Inflation and cosmological perturbation theory

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by

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CERTIFICATE

This is to certify that the project titled **Inflation and cosmological perturbation theory** is a bona fide record of work done by **Sayantan Auddy** towards the partial fulfillment of the requirements of the Master of Science degree in Physics at the Indian Institute of Technology, Madras, Chennai 600036, India.

(L. Sriramkumar, Project supervisor)

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ABSTRACT

The aim of the project is to understand the need for the epoch of the inflation during the early stage of the radiation dominated era, and describe how inflation is typically achieved using scalar fields. In this context we begin by studying the hot big bang model and understand its shortcomings such as the horizon and flatness problems. We shall then go on to analyse as to how a brief period of accelerated expansion, viz. inflation, can help in overcoming these difficulties. Inflation is typically achieved using scalar fields, and it is the quantum components of the scalar fields which are responsible for the generation of the perturbation in the early universe. Finally, the goal will be to understand the characteristics and the evolution of the perturbations in the Friedmann universe.

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Chapter 1

Introduction

Currently, cosmology is one of the most promising areas of physical science that has inspired many to devote their mind and thoughts into it. Cosmology forms a nice balance between observation and theory, where theory is formulated based on the observational data, which in turned is interpreted using correct theory. This discipline of physics is based on the endeavour to understand the origin, evolution and the future of this ever evolving universe. The complex laws governing the dynamics of this evolving universe still eludes bright minds, thus making cosmology that much more attractive and acceptable. With accurate and advanced data available from observations, in the recent times, cosmology has become a precision science.

1.1 The hot big bang model

Modern cosmology is based on the prevailing theory about the origin and the evolution of the universe, known as the hot big bang model [1]. The fundamental development of the theory is based on the key ideas of the general theory of relativity and cosmological principle [2]. General relativity generalises special relativity and the Newton's law of universal gravitation to describe gravity as the geometric property of space and time. The cosmological principle, in its modern form, states that on a sufficiently large scale, i.e. of the order of $100\,\mathrm{Mpc}$, the matter content of the universe in homogeneous and isotropic. This is, in essence, a generalisation of the Copernican principle that the earth is not at the centre of the solar system. In the same notion there is no specially favoured position in the universe. This is, strongly supported by a variety of observations and the most overwhelming among them has been the nearly identical temperature of the cosmic microwave background radiation coming from the different parts of the sky. This is the best available theory describing our cosmos, thus inevitably theoretical cosmology in the recent years have developed on this foundation.

Over the years, the hot big bang model has survived numerous scientific interventions, because of three strong and significant observational triumphs. The first of these observations essentially provides evidence that the universe is expanding. It is suggested by the fact that the all galaxies have a redshift that is proportional to their distance. It is true that all objects in relative motion with respect to earth, the point of observation, will exhibit a Doppler shift. But, surprisingly majority of the observed objects exhibit redshift, and not a blue shift, which is suggestive of the fact that they are receding away from us. Moreover as the redshift is proportional with the distance of the object (for suitable small distances), it can be concluded that recession speed enhances with distance. However local systems like our galaxy do not feel this expansion due to the local gravitational attraction. Otherwise it would have been next to impossible to measure the redshift. Edwin Hubble was the first to observe this phenomenon in the year 1927, and gave the relation

$$v = H_0 d$$

where v is the velocity of the galaxy and d its distance, a relation commonly known as the Hubble's law [2, 3] and H_0 is the Hubble parameter today.

The next most important observational fact is the presence of almost perfectly isotropic background radiation of photons from our relic past. The isotropy of the radiation is about 1 part in 10^5 and is known as the Cosmic Microwave Background Radiation (CMB) [4]. The temperature of the microwave is measured to be 2.728 $\pm$ 0.004 K and is, in fact, a nearly perfect black body spectrum. CMB are the relic radiation reaching us from the era of decoupling when photon ceases to interact with matter as radiation density falls below a certain level. This reflects the fact that radiation density falls faster than the matter density and CMB is precisely the radiation reaching us from around that time of transition. The CMB was discovered by the American radio astronomer Arno Penzias and Robert Wilson in the year 1964. Since then, observational data from Cosmic Background Explorer Satellite (COBE) and Wilkinson Microwave Anisotropy Satellite (WMAP) have enlightened our understanding of this background radiation. While fairly isotropic, these satellites provide enough evidence for the presence of anisotropies in the CMB. These observed patterns of the fluctuation provide a direct snapshot of the early universe.

In addition to these, another remarkable achievement of modern cosmology is the theory of Big Bang Nucleosynthesis (BBN), which explains the abundance of very light elements in the early primordial universe. It is believed that the process of necleosynthesis was triggered within the first three minutes after the big bang. Observations as well as the theory suggest that the ratio of the number density of the baryons to the photons remains constant over time, thus allowing us to determine the baryon density at a given time. BBN is in good agreement with both the theoretical understanding and the observational measurement and thus is regarded as a strong evidence towards the support of the hot big bang model.

Thus these three highly successful observational triumphs strongly support the hot big bang model. However, despite the success of the model in explaining these results from the different observations, the model has serious drawbacks. According to the model, the CMB photons arriving at us today from widely separated directions of the sky could not have interacted at the time of decoupling. However, observations suggest that the CMB photons are highly isotropic, as mentioned earlier. Even the photons observed from exactly diametrically opposite ends of the sky have nearly identical temperature. Thus this leads to serious concerns about the validity of this theory. Later we will discuss as to how inflation can be used to resolved this so called horizon problem and overcome the drawbacks.

1.2 The scope of this project

In the succeeding chapters, the main focus will be to discuss the problems with the hot big bang model. We will broadly classify the drawbacks as the horizon and the flatness problems and discuss how we are able to solve them using the inflationary paradigm. However before we go on to the details, we shall discuss the dynamics of the Friedmann-Robertson-Walker (FRW) metric. The next chapter is dedicated to understanding the arguments which lead to the development of the FRW metric. Then we study the dynamics by solving the Einstein equations and assuming ideal fluid to be the source we arrive at the Friedmann equations. The evolution of the universe is divided into radiation and matter dominated eras and we impose condition on the equation of state to arrive at the dependence of the scale factor on time. In the following chapter, we discuss in detail the problem associated with the big bang model. In the fourth chapter, we shall describe the brief period of accelerated expansion,viz. inflation. We shall then go onto discuss how inflation overcomes the horizon problem and how it is achieved using scalar fields. We then introduce the Potential Slow Roll (PSR) parameters and the Hubble Slow Roll (HSR) parameters and successfully show how the smallness of these parameters satisfies the conditions for inflation.

The last chapter is mainly dedicated to understanding cosmological perturbation theory. We will discuss the evolution of the scalar perturbation at the super-Hubble scales during both radiation and matter dominated eras, as it is mainly responsible for the anisotropies in the CMB. The perturbation is considered only in the linear order as deviation from homogeneity is very small in the early phase. Finally, we shall discuss some aspects of vector and tensor perturbations.

1.3 Notations and conventions

We shall mention here the various conventions and the notation that will be used throughout the following chapters. All the analysis are done in $(3 + 1)$ dimensions, and we will adopt the metric signature of $(+, -, -, -)$. The Greek indices will represent the spacetime coordinates, while the Latin indices shall denote spatial coordinates only. Both cosmic time t and conformal time τ are used depending upon the need. Further, differentiation with respect to cosmic time coordinate will be referred as an over-dot, while an over-prime denotes derivative with respect to conformal time coordinates of the Friedmann metric. Throughout the text, we shall work with the spatially flat Friedmann model.

Chapter 2

The Friedmann universe

In this chapter, we shall begin by arriving at the FRW metric using the cosmological principles as the guiding tool. We then go on to discuss the dynamics of the Friedmann model and the various epochs of the early and the present universe.

2.1 The Friedmann-Robertson-Walker metric

To construct the simplest model of the universe, we begin with the assumption that the universe is homogeneous and isotropic, i.e. there is no preferred position and direction in space respectively. As discussed earlier, this is based on the cosmological principle, which largly is the result of the observation of the CMB, which is isotropic to 1 part in 10^5. A generic space time interval can be written as :

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{tt} dt^2 + 2g_{ti} dt dx^i + g_{ij} dx^i dx^j. \tag{2.1}$$

Isotropy of space implies that the g_{ti} component must be zero, otherwise a non zero three vector, say v_i, identifies a specific direction in space, which violates isotropy. In the coordinate system of the fundamental observer, we can label the spacelike surfaces using proper time of the clock carried by them. This implies that we can set $g_{tt} = 1$. The metric now reduces to the form:

$$ds^2 = dt^2 - dl^2, \tag{2.2}$$

where dl^2 is the spatial part of the metric that can be expressed as

$$\mathrm{d}l^2 = e^{\lambda(r)}\, \mathrm{d}r^2 + r^2\, \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right) = l_{ij} dx^i dx^j. \tag{2.3}$$

The quantity $\lambda(r)$ is independent of ϕ and θ due the spherical symmetry. Now, a space of constant curvature is characterized by the following Riemann tensor [2]

$$R_{ijkl} = \kappa(l_{ik}l_{jl} - l_{il}l_{jk}). \tag{2.4}$$

The Ricci tensor corresponding to the Reimann tensor Eq. (2.4) is given by

$$R_{ij} = 2\kappa l_{ij}. \tag{2.5}$$

Now calculating the non trivial component of the Ricci tensor and using the above equation (2.5), we arrive at the following equations:

$$\frac{1}{r}\frac{\mathrm{d}\lambda}{\mathrm{d}r} = 2\kappa e^{\lambda}, \tag{2.6}$$

and

$$1 + \left(\frac{r}{2}\right)\left(\frac{d\lambda}{dr}\right) e^{-\lambda} - e^{-\lambda} = 2\kappa r^2. \tag{2.7}$$

On integrating these equations , we obtain that

$$e^{-\lambda(r)} = (1 - \kappa r^2). \tag{2.8}$$

Upon substituting this $e^{-\lambda(r)}$ in Eq (2.3), we obtain the spatial component of the line element to be

$$\mathrm{d}l^2 = \frac{\mathrm{d}r^2}{(1-\kappa r^2)} + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right) = l_{ij}dx^i dx^j. \tag{2.9}$$

Consequently this leads to the final FRW metric:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{(1-\kappa r^2)} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)\right]. \tag{2.10}$$

Here a(t) is the scale factor determining the expansion rate the universe. This is only dependent on time and is independent of any spatial components. However such geometric considerations alone does not allow us to determine the value of κ and the form of the function a, known as the expansion factor. Thus we need to consider the dynamics of the Friedmann model in order to understand the evolution of the universe.

