NON-PERTURBATIVE EFFECTS IN QUANTUM FIELD THEORY

A project report
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in
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by

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under the guidance of
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CERTIFICATE

This is to certify that the project titled **Non-Perturbative effects in quantum field theory** is a bona fide record of work done by **Somdutta Ghosh** towards the partial fulfillment of the requirements of the Master of Science degree in Physics at the Indian Institute of Technology, Madras, Chennai 600036, India.

(L. Sriramkumar, Project supervisor)

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ABSTRACT

In this project we intend to study a couple of non-perturbative effects in quantum field theory, specifically, the Vacuum polarization such as the Casimir effect and the Particle production such as the Schwinger mechanism. We begin with the study of basic quantum field theory. Throughout this study, we focus on quantum scalar field, that is, the Klein-Gordon field. For studying Casimir effect, we first calculate the vacuum fluctuation energy for a simple system which has one spatial and one time dimension (that is, 1+1 dimensional system). This is followed by a more realistic case where we consider a real system which has three spatial and one time dimension (3+1 dimensional system). To study particle production, we consider a scalar quantum field in a classical electromagnetic background. The vector potential describing the field has spatial time dependence along a single direction. We quantise the scalar field in this time dependent gauge. We then use special functions and their properties to calculate the Bogolubov coefficients. From the value of these coefficients we can read off the number of particles produced by the quantum scalar field in a constant electromagnetic background described by the time dependent gauge.

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Chapter 1

Introduction

We study field theory in the hope that it will shed light on the interactions of the fundamental particles of matter. Non-Relativistic quantum mechanics helped us decipher a lot about atomic physics. However, certain fundamental phenomena, such as the interaction between atoms and the photon, the anomalous magnetic moment of electrons, pair production, spontaneous emission, etc. mark the limitation of this theory. To explain these a relativistic generalization of non-relativistic quantum mechanics was required.

The relativistic theory of quantum mechanics, though already in existence owing to the work of Klein and Gordon for scalar fields and Dirac for spinor fields was not sufficient. The negative energy states and the lack of positive definite probability density proved this. These were the conditions which demanded a quantum theory of fields.

1.1 Notations

We shall work mostly in $(3+1)$-spacetime dimensions except for section (2.1) where we consider n-dimensional spacetime and in section (2.2) we work in $(1+1)$ dimensional spacetime. The metric signature adopted is $(+,-,-,-)$. The Greek indices shall denote the spacetime coordinates and the Latin indices shall denote the spatial coordinates.

In this notation, the interval between two points in $(3+1)$ Minkowski spacetime is

$$ds^2 = dx^\mu dx_\mu = c^2 dt^2 - dx^2 - dy^2 - dz^2. \tag{1.1}$$

1.2 Klein-Gordon Equation

Consider a spin 0 particle. The energy-momentum 4-vector of the particle is [1]

$$p^\mu = (\frac{E}{c}, \vec{p}), \tag{1.2}$$

where, $\vec{p}$ is the 3-momentum.
The invariant is

$$p^2 = p^\mu p_\mu = \frac{E^2}{c^2} - \vec{p}.\vec{p} = m^2 c^2. \tag{1.3}$$

Now, as in quantum theory

$$E \longrightarrow i\hbar\frac{\partial}{\partial t}\, , \vec{p} = -i\hbar\vec{\nabla}.$$

Substituting the operators in equation (1.3), we get in units of $\hbar = 1$, $c = 1$

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi(\vec{x}, t) + m^2\phi(\vec{x}, t) = 0,$$

$$(\Box + m^2)\phi(\vec{x}, t) = 0, \tag{1.4}$$

where, $\Box$ is called the d'Alembertian.
The equation (1.4) is called the Klein-Gordon equation. It is a relativistic version of the Schroedinger equation. It is also second order in space and time and, therefore, Lorentz covariant.

1.3 Probability Density

We know that the Klein-Gordon equation is a relativistic equation. So, the probability density should obey the 4-vector transformation law. So it is defined as,

$$j^\mu = (\rho, \vec{j}), \tag{1.5}$$

where ρ is the time component and $\vec{j}$ is the spatial component.
We define, [1]

$$\rho = \frac{i\hbar}{2m}(\phi^*(\vec{x}, t)\frac{\partial\phi(\vec{x}, t)}{\partial t} - \phi(\vec{x}, t)\frac{\partial\phi^*(\vec{x}, t)}{\partial t}) \tag{1.6}$$

and

$$\vec{j} = -\frac{i\hbar}{2m}(\phi^*\vec{\nabla}\phi - \phi\vec{\nabla}\phi^*). \tag{1.7}$$

So, from equation (1.5), we get

$$j^{\mu} = \frac{i\hbar}{m}[\phi^*(\vec{x},t)(\overleftrightarrow{\partial_0}, \overleftrightarrow{\nabla})\phi(\vec{x},t)] = \frac{i\hbar}{m}\phi^*(\vec{x},t)\overleftrightarrow{\partial_\mu}\phi(\vec{x},t), \tag{1.8}$$

where,

$$A\overleftrightarrow{\partial^\mu}B = \frac{1}{2}[A\,\partial^\mu B - (\partial^\mu A)\,B]. \tag{1.9}$$

Therefore, we have the continuity equation

$$\partial_\mu j^\mu = \frac{i\hbar}{2m}[\phi^*(\vec{x},t)\Box\phi(\vec{x},t) - \phi(\vec{x},t)\Box\phi^*(\vec{x},t)] = 0. \tag{1.10}$$

