M$, t becomes infinity which denotes that the particle will never pass the horizon. This shows that the singularity at the $r = 2M$ is dependent on the coordinate basis we have chosen.

2.2.2 Precession of the perihelion of Mercury

For a one body system having a central gravitating object producing a spherically symmetric gravitational field, the appropriate solution of the metric around such a system is described by the Schwarzschild solution. Differentiating both sides of Eq. (2.2) with respect to proper time τ, we get, for timelike geodesics:

$$1 = \left(1 - \frac{2M}{r}\right)\dot{t}^2 - \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 - r^2(\dot{\theta}^2 + sin^2\theta\dot{\phi}^2), \tag{2.10}$$

where, as mentioned before, an overdot represents differentiation with respect to τ.

The timelike Killing vector ξ_t and the rotational Killing vector ξ_ϕ lead to:

$$\left(1 - \frac{2M}{r}\right)\dot{t} = \frac{\tilde{E}}{mc^2} = E \tag{2.11}$$

$$r^2\dot{\phi} = \frac{\tilde{L}}{mc^2} = L, \tag{2.12}$$

where L and E represents angular momentum and energy per unit mass, respectively. Using equations (2.11) and (2.12) in (2.10) and taking θ to be constant we get:

$$1 = \left(1 - \frac{2M}{r}\right)^{-1}E^2 - \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 - r^{-2}L^2 \tag{2.13}$$

Upon changing to variable $r = 1/u$, this equation takes the form:

$$\frac{d^2u}{d\phi^2} + u = \frac{M}{L^2} + 3Mu^2. \tag{2.14}$$

This equation differs from the Newtonian result due to the presence of the second term on the right hand side. For planetary motion this last term is very small and so we can solve this differential equation by perturbative method under some assumptions. First we solve the equation

$$\frac{d^2u}{d\phi^2} + u = 0, \tag{2.15}$$

by substituting $u = Ae^{-\alpha\phi}$. Using proper boundary condition we get $u = C\cos\phi$, where C is a constant.

Taking the particular solution of the equation

$$\frac{d^2u}{d\phi^2} + u = \frac{M}{L^2}, \tag{2.16}$$

the total solution of the above equation can be obtained to be

$$u = \frac{M}{L^2}(1 + e\,\cos\phi) \tag{2.17}$$

where $e = CL^2/m$ is the eccentricity of the orbit. Substituting this value of u in equation (2.14) we can expand the last term and write the full equation as:

$$\frac{d^2u}{d\phi^2} + u = \frac{M}{L^2} + \frac{3M^3}{L^4}(1 + e^2\,\cos^2\phi + 2\mathrm{e}\cos\phi). \tag{2.18}$$

The solution to this equation can be arrived at by adding solution of the particular integrals to the general solution. Due to the correction term on right hand side there are three particular integrals whose solutions are

$$\begin{aligned} u &= \frac{3M^3}{L^4}, && (2.19)\\ u &= -\frac{3M^3}{6L^4}\cos 2\phi + \frac{1}{2}\frac{3\mathrm{M}^3}{\mathrm{L}^4}, && (2.20)\\ u &= \frac{1}{2}\frac{3m^3}{L^4}\phi\sin\phi. && (2.21) \end{aligned}$$

The first two terms add a small constant to the solution (2.17). Only the third term contributes significantly and this has been also experimentally observed. Taking into account all these facts we can write the final solution of Eq. (2.14) as

$$u = \frac{M}{L^2}(1 + e\ \cos\phi) + \frac{3\mathrm{M}^3}{\mathrm{L}^4}\phi\sin\phi. \tag{2.22}$$

For small ϕ, $\sin\phi \sim \phi$ and $\cos\phi \sim 1$ Under this approximation the total solution is given by:

$$u = \frac{M}{L^2}\left[1 + e\ \cos\left(1 - \frac{3\mathrm{M}^2}{\mathrm{L}^2}\right)\phi\right]. \tag{2.23}$$

This relation shows that u is a periodic function of ϕ and the periodicity is

$$\frac{2\pi}{\left(1 - \frac{3M^2}{L^2}\right)}. \tag{2.24}$$

This periodicity is clearly greater than 2π and so it can be concluded that 'r' traces out an approximate ellipse that rotates about its foci. The amount of precession per rotation is given as

$$\Delta = \frac{2\pi}{\left(1 - \frac{3M^2}{L^2}\right)} - 2\pi = \frac{6\pi M^2}{L^2}. \tag{2.25}$$

Inserting all the constants which we have set to unity we obtain that

$$\Delta = \frac{6\pi G^2 M^2 m^2}{\tilde{L}^2 c^2}. \tag{2.26}$$

2.2.3 Bending of light

For null geodesics $ds^2 = 0$ and thus the constants obtained by using the Killing symmetries of the Schwarszchild metric get modified as the derivative of Eq. (2.2) is no more with respect to the arc length, but wit respect to a suitable affine parameter.

Using this fact and equation (2.14), we can write the equation governing the trajectory of a photon as

$$\frac{d^2u}{d\phi^2} + u = 3Mu^2. \tag{2.27}$$

Setting the right hand side to zero and using the boundary condition that $\phi = 90^0$ when the photon hits the surface of the central mass we get

$$u = \frac{\sin\phi}{R} \tag{2.28}$$

Here R can be regarded as the distance of closest approach for the photon to the central potential or the radius of the central gravitating object. Substituting this value of u in equation (2.27) we get

$$\frac{d^2u}{d\phi^2} + u = \frac{3M\sin^2\phi}{R^2} = \frac{3M}{R^2}(1 - \cos^2\phi). \tag{2.29}$$

This equation has two particular solutions given by

$$u = \frac{3M}{R^2}, \tag{2.30}$$

$$u = \frac{3M}{6R^2}\cos2\phi. \tag{2.31}$$

Summing up everything the total solution is

$$u = \frac{\sin\phi}{R} + \frac{M}{2R^2}\cos2\phi + \frac{3\mathrm{M}}{2\mathrm{R}^2}. \tag{2.32}$$

From the above expression we see that at large 'r' the value of ϕ should be small. In the limit $u \to 0$, $sin\phi \sim \phi$, $cos\phi \sim 1$, we get the angle of deviation viz.ϕ_∞ to be

$$\phi_\infty = -\frac{2M}{R}. \tag{2.33}$$

Chapter 3

Stationary blackholes

By definition, a stationary spacetime exhibits a one parameter group of isometries whose orbits are timelike. This implies that the sliding of the metric along a timelike killing vector field, say ξ_t, preserves the metric. In this chapter we shall consider axisymmetric spacetimes. By this we mean that the spacetime must exhibit a spacelike Killing vector ξ_ϕ whose orbits are closed. For a spacetime to be both stationary and axisymmetric the Killing vectors , ξ_t and ξ_ϕ, must commute with each other. This extra condition implies that we can treat ξ_t and ξ_ϕ as coordinate vectors since the Lie bracket of two coordinate vector is always zero[2].

By using the above mentioned conditions we can write the general form of a stationary and axisymmetric metric as:

$$ds^2 = \sum_{ab} g_{ab} dx^a dx^b \tag{3.1}$$

where the chosen coordinate system is $(x^0 = t, x^1 = \phi, x^2, x^3)$ and due to the symmetries of the spacetime g_{ab} is independent of t and ϕ. Thus we have ten unknown variables in two functions that is needed to be solved. Using certain theorems and few other assumptions helps one to reduce the problem into a much more convenient form. In the next section we see how this reduction of variables is done to lead to a form for the metric describing a stationary axisymmetric spacetime.

3.1 Metric describing a rotating black hole

The general form of the metric given in (3.1) can be simplified considerably using the theorem which states that the two dimensional subspace spanned by vectors orthogonal to the coordinate vectors ξ_t and ξ_ϕ is tangent to the two dimensional surface, i.e. integrable. For this theorem we can choose a set of coordinate (x^2, x^3) and span the two dimensional surface, without having any crossterm with t or ϕ. This reduces the number of independent variables in the metric from ten to six. Making the choice of coordinate $x^2 = \rho$ and $x^3 = z$ we can write the most general metric for a stationary axisymmetric metric as

$$ds^2 = -V(dt - wd\phi)^2 + V^{-1}\rho^2 d\phi^2 + \Omega^2(d\rho^2 + \Lambda dz^2). \tag{3.2}$$

Here we have made the choice $\rho^2 = VX + W^2$ and $w = W/V$

Taking $\rho^2 = r^2 + a^2\cos^2\theta$, defining $\Delta = r^2 - 2 + a^2$ and using the metric (3.2) we get the metric of a rotating black hole in vacuum as;

$$ds^2 = -\frac{\rho^2}{\sum^2}\Delta dt^2 + \frac{\sum^2 \sin^2\theta}{\rho^2}(d\phi - \omega dt)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \tag{3.3}$$

where

$$\sum{}^2 = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta \quad \text{and} \quad \omega = \frac{2\mu \mathrm{ra}}{\sum^2}. \tag{3.4}$$

Looking at this metric in the asymptotic limit determined by the condition $r \to \infty$ we can identify a as the angular momentum per unit mass.

The metric given in (3.3) describes the spacetime around a rotating black hole and is known as the Kerr metric.