2.2 Dynamics of the Friedmann Universe

The main aspect of General Relativity is that it connects the geometry to the matter or the energy content of the universe. This is described by the Einstein equation [3]

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = (8\pi G)T_{\mu\nu}, \tag{2.11}$$

where $G_{\mu\nu}$ is the Einstein Tensor , $R_{\mu\nu}$ is the Ricci tensor, the Ricci Scalar

$$R = g^{\mu\nu}R_{\mu\nu}, \tag{2.12}$$

is the contracted Ricci tensor and $T_{\mu\nu}$ is the symmetric tensor describing the matter content of the universe. Also G is the Newton gravitational constant.

Based on the cosmological principle of isotropy of spacetime it is required that the component T^i_0 of the stress energy tensor must vanish and T^i_j must be diagonal. Likewise homogeneity implies that all the components must the independent of any spatial direction. Thus the stress-energy tensor can be represented as:

$$T^\mu_\nu = \text{diag.}\,(\rho, -p, -p, -p). \tag{2.13}$$

Here we have assumed the source to an ideal fluid with energy density ρ and pressure p. The Einstein tensor on the left side of the Eq. (2.12) can be calculated for the FRW metric and one obtains the following non trivial components [3]:

$$G^t_t = \tfrac{3}{a^2}(\dot{a} + \kappa) \qquad G^i_j = \tfrac{1}{a^2}(2a\ddot{a} + \dot{a}^2 + \kappa^2)\delta^i_j. \tag{2.14}$$

Upon using the above stress energy tensor as the source and the expressions for the Einstein tensor we arrive at the following two Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} = \frac{8\pi G}{3}\rho, \tag{2.15}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} = -(8\pi G)p. \tag{2.16}$$

Substituting Eq. (2.15) in equation Eq. (2.16) we obtain the following relation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3p\right). \tag{2.17}$$

The Friedmann equation combined with the equation of state describing the ideal fluid is used to arrive at the scale factor a.

The Friedmann equation (2.15) can be written in term of critical density $\rho_c(t)$ and density parameter $\Omega(t)$. Rearranging Eq. (2.15) we get

$$\frac{\kappa}{a^2} = \frac{\dot{a}^2}{a^2}\left[\frac{\rho}{3H^2/8\pi G} - 1\right]. \tag{2.18}$$

Using the definitions:

$$\rho_c(t) = \frac{3H^2}{8\pi G} \qquad \textit{and} \qquad \Omega(t) = \frac{\rho}{\rho_c}, \tag{2.19}$$

we arrive at

$$\frac{\kappa}{a_0^2} = H_0^2(\Omega - 1). \tag{2.20}$$

Now for ordinary matter we have $(\rho + 3p) > 0$. Differentiating the Friedmann equation (2.15) we arrive at the following expression

$$\frac{d}{dt}(\rho a^3) = -p\frac{da^3}{dt}, \tag{2.21}$$

which can be further simplified using the equation (2.16) as

$$\frac{d}{da}(\rho a^3) = -(3a^2 p). \tag{2.22}$$

If we now consider a equation of state of the form $p = w\rho$ where w is a constant, then equation (2.22) can be solved to arrive at the following behavior for energy the density

$$\rho \propto a^{-3(1+w)}. \tag{2.23}$$

Depending on the value of w, which specifies the equation of state governing the dynamics of the universe, it could be specified if the universe was radiation or matter dominated and consequently the dependence of density parameter ρ on the scale factor a in these eras can be obtained. If we consider the universe to be flat, which is strongly supported by observations, then $\kappa = 0$, and the Friedmann equation reduces to

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho. \tag{2.24}$$

In the following section we will find out how the scale factor depends of time as we change the value of w.

2.3 Different epochs

From the previous section, we have seen that for the spatially flat universe, we have the following equation governing the scale factor:

$$\frac{\dot{a}^2}{a^2} \propto a^{3(1+w)}, \tag{2.25}$$

Integrating this equation leads to:

$$a(t) \propto t^{\frac{2}{3(1+w)}}. \tag{2.26}$$

1.When $w = 0$; the equation of state $p = 0$, corresponds to non relativistic matter. Solving Eq. (2.22) using equation $w = 0$ we get

$$\rho_{NR} = \frac{\rho_0}{a^3}, \tag{2.27}$$

and setting the value $w = 0$ to Eq. (2.26), we obtain

$$a \propto t^{\frac{2}{3}} \tag{2.28}$$

2. When for $w = \frac{1}{3}$; the equation of state is given by $p = \rho/3$ and we have the radiation dominated epoch in such a case. Solving Eq. (2.22) we obtain the dependence of the density parameter on the scale factor during the radiation dominated era to be

$$\rho_{rad} \propto \frac{\rho_0}{a^4}, \tag{2.29}$$

And the corresponding scale factor is obtained using Eq. (2.26) to be

$$a \propto t^{\frac{1}{2}}. \tag{2.30}$$

3. Certain other values of w are also of special importance. For instance, when $w = -1$, Eq. (2.23) yields $\rho = constant$ and the pressure $p = -\rho$ is negative, as we must have $\rho > 0$. On solving Eq. (2.25) we get the so called de sitter universe with the scale factor varying as $a(t) \propto e^{\lambda t}$.

Chapter 3

Drawbacks of the hot big bang model

The hot big bang model has been very successful in explaining various observational results. However, in spite of its success there are certain major drawbacks where the model fails to match up with observations. For instance, observations suggest that the CMB photons reaching us from diametrically opposite ends of the sky have almost identical temperature. But, in contrary, the theory indicates that the photons from widely separated direction of the sky could not have interacted at the time of decoupling. In this chapter, we shall discuss these issues in detail and subsequently build up the platform for a new paradigm that has to be added to the big bang model.

3.1 The horizon problem

The horizon problem of the standard big bang model identifies that different regions of the sky could not have been causally connected because of the finite speed at which information can travel. However observation from the CMB reveals that regions sufficiently separated have identical temperature and physical properties. To understand this problem we shall start with the following spatially flat, i.e. $\kappa = 1$, FRW line element :

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)[\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)]. \tag{3.1}$$

Now if we insert the condition of the radial null geodesic, viz.

$$ds^2 = \mathrm{d}\theta^2 = \mathrm{d}\phi^2 = 0, \tag{3.2}$$

into Eq. (3.1), we arrive at

$$dr = \frac{\mathrm{d}t}{a(t)}. \tag{3.3}$$

We then go on to define the horizon, as the physical distance traveled by light from the big bang singularity at $t = 0$, till the present day corresponding to time t. In term of scale factor we have

$$h(t) = a(t) \int_0^t \frac{dt}{a(t)}. \tag{3.4}$$

We now use this definition of the horizon to calculate the dimensions of the backward and the forward light cones. We define the forward light cone as the distance traveled by light from the big bang singularity at $t = 0$ till the era of the decoupling denoted as $t = t_{dec}$. Similarly, we define the backward light cone as the distance spanned by light from the present epoch to decoupling. We further assume that the universe was radiation dominated before decoupling, so that the linear dimension of the horizon at the decoupling turns out to be

$$h_f(t_{dec}, 0) = a(t) \int_0^{t_{dec}} \frac{dt}{a(t)}. \tag{3.5}$$

For radiation dominated epoch, the scale factor is given by Eq. (2.30). Upon substituting the quantity in the above expression, we get

$$h_f(t_{dec}, 0) = \left(\frac{t_{dec}}{t_0}\right)^{\frac{1}{2}} \int_0^{t_{dec}} \frac{t_0 \mathrm{d}t}{t} = 2t_{dec}. \tag{3.6}$$

If the universe is dominated by non relativistic matter from the time of decoupling to the present epoch, the horizon represents the size of the region on the surface of the last scattering from which we receive the CMB. It is given by the following expression

$$h_b(t_0, t_{dec}) = a(t) \int_{t_{dec}}^{t_0} \frac{dt}{a(t)}, \tag{3.7}$$

with the scale factor for the matter dominated universe given by the Eq. (2.28). Upon evaluating the above integral we obtain the backward light cone to be

$$h_b(t_0, t_{dec}) = a_{dec} t_o^{\frac{2}{3}} \int_{t_{dec}}^{t_0} \frac{dt}{t^{\frac{2}{3}}} \simeq 3(t_{dec}^2 t_0)^{\frac{1}{3}}. \tag{3.8}$$

We now consider the ratio, say R, of the linear dimensions of the backward and the forward light cone, we find that

$$R = \frac{h_b}{h_f} = \frac{3}{2} \left(\frac{t_o}{t_{dec}}\right)^{\frac{1}{3}}. \tag{3.9}$$

We take into account the observational value of t_o to be of the order 10^{10} and the corresponding t_{dec} of the order 10^5, then on substituting this in the above expression we obtain [8]

$$R = \left(\frac{3}{2}\right) 10^{\frac{5}{3}} \simeq \ 70 \tag{3.10}$$

In other words, the dimension of the backward light cone is about 70 times more than that of the forward light cone at the time of the decoupling, which necessarily implies that every point of the sky is not causally connected. But this clearly contradicts the fact that observation from the CMB is nearly isotropic. This is the statement of the horizon problem [5].

