Big Bang Nucleosynthesis

A project report
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Master of Science
in
Physics
by

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under the guidance of
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CERTIFICATE

This is to certify that the project titled **Big Bang Nucleosynthesis** is a bona fide record of work done by **Simran Singh** towards the partial fulfillment of the requirements of the Master of Science degree in Physics at the Indian Institute of Technology, Madras, Chennai 600036, India.

(L. Sriramkumar, Project supervisor)

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ABSTRACT

We live in a universe where the observed abundances of light elements like helium and deuterium cannot be explained by stellar processes. The synthesis of these light elements took place very early on, about a hundred seconds after the big bang. It is one of the greatest triumphs of cosmology to account for the present helium and deuterium abundances. Big bang nucleosythesis is a research area where a lot of theoretical and computational research has been done yielding experimentally verifiable results, and still research continues to improve accuracy of those results. The aim of this project was to firstly study the basics of cosmology required to understand the thermal history of the universe and secondly to build theoretical arguments to explain the presently observed helium abundance.

Contents

Chapter 1

Introduction

1.1 The hot big bang model

The discovery of the cosmic microwave background radiation (CMBR) in 1965 by Penzias and Wilson led to the acceptance of the big bang theory as the most likely theory of the origin of our universe. This radiation followed a black-body spectra with a temperature of 2.7 $^\circ$K. Since this radiation was isotropic (temperature measured in all directions was nearly the same), its origin could not have been pointed out to a single source, and it is thought to be a relic of the big bang. Early in the universe, a few seconds after the big bang, photons were in thermal equilibrium with muons, electrons etc and their anti-particles. Since photons were strongly interacting with these relativistic particles, the universe would have been opaque to optical radiation at that time. Only when the universe cooled to temperatures such that these interactions went out of equilibrium, would photons become free to propagate and the universe would become transparent. CMBR is thought to be that radiation, only red-shifted. Based on the relations between cosmological red-shift, the scale factor and temperature of this radiation, we can conclude that the universe was very hot at the time when radiation became free to propagate and was even hotter and denser before that.

The importance of giving this information about the hot big bang model is that nucleosynthesis took place in a radiation dominated era, in which the most important contribution to energy density came from photons and other relativistic particles.

1.2 Outline of big bang nucleosynthesis (BBN)

For the purpose of discussing nucleosynthesis, we can start with an initial temperature of $T \simeq 10^{12}\ ^{\circ}K$, which corresponds to an energy of ~86 MeV. Since this is very less than the rest mass of protons and neutrons, these species were already present during this time and were equally abundant. Equally abundant because the mass difference between them ($m_n - m_p = 1.293$ MeV) is too small compared to the temperature considered and hence energetics cannot favour protons over neutrons at this temperature.

Till the universe was about a hundred seconds old, neutrons and protons were in equilibrium via weak interactions ($p+e^- \rightleftharpoons n+\nu_e$ and $n+e^+ \rightleftharpoons p+\overline{\nu}_e$). The rates of these reactions are obtained numerically, but in some limit can be approximated as an analytic expression (see section 3.8). A little later these rates fell below the expansion rate of the universe and hence 'froze out'. This means that the abundance ratios became fixed as the reactions could no longer take place. But the abundance of neutron could not remain fixed because of the mass difference between proton and neutron and hence neutrons started beta-decaying to protons ($n \rightarrow p + e^- + \overline{\nu}_e$).

The only thing that could save neutrons was for them to combine with protons and form nuclei. A big problem called 'deuterium bottleneck' occurred at that stage (approximately ~ 180 sec after the big bang). What happened was that, the 'entropy' (ratio of number density of photons to the number density of baryons) of the universe was very high and nucleosynthesis could only proceed via two-body interactions. The only nucleus that could be formed with two body interactions of proton and neutron was deuteron. The problem is that the binding energy of deuteron was too low (~ 2.22 MeV) to survive photo-dissociation. And unless deuteron formed, the other light nuclei (H^3, He^3, He^4) could not form. Till the temperature of the universe fell below 0.2 MeV (why not 2.22 MeV is explained in detail in Chapter 3), deuteron abundance was not large enough to proceed with further nucleosynthesis. This is why this era was termed as 'deuterium bottleneck'. Once deuteron could form stably (without photo-dissociating), all the remaining neutrons combined to form He^4 nucleus (since it is the most stable light nuclei). A little later beryllium and lithium were also formed. The chain of reactions taking place is as follows :

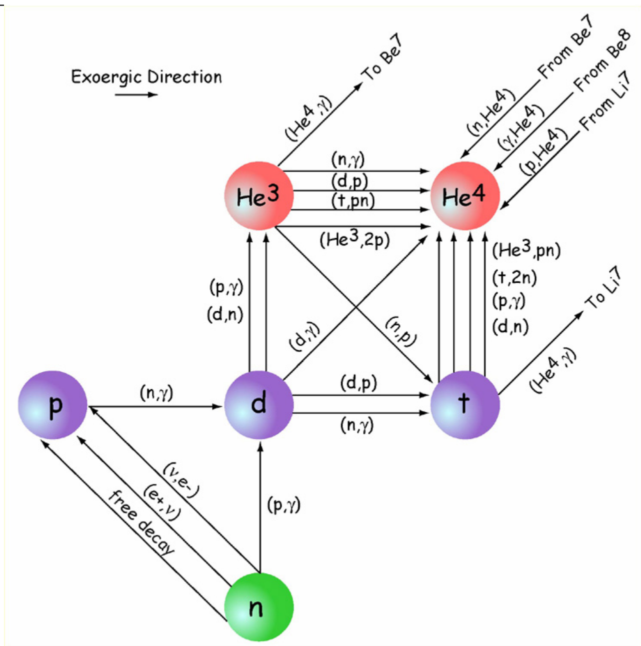

Figure 1.1: This image has been adapted from ref. [1]

1.3 Importance of observations in BBN

In this section only a very brief outline is provided of the importance of observational tools in determining the final abundance of light nuclei. For a detailed discussion, see refs. [4] and [5].

When calculating the abundances of light nuclei, either numerically or analytically, some parameters have to be specified. These include the lifetime of neutron (τ_n) and the entropy of the universe (η) (or baryon density parameter ($\Omega_B h^2$)). Another parameter is the relativistic degrees of freedom (g^*). The observed abundance of helium restricts the number of light neutrinos to three (see ref. [4]), and hence puts a constraint on g^*. Similarly, observations of primordial abundances of some light nuclei put a bound on the value of $\Omega_B h^2$ and using that we can arrive at the correct theoretical picture of BBN.

Now, there is a big problem in the determination of primordial abundances of light nuclei. Since these elements are also produced in stars, the measured abundance will not be solely due to BBN, but will have contributions from stellar synthesis. Two things can be done about this[1] : the first is to look at the spectra of very old stellar structures like quasars

[1]See $http://pdg.lbl.gov/2014/reviews/rpp2014-rev-bbang-nucleosynthesis.pdf$

(which were very abundant in the early universe), so that the spectra of these objects will allow us to study the abundances of elements in the early universe. The second is to look at deuterium abundance. This is because the most significant source of deuterium is BBN, since it is only destroyed inside stars and hence whatever value is obtained for its abundance will be a lower limit from BBN.

A note on the units : For the purpose of writing this report, natural units were used, i.e $c = \hbar = k_B = 1$.

Chapter 2

Basic Cosmology

This chapter describes the symmetries and geometry of the Friedmann model which describes our universe on large scales. It also discusses the evolution of energy density of its constituents and the various epochs which our universe has gone through. On very large scales ($\sim$100 Mpc), the universe can be considered to be homogeneous and isotropic. But this translational invariance is only for the three dimensional space, because the universe is evolving in time.

As has been observationally confirmed, our universe is expanding. The evolution of the universe is determined by Einstein's general theory of relativity. We associate a parameter called 'scale factor' with the expansion of the universe. This parameter relates coordinate distances to physically distances. It is denoted by '$a(t)$'.

Another quantity which is directly measurable is defined using the scale factor. It is called the Hubble parameter and is defined as : $H(t) = \dot{a}(t)/a(t)$. Hence, $H(t)$ determines the rate of expansion of the universe. It is a very important quantity in determining the abundance of various elements in the universe. This is because, for any reaction that is producing/destroying elements, its rate competes with the expansion rate $H(t)$. The value of the Hubble parameter today is called the Hubble constant and is denoted by 'H_0'. Time evolution of the Hubble parameter during various epochs can be arrived at by solving the following equation (the Friedmann equation).

$$H^2(t) = \frac{8\pi G}{3}\left[\rho(t) + \frac{\rho_{cr} - \rho_0}{a^2(t)}\right] \tag{2.1}$$

Note that, in arriving at the above equation the scale factor today (a_0) is taken to be unity. In the above equation, $\rho(t)$ is the energy density due to all the constituents of the universe at any given time, ρ_0 is the energy density today and ρ_{cr} is the critical density given by :

$$\rho_{cr} \equiv \frac{3{H_0}^2}{8\pi G}$$

Because our universe is dynamic, the energy density $\rho(t)$ continuosly changes in form. There have been eras of radiation domination and matter domination, where $\rho(t)$ takes different forms and hence $H(t)$ varies with time. Therefore, $\rho(t)$, in general, is a sum of several different components, which may evolve differently with time. Some aspects of cosmology have been discussed below relevant to BBN.