3.2 General properties of the Kerr metric

In the so called Boyer-Lindquist coordinates, the Kerr spacetime is described by the line element [23]

$$\begin{aligned} ds^2 = -\left(1 - \frac{2\mu r}{\rho^2}\right)dt^2 - \frac{4\mu ar\sin^2\theta}{\rho^2}dtd\phi + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \\ + \left(r^2 + a^2 + \frac{2\mu ra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta d\phi^2. \end{aligned} \tag{3.5}$$

In the limit $a \to 0$ this metric reduces to the Schwarzschild metric as expected and in the limit $\mu \to 0$, it corresponds to the flat spacetime metric thus giving the interpretation of μ as the mass parameter M of the Schwarzschild metric [1]. In the limit $\mu \to 0$ the metric reduces to

$$ds^2 = -dt^2 + \frac{\rho^2}{r^2 + a^2}dr^2 + \rho^2 d\theta^2 + (r^2 + a^2)sin^2\theta d\phi^2 \tag{3.6}$$

Performing the coordinate transformation

$$x = \sqrt{(r^2 + a^2)}\mathrm{sin}\theta\mathrm{cos}\phi; \qquad \mathrm{y} = \sqrt{(\mathrm{r}^2 + \mathrm{a}^2)}\mathrm{sin}\theta\mathrm{sin}\phi; \qquad \mathrm{z} = \mathrm{rcos}\theta \tag{3.7}$$

the above metric reduces to the known Cartesian form. This coordinate transformation also gives that in the plane given by $\theta = \pi/2$, $r = 0$ corresponds to a disk of radius 'a'.

As mentioned before, since the Kerr metric is axially symmetric and stationary, $\xi_t = \partial/\partial t$ and $\xi_\phi = \partial/\partial\phi$ are two Killing vectors of the metric. We can relate the norm of these Killing vectors with three metric components and arrive at some interesting results about the spacetime. The metric becomes ill defined at $\rho = 0$. The condition $\rho = 0$ also gives

$$r^2 + a^2 cos^2\theta = 0 \tag{3.8}$$

When $\theta = \pi/2$ this equation gives $r = 0$. Since $r = 0$ in the $\theta = \pi/2$ plane corresponds to a disk of radius a, the condition $\rho = 0$ gives a ring like singularity.

At the event horizon, the norm of the $r = constant$ hypersurface becomes null. The normal to the $r = constant$ surface is given by $\partial_a r$. The normal becomes null when $g^{ab}\partial_a r \partial_b r = 0 = g^{rr}$. The condition $g^{rr} = 0$ leads to $\Delta = 0$ and thus the solution to the equation $\Delta = 0$ gives the location of the event horizon. On solving the quadratic equation given by $\Delta = 0$ we find that the location of the event horizon to be

$$R_H = \mu + \sqrt{\mu^2 - a^2} \tag{3.9}$$

Here we ignored the second root, which corresponds to the interior horizon.

If we consider trajectory of observers moving with 4-velocity u along the timelike killing vector ξ_t then we can define the relation $\xi_t = Ru$. R is the normalization constant defined by $R^2 = -\xi_t\xi^t$. Using this relation and the frequency of an observer moving in spacetime with momentum p the surface of infinite redshift is given by the condition $R = 0$. Now $R^2 = -\xi_t\xi^t = -g_{tt}\xi^t\xi^t = 0 = g_{tt}$. Solution to the equation $g_{tt} = 0$, which is given as

$$R_E = \mu + \sqrt{\mu^2 - a^2cos^2\theta} \tag{3.10}$$

gives the location of the surface of infinite redshift in case of the Kerr metric. As we can see, in the case of Kerr black holes, the event horizon is not the surface of infinite redshift. The surface of infinite redshift is known as the ergosurface and the region between the event horizon and the ergosurface is known as ergosphere. Thus, it is seen clearly that the timelike Killing vector field ξ_t does not generate the event horizon, instead the event horizon is generated by the Killing vector $\xi_t + \Omega_H\xi_\phi$, where Ω_H can be interpreted as the angular velocity of the horizon.

3.2.1 Static limit

A static observer i.e an observer whose r, θ, ϕ are fixed, will have a four velocity proportional to the timelike Killing vector $C\xi_t$. Where C is the normalization constant and given by $C = (-g_{tt})^{\frac{1}{2}}$. As seen earlier ξ_t becomes null on the ergosurface and it is spacelike in the ergosphere. This denotes that a static observer cannot remain static inside the ergosphere. The dragging of the inertial frame inside the ergosphere forces an observer to rotate with the blackhole.

Zero angular momentum observers (ZAMOs) can be defined as the observers satisfy the condition

$$L = u^a\xi_\phi = g_{\phi t}\dot{t} + g_{\phi\phi}\dot{\phi} = 0. \tag{3.11}$$

From this equation we get

$$\Omega = \frac{d\phi}{dt} = \frac{-g_{t\phi}}{g_{\phi\phi}}, \tag{3.12}$$

where Ω is the angular velocity of the ZAMOs. Using the explicit form of the Kerr metric (3.3) we get $\Omega = \omega$. Thus the ZAMOs rotate with the black hole and the angular velocity increases as one moves closer to the black hole (as $\omega \propto r$).

3.2.2 Stationary observers

We consider an observer moving with angular velocity Ω such that it does not feel the variation of the gravitational field around the black hole. The four velocity of such an observer is given as

$$u = C(\xi_t + \Omega\xi_\phi), \tag{3.13}$$

The normalization constant C is found to be

$$C = (-g_{tt} - 2\Omega g_{t\phi} - \Omega^2 g_{\phi\phi})^{1/2}. \tag{3.14}$$

A stationary observer can exist in this spacetime when the four velocity (3.13) is timelike. It fails to remain timelike when the norm C changes sign and thus we get a limiting condition governed by the sign of C for the existence of stationary observer in Kerr spacetime. Using the fact that the sign of C must be positive, we get the equation

$$g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi} < 0. \tag{3.15}$$

This equation has two roots and this limits the value of Ω as

$$\Omega_- < \Omega < \Omega_+ \tag{3.16}$$

where $\Omega_\pm$ is given as

$$\Omega_\pm = \omega \pm \sqrt{\omega^2 - (g_{tt}/g_{\phi\phi})}. \tag{3.17}$$

For a static black hole $\Omega = 0$, this ensures that Ω_- should change sign at R_E. As an observer moves further into the ergosphere Ω_- increases while Ω_+ decreases. Using the explicit metric components from (3.3) and using (3.17) we get $g_{tt}/g_{\phi\phi} = \omega^2$ at the event horizon. Thus when the event horizon is reached $\Omega_- = \Omega_+ = \Omega_H$ and the observer is forced to rotate around the black hole with an angular velocity ω, where ω is given by

$$\omega = \frac{a}{R_H^2 + a^2}. \tag{3.18}$$

From the above analysis, one can also come across the fact that the Killing vector defined as

$$\xi_k = \xi_t + \Omega_H \xi_\phi \tag{3.19}$$

becomes null on the event horizon and thus for Kerr black holes the event horizon is a Killing horizon generated by ξ_k

3.3 Penrose Process

The conserved energy of a particle can be defined as $E = -p^\mu \xi_t$. As it was seen earlier the Killing vector ξ_t is not timelike throughout the spacetime and changes sign as one enters the ergosphere. Thus inside the ergosphere of a Kerr blackhole particles can have negative energy [23]. Energy of a particle must be positive definite asymptotically and so it is not possible for a particle to have negative energy while entering or leaving the ergosphere. A particle having positive energy can enter the ergosphere and then break into two parts, one part having negative energy while the other part can have positive energy. This mechanism can be adopted suitably for explanation of the existence of negative energy particle within the ergosphere. The part having negative energy can fall into the event horizon while the part having positive energy escapes out to infinity.

Let the initial energy of the particle be E. Once the particle enters the ergospehere and splits up into two, let the energy of the two parts be E_-(negative energy part) and E_+ (positive energy part). From the conservation of energy we get $E_+ > E$ and thus it is possible to extract energy from a rotating blackhole and transfer them to infinity by this process. The existence of the particles having negative energy inside the ergosphere can be explained in a better way by considering the scalar product of the $4-$momentum with itself and using $E = -p^\mu \xi_t = -p_t$.

For the Penrose process both the angular momentum (L) and mass (M) of the Kerr blackhole decreases. The dot product of the $4-$momentum with the Killing vector ξ_k must be negative outside the event horizon since ξ_k is timelike in this region. Using $E = -p^\mu \xi_t$ and $L = p^\mu . \xi_\phi$ one can write

$$p^\mu . \xi_t = p^\mu . \xi_t + \Omega_H p^\mu . \xi_\phi = -E + \Omega_H L < 0. \tag{3.20}$$

When a particle enters the ergo region, the mass and angular momentum changes by $\delta M = E$ and $\delta J = L$. Using the above equation and (3.18) one can write

$$\delta M > \frac{a \delta J}{R_H^2 + a^2}. \tag{3.21}$$

Using the two dimensional metric describing the event horizon and let σ be the determinant of this metric, the area of the event horizon can be defined as

$$A = \int \int \sqrt{\sigma} d\theta d\phi = 4\pi(R_H^2 + a^2). \tag{3.22}$$

Taking variation of this area we get

$$\delta A = 8\pi(R_H \delta R_H + a\delta a). \tag{3.23}$$

Using equation (3.9) and considering $J = Ma$ the variation in the area of the event horizon can be written as

$$\delta A = 8\pi \frac{a}{\Omega_H \sqrt{M^2 - a^2}} (\delta M - \Omega_H \delta J). \tag{3.24}$$

For particle entering the ergosphere it was obtained previously, $(\delta M - \Omega_H \delta J) > 0$. Thus for the Penrose process the area of the event horizon increases and when the above equation is written in the form

$$\delta M = \frac{\kappa}{8\pi G} \delta A + \Omega_H \delta J. \tag{3.25}$$

It can be interpreted as the first law of black hole thermodynamics.