This very issue can be analysed from a different perspective. In a power law expansion ie. $a(t) \propto t^f$, the Hubble radius which is defined as $d_H = H^{-1} = (\frac{\dot{a}}{a})^{-1}$ takes the form $d_H \simeq t^{-1}$, or it goes as $a^{\frac{1}{f}}$. However the physical wavelength λ_p always grows as the scale factor a, i.e. $\lambda_p \propto a$. Thus, as we go back in time, the physical wavelength grows faster than the corresponding Hubble radius d_H in both radiation and matter dominated epoch, as shown in the Figure 3.1, for which the f value are $1/2$ and $2/3$ respectively. Thus the perturbation in the early universe must be correlated on scales larger than the Hubble radius so as to lead to the anisotropies that are observed in the CMB.

3.2 The flatness problem

Another essential drawback of the hot big bang model is the flatness problem [10] which is a cosmological fine tuning issue. It can be illustrated using Eq. (2.20). In standard cosmology we describe the scale factor as $a \propto t^f$, where f is equal to $2/3$ and $1/2$ for the matter and the radiation dominated eras respectively. As a consequence, we have for the radiation dominated era

$$(\Omega - 1) \propto t, \tag{3.11}$$

and for the matter dominated era

$$(\Omega - 1) \propto t^{\frac{2}{3}}. \tag{3.12}$$

Thus, it is evident that $|(\Omega - 1)|$ will diverge with time. But the present cosmological observations suggest that the density is very close to one. Since the total density departs rapidly from this critical value with time, the universe in the early times must have a density even

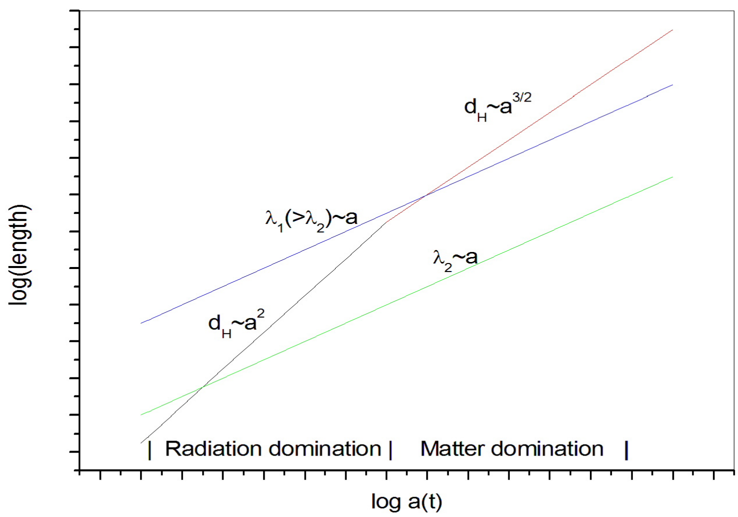

Figure 3.1: The Hubble radius d_H is plotted as the function of the scale factor a on the logarithmic scale for both the radiation and matter dominated epochs. The behavior of the physical wavelength λ_1 and λ_2 is also shown, where $\lambda = a/k$. It is evident that the physical modes enter the Hubble radius during the radiation or the matter dominated era and as one goes back in time these are outside the Hubble radius and there is no causal contact at early times.

closer to critical value. Effectively this demands extreme fine tuning of Ω such that it is very close to 1 in the early phase.

Chapter 4

The inflationary paradigm

As we have seen in the previous section, the hot big bang model suffers many drawbacks which lacks satisfactory solutions. The horizon problem is arguably the most significant of all. The inflationary model aids in resolving these issues and come up with a proper explanation for these puzzles. In the subsequent section we shall discuss how inflation resolves the horizon problem and then we shall go on to analyse how scalar fields can be used to achieve inflation. Finally we will discuss the slow roll inflationary scenario.

4.1 Resolving the horizon problem

It should be evident from the discussion in previous chapter, that the physical modes are outside the Hubble radius at sufficiently early times and enters the Hubble radius only during the radiation or matter dominated epochs. In this context see Figure 3.1. However for the inhomogeneities that we observe in the CMB sky to be causally connected these length scales must be inside the Hubble scales. For this to happen, the physical wavelength should grow slower than the corresponding Hubble radius at early times, or in other words, we should have [11]

$$-\frac{d}{dt}\left(\frac{\lambda_p}{d_H}\right) < 0. \tag{4.1}$$

For $d_H = a/\dot{a}$ and $\lambda \propto a$ this expression reduces to

$$\ddot{a} > 0. \tag{4.2}$$

Thus in order to have a causal connection between the primordial perturbations which are essentially responsible for the generation of the anisotropies in the CMB, the universe needs to have a phase of accelerated expansion during the early part of the radiation dominated epoch.

In Figure 4.1, the physical wavelength for different modes and the Hubble radius during the inflation and the radiation era are plotted against the scale factor a. Both the axes are in log scale. As mentioned previously, in the power law expansion the scale factor a goes as t^f , and during inflation $f > 1$. The physical wavelength $\lambda_p \propto a$ in all the epochs, whereas d_H behaves as $a^{\frac{1}{f}}$ and a^2 during the inflation and the radiation dominated epoch respectively. There the straight lines of the physical wavelength always have unit slope while the Hubble radius d_H is less than unity and 2 for the inflationary and the radiation dominated eras respectively. Thus it it evident from the Figure 4.1, that λ_p leaves the Hubble radius in the early phase of the radiation dominated phase and will only be inside the Hubble radius in the early phase if there is a period of inflation.

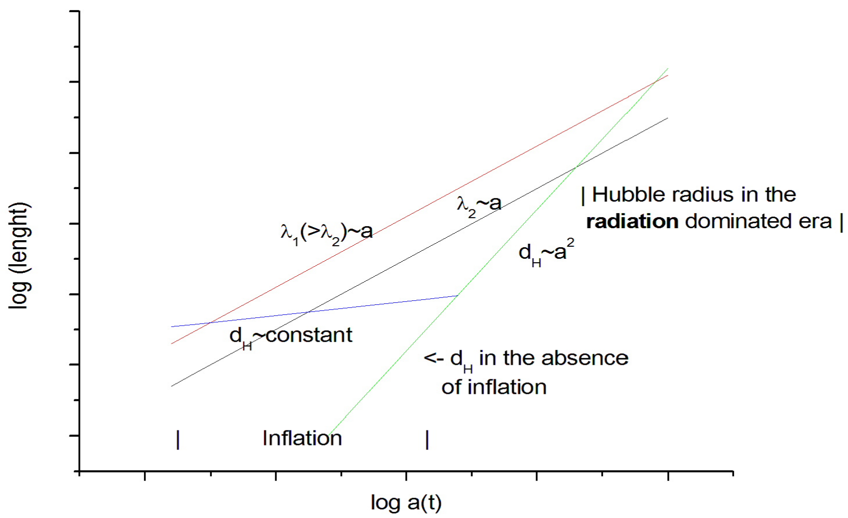

Figure 4.1: The evolution of the Hubble radius d_H and the physical wavelength λ_1 and λ_2 is plotted as the function of the scale factor a on the logarithmic scale for inflationary and the radiation dominated epochs. As discussed in the text the physical modes enters the Hubble radius during inflation as the slope of the Hubble length is much less than unity during inflation. Thus inflation helps to bring the modes into causal contact at early times

4.2 Driving inflation with scalar fields

The Einstein equation corresponding to the line element Eq. (2.10) and the stress energy tensor Eq. (2.13) results in the following Friedmann equations

$$H^2 = \left(\frac{8\pi G}{3}\right)\rho, \tag{4.3}$$

$$\left(\frac{\ddot{a}}{a}\right) = -\left(\frac{4\pi G}{3}\right)(\rho + 3p). \tag{4.4}$$

These equations are obtained from Eq. (2.15) and Eq. (2.16), by making use of the relation $H = (\dot{a}/a)$ and the condition that the Friedmann universe is flat, i.e. $\kappa = 0$. In order to satisfy condition (4.2), $(\rho + 3p)$ must be negative. However ordinary matter, for which $p = 0$ and for radiation $p = \rho/3$, the condition for inflation is not achievable. Thus we need to look for alternative sources in order to achieve inflation.