2.1 The Friedmann-Robertson-Walker (FRW) metric

In this section we will show how the FRW metric can be arrived at purely by symmetry considerations. Since we have assumed that the universe is homogeneous and isotropic, we can start by demanding that the universe is spherically symmetric. Since this symmetry is that of $\mathbb{S}^2$, we can write the spatial part of the metric in the following form:

$$dl^2 = \lambda^2(r)dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{2.2}$$

where an additional parameter called $\lambda(r)$ is introduced, which depends only on r, so that later we can use this to arrive at the homogeneity condition. One factor which will make this metric different from others is that our space-time is not *static*, it is continously expanding. Hence the *spatial* part of the metric should also depend on time. The simplest modification to Eq (2.2), will be to multiply it's LHS with a function $a^2(t)$, which depends only on time. This will ensure isotropy and homogeneity at all times.

To arrive at the form of $\lambda(r)$, we can impose the condition of homogeneity. For a homogeneous Universe, the *curvature* of space should be independent of r. The only scalar which determines curvature is $\mathcal{R}$ (Ricci scalar for the constant time hypersurface), hence we have to calculate $\mathcal{R}$ and demand its independence of r. For the spatial line element above, $\mathcal{R}$ is given by the following:

$$\mathcal{R} = \frac{3}{2a^2(t)r^3}\frac{d}{dr}\left[r^2\left(1 - \frac{1}{\lambda^2(r)}\right)\right] \tag{2.3}$$

Clearly the only way $\mathcal{R}$ can be independent of r is when the LHS is a constant. Hence, equating it to a constant (say k) and intergrating we get :

$$r^2\left[1 - \frac{1}{\lambda^2(r)}\right] = kr^4 + Const.of Integration \tag{2.4}$$

Demanding that $\lambda(r)$ be well behaved at $r = 0$, we can set the constant of integration to zero. We then invert the above relation to obtain $\lambda(r)$ as follows :

$$\lambda(r) = \frac{1}{1 - kr^2}$$

the spatial line element then becomes,

$$dl^2 = a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)\right] \tag{2.5}$$

and hence the space-time metric is

$$ds^2 = c^2dt^2 - a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)\right]. \tag{2.6}$$

2.2 Geometry of the Friedmann universe

When written in the above coordinate system (t, r, θ, ϕ), it is not easy to see from the metric what values the parameter k should take and what geometry choosing a particular k will correspond to. Hence, a coordinate transformation of the form is chosen :

$$\chi = \int \frac{1}{\sqrt{1 - kr^2}} dr$$

Now the above integration can be performed to give

$$\chi(r) = \begin{cases} sin^{-1}r, & \text{if k = 1} \\ r & \text{if k = 0} \\ sinh^{-1}r & \text{if k = -1} \end{cases} \tag{2.7}$$

First consider the case in which $k = 1$. In this case the spatial line element becomes

$$dl^2 = a^2(t)[d\chi^2 + sin^2\chi(d\theta^2 + sin^2\theta d\phi^2)] \tag{2.8}$$

Hence, $k = 1$ corresponds to the geometry of spatial hypersurfaces of the Friedmann universe to be that of $\mathbb{S}^3$. Since $\mathbb{S}^3$ can be parametrised by three angles and has a *finite* volume, such a universe is called *closed*.

For $k = 0$, the metric reduces to that of *flat* space-time and it's spatial part is given by :

$$dl^2 = a^2(t)[dr^2 + r^2\chi(d\theta^2 + sin^2\theta d\phi^2)] \tag{2.9}$$

Now, for $k = -1$, the spatial metric is given by :

$$dl^2 = a^2(t)[d\chi^2 + sinh^2\chi(d\theta^2 + sin^2\theta d\phi^2)] \tag{2.10}$$

This has the geometry of a three dimensional hyperboloid and hence has *infinite* volume and is termed *open*.

As has been observationally confirmed by WMAP data, our universe (atleast the observable part) happens to be *spatially flat* with a 0.4% margin for error (see ref. [9]).

2.3 Dynamics of the Friedmann universe

We can study the evolution of the Friedmann universe by solving the Einstein's equations for the metric given by Eq. (2.6) and a given distribution of sources. Since the universe is homogenous and isotropic, a perfect fluid model is used to desribe the distribution of matter. Hence the stress energy tensor describing such a model is

$$T^\mu_\nu = diag[\rho(t), -P(t), -P(t), -P(t)], \tag{2.11}$$

where $\rho(t)$ is the energy density of the constituents of the universe and $P(t)$ is the pressure. Einstein's equations read as follows :

$$G^\mu_\nu = 8\pi G T^\mu_\nu \tag{2.12}$$

Now from the metric we need to determine G^μ_ν.

2.3.1 Determining G^μ_ν

G^μ_ν is given by

$$G^\mu_\nu \equiv R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu \mathcal{R} \tag{2.13}$$

where $R_{\nu\alpha}$ is the Ricci tensor and is defined as:

$$R_{\mu\nu} \equiv \Gamma^\lambda_{\mu\nu,\lambda} - \Gamma^\lambda_{\mu\lambda,\nu} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho} \tag{2.14}$$

Using the Friedmann metric (Eq 2.6), the various Christoffel symbols can be calculated as described below.

In general, there will be $n^2(n+1)/2$ symbols to be evaluated, but because of the symmetries of the space-time, many of these will be zero. In this case of four dimensional space-time, out of forty Christoffel symbols, only thirteen are non-trivial. These are obtained by using the following expression for Christoffel symbols :

$$\Gamma^{\lambda}_{\mu\nu} = \frac{g^{\lambda\gamma}}{2}\left[g_{\gamma\mu,\nu} + g_{\gamma\nu,\mu} - g_{\nu\mu,\gamma}\right] \tag{2.15}$$

Using this, the following non-trivial Christoffel symbols result (repeated indices to be summed over and i,j,k run over 1,2,3) :

$$\begin{aligned}
\Gamma^{0}_{ii} &= \frac{\dot{a}}{a} g_{ii} \\
\Gamma^{i}_{0i} &= \frac{\dot{a}}{a} \\
\Gamma^{1}_{11} &= \frac{2kr}{1-kr^2} \\
\Gamma^{1}_{22} &= -r(1-kr^2)^2 \\
\Gamma^{1}_{33} &= -r \times sin\theta(1-kr^2)^2 \\
\Gamma^{2}_{21} &= \frac{1}{r} \\
\Gamma^{2}_{33} &= -sin\theta cos\theta \\
\Gamma^{3}_{13} &= \frac{1}{r} \\
\Gamma^{3}_{32} &= cot\theta
\end{aligned} \tag{2.16}$$

Using Eq. (2.14), various components of $R_{\alpha\beta}$ can be calculated as follows :

$$R_{00} = \Gamma^{\mu}_{00,\mu} - \Gamma^{\mu}_{0\mu,0} + \Gamma^{\mu}_{\mu\nu}\Gamma^{\nu}_{00} - \Gamma^{\mu}_{0\nu}\Gamma^{\nu}_{0\mu} \tag{2.17}$$

Using the expressions (Eq 2.16),

$$\begin{aligned}
R_{00} &= 0 - 3\frac{\ddot{a}}{a} + 3\left(\frac{\dot{a}}{a}\right)^2 + 0 - \frac{\dot{a}}{a}\Gamma^{\mu}_{0\mu} \\
&= -3\frac{\ddot{a}}{a} + 3\left(\frac{\dot{a}}{a}\right)^2 - 3\left(\frac{\dot{a}}{a}\right)^2 \\
&= -3\frac{\ddot{a}}{a}
\end{aligned}$$

Now evaluating R_{ii} (We have only shown for R_{11}, the rest will follow similarly):

$$R_{11} = \Gamma^{\mu}_{11,\mu} - \Gamma^{\mu}_{1\mu,1} + \Gamma^{\mu}_{\mu\nu}\Gamma^{\nu}_{11} - \Gamma^{\mu}_{1\nu}\Gamma^{\nu}_{1\mu} \tag{2.18}$$