3.4 Super-radiance

Study of particles in the ergo region showed that energy can be extracted from the Kerr black hole. When a similar analysis is done by considering free scalar fields propagating in the ergosphere the phenomenon of super-radiance takes place.

Equation of motion of a massless scalar field in Kerr spacetime is given by

$$\partial_a(\sqrt{-g} g^{ab} \partial_a \Phi) = 0 \tag{3.26}$$

Using the explicit form of the Kerr metric this equation can be expanded as

$$\left[-\frac{(r^2 + a^2)^2}{\Delta} + a^2 sin^2\theta\right] \frac{\partial^2 \Phi}{\partial t^2} - \frac{arM}{\Delta} \frac{\partial^2 \Phi}{\partial t \partial \phi} + \left(\frac{1}{sin^2\theta} - \frac{a^2}{\Delta}\right) \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial}{\partial r}\left(\Delta \frac{\partial \Phi}{\partial r}\right) + \frac{1}{sin\theta} \frac{\partial}{\partial \theta}\left(sin\theta \frac{\partial \Phi}{\partial \theta}\right) = 0. \tag{3.27}$$

In this wave equation if we substitute

$$\Phi = e^{-i\omega t} e^{im\phi} R(r) S(\theta), \tag{3.28}$$

we can separate out the 'r' dependent part as

$$\frac{\partial}{\partial r}\left(\Delta \frac{\partial R}{\partial r}\right) + \left[\omega^2 \frac{(r^2+a^2)^2 - 4arM\omega m + a^2 m^2}{\Delta} - D\right] R = 0, \tag{3.29}$$

where D is the separation constant.

This equation can be simplified by introducing a new coordinate defined by

$$\frac{dr^*}{dr} = \frac{(r^2+a^2)}{\Delta}. \tag{3.30}$$

In terms of the variable r^*, the redefined equation in the asymptotic limit $(r \to \infty)$ reduces to

$$\frac{d^2R}{dr^{*2}} + \frac{2}{r}\frac{dR}{dr^*} + \omega^2 R = 0. \tag{3.31}$$

This equation is similar to the radial part of the spherical wave equation defined in flat space time. Solving this equation we get

$$R \sim \frac{e^{\pm i\omega r^*}}{r} \tag{3.32}$$

The two values correspond to the ingoing and the outgoing wave modes in the asymptotic limit.

Similarly taking the near horizon limit defined by $\Delta \to 0$ equation (3.29) can be written as

$$\frac{d^2R}{dr^{*2}} + \left[\omega^2 - \frac{4aR_H M\omega m + a^2 m^2}{(R_H^2+a^2)^2}\right] R = 0 \tag{3.33}$$

This equation can be simplified by using (3.9) and (3.18) to yeild

$$\frac{d^2R}{dr^{*2}} - (\omega - m\Omega_H)^2 R = 0. \tag{3.34}$$

This equation has the solution

$$R \sim e^{[\pm i(\omega - m\Omega_H)r^*]} \tag{3.35}$$

The wave mode with the minus sign in the exponent corresponds to the ingoing mode near the horizon and the other one, is the outgoing mode. For the energy momentum tensor for the above scalar field is

$$4\pi T_{ab} = \partial_a \Phi \partial_b \Phi - \frac{1}{2} g_{ab} [\partial_d \Phi \partial^d \Phi] \tag{3.36}$$

The energy flux is given as $P_b = T_{ab}\xi_t = T_{at}$. The flux of energy coming out or going into the horizon can be calculated by integrating p_b over the two surface describing the horizon for the given wave mode. The total energy flux is given as

$$P = \int T_t^r \sqrt{-g} d\theta d\phi \tag{3.37}$$

Using the expression of the energy momentum tensor and the wave modes given in (3.35) the energy flux through the horizon is given as

$$\frac{dE}{dt} = C\omega(\omega - m\Omega_H), \tag{3.38}$$

where C is a constant. The sign of $(\omega - m\Omega_H)$ determines whether energy is flowing into the horizon or flowing out through the horizon. This shows that for wave modes having frequency in the range $0 < \omega/m < \Omega_H$ energy comes out of the black hole and this phenomenon is called super-radiance.

Chapter 4

Quantum field theory in flat spacetime

Let us consider a $(1+1)$ dimensional system consisting of a real massless scalar field, $\Phi(x,t)$, satisfying the field equation

$$\partial^a \partial_a \Phi = 0. \tag{4.1}$$

This field equation can be derived from the action of the system which is given as

$$S = \frac{1}{2} \int d^2x \sqrt{-g} g_{ab} \partial^a \Phi \partial^b \Phi. \tag{4.2}$$

In the process of canonical field quantization [7] we define Φ and its conjugate momentum, $\pi(x,t)$, as an operator satisfying the equal time commutation relation given as

$$\begin{aligned} \Big[\Phi(x,t), \Phi(x^{'},t)\Big] &= 0, & (4.3)\\ \Big[\pi(x,t), \pi(x^{'},t)\Big] &= 0, & (4.4)\\ \Big[\Phi(x,t), \pi(x^{'},t)\Big] &= i\delta(x - x^{'}). & (4.5) \end{aligned}$$

Using (4.1) we can define the Klein-Gordon inner product calculated on a spacelike hypersurface as

$$(\Phi_1, \phi_2) = -i \int d\Sigma^a \sqrt{-g_\Sigma} (\Phi_1 \overleftrightarrow{\partial_a} \phi_2^*) \tag{4.6}$$

4.1 Quantization in Minkowski coordinates

We take the Minkowski line element in $(1+1)$ dimensions which is given by

$$ds^2 = dt^2 - dx^2. \tag{4.7}$$

For this line element, equation (4.6) can be written as

$$(\partial_t^2 - \partial_x^2)\Phi = 0. \tag{4.8}$$

The solution of this equation after proper normalization is give by

$$u_k = \frac{1}{\sqrt{4\pi\omega}} e^{-i(\omega t - kx)}. \tag{4.9}$$

These modes are defined to be positive frequency modes since they are eigenfunctions of the operator $(\partial/\partial t)$ with positive eigen values. Using (4.1) and calculating the scalar product on $t = constant$ hypersurface we define a set of orthogonality relation between the modes u_k and their complex conjugates u_k^* as

$$(u_k, u_{k'}) = \delta(k - k^{'}); \qquad (u_k^*, u_{k'}^*) = -\delta(k - k^{'}); \qquad (u_k, u_{k'}^*) = 0 \tag{4.10}$$

From the above relations it is clear that the normal modes defined in (4.9) and their complex conjugates form a complete orthonormal basis which can be used for expanding the scalar field as

$$\Phi(x,t) = \int dk \left(\hat{a}_k u_k(t,x) + \hat{a}_k^\dagger u_k^*(t,x) \right), \tag{4.11}$$

where $\hat{a}_k$ and $\hat{a}_k^\dagger$ are the annihilation and the creation operators which satisfy the standard commutation relations. Using the annihilation operator the Minkowski vacuum state can be defined as

$$\hat{a}_k |0_M\rangle = 0 \tag{4.12}$$

From the Minkowski vacuum state defined above we can obtain the multi particle states by repeated application of the creation operator, $\hat{a}_k^\dagger$.

4.2 Quantization in accelerated frame

In a similar way as in the previous section, we can write the field equation in an accelerated frame of reference (Rindler coordinates) and quantize the system. But before doing so let us set up the Rindler coordinates which describes an accelerated observer [24].

For an observer travelling with uniform acceleration,say, a, the equation of motion is given as

$$\frac{d}{dt}(\gamma u) = a, \tag{4.13}$$

where u is the velocity of the observer and $\gamma = 1/\sqrt{1-u^2}$. Integrating Eq. (4.13) and using the boundary condition,$u = 0$ at $t = 0$ gives

$$u = at\sqrt{1+a^2t^2}. \tag{4.14}$$

We integrate Eq. (4.14) to get x in terms of t as

$$x = \frac{1}{a}\sqrt{1+a^2t^2}, \tag{4.15}$$

where we have used the condition that $x = 1/a$ at $t = 0$. The proper time τ of an observer set in the accelerated frame is related to the Minkowski time t as follows

$$d\tau = dt\sqrt{1-v^2}. \tag{4.16}$$

We integrate this equation and get

$$\tau = \frac{1}{a}\mathrm{sinh}^{-1}(\mathrm{at}). \tag{4.17}$$

Using Eqs. (4.15) and (4.17), we can write

$$x = \frac{1}{a}\mathrm{cosh}(\mathrm{a}\tau); \quad \text{and} \quad \mathrm{t} = \frac{1}{\mathrm{a}}\mathrm{sinh}(\mathrm{a}\tau) \tag{4.18}$$

Using the fact that any two dimensional coordinate system is conformally flat we can write the relation between the coordinates of an accelerated frame (τ, ξ) and Minkowski coordinate (x, t) as

$$t = \frac{1}{a} e^{\xi a} \sinh(a\tau), \quad \text{and} \quad \mathrm{x} = \tfrac{1}{\mathrm{a}} \mathrm{e}^{\xi \mathrm{a}} \cosh(\mathrm{a}\tau). \tag{4.19}$$

In these newly defined coordinate the flat space line element takes the form

$$ds^2 = e^{2g\xi}(d\tau^2 - d\xi^2). \tag{4.20}$$

From the transformation defined in (4.19) we see that in the range $-\infty < \tau < \infty$ and $-\infty < \xi < \infty$ the coordinates only cover the right wedge of the two dimensional Minkowski spacetime. Thus the Rindler coordinates are incomplete and we can infer this by saying that the accelerated observer cannot observe more than $1/a$ in the direction opposite to its direction of motion. Since events beyond the right wedge cannot be observed we can think of it as a horizon.