We will go on to consider a scalar field ϕ called the inflaton, which is governed by the action [5]

$$S[\phi] = \int d^4x\sqrt{-g}L = \int d^4x\sqrt{-g}\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right]. \tag{4.5}$$

The symmetries of the Friedmann background suggest both homogeneity and isotropy which implies that the scalar field ϕ is independent of the spacial coordinates. Using the Euler Lagrangian equation

$$\partial_\mu\frac{\partial(\sqrt{-g}L)}{\partial\partial_\mu} - \frac{\partial(\sqrt{-g}L)}{\partial\phi} = 0, \tag{4.6}$$

the equation of motion of the scalar field ϕ in the Friedmann background can be obtained to be

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = 0, \tag{4.7}$$

where $V_\phi = dV/d\phi$, and in the FRW metric $\sqrt{-g} = a^3$. Further, the general expression for the stress energy tensor is [10]

$$T^\mu_\nu = \partial^\mu\phi\partial_\nu\phi - \delta^\mu_\nu\left[\frac{1}{2}\partial_\lambda\phi\partial^\lambda\phi - V(\phi)\right]. \tag{4.8}$$

Using the symmetry condition of the FRW background, the energy density and the pressure associated with the scalar field are defined as $T^0_0 = \rho$ and $T^i_j = -p\delta^i_j$ and thus we obtain [12]

$$\rho = \left[\frac{\dot{\phi}^2}{2} + V(\phi)\right], \tag{4.9}$$

$$p = \frac{\dot{\phi}^2}{2} - V(\phi). \tag{4.10}$$

Using the expression of the energy density and pressure the condition for the inflation, viz. $(\rho + 3p)$ reduces to

$$\dot{\phi}^2 < V(\phi). \tag{4.11}$$

This is precisely the condition, where the potential energy of the scalar field dominates the kinetic energy, to achieve inflation. Using the expression (4.9) and (4.10) the Friedmann equations can be rewritten in terms of the scalar field ϕ and its corresponding potential $V(\phi)$

$$H^2 = \frac{1}{3M_p^2}\left[\frac{\dot{\phi}^2}{2} + V(\phi)\right], \tag{4.12}$$

$$\dot{H} = -\frac{1}{2M_p^2}\dot{\phi}^2, \tag{4.13}$$

where we have set $M_p^2 = 1/8\pi G$. The above equations can be further simplified to express the scalar field ϕ and the potential $V(\phi)$ as a function of cosmic time as follows [5, 3]

$$\phi(t) = \sqrt{2}M_p \int dt\sqrt{(-\dot{H})}, \tag{4.14}$$

$$V(t) = M_p^2(3H^2 + \dot{H}). \tag{4.15}$$

4.3 Slow roll inflation

It is seen previously that the condition for the inflationary universe model is based upon the possibility of slow evolution of some scalar field ϕ in a given potential $V(\phi)$, given by the Eq. (4.11). In the present literature two different types of slow-roll-approximation exists. The first form called the potential slow roll (PSR) approximation which places restriction on the form of the potential and requires the additional condition of slow evolution of the scalar field along the attractor solution. The other form, known as the Hubble slow-roll parameters (HSR) places conditions on the evolution of the Hubble parameter during inflation. The (HSR) parameters have distinct advantage over the PSR parameters. In the following sections, we shall go onto discuss the Hamilton-Jacobi formulation of inflation and then discuss both PSR and HSR approximations in some detail.

4.3.1 The equation of motion

Here we derive a very useful alternative form of Eq. (4.12). We rewrite Eq. (4.13) in a slightly different form, as we change the time derivative of the Hubble parameter into derivative with respect to the scalar field ϕ and arrive at the equation

$$H_\phi = -\frac{\dot{\phi}}{2M_p^2}. \tag{4.16}$$

Substituting this expression of $\dot{\phi}$ in Eq. (4.12) we arrive at the so called Hamilton-Jacobi equation for inflation [13], viz.

$$H_\phi^2 = \frac{3H^2}{2M_p^2} - \frac{V}{2M_p^4}. \tag{4.17}$$

4.3.2 The potential slow roll parameters

For a given potential $V(\phi)$, the slow roll condition can be realised with the following condition on these two dimensionless parameters:

$$\epsilon_v \ll 1 \qquad\qquad and \qquad\qquad \eta_v \ll 1, \tag{4.18}$$

where the quantity ϵ_v and η_v are referred to as the Potential Slow Roll (PSR) parameters [13, 5] and are defined as

$$\epsilon_v = \frac{M_p^2}{2}\left(\frac{V_\phi}{V}\right)^2, \tag{4.19}$$

$$\eta_v = M_p^2\left(\frac{V_{\phi\phi}}{V}\right), \tag{4.20}$$

and $V_{\phi\phi} = (d^2V/d\phi^2)$. The smallness of these PSR parameters are equivalent to neglecting the kinetic energy term in the Friedmann equation (4.12) and the acceleration term in the equation of motion of the scalar field (4.7). The PSR parameters only restrict the form of the potential V but not the properties of the dynamical solutions. The smallness of ϵ_v and η_v does not restrict the value of the $\dot{\phi}$, which essentially governs the size of the kinetic energy term. Thus is addition to the smallness of the PSR parameters these require the additional condition that the scalar field evolves slowly along the attractor solution, determined by the equation

$$\dot{\phi} \simeq \frac{-V_\phi}{3H}. \tag{4.21}$$

4.3.3 The Hubble slow roll parameters

If $H(\phi)$ is considered as the primary quantity, then the Hubble Slow Roll (HSR) parameters [13, 5] are a better choice, than the PSR parameters, to describe the slow roll approximation, as they do not require any additional condition. The dimensionless HSR parameters ϵ_H and η_H are defined as follows:

$$\epsilon_H = 2M_p^2 \left(\frac{H_\phi}{H}\right)^2, \tag{4.22}$$

$$\eta_H = 2M_p^2 \left(\frac{H_{\phi\phi}}{H}\right). \tag{4.23}$$

where $H_{\phi\phi} = d^2H/d\phi^2$. By using Eqs. (4.3) and (4.16) these two relations can be expressed as

$$\epsilon_H = \frac{6M_p^4 H_\phi^2}{\rho} = \left(\frac{3\dot{\phi}^2}{2\rho}\right) = -\left(\frac{\mathrm{d}\ln H}{d\ln a}\right), \tag{4.24}$$

$$\eta_H = -\left(\frac{\ddot{\phi}}{H\dot{\phi}}\right) = -\left(\frac{\mathrm{d}\ln H'}{d\ln a}\right). \tag{4.25}$$

The condition that $\epsilon_H \ll 1$ is exactly the the condition required to ignore the kinetic energy term in the total energy of the scalar field and the second condition. The condition $\eta_H \ll 1$ corresponds to neglecting the acceleration term in the equation of motion of the scalar field (4.7).

Finally the inflationary condition is $\ddot{a} > 0$ is precisely satisfied by $\epsilon_H < 1$. Thus it is evident that all the necessary dynamical information is expressed in the HSR parameters and it need not be supplemented by additional assumptions.

4.3.4 Solution in the slow roll approximation

The equation of motion (4.7) of the scalar field ϕ can be rewritten as

$$3H\dot{\phi}\left(1 - \frac{\ddot{\phi}}{3H\dot{\phi}}\right) = -V_\phi. \tag{4.26}$$

Upon using Eq. (4.24) in the above expression, we get

$$3H\dot{\phi}\left(1 - \frac{\eta_H}{3}\right) = -V_\phi. \tag{4.27}$$

Now the smallness of the HSR parameter $\eta_H \ll 1$, reduces this equation, at the leading order, to

$$3H\dot{\phi} \simeq -V_{\phi}. \tag{4.28}$$

Similarly the Friedmann Eq. (4.3) can be written as

$$H^2\left[1-\frac{1}{3}\left(\frac{3\dot{\phi}^2}{2\rho}\right)\right] = \frac{V(\phi)}{3M_p^2}, \tag{4.29}$$

and using Eq. (4.24) in this expression we arrive at

$$H^2\left[1-\frac{1}{3}\epsilon_H\right] = \frac{V(\phi)}{3M_p^2}. \tag{4.30}$$

Again, the smallness of the other HSR parameter $\epsilon_H \ll 1$, reduces this Friedmann equation, at the leading order, to

$$H^2 \simeq \frac{V(\phi)}{3M_p^2}. \tag{4.31}$$

In what follows, we shall discuss the solution of the above differential equations to obtain the expression for the evolution of the scalar field in the slow roll limit [12, 5]. We introduce the large field model which is defined as

$$V(\phi) = V_o\phi^n \tag{4.32}$$

here V_o is a constant and $V(\phi)$ is positive for all values of n as we consider only positive scalar field ϕ. This is called the large field model because the conditions for the slow roll inflation (4.18) are satisfied only when the value of the field ϕ is much larger than M_p. We will now obtain the solution to the scalar field in this limit for the large field model. Substituting V_ϕ, when $n \neq 4$, in the differential Eq. (4.28) and using Eq. (4.31) we get

$$\dot{\phi} = -\frac{\sqrt{V}_o\phi^{\frac{n}{2}-1}M_p}{\sqrt{3}} \tag{4.33}$$

On integrating this equation with the condition, that at initial time t_i the value of the scalar field is ϕ_i, we arrive at

$$\phi^{\left(2-\frac{n}{2}\right)}(t) \simeq \phi_i^{\left(2-\frac{n}{2}\right)} + M_p(t-t_i)\sqrt{\frac{V_o}{3}}\left[\frac{n(n-4)}{2}\right]. \tag{4.34}$$