Here, simplification is possible because most of the Γ's vanish, and many others cancel giving:

$$R_{11} = \Gamma^0_{11,0} - \Gamma^2_{12,1} - \Gamma^3_{13,1} + \Gamma^2_{21}\Gamma^1_{11} + \Gamma^3_{31}\Gamma^1_{11} - \Gamma^2_{12}\Gamma^2_{12} - \Gamma^3_{13}\Gamma^3_{13} \tag{2.19}$$

Substituting the values for the relevant Γ's, we get :

$$R_{11} = [2\dot{a}^2 + a\ddot{a} + 2]/(1 - kr^2)^2 \tag{2.20}$$

Pulling out a factor of a^2 and comparing with the metric coefficients, this can be written as :

$$R_{11} = -\left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2}\right] g_{11}$$

Similarly this can be shown for the other components also, and the general expression for R_{ij} then becomes:

$$R_{ij} = -\left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2}\right] g_{ij} \tag{2.21}$$

Now if we try to calculate the Ricci scalar $\mathcal{R} = g^{\mu\nu}R_{\mu\nu}$, we get the following :

$$\mathcal{R} = -6\left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2}\right] \tag{2.22}$$

Plugging Eqs. (2.21) and (2.22) into Eq. (2.13), we get $G^0_0 = 3(\dot{a}^2 + k)/a^2$. And using the value of G^0_0 into Eq. (2.12) and re-arranging, we get the Friedmann equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) - \frac{k}{a^2} \tag{2.23}$$

By defining $3k/8\pi G$ as $\rho_{cr} - \rho_0$, the above equation becomes the standard Friedmann equation (Eq. 2.1) . In the same way as k defines the geometry of space, the quantity $(\rho_{cr} - \rho_0)$ also defines the geometry of space.

Also using the expression for R_{ij}, we can evaluate G^i_j and obtain the other three equations as:

$$\begin{aligned} G^i_j &= \frac{1}{a^2}(2a\ddot{a} + \dot{a} + k)\delta^i_j \\ \frac{2\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} &= -8\pi G p \end{aligned} \tag{2.24}$$

2.4 Constituents of the universe

Consituents of the observable universe can be divided into two broad categories : Matter and Radiation. Matter includes non-relativistic particles at any given time. Photons and other relativistic particles constitute radiation. In the early universe (a few milli-seconds after the big bang), there were photons, leptons and quarks and their anti-particles. As the universe expanded and cooled, composite particles started forming (baryons, mesons). All constituents were in thermodynamic equilibrium with each other initially. The constituents relevant to the BBN era are : electrons, positron, neutrons, protons, electron neutrinos and photons.

Neutrons and protons were highly non-relativistic during the BBN, such that their energy density did not contribute significantly to the total energy density in the Friedmann equation (see section 2.5.2).

Electrons and positrons were relativistic and contributed to the energy density. They remained coupled to photons till nucleosynthesis began. As the temperature fell below their rest mass energy (~ 0.5 MeV), they annihilated giving their energy to photons.

Neutrinos were ultra-relativistic during BBN, in fact they are relativistic even today, when the temperature is ~ 2.7K.

2.5 Evolution of energy density

The reason there is a necessity to discuss the time(or temperature) evolution of energy density is as follows. As the universe expands, the number density of its constituents is constantly changing. Depending on the temperature of the universe (i.e the temperature of photons) and their rest mass energies, some particles become non-relativistic, while some annihilate with their anti-particles to supply energy to the radiation they are interacting with and others bind together to give complex particles. Another possibility is that of decoupling, when reactions coupling two or more species go out of equilibrium, and if for a particular species this was the sole reaction keeping it in equilibrium with the rest of the constituents of the universe, that species 'decouples' and its energy density evolves independently of others.

Hence, energy cannot be simply approximated as a single function of time (temperature) over a large interval of time, but needs to be evaluated for separate time intervals. In general this requires us to numerically integrate complicated functions of temperature.

Considering a homogenous and isotropic universe, the following relations (between energy density (ρ), pressure (P)) can be written for the Friedmann model of the universe beginning with the equation of continuity (a small derivation follows) :

The equation of continuity can be derived using the conservation of stress energy tensor (T^{μ}_{ν}). That is, by demanding that it's covariant derivative vanishes. The covariant derivative is defined as follows :

$$T^{\mu}_{\nu;\mu} = T^{\mu}_{\nu,\mu} + \Gamma^{\mu}_{\alpha\mu} T^{\alpha}_{\nu} - \Gamma^{\alpha}_{\nu\mu} T^{\mu}_{\alpha} \tag{2.25}$$

By setting the covariant derivative of the stress energy tensor to zero, we will arrive at four equations. The zeroth component gives the continuity equation as :

$$\begin{aligned} T^{0}_{0,0} + \Gamma^{\mu}_{0\mu} T^{0}_{0} - \Gamma^{\alpha}_{0\mu} T^{\mu}_{\alpha} &= 0 \\ \frac{\partial \rho}{\partial t} + \frac{\dot{a}}{a}(3\rho + 3P) &= 0 \end{aligned} \tag{2.26}$$

Multiplying the above equation by a^3 and combining it with ρ inside the partial derivative, we obtain:

$$a^{-3} \frac{\partial(\rho a^3)}{\partial t} = -3\frac{\dot{a}}{a} P \tag{2.27}$$

If we apply this equation to matter, we will obtain the energy density of matter as a function of scale factor. Since matter effectively exerts no pressure, taking $P = 0$, we get

$$\frac{\partial(\rho a^3)}{\partial t} = 0,$$

implying that

$$\rho_m \propto a^{-3}$$

Next, applying Eqn (2.27) to radiation and using $\rho = 3P$, we get

$$\begin{aligned} a^{-3} \frac{\partial(\rho a^3)}{\partial t} &= -\frac{\dot{a}}{a}\rho \\ \implies \frac{\partial \rho}{\partial t} &= -4\frac{\dot{a}}{a} \\ \implies \frac{\partial(\rho a^4)}{\partial t} &= 0, \end{aligned}$$

which implies that $\rho \propto a^{-4}$. Thus the *inverse* relation between energy density of matter and radiation with scale factor shows how the energy per particle *decreases* as the universe expands.

Now we need to derive the energy density of matter and radiation as functions of temperature. This can be done using principles of equilibrium statistical mechanics. In this, if we know the distribution function for a particular type of particle, which gives the number of those particles in any region of phase space, we just need to multiply it with the energy of particles in that region of phase space and integrate over all momentum values to get the average energy density :

$$\rho_i = g_i \int \frac{d^3p}{(2\pi)^3} f_i(\vec{x}, \vec{p}) E(p) \tag{2.28}$$

We can calculate this expression for different types of particles to obtain the relation between energy density and temperature.

2.5.1 Radiation

For photons, since they follow Bose Einstein statistics, the distribution function is given by

$$f_\gamma = \frac{1}{e^{(E-\mu)/T} - 1}$$

with $\mu = 0$ and because they are massless, they have two degrees of freedom (the two polarization states), which gives $g_\gamma = 2$. The energy per photon is simply given as $E(p) = p$ (in units with $c = 1$), and the energy density of radiation is evaluated as

$$\rho_\gamma = 2 \int \frac{d^3p}{(2\pi)^3} \frac{p}{e^{p/T} - 1}$$

which can be evaluated to give :

$$\rho_\gamma = g_\gamma \frac{T^4}{30} \tag{2.29}$$

The derivation of the energy density for the case of non-degenerate, relativitic Bose and Fermi gases is shown below :

The approximations used are :

1. Mass of particle (m) is much less than temperature of universe ($T >> m$), hence $E \simeq p$.
2. In the non-degenerate case, chemical potential of particles may be neglected ($\mu = 0$).

The standard expression of energy density ($+$ sign for fermions and $-$ for bosons):

$$\rho = \frac{g}{2\pi^2}\int_m^{+\infty} \frac{(E^2 - m^2)^{1/2}E^2}{exp[(E-\mu)/T] \pm 1} dE$$

reduces to the following form under the given approximations :

$$\rho = \frac{g}{2\pi^2}\int_0^{+\infty} \frac{p^3}{exp[p/T] \pm 1} dp \tag{2.30}$$

We can carry out the bosonic integral using the following trick (perturbative expansion in $e^{-p/T}$) :

$$\begin{aligned}\frac{1}{e^{p/T} - 1} &= e^{-p/T}(1 - e^{-p/T})^{-1} \\ &= e^{-p/T}(1 + e^{-p/T} + e^{-2p/T} + ..)\end{aligned}$$

Using this the above inetgral can be converted to a sum over integrals which can be integrated, and the resulting summation turns out to be a Reimann zeta function:

$$\begin{aligned}\rho(T) &= \Sigma_l \frac{g}{2\pi^2} T^4 \int_0^{+\infty} x^3 e^{-(l+1)x} dx \\ &= \Sigma_l \frac{g}{2\pi^2}\frac{T^4}{(l+1)^4}\Gamma(4) \\ &= \frac{\pi^2 g T^4}{30}\end{aligned} \tag{2.31}$$

The fermionic integral need not be evaluated, it is directly obtained from the result of the bosonic interal usign the following trick. Since,

$$\zeta(n) = 1 + \frac{1}{2^n} + \frac{1}{3^n} + ...$$

Then, if we are required to evaluate the following summation:

$$S(n) = 1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n}...$$

It can be manipulated to arrive at the following :

$$S(n) = \left(1 - \frac{1}{2^{n-1}}\right)\zeta(n)$$

Since it is $S(n)$ and not $\zeta(n)$, that will appear in the fermionic integral, this result is useful.