In the Rindler coordinates the field equation (4.1) takes the form

$$(\partial_\tau^2 - \partial_\xi^2)\Phi(\tau, \xi) = 0. \tag{4.21}$$

Solving this equation we can write the wave modes after proper normalization as

$$v_{\tilde{k}}(\tau, \xi) = \frac{1}{\sqrt{4\pi\tilde{\omega}}} e^{-i(\tilde{\omega}\tau - \tilde{k}\xi)} \tag{4.22}$$

Using (4.6) the orthogonality relation between these modes and their complex conjuate $v_{\tilde{k}}^*$ can be defined as

$$(v_{\tilde{k}}, v_{\tilde{k}'}) = \delta(\tilde{k} - \tilde{k}^{'}); \qquad (v_{\tilde{k}}^*, v_{\tilde{k}'}^*) = -\delta(\tilde{k} - \tilde{k}^{'}); \qquad (v_{\tilde{k}}, v_{\tilde{k}'}^*) = 0. \tag{4.23}$$

As these modes and their complex conjugate form a complete basis, the expansion of the scalar field in terms of these modes are given by

$$\Phi(\tau, \xi) = \int d\tilde{\omega}(\hat{b}_{\tilde{k}} v_{\tilde{k}}(\tau, \xi) + \hat{b}^{\dagger}_{\tilde{k}} v^{*}_{\tilde{k}}(\tau, \xi)) \tag{4.24}$$

Here $\hat{b}_{\tilde{k}}$ and $\hat{b}_{\tilde{k}^{\dagger}}$ are the annihilation and the creation operators defined in the Rindler coordinated, which satisfy the standard commutation relation. The vacuum state in this new coordinate system can be defined as

$$\hat{b}_{\tilde{k}}|0_R\rangle = 0 \tag{4.25}$$

where $|0_R\rangle$ is refered to as the Rindler vacuum.

4.3 Unruh effect

The fact that the vacuum in Minkowski space $|0_M\rangle$ appears to be a thermal state when viewed by an accelerated observer is known as Unruh effect. For derivation of the Unruh effect we need to express the Minkowski modes in terms of the Rindler modes by means of the so called Bogolubov transformations.

As both sets of normal modes, u_k and $v_{\tilde{k}}$ are complete we can express one of them in terms of the other as

$$v_{\tilde{k}}(\tau, \xi) = \int dk \left(\alpha(k, \tilde{k}) u_k(t, x) + \beta^{*}(k,) u^{*}_k(t, x) \right) \tag{4.26}$$

and

$$u_k(t, x) = \int d\tilde{k} \left(\alpha^{*}(k, \tilde{k}) v_{\tilde{k}}(\tau, \xi) - \beta(k,) v^{*}_{\tilde{k}}(\tau, \xi) \right) \tag{4.27}$$

The quantities $\alpha(k, \tilde{k})$ and $\beta(k, \tilde{k})$ are known as the Bogolubov coefficients [19, 20]. We can use the inner product defined in (4.6) and the orthonormality conditions of the mode to express the Bogolubov coefficients as follows

$$\alpha(k, \tilde{k}) = (v_{\tilde{k}}, u_k) \qquad\qquad \beta(k, \tilde{k}) = -(v_{\tilde{k}}, u^{*}_k) \tag{4.28}$$

The annihilation operators $\hat{a}_k, \hat{a}^\dagger_k$ and $\hat{b}_{\tilde{k}}, \hat{b}^\dagger_{\tilde{k}}$ can be related using the Bogolubov coefficients as

$$\hat{a}_k = (u_k, \Phi(\tau, \xi)) = \int d\tilde{\omega}(\alpha(k, \tilde{k})\hat{b}_{\tilde{k}} + \beta^*(k\tilde{k})\hat{b}^\dagger_{\tilde{k}}) \tag{4.29}$$

and

$$\hat{b}_{\tilde{k}} = (v, \Phi(t, x)) = \int d\omega(\alpha^*(k, \tilde{k})\hat{a}_k - \beta^*(k, \tilde{k})\hat{a}^\dagger_k). \tag{4.30}$$

Using the commutation relations

$$\left[\hat{a}_k, \hat{a}^\dagger_{k'}\right] = \delta(k - k^{'}), \qquad \left[\hat{b}_{\tilde{k}}, \hat{b}^\dagger_{\tilde{k}'}\right] = \delta(\tilde{k} - \tilde{k}^{'}) \tag{4.31}$$

and from the expression for the creation and annihilation operator in Eqs. (4.29) and (4.30), we can arrive at the relations

$$\int dk\left(\alpha^*(k, \tilde{k}^{'})\alpha(k, \tilde{k}) - \beta^*(k, \tilde{k}^{'})\beta(k, \tilde{k})\right) = \delta(\tilde{k} - \tilde{k}^{'}), \tag{4.32}$$

$$\int dk\left(\alpha(k, \tilde{k})\beta(k, \tilde{k}^{'}) - \beta(k, \tilde{k})\alpha(k, \tilde{k}^{'})\right) = 0. \tag{4.33}$$

Next we calculate the Bogolubov coefficients between the Minkowski modes and the Rindler modes using Eq. (4.28). Computing the inner product on the $\tau = 0$ hypersurface we get

$$\alpha(k, \tilde{k}) = \frac{1}{4\pi\sqrt{\omega\tilde{\omega}}}\int d\xi(\omega e^{a\xi} + \tilde{\omega})e^{i\tilde{\omega}\xi}exp\left[-i\left(ka^{-1}e^{a\xi}\right)\right], \tag{4.34}$$

$$\beta(k, \tilde{k}) = \frac{1}{4\pi\sqrt{\omega\tilde{\omega}}}\int d\xi(\omega e^{a\xi} - \tilde{\omega})e^{i\tilde{\omega}\xi}exp\left[i\left(ka^{-1}e^{a\xi}\right)\right]. \tag{4.35}$$

When the Minkowski modes are expressed in terms of the Rindler modes, if the coefficient $\beta(k, \tilde{k})$ is non-zero then we can see from (4.30) the Minkowski vacuum will not be annihilated by the annihilation operator defined in the Rindler coordinate. The Bogolubov coefficients can be calculated by performing the integrals given in (4.34) and (4.35). At first we substitute $z = e^{a\xi}$ and the integrals reduces to

$$\alpha(k,\tilde{k}) = \frac{a^{-1}}{4\pi\sqrt{\omega\tilde{\omega}}}\int \mathrm{d}z(\omega z+\tilde{\omega})z^{ilg^{-1}-1}e^{-ikzg^{-1}}, \tag{4.36}$$

$$\beta(k,\tilde{k}) = \frac{a^{-1}}{4\pi\sqrt{\omega\tilde{\omega}}}\int \mathrm{d}z(\omega z-\tilde{\omega})z^{ilg^{-1}-1}e^{ikzg^{-1}}. \tag{4.37}$$

Using the known identity given by

$$\int_0^\infty x^{s-1}e^{-bx}dx = \exp(-\mathrm{slnb})\Gamma\left(\mathrm{s}\right) \tag{4.38}$$

and using proper cut-off which is set to zero finally, the integrals can be evaluated to give

$$\alpha(k,\tilde{k}) = \frac{1}{4\pi a}\sqrt{\frac{\omega}{\tilde{\omega}}}(\omega\tilde{k}+k\tilde{\omega})\left(\frac{k}{a}\right)^{-i\tilde{k}a^{-1}}\Gamma(-i\tilde{k}a^{-1})e^{\pi\tilde{k}/2a} \tag{4.39}$$

and

$$\beta(k,\tilde{k}) = -\frac{1}{4\pi a}\sqrt{\frac{\omega}{\tilde{\omega}}}(\omega\tilde{k}+k\tilde{\omega})\left(\frac{k}{a}\right)^{-i\tilde{k}a^{-1}}\Gamma(-i\tilde{k}a^{-1})e^{-\pi\tilde{k}/2a}. \tag{4.40}$$

From the above expressions we get the relation

$$\beta(k,\tilde{k}) = -\alpha(k,\tilde{k})e^{-\pi\tilde{k}/a}. \tag{4.41}$$

The number operator can be defined in Rindler coordinates to be $\hat{b}_{\tilde{k}}\hat{b}^{\dagger}_{\tilde{k}}$. The expectation value of this Rindler number operator in Minkowski vacuum is given as

$$\langle 0_M|N_R|0_M\rangle = \langle 0_M|\hat{b}_{\tilde{k}}\hat{b}^{\dagger}_{\tilde{k}}|0_M\rangle = \int dk|\beta(k,\tilde{k})|^2 \tag{4.42}$$

where we use Eq. (4.30) for arriving at the final expression. Using equation (4.32) we get

$$\langle 0_M|N_R|0_M\rangle = \frac{1}{2\pi k}\int\left[\frac{a^{-1}dk}{exp(2\pi\tilde{\omega}a^{-1})-1}\right] \tag{4.43}$$

Thus it is clear that the Rindler number operator in $|0_M\rangle$ state gives a thermal spectrum at temperature a$/2\pi$. This also shows that the quantization in Minkowski and the Rindler coordinates are inequivalent which we shall further discuss in the next section.