Similarly solving the differential Eq. (4.28) and using Eq. (4.31), but this time for $n = 4$, we arrive at

$$\dot{\phi} = -\frac{\sqrt{V}_o \phi 4 M_p}{\sqrt{3}}. \tag{4.35}$$

Again integrating with the same initial conditions, one finds

$$\phi(t) \simeq \phi_i exp\left[-\sqrt{\frac{V_o}{3}}(4M_p)(t - t_i)\right]. \tag{4.36}$$

In the slow roll approximation, the number of e-folds in the inflationary regime can be expressed as

$$N = \ln\left(\frac{a}{a_i}\right) \simeq -\left(\frac{1}{M_p^2}\right)\int_{\phi_i}^{\phi} d\phi \left(\frac{V}{V_\phi}\right), \tag{4.37}$$

where we have made use of the relation obtained by dividing Eq. (4.31) by Eq. (4.28). For the large field model (4.32) the evolution of the scalar field is obtained in terms of the e-folds to be:

$$\phi^2(N) - \phi_i^2(N) = -(2M_p^2 n)N \tag{4.38}$$

4.4 The slow roll attractor

Many inflationary models permits attractor solution. In this section we will illustrate this phenomenon with a specific example. For a scale factor a the amount of expansion during a small interval of time, dt, can be expressed in terms of e-folds and can be defined as

$$dN = Hdt \tag{4.39}$$

With the above relation one can show that, $\dot{\phi} = H\phi_N$. Further $\ddot{\phi} = HH_N\phi + H^2\phi_{NN}$.

Replacing the above relations in the equation of motion for the scalar fields ϕ (4.7) we get:

$$\phi_{NN} + \left(\frac{H_N}{H}\right)\phi_N + 3\phi_N + \left(\frac{V_\phi}{H^2}\right), \tag{4.40}$$

Using Friedmann Eq. (4.12) we get

$$H^2\left(3 - \frac{\phi_N^2}{2}\right) = V. \tag{4.41}$$

and

$$\frac{H_N}{H} = -\frac{\phi_N^2}{2}. \tag{4.42}$$

Substituting these expressions in the Eq. (4.40), we arrive at the equation of motion for the scalar field to be

$$\phi_{NN} + \left(3 + \frac{\phi_N^2}{2}\right)\left(\frac{V_\phi}{V} + \phi_N\right) = 0. \tag{4.43}$$

Now for $V = \frac{1}{2}m^2\phi^2$ the above equation reduces to

$$\phi_{NN} + \left(3 + \frac{\phi_N^2}{2}\right)\left(\frac{2}{\phi} + \phi_N\right) = 0. \tag{4.44}$$

The phase diagram, arrived at numerically, is presented in the Figure 4.2

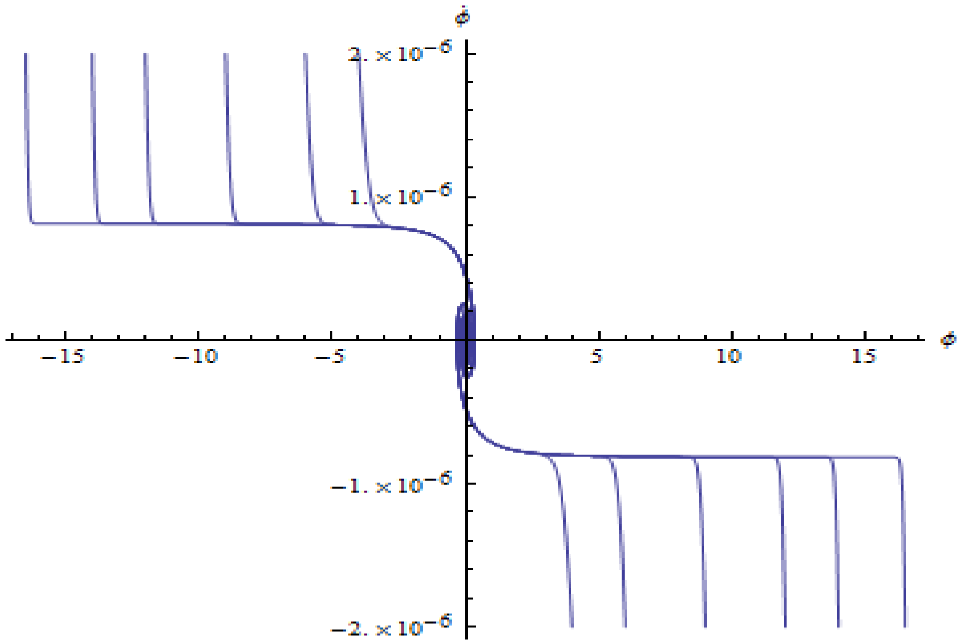

Figure 4.2: The Figure represents the phase space diagram of the inflaton field ϕ for the potential $V = \frac{1}{2}m\phi^2$ .The field oscillates around the origin and finally goes to zero.The plots are generated for different initial conditions

Chapter 5

Cosmological perturbation theory

5.1 Introduction to perturbations

Cosmological principle assumes that the universe is homogeneous and isotropic. But a quick glance around us clearly suggests that this is not really true. However, CMB data indicates deviations from isotropy and homogeneity are small, i.e. 1 part in 10^5, in the early stages of the universe. Thus we shall use perturbative techniques to understand the generation and evolution of such anisotropies in the early universe.

To begin our discussion we redefine the background metric [12] by introducing perturbations

$$g_{\mu\nu} = g^{(0)}_{\mu\nu}(t) + \epsilon g^{(1)}_{\mu\nu}(t,x) + \epsilon^2 g^{(2)}_{\mu\nu}(t,x) + ... \tag{5.1}$$

Here $g^{(o)}_{\mu\nu}(t)$ is the standard background FRW metric and $g^{(1)}_{\mu\nu}(t,x)$ is the 1st order perturbed metric. The quantity ϵ is just a bookkeeping parameter to specify the order of perturbation. The dependence of $g^{(1)}_{\mu\nu}(t,x)$ on spatial coordinate shows the deviation from the cosmological principle. Now, since the amplitude of inhomogeneity is small in the early epoch, perturbation is considered only in the linear order. The metric perturbation can be decomposed based on the local rotation of the spacial coordinates on the hypersurface of constant time. Thus perturbations can be classified as scalars, vectors and tensors [10].

The vector perturbations are generated from rotational velocity fields and are essentially spin 1 modes. Tensor perturbations corresponds to gravitational waves and can exist even in vacuum. Finally, the scalar perturbations are essentially responsible for the generation of anisotropies and inhomogeneities in the universe. These remain invariant under rotations and thus have zero spin.

5.1.1 Decomposition theorem

It is evident that the perturbed metric can be decomposed into different types of perturbations. In order to determine the evolution of the perturbed quantities we need to take into account the perturbed Einstein equation and equate both sides of same order. Therefore, we may expand the Einstein tensor and the stress energy tensor as [12]

$$G_{\mu\nu} = G^{(0)}_{\mu\nu} + \epsilon G^{(1)}_{\mu\nu} \qquad T_{\mu\nu} = T^{(0)}_{\mu\nu} + \epsilon T^{(1)}_{\mu\nu}. \tag{5.2}$$

Then we may identify the terms of the same order as follows

$$G^{(0)}_{\mu\nu} = 8\pi G T^{(0)}_{\mu\nu} \qquad G^{(1)}_{\mu\nu} = 8\pi G T^{(1)}_{\mu\nu}. \tag{5.3}$$

The source of the metric perturbation, i.e. stress-energy tensor can be classified into scalar, vector and tensor perturbations as well. For example, the perturbed inflaton $\delta\phi$ are scalar sources but the velocity fields with vortices are the vector sources. The decomposition theorem states that at the linear order the different types of the metric perturbations are affected only by the source of same type. The scalar, vector, tensor metric perturbations are affected by scalar, vector and tensor sources, respectively, and thus these perturbations can be studied independently.

5.1.2 The number of independent degrees of freedom

The perturbed metric tensor $g^{1}_{\mu\nu}(t,x)$ can be decomposed as [5]

$$g^{(1)}_{\mu\nu}(t,x) = g^{(1)}_{00} + g^{(1)}_{0i} + g^{(1)}_{ij}. \tag{5.4}$$

Here $g^{(1)}_{00}$ is scalar and can be termed as A. According to the Helmholtz theorem any vector v_i can be decomposed as gradient of a scalar and a divergence free vector, $v_i = \partial_i v + u_i$ where u_i is a divergence free vector, i.e. $\partial_i u^i = 0$ and v is some scalar. In the similar fashion $g^{(1)}_{0i}$ can be decomposed as

$$g^{(1)}_{0i} = \partial_i B + Q_i. \tag{5.5}$$

where B is a scalar and $\partial_i Q^i = 0$. Similarly the tensor perturbation of the spatial components of the metric tensor, viz. $g^{(1)}_{ij}$, can be decomposed as

$$g^{(1)}_{ij} = \psi\delta_{ij} + (\partial_i D_j + \partial_j D_i) + \left[\left(\frac{1}{2}\right)(\partial_i\partial_j +_j \partial_i) - \left(\frac{1}{3}\right)\delta_{ij}\partial^k\partial^k\right] E + \mathcal{H}_{ij}, \tag{5.6}$$

where once again D_i is a divergence free vector and and E is a scalar. The function $\mathcal{H}_{ij}$ is a symmetric tensor satisfying the conditions, $\delta^i_j H^j_i = 0$, $\partial^j \mathcal{H}_{ij} = 0$.