2.5.2 Matter

The contribution to the energy density of non-relativistic particles is exponentially suppressed compared to radiation. It scales as the following :

$$\rho(t) \propto e^{-m/T}T^{3/2}$$

Hence the energy contribution due to non-relativistic matter can be ignored in comparison to radiation and relativistic particles.

This form of energy density can be arrived at by the following method. The approximations used will be:

1. Mass of particle (m) is much greater than the temperature of the universe $(m/T >> 1)$, hence
$$(p^2 + m^2)^{1/2} \approx m + \frac{p^2}{2m}$$
The factor of $p^2/(2m)$ is to be retained in the exponential, but neglected otherwise.

2. Maxwell-Boltzmann distribution can be used to describe such particles (because of the observation : E($\sim$m) $>>$ T)

The integral to be solved then becomes:

$$\rho(T) = \frac{2mg}{\pi} e^{(-m+\mu)/T} \int_0^{+\infty} p^2 e^{-p^2/(2m)} dp \tag{2.32}$$

Which is a straight-forward integral to perform and yields :

$$\rho(T) = \frac{mg(mT)^{3/2}}{(2\pi)^{3/2}} e^{(-m+\mu)/T}$$

2.6 Time evolution of the scale factor

We can re-write Eq. (2.23) in the following form

$$\rho a^3 = \frac{3}{8\pi G} a((\dot{a})^2 + k).$$

Differentiating the above expression we get :

$$\frac{d(\rho a^3)}{dt} = \frac{3}{8\pi G}[\dot{a}^3 + \dot{a}k + 2\ddot{a}\dot{a}a] \tag{2.33}$$

Substituting the expression for $2\ddot{a}a$ from Eq. (2.24) into Eq. (2.33), we obtain:

$$\begin{aligned} \frac{d(\rho a^3)}{dt} &= -3Pa^2\dot{a} \\ \frac{da}{da}\frac{d(\rho a^3)}{dt} &= -3Pa^2\frac{da}{dt} \\ \frac{d(\rho a^3)}{da} &= -3Pa^2 \end{aligned} \tag{2.34}$$

Hence, in principle, if we know the equation of state ($P = P(\rho)$), we can solve the above equation to arrive at the relation between the scale factor (a) and time. Below, for the case of *flat* universe (k = 0), using the Friedmann equation, the dependence of a(t) on time is shown for radiation and matter.

1. **Radiation dominated era** : The equation of state for photons is : $P = \rho/3$. Using this in Eq.(2.34) and using Eq. (2.23), we obtain :

$$a(t) \propto t^{1/2}$$

2. **Matter dominated era** : The pressure exerted by matter is taken to be zero. Hence the equation of state is P = 0. Using this we get :

$$a(t) \propto t^{2/3}$$

Another parameter of interest when discussing the evolution of the universe is the *density parameter*. It is defined as:

$$\Omega(t) = \frac{\rho(t)}{\rho_{cr}}$$

Where $\rho(t)$ is the total energy density at any given time and ρ_{cr} is the critical density defined at the beginning of the chapter.

2.7 Dependence of temperature on scale factor

The importance of determining temperature as a function of time or scale factor is realised in determining the abundances of light nuclei numerically. This is because, the differential equations that need to be solved for each element are first order in time, but the coefficients of mass fractions appearing on the LHS (the reaction rates) can only be determined numerically as functions of temperature. Hence a conversion factor is required between time and temperature.

Once we know the energy density as a function of scale factor and also energy density as a function of temperature, we can find the dependence of temperature on scale factor. In the case of radiation (as seen in the preceding section),

$$\rho(t) \propto T^4$$

and

$$\rho(t) \propto a^{-4},$$

hence

$$T \propto a^{-1}.$$

Now, the key point here is that we do not need to establish the relation between temperature and scale factor for each type of particle present in the universe. This is because almost all particles ('almost' because neutrinos decouple very early before nucleosynthesis) are interacting with radiation and hence almost all constituents (this statement depends on the era, but for nucleosythesis, the universe is radiation dominated) are in thermal equilibrium with photons. But as can be guessed, the fall of radiation energy with temperature is *not* strictly $1/T$. This is because, whenever particles-antiparticles which were in equilibrium with photons annihilate, they supply energy to the photons, thereby slowing the rate of cooling of photons.

Chapter 3

Helium Synthesis

In this chapter arguments leading upto the final abundance of Helium-4(He^4) are given. The first part is a semi-analytic method of arriving at the He^4 abundance and the second part describes our attempts to arrive at the results purely numerically. Dependance of abundances of light nuclei on observational constraints is also discussed.

3.1 Broad outline for helium abundance

The analysis starts at an approximate temperature of $T \approx 10^{12}$ °K. At this temperature neutrons and protons would already have been formed (because $T < m_p c^2$), and were in equilibrium via the weak interactions $n + e^+ \rightleftharpoons \overline{\nu_e} + p$ and $n + \nu_e \rightleftharpoons e^- + p$. At approximately 10^9 °K, the rate of these reactions falls below the expansion rate of the universe determined by $H = \dot{a}(t)/a(t)$. When this happens the reactions cannot proceed fast enough to maintain equilibirum and the abundance of neutron and proton is fixed. This abundance is only changed via the free neutron decay into protons ($n \rightleftharpoons p + e^- + \overline{\nu}_e$).

As the universe cools, ideally elements should start forming in the decreasing order of their binding energy (i.e He^4 first, then Helium-3 (He^3), Tritium (T) and Deuterium (D)). But due to the low baryon to photon ratio, or the high entropy of the universe, only two body interactions are energetically favoured and only those interactions proceed with high enough rates. This means that neutrons and protons cannot directly combine to give a He^4 nucleus, but there is a two body interaction network (ref. [1]) which must be followed. The first step of this synthesis is the formation of the deuteron nucleus. The main contribution to deuteron abundance is the reaction : $p + n \rightleftharpoons d + \gamma$. This reaction stays in equilibrium nearly throughout the nucleosynthesis era, which means that any deuteron produced is subsequently 'photo-dissociated'. Even when the temperature falls below the binding energy of deuteron, because of the high entropy of the universe, the nucleus is dissociated until the point at which the temperature drops below nearly a tenth of the binding energy of deuteron (see section 3.2.2).

Hence around the temperature when deuteron can form, it is possible for the remaining neutrons at that temperature to become part of the helium-4 nuclei. The aim of this analytic argument is to deduce that temperature and find the corresponding neturon abundance to arrive at the lower limit of helium-4 abundance. This argument is semi analytic because the 'freeze-out' (the temperature at which the weak interactions responsible for neutron-proton interconversions fall out of equilibrium) mass fraction of neutron is determined numerically.

3.2 Semi-analytic determination of helium abundance

3.2.1 Weak interaction freeze-out and neutron abundance

This section gives the numerical determination of freeze-out temperature and mass fraction of neutron.