4.4 Inequivalent quantization and correlation in vacuum

As mentioned previously, the Rindler coordinates cover only a part of the Minkowski space and an observer in Rindler spacetime cannot obtain any information from the region beyond the Rindler horizon. Defining a new set of coordinates as

$$t + x = \tilde{v} \;=\; g^{-1}e^{gv}, \tag{4.44}$$

$$t - x = \tilde{v} \;=\; -g^{-1}e^{gu}, \tag{4.45}$$

where $v = \xi + \tau$ and $u = \xi - \tau$. In these coordinates, the Rindler line element (4.20) takes the from

$$ds^2 = e^{2g\xi}dudv \tag{4.46}$$

and the general solution of these wave equation is in these coordinate can be written as $P(u) + F(v)$. The outgoing mode i.e. the mode dependent on u when expressed in terms of the Minkowski null coordinates defined in (4.45) is given as

$$p = \exp[\mathrm{i}\frac{\omega}{\mathrm{g}}\ln(-\tilde{\mathrm{u}})]. \tag{4.47}$$

Clearly this field mode cannot be defined for all values of $\tilde{u}$ as $p = 0$ for $\tilde{u} > 0$.

The Rindler spacetime exhibits a Killing vector given by $\chi = \partial/\partial\tau$. In terms of the null coordinates defined for Minkowski space χ takes the form

$$\chi = g(\tilde{v}\partial_{\tilde{v}} - \tilde{u}\partial_{\tilde{u}}) \tag{4.48}$$

It is easily seen that the field mode defined in (4.47) is a positive frequency mode with respect to ξ but it is not a purely positive frequency mode with respect to the timelike Killing vector ($\partial/\partial t$) defined for Minkowski space. The field mode p is not purely positive frequency with respect to u coordinate is also evident from the fact that p vanishes for $u > 0$ and a purely positive frequency mode cannot vanish on any open interval.

For expressing the Minkowski wave modes in terms of the Rindler modes we can define p for $\tilde{u} > 0$ by analytic continuation. If we take the branch cut of $\ln(\tilde{u})$ on the upper half plane we can define a function $\ln(\tilde{u} + i\pi)$ which is analytic for positive values of $\tilde{u}$ and it agrees with $\ln(-\tilde{u})$ on the negative real axis. Using this we can define a new mode which will have purely positive $\tilde{u}$ frequency. This mode is defined as

$$h = p(\tilde{u}) + e^{\frac{-\pi\omega}{g}} p(-\tilde{u}), \tag{4.49}$$

and the annihilation operator defined by $a(h) = (h, \phi)$ gives $a(h)|0\rangle_M = 0$. By using linearity of the Klein-Gord0n product we can express the annihilation operator defined above in terms of the Rindler annihilation and creation operator as

$$a(h) = a(p) + e^{-\pi\omega/g} a(\tilde{p}), \tag{4.50}$$

where $\tilde{p}(\tilde{u}) = p(-\tilde{u})$ and $a(\tilde{p}) = -a^{\dagger}(\tilde{p}^{*})$ Using eq. (4.50) the expectation value of the number of particle in Minkowski space as detected by an Rindler observer can be calculated after normalizing the modes properly and a Planckian spectrum can be obtained with the temperature $a/2\pi$.

Chapter 5

Quantum field theory in curved spacetimes

Previously we saw that a rotating black hole exhibits the phenomenon of super radiant scattering which involves stimulated emission when a scalar field is incident on a rotating black hole. This suggests that when quantum fields are studied around a rotating black hole it should also exhibit spontaneous emission. Remarkably it was found by Hawking that even static black holes exhibit this phenomenon of spontaneous radiation in presence of quantum fields around the black hole. This phenomenon of spontaneous emission of particle from a blackhole is known as Hawking radiation.

When a free quantized scalar field passes through the interior of a collapsing star its modes gets red-shifted. As this field crawls out of the surface of the star undergoing collapse the extent of red-shif increases. On performing the Bogolubov transformation between the standard outgoing Minkowski field modes and the red-shifted modes emerging from the star we get a Planckian spectrum of particle. Thus it implies that the initial "in vacuum" state contains a thermal flux of outgoing particles at late times. We work in $(1+1)$ dimension by choosing a two dimensional metric corresponding to a spherical collapsing object. This is done to avoid the difficulty of dealing with complex algebra while we get the same result when extended to $(3+1)$ dimensions. The way in which the star is collapsing is also kept arbitrary as it does not effect the final result as long as it asymptotically settles down to a Schwarzschild black hole.

5.1 Particle production by black hole

It is assumed that when the star started collapsing in the past, spacetime was nearly flat and thus the Minkowski vacuum is a good approximation describing such a state. When the star has collapsed sufficiently and formed a black hole, the exterior spacetime of the star is described by the Schwarzschild metric and in this region we also need to define a new vacuum state. In this section we will calculate the Bogolubov transformation between the "in" and "out" vacuum state to obtain the thermal spectrum of particle at late times far away from the black hole.

Now we look at mode solutions of the standard Klein-Gordon equation for Schwarzschild spacetime. The Klein-Gordn equation can be written as

$$\frac{1}{\sqrt{-g}}\partial_a\left[-gg^{ab}\partial_b\phi\right]=0 \tag{5.1}$$

where g is the determinant of the Schwarzschild metric. For solving this equation we use the trial solution

$$R(r)\Theta(\theta\phi)\left(Ae^{-i\omega t}+A^*e^{i\omega t}\right) \tag{5.2}$$

In the asymptotic region defined by $r\to\infty$ the radial part of Eq. (5.1) is simply given as $e^{\pm i\omega r}$. We change variable r to r^*, where r^* is defined as the tortoise coordinates (given in Appendix A). In terms of these redefined coordinates the solution of Eq. (5.1) is given as

$$\frac{1}{\sqrt{2\pi\omega}}e^{-i\omega(t-r^*)/rY_{lm}(\theta\phi)} = \frac{1}{\sqrt{2\pi\omega}}e^{i\omega u/rY_{lm}(\theta\phi)}, \tag{5.3}$$

$$\frac{1}{\sqrt{2\pi\omega}}e^{-i\omega(t+r^*)/rY_{lm}(\theta\phi)} = \frac{1}{\sqrt{2\pi\omega}}e^{i\omega v/rY_{lm}(\theta\phi)}. \tag{5.4}$$

When working in (1+1) dimensions and neglecting the effect of back scattering of field modes these mode solutions reduces to the standard flat space form for large distances. We can define a vacuum with respect to these modes as $a|0_M\rangle=0$ where a is the annihilation operator defined with respect to the modes given in Eqs.(5.3)and (5.4). This suggests that there is no incoming radiation from $\mathcal{I}^-$. Due to the presence of the collapsing star these modes will get red-shifted which otherwise would have propagated in the same initial form.

We now compute the red-shifted modes reaching $\mathcal{I}^+$ after passing through the collapsing star. In $(1+1)$ dimensions the spacetime in the exterior region of the collapsing star can be best described by the Schwarzschild metric. We take an arbitrary form of the metric defined in terms of the null coordinates as

$$ds^2 = C(r)dudv \tag{5.5}$$

where

$$u = t - (r^* - R_0^*), \tag{5.6}$$

$$v = t + (r^* - R_0^*), \tag{5.7}$$

where R_0^* is a constant and r^* is defined as

$$r^* = \int C(r)^{-1} dr. \tag{5.8}$$

This arbitrary metric is assumed to be asymptotically flat and this is given by the condition $C(r) \to 1$ in the limit $r \to \infty$. The interior spacetime of the collapsing star is defined by a metric which in a arbitrary form is given as

$$ds^2 = A(U,V)dUdV \tag{5.9}$$

and

$$U = \tau - (r - R_0), \tag{5.10}$$

$$V = \tau + (r - R_0), \tag{5.11}$$

R_0 and R_0^* are related in the same way as r and 'r^*'. It is assumed that at $\tau = 0$ the star is at rest and the surface of the star is given by $r = R_0$. To depict the scenario of a wave entering the collapsing star and emerging out we assume that the wave gets reflected at $r = 0$, which is the center of the star, and restrict the treatment to only positive values of r. To achieve this we need to impose the boundary condition $\phi = 0$ at $r = 0$.