Thus it is evident that the decomposition of the perturbed metric results in four unknown scalar function: A, B, ψ and E. It also depends on two divergence free vectors Q_i and D_i and a transverse and traceless tensor $\mathcal{H}_{ij}$.

Having decomposed the metric we will now calculate the degrees of freedom of the perturbed symmetric metric tensor $g^{(1)}_{\mu\nu}(t, x)$. If we generalise this discussion for $(N + 1)$ spacetime dimension then we find that the two divergence free spatial vectors Q_i and D_i, each having N spatial degrees of freedom, have $[2(N - 1)]$ degrees. The four scalars mentioned above add up to 4 degrees of freedom. Lastly the tensor $\mathcal{H}_{ij}$, which being symmetric has $N(N + 1)/2$ degrees of freedom, but on imposing the traceless (which corresponds to $1 constraint$) and the transverse condition ($N constraints$) the independent degrees reduce to

$$\frac{N(N+1)}{2} - (N+1) = \frac{(N+1)(N-2)}{2}. \tag{5.7}$$

Upon adding degrees of all the individual components we get [5]

$$4 + 2(N-1) + \frac{(N+1)(N-2)}{2} = \frac{(N+1)(N+2)}{2} \tag{5.8}$$

which is essentially the degrees of freedom associated with the perturbed metric $g^{(1)}_{\mu\nu}(t, x)$ in $(N + 1)$ spacetime dimensions.

Let us now understand the degrees of freedom associated with the coordinate transformations. For the perturbed metric the $(N + 1)$ coordinate transformation can be explicitly expressed in terms of scalars, say, δt and δx as [5]

$$t \to t + \delta t \quad and \quad x^i \to x^i + \partial^i \delta x. \tag{5.9}$$

In the same manner there could a coordinate transformation in terms of a divergence free vector δx^i as

$$t \to t \quad and \quad x^i \to x^i + \delta x^i. \tag{5.10}$$

Again, let us count the number of independent degrees of freedom associated with the perturbed metric tensor, but this time incorporating the coordinate degrees of freedom and subtracting them from the total number of degrees. Along with the four independent scalar functions A, B, ψ and E in the perturbed metric there exist two scalar degrees associated with the coordinate transformation. Thus the effective scalar degree reduces to $(4-2)=2$. Also, the divergence free vector δx^i introduced through the coordinated transformation Eq. (5.10) has $(N-1)$ degrees. This, when subtracted from the $2(N-1)$ degrees of the two divergence free spacial vectors Q_i and D_i of the perturbed metric, leads to $(N-1)$ true degrees of freedom. Adding to this the number of degrees of the tensor perturbation Eq. (5.7) we obtain

$$2+(N-1)+\frac{(N+1)(N-2)}{2}=\frac{N(N+1)}{2} \tag{5.11}$$

which is actually the number of degrees describing the perturbed metric.

5.2 Scalar perturbations

In order to derive the equations governing the evolution of each of the perturbations it is convenient to work in a particular gauge. In the remaining section we will work in the longitudinal gauge which essentially corresponds to $A \propto \Phi$, $\psi \propto \Psi$ and the other two scalar are set to zero, i.e. $B=E=0$. In this particular gauge, the Friedmaan line element is given by [10]

$$ds^2=(1+2\Phi)dt^2-a^2(t)(1-2\Psi)\delta_{ij}dx^i dx^j. \tag{5.12}$$

We can rewrite the metric in terms of the conformal time coordinate as

$$ds^2=a(\eta)^2[(1+2\Phi)d\eta^2-(1-2\Psi)\delta_{ij}dx^i dx^j]. \tag{5.13}$$

The perturbed metric can be represented as

$$g^{(1)}_{\mu\nu}=a^2\begin{bmatrix}(1+2\Phi) & 0\\ 0 & -(1-2\Psi)\delta_{ij}\end{bmatrix}. \tag{5.14}$$

Now in order to obtain the inverse metric ie $g^{\mu\nu}$ at the linear order we need to solve the equation

$$(g^{(0)\mu\beta}+g^{(1)\mu\beta})(g^{(0)}_{\nu\beta}+g^{(1)}_{\nu\beta})=\delta^{\mu}_{\nu}. \tag{5.15}$$

where $g^{(0)\mu\beta}$ is the unperturbed Friedmann metric:

$$g^{(0)\mu\nu}=\frac{1}{a^2}\begin{bmatrix}1 & 0\\ 0 & -\delta_{ij}\end{bmatrix}. \tag{5.16}$$

Substituting this is in Eq. (5.15) and restricting oneself to the linear order, we finally arrive at the perturbed metric tensor to be

$$g^{(1)\mu\nu} = \frac{1}{a^2}\begin{bmatrix}(1-2\Phi) & 0 \\ 0 & -(1+2\Psi)\delta_{ij}\end{bmatrix}. \tag{5.17}$$

In order to arrive at the Einstein equation we shall evaluate the perturbed components of the Ricci scalar and the scalar curvature, at the linear order, to obtain the expression of the perturbed Einstein tensor, viz. $G^{(1)\mu}_{\ \ \nu}$ [8, 10] The perturbed Ricci scalars, considering only the first order variation is given by

$$R^{(1)}_{00} = 3\frac{a'}{a}\Psi' + 3\frac{a'}{a}\Phi' + \partial^i\partial_i\Phi + 3\Psi'', \tag{5.18}$$

$$R^{(1)}_{0i} = 2\frac{a'}{a}\partial_i\Phi + 2\partial_i\Psi', \tag{5.19}$$

$$\begin{aligned} R^{(1)}_{ij} = &\left[-\frac{a'}{a}\Phi' - 5\frac{a'}{a}\Psi' - 2\frac{a''}{a}\Phi - 2\left(\frac{a'}{a}\right)^2\Phi - 2\frac{a''}{a}\Psi \right.\\ &\left. -2\left(\frac{a'}{a}\right)^2\Psi - \Psi'' + \partial^k\partial_k\Psi\right]\delta_{ij} + \partial_i\partial_j\Psi - \partial_i\partial_j\Phi. \end{aligned} \tag{5.20}$$

The background values of the Ricci scalars are

$$R^{(0)}_{00} = -3\frac{a''}{a} + 3\left(\frac{a'}{a}\right)^2, \tag{5.21}$$

$$R^{(0)}_{ij} = \left[\frac{a''}{a} + \left(\frac{a'}{a}\right)^2\right]\delta_{ij}. \tag{5.22}$$

The perturbed value of the scalar curvature is

$$R^{(1)} = \frac{1}{a^2}\left(6\Psi'' + \partial^i\partial_i\Phi + 6 - \frac{a'}{a}\Phi' + 18\frac{a'}{a}\Psi' + 12\frac{a''}{a}\Phi - \partial^i\partial_i\Psi\right). \tag{5.23}$$

The background value of the scalar curvature is

$$R^{(0)} = -\frac{6}{a^2}\frac{a''}{a}. \tag{5.24}$$

Finally, the Einstein tensor can be evaluated at the first order in the perturbations to be

$$G^{(1)0}_{\ \ \ 0} = -\frac{6}{a^2}\mathcal{H}\left(\mathcal{H}\Phi + \Psi'\right) + \left(\frac{2}{a^2}\right)\partial_i\partial^i\Psi, \tag{5.25}$$

$$G^{(1)0}_{\ \ i} = \frac{2}{a^2}\partial_i\left(\mathcal{H}\Phi + \Psi'\right), \tag{5.26}$$

$$\begin{aligned} G^{(1)i}_{\ \ j} = &-\left(\frac{2}{a^2}\right)\left[\Psi'' + \mathcal{H}(2\Psi' + \Phi') + (2\mathcal{H}' + \mathcal{H}^2)\Phi \right. \\ &\left. + \partial_i\partial^i(\Phi - \Psi)\right]\delta^i_j + \frac{1}{a^2}\partial^i\partial_j(\Phi - \Psi), \end{aligned} \tag{5.27}$$

where $\mathcal{H} = a'/a$.