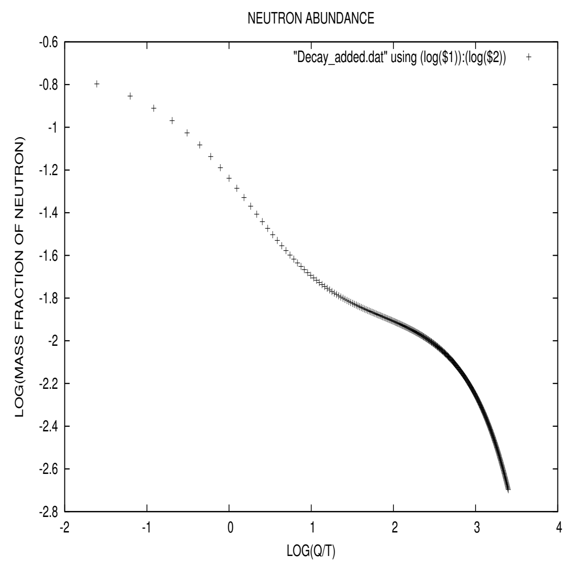

Figure 3.1: The log scale plot of evolution of neutron abundance (with the decay added)

As can be seen from the second plot, at the onset of freeze-out, the neutron abundance

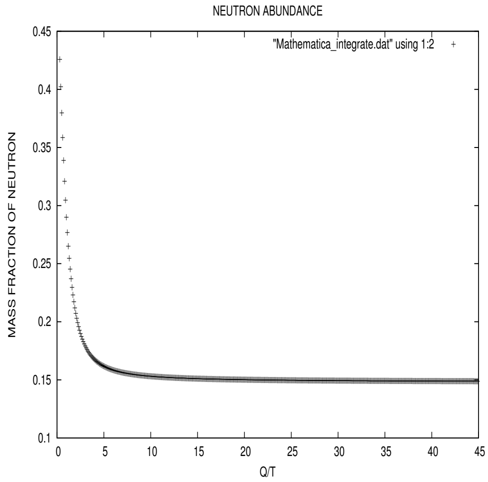

Figure 3.2: This plot of neutron abundance showing freeze-out mass fraction (without decay)

in principle becomes fixed at ~ 0.16, but in reality decays exponentially via beta decay. Its abundance is determined by the following :

$$X_n = 0.16e^{-t/\tau_n} \tag{3.1}$$

where $\tau_n \approx 881$ sec. We need to determine the temperature at which deuteron starts forming stably. Because it is at this temperature, that bound states of neutron and proton do not dissociate.

3.2.2 Effect of entropy on the temperature at which stable nuclei form

To arrive at the relation between temperature, binding energy and entropy, we have to assume that the reactions coupling concerned nuclei could maintain equilibrium. Since the temperature concerning nucleosynthesis era is much smaller than the rest mass energy of nucleons, non-relativistic limit is considered ($T_{N_{rest}} \approx 10^{14}$ °K and $T_{maxBBN} \approx 10^{12}$ °K).

Also, during the nucleosynthesis era, the rates for nuclear reactions should be fast enough to proceed with the production of light elements, hence nuclear statistical equilibrium (NSE) is assumed. In this limit, the number density of a nucleus N_A with mass number A and atomic number Z is given by the following :

$$
\begin{aligned}
n_A &= g_A \times \left(\frac{m_A T}{2\pi}\right)^{3/2} \times e^{\frac{-(m_A-\mu_A)}{T}} \\
n_p &= 2 \times \left(\frac{m_p T}{2\pi}\right)^{3/2} \times e^{\frac{-(m_p-\mu_p)}{T}} \\
n_n &= 2 \times \left(\frac{m_n T}{2\pi}\right)^{3/2} \times e^{\frac{-(m_n-\mu_n)}{T}}
\end{aligned}
\tag{3.2}
$$

Because the chemical potential is a conserved but unkown quantity for reactions in equilibrium, the above relation can be inverted to obtain an expression for μ_A and it can be eliminated using the following relations :

$$
\begin{aligned}
\mu_A &= Z \times \mu_p + (A-Z) \times \mu_n \\
\Rightarrow e^{\frac{\mu_A}{T}} &= e^{\frac{Z\times\mu_p+(A-Z)\times\mu_n}{T}} \\
&= (e^{\frac{\mu_p}{T}})^Z \times (e^{\frac{\mu_n}{T}})^{A-Z}
\end{aligned}
\tag{3.3}
$$

From Eq. (3.2), substituting the expression for $e^{\mu_A/T}$, $(e^{\mu_n/T})^{A-Z}$ and $(e^{\mu_p/T})^Z$ in Eq. (3.3) , we obtain

$$
\begin{aligned}
\frac{n_A}{g_A} \times \left(\frac{2\pi}{m_A \times T}\right)^{3/2} \times e^{\mu_A/T} = \left(\frac{n_p}{2} \times \left(\frac{2\pi}{m_p \times T}\right)^{3/2} \times e^{\mu_p/T}\right)^Z \times \\
\left(\frac{n_n}{2} \times \left(\frac{2\pi}{m_n \times T}\right)^{3/2} \times e^{\mu_n/T}\right)^{A-Z}
\end{aligned}
\tag{3.4}
$$

For mathematical convenience, we can set $m_p \approx m_n \approx m_B$ (since we are only interested in obtaining the functional dependence of $T_{nucleosynthesis}$ on B_A and η). Also, the binding energy B_A is retained in the exponential but using $m_A \approx Zm_p + (A-Z)m_n \approx A \times m_B$ in the pre-factors of the exponential and after some algebraic manipulations we obtain :

$$
n_A = A^{3/2}\frac{g_A}{2^A} \times \left(\frac{2\pi}{m_B T}\right)^{3(A-1)/2} \times (n_p)^Z (n_n)^{A-Z} \times e^{B_A/T}
\tag{3.5}
$$

To introduce the baryon to photon ratio, consider the abundance of any nucleus A. $X_A = A \times n_A/n_B$, where n_B is the total baryon number density. And since the baryon to photon ratio is given by $n_B/n_\gamma = \eta$, we can write $n_A = \eta n_\gamma\ X_A/A$, $n_p = \eta n_\gamma X_p$ and $n_n = \eta n_\gamma X_n$. Substituting these expressions for n_A, n_p and n_n, we obtain :

$$
X_A = A^{5/2}\frac{g_A}{2^A} \times \left(\frac{2\pi}{m_B T}\right)^{3(A-1)/2} \times (X_p)^Z (X_n)^{A-Z} \times (\eta n_\gamma)^{A-1} \times e^{B_A/T}
\tag{3.6}
$$

Now, substituting the expression for n_γ i.e $n_\gamma = 2(\zeta(3)/\pi^2)T^3$, we get :

$$X_A = \pi^{(1-A)/2}\zeta^{A-1}2^{(3A-5)/2}A^{5/2}\frac{g_A}{2^A} \times (\frac{T}{m_B})^{3(A-1)/2} \times (X_p)^Z(X_n)^{A-Z} \times (\eta)^{A-1} \times e^{B_A/T} \quad (3.7)$$

Let $f_A = \pi^{(1-A)/2}\zeta^{A-1}2^{(3A-5)/2}A^{5/2}\frac{g_A}{2^A}$, then the above equation becomes :

$$X_A = f_A \times (\frac{T}{m_B})^{3(A-1)/2} \times (X_p)^Z(X_n)^{A-Z} \times (\eta)^{A-1} \times e^{B_A/T} \quad (3.8)$$

As can be seen, for any $A > 1$, $X_A \propto \eta^{A-1}$ and η is of the order of 10^{-10}. The only other factor which can compete with η is $e^{B_A/T}$. This clearly shows that only when the temperature falls *much* below B_A, will the abundance of the nucleus A, i.e X_A be significant enough to affect nucleosynthesis. The above equation can be expressed in its more common form :

$$T_A \approx \frac{B_A}{(A-1)(ln(\eta^{-1}) + 1.5ln(m_B/T))} \quad (3.9)$$

And since deuteron formation has to precede the formation of other nuclei, only when $T << 2.2MeV$ will nucleosynthesis actually begin. After showing the dependence of $T_{nucleosynthesis}$ on B_A and η, we now have to determine the temperature when deuteron formation rate exceeded its dissociation rate.

3.3 Arriving at helium abundance

According to the binding energy curve, there is no stable element which could be formed with mass number greater than four during nucleosynthesis era, i.e the elements in the gap from mass numbers five to eight cannot be synthesised with the high entropy or low baryon number density during the first three minutes after big bang (The gap is bridged only in stars, see ref. [3]). This means that around the time when deuteron abundance became high enough to proceed with nucleosynthesis, almost all free neutrons combined to form the He^4 nucleus. This can be seen as :

We know that the mass fraction of any baryon is given as

$$X_A = \frac{n_A \times A}{n_B}, \quad (3.10)$$

where n_B is the total baryon number density, n_A is the number density for the nucleus A, and A is the mass number of that particular nuclear species. Also since He^4 is the most stable nucleus formed during BBN, we can assume that all neutrons end up in a helium nucleus. Thus if the number density of neutrons is n_n, then the number density of helium is proportional to $n_n/2$ (since two neutrons are required to make a helium nucleus). Hence if, the mass fraction of neutron is

$$X_n = \frac{n_n}{n_B}, \tag{3.11}$$

then the mass fraction of helium is :

$$X_{He^4} = 4 \times \frac{n_{He^4}}{n_B} \Rightarrow X_{He^4} = 4 \times \frac{n_n}{2n_B} \Rightarrow X_{He^4} = 2X_n \tag{3.12}$$

Since we know that, before nucleosynthesis began, but after the weak interactions froze-out, the neutrons were freely decaying into protons and their mass fraction was determined by the relation given by Eq. (3.1). So, if we determine the temperature at which nucleosynthesis begins and calculate the neutron abundance at that temperature, we will obtain the helium abundance.