We denote the relation between the interior and the exterior coordinates of the star, ignoring any reflection at the surface of the star, in an arbitrary functional form as

$$U = \alpha(u) \tag{5.12}$$

$$v = \beta(V) \tag{5.13}$$

Using Eqs. (5.10) and (5.11), we can define the centre $(r = 0)$ of the radial coordinate by the line

$$V = U - 2R_0 \tag{5.14}$$

For $\tau > 0$ the star starts collapsing along the worldline $r = R(\tau)$. Matching the interior and the exterior metric along this collapsing surface we get

$$\frac{dU}{du} = \frac{C(1-\dot{R})}{[AC(1-\dot{R}^2)+\dot{R}^2]-\dot{R}} \tag{5.15}$$

$$\frac{dv}{dV} = \frac{[AC(1-\dot{R}^2)+\dot{R}^2]+\dot{R}}{C(1+\dot{R})} \tag{5.16}$$

where C is evaluated at $r = R(\tau)$ and the overdot represents differentiation with respect to τ. When the star collapses to a sufficiently small region in spacetime, it forms a black hole. At this point the surface of the collapsing star coincides with the event horizon of the black hole which is given by $C = 0$. When this condition is used Eqs.(5.15) and (5.16) reduce to

$$\frac{dU}{du} \sim (\dot{R}-1)C(R)/2\dot{R} \tag{5.17}$$

$$\frac{dv}{dV} \sim A(1-\dot{R})/2\dot{R}, \tag{5.18}$$

and calculating the second limit using the standard L Hospital‘s rule. Now, near the event horizon, $R(\tau)$ can be expanded as

$$R(\tau) = R_h - \dot{R}(\tau_h)(\tau_h - \tau) + O([\tau_h - \tau]^2), \tag{5.19}$$

where we have defined $R = R_h$ at the horizon.Using this and integrating Eq. (5.17) we get

$$U = De^{-\kappa u} + constant. \tag{5.20}$$

we define κ as the surface gravity of the black hole and it is given as

$$\kappa = \frac{1}{2}\frac{\partial C}{\partial r} \quad \text{calculated at} \quad r = R_h \tag{5.21}$$

Integrating Eq.(5.18) we see that the relation between v and V is linear.

The mode functions are given by the solution of the equation (5.1) with the boundary condition that $\phi = 0$ at $r = o$.We mentioned previously (5.14) how the center of the radial coordinate system is defined. Using these facts we can write the solution to the field mode as

$$\frac{1}{\sqrt{4\pi\omega}}(e^{-i\omega v} - e^{-i\omega\beta[\alpha(u)-2R_0]}) \tag{5.22}$$

where we defined v at $r = 0$ using (5.14) to be

$$v = \beta[V] = \beta[\alpha(u) - 2R_0]. \tag{5.23}$$

This solution shows how the outgoing modes get complicated due to the red-shifting. Using (5.20) we can write this outgoing mode as

$$f_\omega = \frac{1}{\sqrt{4\pi\omega}}e^{-i\omega(a\exp[-\kappa \mathrm{u}]+\mathrm{b})}. \tag{5.24}$$

This mode can be expressed in terms of the standard outgoing modes(i.e the modes which have not suffered the red-shifting) as

$$f_\omega = \frac{1}{2\pi}\int \left[\alpha_{\omega\Omega}e^{-i\Omega u} + \beta_{\omega\Omega}e^{i\Omega u}\right] d\Omega. \tag{5.25}$$

The particle content of this outgoing modes can be calculated in a similar way as in section 4.3 by calculating the Bogolubov coefficients and we get

$$\langle N_\Omega \rangle = |\beta_{\omega\Omega}|^2 = \frac{2M}{exp[8\pi M\Omega] - 1}, \tag{5.26}$$

where we have made use of the fact that the surface gravity, $\kappa = 1/4M$ and M as the mass of the black hole. Thus equation (5.26) gives a thermal spectrum of particles at the temperature of

$$T = \frac{\kappa}{2\pi} \tag{5.27}$$

This confirms the result that a particle detector would not detect any particle at $\mathcal{I}^-$ as spacetime is asymptotically flat but at $\mathcal{I}^+$ due to the complicated phase factor of the modes due to the red-shift the particle detector will register particle for this state. From Eq. (5.26) we can see that the wavelength of the emitted particles are nearly order of 'M'. Thus we cannot investigate the origin of these particles near the horizon as we cannot localize a quanta within one wavelength. Thus the concept of particle is globally defined and particles are observed at $\mathcal{I}^+$. One plausible way of explaining Hawking radiation is by the tunneling mechanism. Due to quantum fluctuations virtual particles and anti particles are continuously created. When the the separation between these virtual particles are of the order of the size of the black hole, strong tidal forces prevents re-annihilation of these pairs. The particle having positive energy escapes out to infinity and contributes to the flux of radiation obtained at $\mathcal{I}^+$ and the other particle having negative energy falls into the black hole singularity. As the black hole absorbs the negative energy particle the energy and the mass of the black hole reduces.

5.2 Hawking radiation – some essential aspects

In this section we look into some features of Hawking radiation and also investigate a major issue related to the way Hawking radiation is derived.

5.2.1 Black hole evaporation

In the previous section we derived Hawking radiation for a two dimensional collapsing star and arrived at the thermal spectrum of outgoing particles (5.26). We also neglected any back-scattering of the mode due the black hole. The later assumption is not valid in $(3+1)$ dimensions as the angular part of the Klein-Gordon equation written in the Schwarzschild coordinates will act as an effective potential which effectively scatter off the incoming waves partially. Taking this fact into account we can modify the spectrum of particle by introducing an absorption factor as

$$N_p = \frac{\Gamma_p}{e^{\omega/T_H} - 1}, \tag{5.28}$$

Where T_H is the Hawking temperature and Γ_p is the factor that indicates the emissivity of the black hole and it is known as 'greybody factor' [7]. The presence of the 'greybody factor' also shows that a black hole does not behave as a perfect black body.

We can calculate the rate of loss of a black hole mass due to Hawking radiation from the flux of the radiant energy. Using Stefan-Boltzmann law we get

$$L = \Gamma\sigma A T_H^4 = \frac{\Gamma}{15360\pi M^2} \tag{5.29}$$

Where L is the flux of energy emitted by the black hole, $\sigma = \pi^2/60$ and 'A' is the area of the black hole which can be calculated as

$$A = 4\pi r_H^2 = 16\pi M^2. \tag{5.30}$$

The rate of loss of mass is related to the energy flux as

$$\frac{dM}{dt} = -L = -\frac{\Gamma}{15360\pi M^2} \tag{5.31}$$

Integrating this equation with the initial condition $M(0) = M_o$ we get

$$M(t) = M_0 \left(1 - \frac{t_l}{t}\right)^{\frac{1}{3}}, \tag{5.32}$$

where t_l is the lifetime of an isolated black hole and it is given as

$$t_l = 5120\pi M_0^3/\Gamma. \tag{5.33}$$

5.2.2 The trans-Planckian issue

It was stated earlier that the model with which we are working, for investigating the Hawking radiation, is asymptoticly flat. For a static observer to detect particle at $\mathcal{I}^+$ the state near the horizon must be vacuum as described by a free-fall observer, i.e. an observe who is falling across the event horizon freely. The generic state which we defined at the past null infinity is Minkowski vacuum and thus one must show that the free-fall vacuum must result from the initial vacuum which we choose. This was done by tracing the v modes backward in time and through the collapsing star to the past null infinity. By doing this the free-fall frequency matches with the Killing frequency at $\mathcal{I}^-$ as we demanded. But tracing the mode backward in time has a subtle problem involved with it [25] [26]. As these modes are propagated backwards they get exponentially blue-shifted with respect to the Killing time. For a outgoing quanta of radiation at a time t after the black hole is formed the amount of blue-shifting of the propagated mode to $\mathcal{I}^-$ is determined by a factor of $e^{\kappa t}$ where κ is the surface gravity of the black hole. This factor can be found from the relation between the frequency of a mode at the past null infinity and the future null infinity. A mode of frequency Ω on $\mathcal{I}^-$ is related to a mode of frequency ω on $\mathcal{I}^+$ as

$$\omega(u, \Omega) = \alpha'(u)\Omega, \tag{5.34}$$

where $\alpha'(u)$ is defined by (5.20). For a Hawking mode of frequency $\sim \kappa$ the typical frequency of the field mode at $\mathcal{I}^+$ is given as

$$\Omega \sim \kappa/\alpha'(u) \propto e^{\kappa u}. \tag{5.35}$$

From this relation we see that for $u \to \infty$ the blue shifting is enormously large. This issue can be sidestepped in several ways and the problem can be resolved in various approaches [27, 28, 29, 30].

Chapter 6

Thermal Green function

For free scalar field we can associate the expectation value of commutator and the anti-commutator with different Green functions. We can write

$$iG(x, x') = \langle 0\phi(x), \phi(x')]|0\rangle$$
$$iG^1(x, x') = \langle 0|\phi(x), \phi(x')|0\rangle, \quad (6.1)$$

where 'G' is known as the Schwinger function and G^1 is called Hadamards elementary function. We can write these Green functions in terms of the positive and negative frequency part as

$$iG(x, x') = G^+(x, x') - G^-(x, x'), \quad (6.2)$$
$$G^1(x, x') = G^+(x, x') + G^-(x, x'), \quad (6.3)$$

where $G^\pm$ are know as the Wightman functions. All the four Green functions introduced till now are defined by expectation value of the field operators in the vacuum state and thus are defined at zero temperature. The thermal Green functions can be obtained by replacing the vacuum expectation value used for defining the zero temperature Green function by the ensemble average over the other pure states.