5.2.1 The perturbed stress energy tensor

From the previous section we have seen that the stress energy tensor for the inflaton field ϕ is given by the expression Eq. (4.8). The corresponding expressions for the background are also given by Eqs. (4.9) and (4.10)

The perturbed stress energy can be expressed as

$$\begin{aligned} T^{(1)}_{\mu\nu} = &\ \partial_\mu\delta\phi\partial_\nu\phi + \partial_\nu\delta\phi\partial_\mu\phi - \delta g_{\mu\nu}\left[\left(\frac{1}{2}\right)g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi)\right] \\ &- g_{\mu\nu}\left[\left(\frac{1}{2}\right)\delta g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + g^{\alpha\beta}\partial_\alpha\delta\phi\partial_\beta\phi - V_\phi(\phi)\delta\phi\right]. \end{aligned} \tag{5.28}$$

In the following section, we shall consider scalar fields and perfect fluids as the only source of perturbation. The perturbed stress energy tensor associated with the scalar field ϕ for the metric Eq (5.13) can be represented as

$$T^{(1)0}_{\ \ 0} = \frac{1}{a^2}(\phi'\delta'\phi + a^2V_\phi\delta\phi - \phi'^2\Phi) = \delta\rho, \tag{5.29}$$

$$T^{(1)0}_{\ \ i} = \frac{1}{a^2}\partial_i(\delta\phi\phi') = \partial_i\delta\sigma, \tag{5.30}$$

$$T^{(1)i}_{\ \ j} = -\frac{1}{a^2}(\phi'\delta'\phi - a^2V_\phi\delta\phi - \phi'^2\Phi)\delta^i_j = -\delta p\delta^i_j. \tag{5.31}$$

Here we have arrived at the mixed form of the stress energy tensor for convenience. The quantities $\delta\rho$, $\delta\sigma$ and δp are the scalar quantities representing the perturbation in the energy density, the momentum flux , and the pressure, respectively.

5.2.2 Equation of motion

The scalar field ϕ does not possess any anisotropic stress. Hence in such a case we have, $\Phi = \Psi$. Upon imposing this condition on the Einstein tensor, $G^{(1)i}_{\ \ j}$, we arrive at the corresponding Einstein's equation, viz. $G^{(1)\mu}_{\ \ \nu} = 8\pi G T^{(1)\mu}_{\ \ \nu}$, which subsequently leads to the equation governing the perturbation. The three Einstein equations can be expressed in terms of the scalar quantities as

$$-\left(\frac{3}{a^2}\right)\mathcal{H}\left(\mathcal{H}\Phi+\Phi'\right)+\left(\frac{1}{a^2}\right)_i\partial^i\Phi=4\pi G\delta\rho, \tag{5.32}$$

$$\left(\frac{1}{a^2}\right)\partial_i\left(\mathcal{H}\Phi+\Phi'\right)=4\pi G(\partial_i\delta\sigma), \tag{5.33}$$

$$\left(\frac{1}{a^2}\right)\left[\Phi''+3\mathcal{H}\Phi'+(2\mathcal{H}'+\mathcal{H}^2)\Phi\right]=4\pi G\delta p. \tag{5.34}$$

Eqs. (5.32) and (5.34) can be combined to arrive at the following equation governing the evolution of the Bardeen potential Φ

$$\Phi''+3\mathcal{H}(1+c_A^2)\Phi'-c_A^2\partial_i\partial^i\Phi+[2\mathcal{H}'+(1+3c_A^2)\mathcal{H}^2]\Phi=(4\pi Ga^2)\delta p^{NA} \tag{5.35}$$

where we have made use of the standard expression

$$\delta p-\delta p^{NA}=c_A^2\delta\rho \tag{5.36}$$

Here δp^{NA} is the non adiabatic pressure perturbation while $c_A^2\equiv p'/\rho'$ represents the adiabatic speed of the perturbation.

5.2.3 Curvature perturbation at super-Hubble scales

At super-Hubble scales the physical wavelengths associated with the perturbations are much larger than the Hubble radius, which implies that $k/\mathcal{H}\ll 1$. Here k refers to the wavenumber of the Fourier modes of the perturbations. We introduce a quantity referred as the curvature perturbation which is a function of the Bardeen potential and its time derivative [15, 14]:

$$\mathcal{R}=\Phi+\left(\frac{2}{3}\right)\frac{\mathcal{H}^{-1}\Phi'+\Phi}{(1+w)} \tag{5.37}$$

where $w=p/\rho$.

Eq. (5.35) can be rewritten in terms of the curvature perturbation defined above. We make use of the background Friedmann equations (2.15) and (2.16) so as to arrive at [5]

$$\mathcal{R}_k'=\left(\frac{\mathcal{H}}{\mathcal{H}^2-\mathcal{H}'}\right)\left[(4\pi Ga^2)\delta p^{NA}-c_a^2k^2\Phi_k\right] \tag{5.38}$$

As stated earlier, on the super-Hubble scales $k/\mathcal{H} \ll 1$, and thus the $c_a^2 k^2 \Phi_k$ term in the above expression can be ignored. Further in the absence of non-adiabatic pressure perturbation, i.e. $\delta p^{NA} = 0$, Eq (5.38) reduces to

$$\mathcal{R}_k' = 0. \tag{5.39}$$

Thus the curvature perturbation $\mathcal{R}_k$ is conserved in the super-Hubble scales when the modes are outside the Hubble radius and in the absence of non-adiabatic perturbation.

5.2.4 Equation of motion for the curvature perturbation

The perturbed Einstein equations, viz. Eqs. (5.32), (5.33) and (5.34) can be rewritten using the expressions of the perturbed stress energy tensor Eqs. (5.29), (5.30) and (5.31) as

$$-\left(\frac{3}{a^2}\right)\mathcal{H}\left(\mathcal{H}\Phi + \Phi'\right) + \left(\frac{1}{a^2}\right)_i \partial^i \Phi = \frac{1}{a^2}(\phi'\delta'\phi + a^2 V_\phi \delta\phi - \phi'^2\Phi), \tag{5.40}$$

$$\left(\frac{1}{a^2}\right)\partial_i\left(\mathcal{H}\Phi + \Phi'\right) = \frac{1}{a^2}\partial_i(\delta\phi\phi'), \tag{5.41}$$

$$\left(\frac{1}{a^2}\right)\left[\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi\right] = -\frac{1}{a^2}(\phi'\delta'\phi - a^2 V_\phi\delta\phi - \phi'^2\Phi). \tag{5.42}$$

For convenience, let us express the equation of motion of the scalar field Eq. (4.7),and the energy Eq. (4.9) and pressure density Eq. (4.9) during inflation in terms of the conformal time coordinates as follows

$$\phi'' + 2\mathcal{H}\phi' + a^2 V_\phi = 0, \tag{5.43}$$

$$\rho = \frac{\phi'^2}{2a^2} + V(\phi), \tag{5.44}$$

$$p = \frac{\phi'^2}{2a^2} - V(\phi). \tag{5.45}$$

We then simplify the perturbed Einstein equations using the above expressions to arrive at the following equation for the Bardeen potential:

$$\Phi'' + 3\mathcal{H}(1 + c_A^2)\Phi' - c_A^2\partial_i\partial^i\Phi + [2\mathcal{H}' + (1 + 3c_A^2)\mathcal{H}^2]\Phi = (1 - c_A^2)\partial_i\partial^i\Phi. \tag{5.46}$$

On comparing the this equation with the more general expression for the evolution of the Bardeen potential Eq. (5.35) we arrive at the relation of the inflaton with the non-adiabatic pressure perturbation

$$\delta p^{NA} = (1 - c_A^2)\left(\frac{\partial_i\partial^i\Phi}{4\pi G a^2}\right). \tag{5.47}$$

With this condition, Eq. (5.38) can be simplified to

$$\mathcal{R}_k' = \left(\frac{\mathcal{H}}{\mathcal{H}^2 - \mathcal{H}'}\right) k^2 \Phi_k. \tag{5.48}$$

Now we introduce a quantity z as

$$z = \left(\frac{a\phi'}{\mathcal{H}}\right). \tag{5.49}$$

On differentiating Eq (5.48) with respect to the conformal time and then simplifying using the background Friedmann equations (2.15) and (2.16), the Bardeen Eq. (5.46) and the defined quantity z one arrives at the following equation of motion describing the evolution of the Fourier modes of the curvature perturbation, due to the scalar field ϕ :

$$\mathcal{R}_k'' + 2\left(\frac{z'}{z}\right)\mathcal{R}_k' + k^2 \mathcal{R}_k = 0. \tag{5.50}$$

Further on introducing the so called Mukhanov-Sasaki variable v, which is defined as $v = \mathcal{R}z$, the above equation reduces to the following differential equation for the Fourier modes of the variable v_k ,

$$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0. \tag{5.51}$$

5.2.5 The Bardeen potential at super-Hubble scale

In this section, we intend to understand the evolution of the Bardeen potential Φ on the super-Hubble scales during the radiation and matter dominated epochs. We have seen that, in the absence of anisotropic stress the Bardeen potentials Φ and Ψ are equal. With this assumption we are able to arrive at the single equation (5.35) governing the evolution of the Bardeen potential. This equation can be modified to get a better physical understanding of the evolution at different epochs. Let us define the variables

$$u = \frac{a^2\theta}{\mathcal{H}}\Phi, \qquad \theta = \left(\frac{1}{a}\right)\left[\frac{\mathcal{H}}{\mathcal{H}^2 - \mathcal{H}'}\right]^{\frac{1}{2}}. \tag{5.52}$$