Now nucleosynthesis will only begin once deuteron abundance is high enough to allow $D + D \to p + T$ and $D + D \to He^3 + n$ to proceed. This is because the rates of these reactions depend on the mass fraction of deuteron at any given time. Also the rate of the only deuteron forming reaction does not depend on its abundance ($p + n \to D + \gamma$). This has an important implication. As temperature falls below T_D and deuteron can form stably to increase its abundance, the rates for the two reactions ($D + D \to p + T$ and $D + D \to n + He_3$) begin to increase, whereas there is no effect on the rate of production of deuteron via $p+n \to D+\gamma$. This means that as soon as deuteron forms, it is used up in the production of heavier elements and hence the abundance of deuteron never gets a chance to increase beyond a certain value ($X_D \approx 10^{-2}$). The rate per neutron for $p + n \to D + \gamma$ is (see ref. [3]):

$$\lambda_D = 2.52 * 10^4 \left(\frac{T}{10^{10}K} \right) (\Omega_B h^2) sec^{-1} \tag{3.13}$$

and equilibrium deuteron abundance is (iff deuteron is in chemical equilibrium, which it is during the nucleosynthesis era) :

$$X_D = 3\sqrt{2} X_p X_n \epsilon e^{\frac{B_D}{K_B T}} \tag{3.14}$$

where ϵ is given by :

$$\epsilon = 1.46 * 10^{-12} (\frac{T}{10^{10}K})^{3/2} (\Omega_B h^2)$$

Using $\Omega_B h^2 = 0.02$, we can see that $10^{-13} < X_D < 10^{-3}$ when 10^{10} °K < T < 10^9 °K, and only when $X_D \approx 10^{-3} - 10^{-2}$ can Λ be large enough to compete with expansion. At the same time we need to consider the expansion of the universe. This is determined by the Friedmann equation (Eq. 3.26). Using Eqs. (3.27) and (3.28), we get :

$$H(t) = \frac{1}{2t}$$

The cross sections for $DDpT$ and $DDnHe_3$ are experimentally determined quantities and the rates per deutron is given as follows :

$$\langle \sigma(d + d \rightarrow T + p)v \rangle \approx 1.8 \times 10^{-17} cm^3/sec$$

$$\langle \sigma(d + d \rightarrow He^3 + n)v \rangle \approx 1.6 \times 10^{-17} cm^3/sec$$

Using Weinberg's notation, let the total rate for these two reations be denoted by Λ, then

$$\begin{aligned} \Lambda &= (\langle \sigma(d + d \rightarrow T + p)v \rangle + \langle \sigma(d + d \rightarrow T + p)v \rangle) X_D n_N \\ \Lambda &\simeq 1.9 \times 10^7 \left(\frac{T}{10^{10} K} \right) (\Omega_B h^2) X_D sec^{-1} \end{aligned} \tag{3.15}$$

Since the total rate Λ is proportional to X_D, it is increasing. There is a very subtle point to note here. In general, reaction rates are functions of temperature and as the temperature falls, the rates decrease, finally becoming equal to the expansion rate, and freeze out occurs. But here, the rates for nucleosynthesis beginning reactions i.e. $DDNHe^3$ and $DDPT$ are already below the expansion rate because of its dependence on X_D.

As the temperature falls below T_D, and deuteron starts forming, this rate actually *increases*, because X_d increases exponentially with falling temperature. For a brief moment, Λ becomes equal to $H(t)$ and this marks the beginning of nucleosynthesis.

The temperature at which this happens is $\sim 10^9$ °K, which according to Eq. (3.28), happens at t = 168 seconds. Plugging this into Eq. (3.1), we obtain $Xn \approx 0.132$. Hence, the helium abundance is $2 * X_n \simeq 0.2644$. This value is in close agreement with recent observational estimates which say that helium abundance is around ~ 0.25 (see ref. [10]).

3.4 Sensititvity of abundances on observational parameters

As expained in Chapter 1 (section 1.5), there are parameters which enter in the BBN light element abundances calculations. These are : the neutron lifetime (τ_n), relativistic degrees of freedom (g^*) and the entropy of the universe (η). It turns out that the dependence of all light elements on these parameters is not the same. Here, we will only discuss the dependence of He^4 abundance on these parameters.

1. τ_n **dependence** : The rate of the weak interactions keeping neutrons and protons in equilibrium is inversely proportional to τ_n. This means that if τ_n is *larger*, these rates will fall below the expansion rate *earlier*, and freeze out will occur at a *higher* temperature. If this happens, then the X_n/X_p ratio will be *larger* at freeze out. This will lead to higher mass fraction of He^4 finally. So there is a sensitive dependence of X_{He^4} on τ_n.

2. g^* **dependence**: The expansion rate of the universe $H(t) \propto g^*$. Hence, *higher* g^* leads to *higher* expansion rate, which again causes freeze-out to occur *earlier* and the same analysis as above follows.

3. η **dependence** : In contrast to the dependence of other elemental abundances on η (Eq. 3.9),He^4 is not so much effected by the high entropy of the universe. This is because, at the time of nucleosynthesis, it was the most stable light element, and almost all neutrons combined to give He^4, irrespective of η.

3.5 Attempts at numerical nucleosynthesis

In this section, the processes involved in the BBN and the approximations and assumptions used for coding are discussed. The set of simultaneous differential equations to be solved are also listed. There were many computational difficulties while trying to integrate these equations. Some of these were related to the fact that different elements were in equilibrium at different time intervals, and for these particular time gaps, subtractive cancellation was occuring and giving erroneous results. This couldn't be overcome in Mathematica in any easy way, because the software didn't allow the user to manipulate time steps involved in numerically solving differential equations. There were other issues like convergence issues because of the presence of some nested numerical integrations.

3.6 Processes and reaction rates

The abundance of all nuclei are calculated in terms of mass fraction defined as :

$$X_i = \frac{A_i n_i}{\rho_b N_A}$$

such that $\Sigma_i X_i = 1$. Here ρ_b is defined in cgs units as : $\rho_b = 0.93 \times 10^{-3} \Omega_b h^2 T_9^3$ The neutron abundance curve can be arrived at by using an analytic expression for the rate of weak reactions, when only the weak reactions were important and only the neutron and proton abundance was important. The derivation and approximations used are given in the subsequent sections.

The relativistic degrees of freedom (g^*) take different values before and after the temperature falls close to or below 1 MeV, since the electron degrees of freedom cannot be counted as relativistic after $T \approx< 1MeV$. The neutron and proton curves are obtained in a separate file starting from $T = 130$ MeV till $T \approx 3$ MeV, using $g^* = 10.75$. The program tries to simulate BBN from $T \approx 3$ MeV onward, using $g^* = 3.36$.
This means that the initial adundance for neutron is taken as 0.16, which is its freeze out value, and decay is added to the differential equation.

The reactions included till now are:

$p + n \rightarrow D + \gamma$

$p + D \rightarrow He^3 + \gamma$

$n + D \rightarrow T + \gamma$

$n + He^3 \rightarrow p + T$

$p + T \rightarrow He^4 + \gamma$

$n + He^3 \rightarrow He^4 + \gamma$

$D + D \rightarrow n + He^3$

$D + D \rightarrow p + T$

$D + D \rightarrow He^4 + \gamma$

$D + He^3 \rightarrow He^4 + p$

$D + T \rightarrow He^4 + n$

$He^3 + He^3 \rightarrow He^4 + p + p$

$T + T \rightarrow He^4 + n + n$

$He^3 + T \rightarrow p + He^4 + n$

$He^3 + T \to He^4 + D$

The rates of these reactions in terms of the variable x (discussed in section 3.9) are given below. The notation used is such that for a reaction $1 + 2 \to 3 + 4$, $[1234]$ denotes the rate for forward reaction and $[3412]$ for the backward reaction. Here symbols represent usual elements, α is for He^4.