If we define a state as $|\psi_i\rangle$, then the probability for a system to be in this state is given as

$$p = e^{-\beta E_i}/\Sigma[e^{-\beta E_i}] \quad (6.4)$$

where $\beta = 1/k_b T$ and E is the energy associated with the states . Using this we can get the ensemble average of any operator 'B' at temperature T as

$$\rangle_\beta = \Sigma p_i \langle\psi_i|B|\psi_i\rangle \quad (6.5)$$

Now we can define the thermal Green functions, $G^{\pm}_{\beta}$, as

$$G^{+}_{\beta} = \langle\phi(x)\phi(x')\rangle_{\beta} \tag{6.6}$$
$$G^{-}_{\beta} = \langle\phi(x')\phi(x)\rangle_{\beta} \tag{6.7}$$

Using Heisenberg's equation the evolution of the field operator can be written as

$$\phi(t,x) = e^{iH(t-t_0)}\phi(t_0,x)e^{-iH(t-t_o)}. \tag{6.8}$$

Using (6.5) we can write

$$\begin{aligned}
G^{+}_{\beta}(t,x;t',x') &= tr[e^{-\beta H}\phi(t,x)\phi(t',x')]/tr[e^{-\beta H}] \\
&= tr[e^{-\beta H}\phi(t,x)e^{\beta H}e^{-\beta H}\phi(t',x')]/tr[e^{-\beta H}] \\
&= tr[\phi(t+i\beta,x)e^{-\beta H}\phi(t',x')]/tr[e^{-\beta H}] \\
&= G^{-}_{\beta}(t+i\beta,x;t',x')
\end{aligned} \tag{6.9}$$

where we have used Eq. (6.8) in the third step. From (6.2) we get

$$iG_{\beta}(x,x') = iG(x,x') \tag{6.10}$$

and this result is justified as the commutator for a free scalar field is a constant number and thus it does not matter in which state we are calculating the expectation value. But from (6.3) we get

$$G^{1}_{\beta}(t,x;t',x') = G^{1}_{\beta}(t+i\beta,x;t',x') \tag{6.11}$$

which shows that the thermal Green function is periodic in imaginary time.

In (1+1) dimensions the Green function calculated for Minkowski spacetime is

$$G_M = \langle 0_M|\phi(x)\phi(x')|0_M\rangle = \frac{1}{4\pi}Ln\left(-(x-x')^2+(t-t')^2\right) \tag{6.12}$$

For an observer traveling along a hyperbolic trajectory given by $\xi = a^{-1}$ we can write transformation equation between the Minkowski coordinate (x, t) and the Rindler coordinate (ξ, τ) as

$$x = a^{-1}\mathrm{sinh\,a}\tau \tag{6.13}$$

$$t = a^{-1}\mathrm{cosh\,,a}\tau \tag{6.14}$$

and using this coordinate transformation we can write (6.12) as

$$G_M = \frac{1}{4\pi}\left(\frac{-1}{a^2}(\mathrm{sinh\,a}\tau - \mathrm{sinh\,a}\tau')^2 + \frac{1}{\mathrm{a}^2}(\mathrm{cosh\,a}\tau - \mathrm{cosha}\tau')^2\right)$$
$$= \frac{1}{4\pi} ln \frac{4}{a^2}\mathrm{sinh}^2\,\mathrm{a}(\tau - \tau'/2)) \qquad program©$$

Although this is periodic in imaginary time with a period of $2\pi/a$ we cannot immediately interpret this Green function as a thermal Green function because it is invariant under translation by $2\pi/a$ in each of its argument which is different from condition given in (6.9). While writing the Minkowski Green function we must consider the fact that the spacetime momentum must lie within the future light cone and also the vacuum cannot have any four momentum.To satisfy these conditions the invariant interval,$(t - t')^2 - (x - x')^2$, must be timelike . Using these facts we can write the Green function for any arbitrary dimension in the integral form as

$$G(x, t; x', t') = \frac{1}{2\pi}\int d^n k\theta(k^0)J(k^2)e^{-ik((x-x')-(t-t'))} \tag{6.16}$$

here $J(k^2)$ is a function which vanishes when k is spacelike. Evaluating this along the hyperbolic trajectory as before, we get

$$G(\tau, \tau') = \frac{1}{2\pi}\int d^n k\theta(k^0)J(k^2)e^{-i(\frac{2}{a}\mathrm{sinh\ a}(\tau-\tau'/2))}. \tag{6.17}$$

This Green function gives a contribution only when $k^0 > 0$. Now for the convergence of the above integral the imaginary part of $\mathrm{sinha}(\tau - \tau'/2)$ must be negative. We can do an analytical continuation by $\tau \to \tau - i\theta$. Using this we can write

$$\mathrm{Sinha}(\tau - \tau'/2) = \mathrm{sinh\ a}(\tau - \tau'/2 - \mathrm{i}\theta/2) \tag{6.18}$$

using the identity $\mathrm{sinh(x-iy) = sinh\,xcos\,y - icosh\,xsin\,y}$ we can see that for the convergence of the integral (6.17) $0 < \theta < 2\pi$. We also know the identity $\mathrm{sinh(x - i\pi) = sinh(-x)}$. If we choose θ as $2\pi/a$ then the above Green function becomes periodic in imaginary time also satisfy the KMS condition

$$G(\tau - i\beta, \tau') = G(\tau', \tau) \tag{6.19}$$

A similar analysis can be done for eternal black holes by looking at the Green function defined in Kruskal coordinate from Schwarszchild coordinate.

Chapter 7

Black hole thermodynamics

The event horizon of a black hole acts as a causal boundary which does not allow any propagation of information from the interior region of the black hole to the outside. This led Bekenstein to propose that we can associate entropy with the horizon of the black hole [16] and it was conjectured by him that this entropy is related to the surface area of the black hole up to some proportionality constant. After this Jim Bardeen, Brandon Carter, and Stephen Hawking proposed four laws governing various properties and behavior of black holes [31]. These laws are analogous to the four laws of classical thermodynamics and at that time people could guess strongly that a black hole acts as a thermodynamic system. We saw previously that in the case of Penrose process and super-radiance we can write a relation between the mass of the black hole, the area of the event horizon and its angular momentum which looks very similar to the first law of thermodynamics, upon interpreting the area of the event horizon as the entropy, the surface gravity as the temperature associated with the horizon. The major break through came in this field after hawking showed that a black hole can emiH thermal flux of particle and the temperature is precisely related to the surface gravity of the black hole in the same was as it was demanded for the laws of black hole thermo dynamics to hold and also the entropy is given as $A/4$, where A is the area of the black hole. It was thus evident that a black hole acts as a thermodynamic system which is in thermal equilibrium with its surrounding. The importance of black hole thermodynamics lies in the fact that we can get an idea about the microscopic degrees of freedom of the space time by investigating and analyzing macroscopic quantities such as entropy. It was also shown later that the field equations governing the dynamics of gravity can be derived by extremising the entropy defined for such a system.

7.1 Zeroth law

The zeroth law of black hole thermodynamics states that the surface gravity of a stationary black hole remains uniform and unchanged over the entire event horizon. There are two ways in which the zeroth law can be proved, each having its own advantages and drawbacks. Firstly we can assume that a spacetime exhibits a bifurcation surface and it can be shown that the zeroth law holds. In the second approach we need to assume that the dominant energy condition holds and using a specific field equation of gravity zeroth law can be proved.

For a stationary axisymmetric black hole the Killing vector generating the horizon can be written as

$$\chi_a = \xi_t + \Omega_H \xi_\phi. \tag{7.1}$$

As this Killing field becomes null on the horizon $(\chi_a \chi^a = 0)$ we can write

$$\chi^b \chi_{a;b} = \kappa \chi_a, \tag{7.2}$$

where κ is a constant, defined as the surface gravity of the black hole. By taking Lie derivative of this equation with respect to the Killing vector field ξ_a we get

$$\kappa, a\xi^a = 0, \tag{7.3}$$

which shows that the surface gravity is constant along the generator of the horizon. Now we need to prove that the surface gravity is also constant along the event horizon (i.e. from one generator to the other).

Using the Killing equation, $\xi_{a;b} = -\xi_{b;a}$, and the Frobenius' theorem, which is given as

$$\xi_{[a} \nabla_b \xi_{c]} = 0, \tag{7.4}$$

we can write (7.2) as

$$\kappa^2 = -\frac{1}{2}\chi_{a;b}\chi^{a;b}. \tag{7.5}$$

To show that κ is constant on a bifurcate killing horizon we take the derivative of (7.5) along the tangent , k^a, to the event horizon. This gives

$$\kappa k^a \underset{a}{\nabla} \kappa = -\frac{1}{2} k^a \underset{a}{\nabla} \underset{b}{\nabla} \chi_c \overset{b}{\nabla} \chi^c. \tag{7.6}$$

Using the known identity

$$\underset{a}{\nabla} \underset{b}{\nabla} \xi_c = -R^d_{bca} \xi_a \tag{7.7}$$

we get

$$\kappa k^a \underset{a}{\nabla} \kappa = \frac{1}{2} k^a R^d_{abc} \xi_d \overset{a}{\nabla} \xi^b. \tag{7.8}$$

$$= 0 \tag{7.9}$$

This shows that the surface gravity defined on a bifurcate killing horizon is constant. All spacetime might not posses a bifurcation surface and so we can derive zeroth law starting from Eq. (7.5) and using Einstein equation and the dominant energy condition, which states that matter should flow along timelike or null world lines.

7.2 First law

We consider a stationary black hole being perturbed by some influx of matter across the horizon and T_{ab} represents the variation of the energy momentum tensor. We assume that once the perturbation is removed the black hole settles down to a stationary state.