Differentiating both the expressions twice and after suitable manipulations the Bardeen equation (5.35), in the Fourier space, reduces to

$$u_k'' + \left(c_A^2 k^2 - \frac{\theta''}{\theta}\right) u_k = 0. \tag{5.53}$$

where we have assumed that the non adiabatic pressure perturbations is absent, i.e. $\delta p^{NA} = 0$. In the super-Hubble limit, i.e. when $k^2 \ll \theta''/\theta$, Eq. (5.53) will have two solutions as follows

$$u_k(\eta) \propto \theta(\eta) \quad and \quad u_k(\eta) \propto \theta(\eta)\int^{\eta}\frac{d\tau}{\theta^2(\tau)}. \tag{5.54}$$

Thus at the leading order the general expression is

$$u_k(\eta) \simeq A_g(k)\theta(\eta)\int^{\eta}\frac{d\tau}{\theta^2(\tau)} + A_d(k)\theta(\eta). \tag{5.55}$$

Here A_g and A$_d$ are arbitary constants associated with the growing and the decaying modes. These constants are functions of k and their values depend on the initial conditions imposed at the early epochs. From the above solution of u_k and using Eq (5.52), the expression for the Bardeen potential Φ_k is given by [12]

$$\Phi_k(\eta) \simeq A_g(k)\left(\frac{\mathcal{H}}{a^2}\right)\int^{\eta}\frac{d\tau}{\theta^2(\tau)} + A_d(k)\left(\frac{\mathcal{H}}{a^2}\right). \tag{5.56}$$

5.2.6 Evolution of Bardeen potential in power law expansion

Let us consider the power law expansion in the conformal time coordinates. It can be expressed as [5]

$$a(\eta) = a_0\eta^{(\beta+1)}, \tag{5.57}$$

where β is a constant defined as

$$\beta = \frac{1-2f}{f-1} \tag{5.58}$$

For the power law expansion, the variable θ reduces to

$$\theta = \frac{1}{a(\eta)}\sqrt{\frac{\beta+1}{\beta+2}}, \tag{5.59}$$

where we have made use of the relations $\mathcal{H} = (\beta+1)/\eta$. In order to determine the evolution of the Bardeen potential at super-Hubble scales, we use the above relations in Eq. (5.56) . We obtain that

$$\Phi_k(\eta) \simeq A_g(k)\frac{(\beta+2)}{(2\beta+3)} + A_d(k)\frac{\beta+1}{a_0^2\eta^{(2\beta+3)}} \tag{5.60}$$

This equation can be further simplified by using the state parameter w [3] defined as

$$w \equiv \frac{p}{\rho} \tag{5.61}$$

On simplifying the above equation using the background Friedmann Eq (2.15) and (2.16) and the power law expansion (5.57), one obtains

$$\beta = \left(\frac{1-3w}{1+3w}\right). \tag{5.62}$$

Replacing the value of β in terms of the equation of state parameter w , we get [5]

$$\Phi_k(\eta) \simeq A_g(k)\left[\frac{3(1+w)}{5+3w}\right] + A_d(k)\left[\frac{2}{(3w+1)a_o^2\eta^{(2\beta+3)}}\right]. \tag{5.63}$$

The first term is actually a constant and it represents the growing mode, whereas the second term is the decaying mode.

Let us now go on to analyse the evolution during different epochs. Inflation corresponds to condition $-\infty < \eta < 0$ and $\beta \leq -2$, with $\beta = -2$ being exponential inflation. Also, radiation and matter dominated eras correspond to $\beta = 0$ and $\beta = 2$, respectively. In these cases the super-Hubble limit represents the early time for which $\eta \to 0$. Therefore the quantity $\eta^{(2\beta+3)}$ tends to zero and the corresponding decaying mode becomes very large in the early times. Thus in order to have finite expression the decaying mode in the Eq (5.63) has to be neglected, so that, on super Hubble scales we have

$$\Phi_k(\eta) \simeq A_g(k)\left[\frac{3(1+w)}{5+3w}\right]. \tag{5.64}$$

It is evident this quantity is strictly zero for $w = -1$, which corresponds to the cosmological constant driven expansion. In other words metric perturbations are not produced by cosmological constant. On super-Hubble scales, Φ_k is a constant and thus from Eq (5.37) we can express the curvature perturbation $\mathcal{R}_k$ as

$$\mathcal{R}_k \simeq \left[\frac{5+3w}{3(1+w)}\right]\Phi_k \simeq A_g(k). \tag{5.65}$$

We have already seen that the curvature perturbation $\mathcal{R}_k$ is conserved at super-Hubble scales and Φ_k is a constant in power law expansion. The Bardeen potential during the matter and radiation dominated epochs are given by [10]

$$\Phi_k^M(\eta) \simeq A_g(k)\left[\frac{3(1+w_M)}{5+3w_M}\right] = \frac{3}{5}A_g(k), \tag{5.66}$$

$$\Phi_k^R(\eta) \simeq A_g(k)\left[\frac{3(1+w_R)}{5+3w_R}\right] = \frac{2}{3}A_g(k), \tag{5.67}$$

where we have made use of the fact that the state parameter during the matter and radiation dominated epochs are $w_M = 0$ and $w_R = \frac{1}{3}$. Thus, if we consider the transition of the Bardeen potential at the super-Hubble scales from the radiation to matter dominated era we see that the potential Φ changes by factor of $(9/10)$. In fact, the pattern of the anisotropies in the CMB and the large scale structures that we observe today, are essentially determined by the spectrum of the Bardeen potential when the modes enters Hubble radius during the matter and radiation dominated epochs.

5.3 Vector perturbations

This section we will arrive at the Einstein equations for the vector perturbation. We will choose a gauge, such that Q_i is zero and $D_i \propto \mathcal{D}_i$, so that the corresponding line element associated with the vector perturbation is given by [5]

$$ds^2 = dt^2 - a^2(t)(\delta_{ij} + (\partial_i \mathcal{D}_j + \partial_j \mathcal{D}_i))dx^i dx^j \tag{5.68}$$

For this metric, the Einstein tensor can be evaluated to be :

$$G^{(1)0}_{\ \ 0} = 0, \tag{5.69}$$

$$G^{(1)0}_{\ \ i} = \frac{1}{2}[\partial_k \partial^k \dot{\mathcal{D}}_i], \tag{5.70}$$

$$G^{(1)i}_{\ \ j} = -\frac{1}{2}\left[(\partial_i \ddot{\mathcal{D}}_j + \partial_j \ddot{\mathcal{D}}_i) + 3H(\partial_i \dot{\mathcal{D}}_j + \partial_j \dot{\mathcal{D}}_i)\right]. \tag{5.71}$$

Thus, it is evident that the non zero components of the Einstein tensor are equal to zero in absence of vector sources. This implies that the metric perturbation $\mathcal{D}_i$ is zero and hence vector perturbations are not generated in the absence of vorticity free sources.

5.4 Tensor perturbations

When the tensor perturbations are included, the FRW metric is described by the line element [5]

$$ds^2 = [dt^2 - a^2(t)(\delta_{ij} + h_{ij})dx^i dx^j] \tag{5.72}$$

where h_{ij} is a traceless, transverse and symmetric tensor and is proportional to $\mathcal{H}_{ij}$. The Einstein tensor can be evaluated corresponding to the above line element in the usual manner.

The component of the first order Einstein tensor are found to be

$$G^{(1)0}_{\ \ \ 0} = 0, \tag{5.73}$$

$$G^{(1)0}_{\ \ \ i} = 0, \tag{5.74}$$

$$G^{(1)i}_{\ \ \ j} = -\left(\frac{1}{2}\right)\left[(\ddot{h}_{ij} + 3H\dot{h}_{ij}) - \left(\frac{1}{a^2}\right)\partial_k\partial^k h_{ij}\right]. \tag{5.75}$$

Further, in the absence of anisotropic stresses the non zero Einstein equation can be expressed in the conformal time coordinate to arrive at the following differential equation of the gravitational waves of amplitude h:

$$h'' + 2\mathcal{H} - \partial_k\partial^k h = 0 \tag{5.76}$$

The two types of polarization associated with the gravitational waves are described by the two degrees corresponding to the traceless and transverse tensor h_{ij}.

Chapter 6

Summary

In this review, we have tried to understand the hot big bang model and the observational evidence that have led to the development of such a cosmological theory. However like every other model this too had many drawbacks which were analysed in detail in chapter 3. We then went on to introduce the inflationary paradigm to resolve the horizon problem associated with the hot big bang model. This precisely suggested that the universe needs to go through a phase of accelerated expansion in the early period of the radiation dominated era such that the primordial fluctuations are intrinsically causal. Following which we discussed driving inflation using scalar field ϕ . We had arrived at the condition $\dot{\phi}^2 < V(\phi)$ for inflation, which demands that the potential energy of the scalar field must dominate the kinetic energy in order to achieve inflation. However further analysis showed that this condition was necessary but suddenly not sufficient to guarantee inflation. The field ϕ needs to roll slowly to achieve a sufficient duration of inflation in order to resolve the horizon problem. This requires additional condition $\ddot{\phi} \ll 3H\dot{\phi}$. These conditions were precisely satisfied by the smallness HSR parameters which we discussed in detail in chapter 4.

Finally, in the fifth chapter, we had reviewed the cosmological perturbation theory. Based on the decomposition theorem, we had studied the scalar, vector and tensor perturbations independently. We had arrived at the equation of motion for the Bardeen potential and discussed its evolution at the super-Hubble scales.

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