$[pnD\gamma] = 2.5 \times 10^4 \rho_b$

$[D\gamma np] = 2.716 \times 10^{11} \times [pnD\gamma] \times \rho_b^{-1} x^{-3/2} e^{-1.72233x}$

$[pDHe^3\gamma] = 366.785 \times \rho_b(x) x^{2/3} e^{-1.508x} [1 + 0.27616x^{-1/3} + 20.55x^{-2/3} + 39.726x^{-1}]$

$[\gamma He^3 pD] = 9.46 \times 10^{11} \times [pDHe^3\gamma] \times \rho_b^{-1} x^{-3/2} e^{-4.25x}$

$[nDT\gamma] = \rho_b[75.5 + 18739.1x^{-1}]$

$T\gamma nD] = 9.46 \times 10^{11} \times [nDT\gamma] \times \rho_b^{-1} \times x^{-3/2} e^{-4.894x}$

$[DDnHe^3] = 6.4 \times 10^7 \times \rho_b \times x^{2/3} e^{[-1.72768x^{1/3}]} \times [1 + 0.24x^{-1/3} + 3.90x^{-2/3} + 6.59x^{-1}]$

$[nHe^3DD] = 1.73 \times [DDnHe^3] \times e^{[-2.53x]} = [DDpT]$

$[pTDD] = 1.73 \times [DDnHe^3] \times e^{[-3.12x]}$

$[He^3pT] = 7.06 \times 10^8 \rho_b$

$[pTnHe^3] = [He^3pT] \times e^{[-0.59x]}$

$[pT\gamma a] = 4720.51 \times e^{-1.56x^{1/3}} \times \rho_b \times x^{2/3} \times [1 + 0.27x^{-1/3} + 2.83x^{-2/3} + 5.27x^{-1} + 11.09x^{-1/3} + 52.49x^{-5/3}]$

$[\gamma\alpha PT] = e^{[-15.33x]} \times [pT\gamma a] \rho_b^{-1} \times x^{-3/2} \times 1.5 \times 10^{12}$

$[nHe^3\alpha\gamma] = 89947.8x^{-1} \times \rho_b$

$[\alpha\gamma nHe^3] = [nHe^3\alpha\gamma] \rho_b^{-1} \times e^{[-15.93x]} \times x^{-3/2} \times 1.51 \times 10^{12}$

$[DD\alpha\gamma] = 3.96 \rho_b x^{2/3} \times e^{[-1.73x^{1/3}]} \times [6.1x^{-2/3} + 10.27x^{-1} + 5.62x^{-4/3} + 24.15x^{-5/3}]$

$[\alpha\gamma DD] = [DD\alpha\gamma] \rho_b x^{-1} x^{-3/2} e^{[-18.46x]} \times 2.61 \times 10^{12}$

$[DHe^3\alpha p] = 4.48 \times 10^7 \rho_b x^{3/2} \times e^{[0.1995x]}$

$[\alpha pDHe^3] = 5.5 \times [DHe^3\alpha p] \times e^{[-14.21x]}$

$[DTn\alpha] = 2.37 \times 10^7 \times x^{3/2} \times e^{[-0.049x]}$

$[\alpha nDT] = 5.5 \times [DTn\alpha] \times e^{[-13.61x]}$

$[He^3He^3\alpha pp] = 1.765 \times 10^7 \times \rho_b e^{-4.968x^{1/3}} \times (1 + 0.083835x^{-1/3}) x^{2/3}$

$[TT\alpha nn] = 1.81 \times 10^8 \times \rho_b e^{-1.975x^{1/3}} \times (1 + 0.21x^{-1/3}) x^{2/3}$

$[He^3T\alpha pn] = 8.3 \times 10^6 \times \rho_b e^{-3.13x^{1/3}} \times (1 + 0.133x^{-1/3}) x^{2/3}$

$[He^3T\alpha D] = 5.75 \times 10^6 \times \rho_b e^{-0.1716x^{1/3}} \times (1 + 0.2698x^{-1/3})$

The equations to be integrated are as follows:

$$\begin{aligned}\frac{dX_n}{dx} &= 1.6x(-(\lambda_d X_n + [pnD\gamma]X_pX_n + \frac{1}{2}[nDT\gamma]X_nX_D + \frac{1}{3}[nHe^3\alpha\gamma])+ \\ &\frac{1}{4}[DDnHe^3]X_DX_D + \frac{1}{6}[DTn\gamma]X_DX_T)\end{aligned} \tag{3.16}$$

$$\begin{aligned}\frac{dX_p}{dx} &= 1.6x(-([npD\gamma]X_nX_p + \frac{1}{2}[pDHe^3\gamma]X_pX_D + \frac{1}{3}[pT\alpha\gamma]X_pX_T+) \\ &+ \lambda_d X_n + \frac{1}{3}[nHe^3pT]X_nX_{He^3} + \frac{1}{4}[DDpT]X_DX_D + \frac{1}{6}[DHe^3p\alpha]X_DX_{He^3})\end{aligned} \tag{3.17}$$

$$\begin{aligned}\frac{dX_D}{dx} &= 1.6x(-([D\gamma np]X_D + [pDHe^3\gamma]X_dX_p + [nDT\gamma]X_nX_D + X_DX_D([DDnHe^3]+ \\ &[DDpT])\frac{1}{2}[DD\alpha\gamma]X_DX_D + \frac{1}{3}[DHe^3\alpha p]X_DX_{He^3} + \frac{1}{3}[DT\alpha n]X_DX_T) + 2[pnD\gamma]X_pX_n+ \\ &\frac{2}{9}[He^3TD\alpha]X_{He^3}X_T)\end{aligned} \tag{3.18}$$

$$\begin{aligned}\frac{dX_T}{dx} &= 1.6x(-([pT\alpha\gamma]X_pX_T + \frac{1}{2}[DT\alpha n]X_DX_T + \frac{1}{3}[He^3T\alpha D]X_{He^3}X_T+ \\ &\frac{1}{3}[TT\alpha nn]X_TX_T) + \frac{3}{2}[nDT\gamma]X_nX_D + [nHe^3pT]X_nX_{He^3} + \frac{3}{4}[DDpT]X_DX_D)\end{aligned} \tag{3.19}$$

$$\begin{aligned}\frac{dX_{He^3}}{dx} &= 1.6x(-([nHe^3pT]X_{He^3}X_n + [nHe^3\alpha\gamma]X_nX_{He^3} + \frac{1}{2}[DHe^3\alpha p]X_DX_{He^3}+ \\ &\frac{1}{3}[He^3He^3\alpha pp]X_{He^3}X_{He^3} + \frac{1}{3}[He^3T\alpha pn]X_{He^3}X_T + \frac{1}{3}[He^3T\alpha D]X_{He^3}X_T)+ \\ &\frac{3}{2}[pDHe^3\gamma]X_pX_D + \frac{3}{4}[DDHe^3n]X_DX_D)\end{aligned} \tag{3.20}$$

$$\begin{aligned}\frac{dX_\alpha}{dx} &= 1.6x(\frac{4}{3}[pT\alpha\gamma]X_pX_T + \frac{4}{3}[nHe^3\alpha\gamma]X_nX_{He^3} + [DD\alpha\gamma]X_DX_D+ \\ &\frac{2}{3}[DHe^3\alpha p]X_DX_{He^3} + \frac{2}{3}[DT\alpha n]X_DX_T + \frac{4}{9}[He^3He^3\alpha pp]X_{He^3}X_{He^3}+ \\ &\frac{4}{9}[TT\alpha pp]X_TX_T + \frac{4}{9}[He^3T\alpha pn]X_{He^3}X_T + \frac{4}{9}[He^3T\alpha D]X_{He^3}X_T)\end{aligned} \tag{3.21}$$

The differential equation for deuteron cannot be solved directly. This is because deuteron follows its equilibrium abundance given by :

$$X_D \approx 10^{-20} \times X_nX_pe^{1.717x} \times x^{3/2}$$

Hence, it either needs to be solved simultaneously as an algebraic equation using this form of X_D, or by the method given in Wagoner (see ref. [1]), which is to equate

$$\frac{dX_D}{dt} = 0,$$

and solve for X_D algebraically.

The initial conditions can be taken as (if starting temperature is $\approx$ 3MeV) : $X_n \approx 0.17$, $X_p \approx 0.83$, $X_{He3} \approx 10^{-23}$, $X_T \approx 10^{-27}$, $X_\alpha \approx 10^{-32}$.

3.7 Approximations

Rates for the weak interactions ($n \to p$ and $p \to n$ interconversions) can either be calculated by numerical integration of rate given in the above section or by approximating the integral by an analytic equation *under the assumptions of* :

1. Non degeneracy of electrons and neutrinos which enables us to use the MB distribution instead of FD,
2. Mass of electron was neglected under the ultra-relativistic assumption.

Even though temperatures in BBN go below 0.5 MeV, assumption (2) is not violated since it was only used to derive the freeze-out mass fraction of neutrons and protons which occurred at 1 MeV. Beyond that neutrons can freely decay into the protons, which is later prevented by formation of heavy nuclei. Both the rates (actual and approximated) were calculated and deviations studied. They only differed at low temperatures, clearly showing where the approximations broke down.