If we define a killing parameter, τ, for the generators of the horizon then from (7.2) we see that, τ, is not an affine parameter along the null geodesic generators of the horizon.We can define an affine parameter, λ along these generators and the relation between both these parameter is

$$\lambda \propto e^{\kappa\tau}. \tag{7.10}$$

For small perturbation of the black hole we can write

$$\triangle M = \int d\tau \int d\sigma^2 \triangle T_{ab} (\xi_t)^a \chi^b, \tag{7.11}$$

$$\triangle J = -\int d\tau \int d\sigma^2 \triangle T_{ab} (\xi_\phi)^a \chi^b, \tag{7.12}$$

where $\triangle M$ and $\triangle J$ are the change in mass and angular momentum of the black hole and $d\sigma^2$ is the differential area element of the horizon. Using Eq. (7.11) and (7.12) we can write

$$\begin{aligned}\triangle M - \Omega\triangle J &= \int T_{ab}((\xi_t)^a + \Omega(\xi_\phi)^a)\chi^b d\sigma^2 d\tau \\ &= \int T_{ab}\chi^a\chi^b d\sigma^2 d\tau \end{aligned} \tag{7.13}$$

Using the Raychaudhuri's equation [2] [23] [32] defined for the change of expansion along a geodesic which is non-affinely parametrized and retaining terms upto first order we get

$$\frac{d\theta}{d\tau} = \kappa\theta - 8\pi T_{ab}\chi^a\chi^b. \tag{7.14}$$

Using this equation from (7.13) we get

$$\begin{aligned}\triangle M - \Omega\triangle J &= -\frac{1}{8\pi}\int \tau d\sigma^2\left(\frac{d\theta}{d\tau} - \kappa\theta\right) \\ &= \frac{\kappa}{8\pi}\int \theta d\tau d\sigma^2 \\ &= \frac{\kappa}{8\pi}\int \frac{1}{\delta\sigma^2}\frac{d(\delta\sigma^2)}{d\tau}d\tau d\sigma^2 \\ &= \frac{\kappa}{8\pi}\delta\sigma^2. \end{aligned} \tag{7.15}$$

Thus we get an expression which is analogous to the first law of thermodynamics. There is a generalized version of the first law which does not takes into account any specific theory of gravity, and is valid for any classical theory of gravity arising from a diffeomorphism invariant Lagrangian [33].

7.3 Second law

This law states that the entropy of the blackhole cannot decrease during any physical process if the null energy condition holds. As we saw that the entropy of a black hole, in general relativity, is related to the surface area of the event horizon, the second law states that the surface area of the black hole cannot decrease during any physical process. We saw this analogy in the case of Penrose process and super-radiance where the surface area of the black hole increased. The second law can be formulated in a mathematical way by looking at the evolution of the black hole surface area using the Raychaudhuri's equation and assuming null energy condition. From Raychaudhuri's equation we get

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma^{ab}\sigma_{ab} - R_{ab}k^a k^b. \tag{7.16}$$

Using Einstein equation we can replace $R_{ab}k^a k^b$ by $T_{ab}k^a k^b$ and in the above equation we have assumed the congruence of the null geodesics to be hypersurface orthogonal. Assuming null energy condition, $T_{ab}k^a k^b > 0$, we see that all the quantities on the right hand side of Eq. (7.16) is positive. Thus we arrive at the condition

$$\frac{d\theta}{d\lambda} < \frac{1}{2}\theta^2. \tag{7.17}$$

Integrating this equation we get

$$\frac{1}{\theta} > \frac{1}{\theta_0} + \frac{1}{2}\lambda. \tag{7.18}$$

Now suppose the expansion, θ_0, is negative at some point of time then from Eq. (7.18) we get that $\theta \sim -\infty$ within some finite value of the affine parameter τ. Now the expansion is given as

$$\theta = \frac{1}{A}\frac{dA}{d\tau}, \tag{7.19}$$

where A is the are of the black hole. For $\theta \sim -\infty$ we get $A = 0$, which implies that there will be caustics formed in the future null direction of the geodesic. Now according to Cosmic censorship theorem, which states that there can be no naked singularities in spacetime or the generators of the event horizon can have no future end points, this is not allowed. Thus the expansion of the event horizon must be always positive which implies that the area of the event horizon always increases.

Chapter 8

Conclusion and summary

In this report, we first investigated various classical aspects of a black hole. When trajectories of particles and photons are studied around a black hole we get some remarkable new predictions which cannot be obtained from Newtonian gravity. The most remarkable feature of Einstein's theory of general relativity is that it allows existence of spacetime having a horizon as a null surface. Predictions such as bending of light around a massive gravitating object and precession of perihelion of Mercury were among the first things to get experimentally verified and general relativity surpassed these tests with a very high degree of accuracy. In various other theories of gravity, such as the Brans Dicke theory, we get the same result with a very high degree of accuracy. In the case of rotating black hole we also saw how we can extract energy from the black hole by means of Penrose process. Considering how the mass of the black hole changes along with its angular momentum, we got a equation which is analogous to first law of thermodynamics provided all the quantities are interpreted properly.

Next we have studied behavior of fields around a black hole and in this context we have investigated super-radiance. In case of super radiance we saw that when scalar fields, having frequency between a particular range, are incident on a rotating black hole it exhibits stimulated emission of radiation where the energy of the outgoing flux is greater than the ingoing flux. Vector fields also exhibit this phenomenon but fermionic fields does not exhibit super-radiance. This directly relates to the fact that fermoinic fields violates the weak energy condition and also the energy current is timelike or null in the ergoregion of a Kerr black hole for a fermionic field. The phenomenon of super-radiance is very important in the context of black holes because it motivates the fact that study of quantized fields might exhibit spontaneous emission.

In the next part of the project we moved into the study of quantum fields in various background spacetime. At first we studied the inequivalent quantization of a scalar field in Minkowski and Rindler coordinates which eventually lead to the so called Unruh effect. We investigated this particular phenomenon using two different approaches. In the first approach we extracted the particle number by performing a Bogolubov transformation between the field modes defined in Minkowski and Rindler spacetime. In the second approach we extended the Rindler modes by analytic continuation and obtained the desired result. Study of quantum fields in curved spacetime lead to the Hawking radiation. For deriving the Hawking radiation we considered a collapsing star which will eventually settle down to a stationary black hole. At infinity, the Killing frequency matches with the frequency described for a Minkowski mode as the spacetime is assumed to be asymptotically flat. Near the horizon, the state is very different from the Minkowski vacuum as it is a vacuum state, known as free fall vacuum, defined by an observer who is freely falling across the event horizon. The fact that the free fall frequency is different from the Killing frequency gives rise to Hawking radiation. In case of Hawking radiation the particles are observed at infinity and as definition of particle is a global concept we do not have much of an idea about the near horizon characteristics of Hawking radiation from this investigation. Studying the expectation value of the stress energy tensor, which is a local object, we can probe the physics near the black hole horizon. For calculating this expectation value choosing the correct state is very important, which in case of a collapsing star (does not have a past horizon), is defined best by the Unruh vacuum. For the case of eternal black holes the state is Hartle-Hawking vacuum. These vacuum states are regular on the horizon and so we do not need to deal with divergences when calculating the expectation value.

The fact that a black hole can emit a thermal distribution of particle and association of entropy with the event horizon of a black hole strengthened the foundation of black hole thermodynamics which was proposed before the discovery of the Hawking radiation. We studied the laws of black hole thermodynamics and show how a black hole can be treated as a thermodynamic system.

APPENDIX A

.1 Kruskal-Szekeres coordinate

As we have seen earlier the Schwazschild metric behaves badly at the $r = 2M$ surface and also by looking at radial null trajectories we have shown that this singularity is just a coordinate singularity. The fact that the $r = 2M$ surface is not a physical singularity can be also verified by extending the spacetime so that it is geodesically complete and is maximal. The maximal analytic extension of the Schwarzschild spacetime was done by Kruskal and we illustrate it in this section.

We define a new coordinate as

$$r^* = \int \frac{rdr}{r - 2M} = r + 2M\ln\left(\frac{\mathrm{r}}{2\mathrm{M}} - 1\right). \tag{1}$$

As we can see from the above equation that r^* changes logarithmically and thus slower than r near the horizon, this is known by the name of tortoise coordinate. We can define a new set of null coordinates as

$$u = (t - r^*) \qquad and \qquad v = (t + r^*). \tag{2}$$

Using these the Schwarzschild metric can be written as

$$ds^2 = (1 - 2M/r\,)du\,dv \tag{3}$$

Using (1) and (2) we can write

$$r - 2M = 2M\exp\left[\frac{\mathrm{v} - \mathrm{u}}{4\mathrm{M}}\right]\mathrm{e}^{-\mathrm{r}/2\mathrm{M}}. \tag{4}$$

We can rewrite (3) using (4) as

$$ds^2 = 2M\,\frac{\exp\left[\frac{\mathrm{v}-\mathrm{u}}{4\mathrm{M}}\right]\mathrm{e}^{-\mathrm{r}/2\mathrm{M}}}{r}\,du\,dv \tag{5}$$

Near the horizon, $r \sim 2M$, this line element is regular but still this coordinate is defined only for $r > 0$. To achieve extension of the coordinate beyond the $r = 0$ point we need to re-parametrize the null geodesics using the coordinate transformation $U = U(u) and V = V(v)$.

For finding out the exact form of the transformation we can calculate the affine parameter along the null geodesic by using the equation

$$E = g_{ab} k^a (\xi_t)^b \tag{6}$$

Using this we can define the coordinate transformation as

$$U = -e^{-u/4M} \tag{7}$$

$$V = e^{v/4M} \tag{8}$$

which transforms the line element into the form

$$ds^2 = \frac{32M^3}{r} e^{-r/2M}\, dU\, dV \tag{9}$$

We can further define two new coordinate as

$$T = (U + V)/2 \qquad and \qquad X = (V - U)/2 \tag{10}$$

to cast the line element defined in (9) as

$$ds^2 = \frac{32M^3}{r} e^{-r/2M} (dT^2 - dX^2) \tag{11}$$

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