3.8 Derivation of the analytic expression for the $n \to p$ and $p \to n$ rate

The entire derivation is done in natural units because the algebra is messy. The Boltzmann equation

$$\begin{aligned}\frac{1}{a^3}\frac{d(na^3)}{dt} = &\int_{-\infty}^{+\infty} \frac{1}{2E_1}\,d^3p_1 \int_{-\infty}^{+\infty} \frac{1}{2E_2}\,d^3p_2 \int_{-\infty}^{+\infty} \frac{1}{2E_3}\,d^3p_3 \int_{-\infty}^{+\infty} \frac{1}{2E_4}\,d^3p_4 \times (2\pi)^4 \times \\ &\delta(E_1 + E_2 - E_3 - E_4) \times \delta^3(p_1 + p_2 - p_3 - p_4) \times \\ &|\mathcal{M}|^2 \times \{f_3 f_4 (1 \pm f_1)(1 \pm f_2) - f_1 f_2 (1 \pm f_3)(1 \pm f_4)\}\end{aligned} \tag{3.22}$$

can be approximated as

$$\begin{aligned}\frac{1}{a^3}\frac{d(na^3)}{dt} = &\int_{-\infty}^{+\infty}\frac{1}{2E_1}d^3p_1\int_{-\infty}^{+\infty}\frac{1}{2E_2}d^3p_2\int_{-\infty}^{+\infty}\frac{1}{2E_3}d^3p_3\int_{-\infty}^{+\infty}\frac{1}{2E_4}d^3p_4\times(2\pi)^4\times\\ &\delta(E_1+E_2-E_3-E_4)\times\delta^3(p_1+p_2-p_3-p_4)\times e^{-(E_1+E_2)/T}\times\left\{\frac{n_3n_4}{n_3^{(0)}n_4^{(0)}}-\frac{n_1n_2}{n_1^{(0)}n_2^{(0)}}\right\}\\ &\times|\mathcal{M}|^2\end{aligned} \tag{3.23}$$

when the distributions are taken as Maxwell Boltzmann, where, using $f(E) \to e^{\frac{\mu}{T}}e^{\frac{-E}{T}}$ and the factor

$$f_3f_4(1\pm f_1)(1\pm f_2) - f_1f_2(1\pm f_3)(1\pm f_4)$$

was written as

$$e^{-(E_1+E_2)/T}\times\left\{\frac{n_3n_4}{n_3^{(0)}n_4^{(0)}}-\frac{n_1n_2}{n_1^{(0)}n_2^{(0)}}\right\}$$

Now, defining a quantity called 'thermally averaged cross-section' ($\langle\sigma v\rangle$) as follows :

$$\begin{aligned}\langle\sigma v\rangle = &\frac{1}{n_1^{(0)}n_2^{(0)}}\int_{-\infty}^{+\infty}\frac{1}{2E_1}d^3p_1\int_{-\infty}^{+\infty}\frac{1}{2E_2}d^3p_2\int_{-\infty}^{+\infty}\frac{1}{2E_3}d^3p_3\int_{-\infty}^{+\infty}\frac{1}{2E_4}d^3p_4\times(2\pi)^4\times\\ &\delta(E_1+E_2-E_3-E_4)\times\delta^3(p_1+p_2-p_3-p_4)\times e^{-(E_1+E_2)/T}|\mathcal{M}|^2\end{aligned} \tag{3.24}$$

Under the approximations we have taken, the rates for both the processes : $n+e^+ \rightleftharpoons \overline{\nu_e}+p$ and $n+\nu_e \rightleftharpoons e^-+p$ are the same, hence we only need to solve for $\langle\sigma v\rangle$ using the above equation for any one of them. Let us perform the calculation of rate for $n+\nu_e \rightleftharpoons e^-+p$. The rate in this case is defined as : $\lambda_{np} = n_{\nu_e}^{(0)}\langle\sigma v\rangle$

Also, under the approximations taken, in the highly non-relativistic limit, $E_1 \approx m_n$ and $E_2 \approx p_\nu$ and $m_n \approx m_p = m$. Similarly we can also take the proton to be non-relativistic and $E_4 \approx m_p$. Now if we perform the hearvy particle integrals, one over the δ^3 function and the other as it is, we will arrive at a factor of $n_n^{(0)}$ in the numerator which will cancel the $n_n^{(0)}$ in the denominator of $\langle\sigma v\rangle$ and we will be left with :

$$\begin{aligned}\lambda_{np} = \frac{\pi}{4m^2}&\int\frac{dp_\nu}{(2\pi)^3\times 2p_\nu}\times|\mathcal{M}|^2\\ &\int\frac{dp_e}{(2\pi)^3\times 2p_e}\delta(m_n-m_p+p_\nu-p_e)\end{aligned} \tag{3.25}$$

Using the standard expression for $|\mathcal{M}|^2 = 32{G_F}^2(1+3{g_A}^2){m_p}^2 p_\nu p_e$ from literature and performing the δ integral over p_e, the integral that remains (with respect to p_ν) is the following:

$$K \times \int dp_\nu {p_\nu}^2({p_\nu}^2 + Q^2 + 2Qp_\nu)e^{-p_\nu/T}$$

Where K is a factor containing the temperature T and the lifetime of neutron (τ_n) . It is trivial to perform this gamma function integration between the limits zero and Q, to yield the following result:

$$\lambda_{np} = \frac{255}{\tau_n x^5}(12 + 6x + x^2)$$

where x = Q/T. This result was in reasonable agreement with the results obtained from numerical integration of reaction rate and was used for the numerical integration purposes.

3.9 A note on integration variable used

Since the rate equations are generally of the form : $\frac{d(X)}{dt} = \lambda_1(T)Y - \lambda_2(T)X$, where λ's are given as functions of temperature and the integration variable is time, another variable - x, is used, which is related to T as: $x = \frac{Q}{T}$. Time and temperature are related via the Friedmann equation:

$$H^2(t) = \frac{8\pi G}{3}\left[\rho(t) + \frac{\rho_{cr} - \rho_0}{a^2(t)}\right]$$

Considering the universe to be flat, $\rho_{cr} = \rho_0$. Then the Friedmann equation becomes :

$$H^2(t) = \frac{8\pi G}{3}\rho(t) \tag{3.26}$$

with

$$\rho(T) = \frac{\pi^2}{30}g^* T^4.$$

Now, writing $H(t) = \dot{a}/a$ and $a = T_0/T$, we obtain $\dot{a}/a = -\dot{T}/T$. Substituting all these relations in eqn 3, after taking the square root, realising that $\dot{T}/T < 0$, we get

$$\frac{\dot{T}}{T} = -\left(\frac{8\pi^3 G g*}{90}\right)^{1/2} T^2 \tag{3.27}$$

solving which we get (since we are working in natural units (c = ℏ = 1), $G \approx 0.015^{-2} sec^{-2}$ MeV^{-4}),

$$t = \frac{1.34}{T^2} \tag{3.28}$$

and

$$\frac{d}{dt} = \frac{1}{1.6x} \times \frac{d}{dx} \tag{3.29}$$

The factor 1.6 is for $g^* = 3.36$, which is assumed after weak interactions fall out of equilibrium. And since the rates in Wagoner's paper (ref. [1]) were given in terms of T_9 ($10^{9\circ}$ K), these were also converted to x, by the relation $T_9 = 11.5942 * Q/x$.

Chapter 4

Summary

We shall now briefly summarize the contents of this report. We started with understanding the Friedmann Model of the universe and the evolution of energy density. We also derived the dependence of the scale factor on time during the different epochs of radiation and matter domination. From this we learnt the expansion rate of the universe. We then argued about what particles will constitute the universe at any given time (temperature) and whether they will be relativistic or not. Then a basic chain of reactions was studied to be able to arrive at the helium abundance. This included deriving the rates of weak interactions that coupled neutrons and protons, and numerically establishing the approximate 'freeze-out' temperature of these reactions and the abundances of neutrons and protons at the time of freeze-out. Then the main phenomena which prevented nucleosynthesis from happening earlier was discussed. This was the 'deuterium bottleneck' problem. The effect of the high entropy of the universe on light element abundances was then discussed and using the arguments of stability of the helium nucleus, its final abundance was deduced. The dependence of this result on obervationally determined parameters like entropy, relativistic degrees of freedom and neutron lifetime was discussed. One of the attempts at arriving at the light element abundances numerically was then briefly discussed.

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[8] See, `http://www.damtp.cam.ac.uk/user/db275/Cosmology/Lectures.pdf/`.

[9] See, `http://map.gsfc.nasa.gov/universe/uni_shape.html/`.

[10] See, `http://www.pas.rochester.edu/~emamajek/memo_Yp.html/`.