The Klein-Gordon equation is a second order differential equation. So, $\phi(\vec{x},t)$ and $\frac{\partial\phi(\vec{x},t)}{\partial t}$ can be fixed arbitarily at a given time. Hence, ρ as given by equation (1.6) is not positive definite and can take negative values. So the interpretation of ρ as a probability density and along with it the interpretation that the Klein-Gordon equation is a single particle equation with wave function $\phi(\vec{x},t)$ also needs to be abandoned. We need to quantise the field and reinterpret the Klein-Gordon equation as a field equation. [1]

Another problem with the Klein-Gordon equation is that equation (1.3) allows both positive and negative energy solutions, that is,

$$E = \pm(m^2c^4 + p^2c^2)^{1/2}. \tag{1.11}$$

While the positive solution is as per our expectation, the negative solution for energy bother us. An interacting particle may exchange energy with its environment and then there is nothing that can stop it from cascading down to infinite negative energy states, emitting an infinite amount of energy in the process. [1]

This of course does not happen, and we once again come to the conclusion that the Klein-Gordon equation cannot be interpreted as a single particle equation.

1.4 Energy of the Klein Gordon Field

In the last section we saw that if we try to interpret the Klein-Gordon equation as a single particle equation we face difficulties. So let us now consider that the Klein-Gordon equation describes the field $\phi(\vec{x},t)$. Klein-Gordon equation has no classical analogue, so $\phi(\vec{x},t)$ is strictly a quantum field, but, to keep matters simple we treat $\phi(\vec{x},t)$ as a classical field. Let us find the energy of the 'classical' Klein Gordon field. [1]

The energy, is given by

$$\mathcal{H} = \int T^{00} d^3x, \tag{1.12}$$

and the momentum is

$$P_i = \int T^{0i} d^3x \tag{1.13}$$

where

$$T^{\mu\nu} = (\partial^\mu \phi(\vec{x}, t))(\partial^\nu \phi(\vec{x}, t)) - g^{\mu\nu}\mathcal{L}, \tag{1.14}$$

is the energy momentum tensor and

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial^\mu \phi(\vec{x}, t))(\partial_\mu \phi(\vec{x}, t)) - \frac{m^2}{2}\phi^2(\vec{x}, t) \\ &= \frac{1}{2}[(\partial_0 \phi(\vec{x}, t))^2 - (\vec{\nabla}\phi(\vec{x}, t))^2 - m^2\phi^2(\vec{x}, t)],\end{aligned} \tag{1.15}$$

is the Lagrangian which gives us the Klein Gordon Equation.
Therefore, the Hamiltonian $\mathcal{H}$ is given by

$$\mathcal{H} = \frac{1}{2}\int d^3x[(\partial_0\phi)^2 - (\vec{\nabla}\phi).(\vec{\nabla}\phi) - m^2\phi^2], \tag{1.16}$$

for a real scalar field $\phi(\vec{x}, t)$.

So, we can see that in this case the energy, that is, the Hamiltonian is positive definite. Thus, the scalar field is no longer plagued by the negative energy problem which is present in the single particle theory.

Now, let us relate the positive definite energy to the energy of the single-particle states. To do this, we need to quantise the field. Field quantisation forces us to reinterpret the field as a quantum system instead of a classical system.

1.5 Field Quantisation

We consider the field $\phi(\vec{x}, t)$ as a Hermitian operator, whose Fourier expansion can be written as [1]

$$\phi(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}[\hat{a}(k)e^{-ikx} + \hat{a}^\dagger(k)e^{ikx}], \tag{1.17}$$

where $\hat{a}(k)$ is called the annihilation operator and $\hat{a}^\dagger(k)$ is called the creation operator and $\omega_k = (\vec{k}^2 + m^2)^{1/2}$.

The measure in the integrand is chosen in such a way that it is relativistically invariant. The quantities k and x are 4-vectors,

$$k = (k_0, \vec{k})$$

and

$$x = (x^0, x^1, x^2, x^3).$$

Here $\phi(\vec{x}, t)$ plays a role in field theory analogous to that played by the position vector, $\vec{x}$, in particle mechanics. So, if we define [1]

$$\pi(\vec{x}, t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\vec{x}, t)}, \tag{1.18}$$

which is analogous to the momentum operator in particle mechanics, we obtain the relations

$$\begin{aligned} [\phi(\vec{x}, t), \pi(\vec{x}', t)] &= i\delta(\vec{x} - \vec{x}'), \\ [\phi(\vec{x}, t), \phi(\vec{x}', t)] &= 0, \\ [\pi(\vec{x}, t), \pi(\vec{x}', t)] &= 0, \end{aligned} \tag{1.19}$$

called the equal time commutation relations (ETCR), analogous to the commutation relation in particle mechanics.

So we can write the field expansion equation (1.17) as

$$\phi(\vec{x}, t) = \int \frac{d^3k}{[(2\pi)^3 2\omega_k]^{1/2}} [u_k(\vec{x}, t)\hat{a}(k) + u_k^*(\vec{x}, t)\hat{a}^\dagger(k)], \tag{1.20}$$

where

$$u_k(\vec{x}, t) = \frac{1}{[(2\pi)^3 2\omega_k]^{1/2}} e^{-ikx}, \tag{1.21}$$

are called the positive frequency solutions.

The $u_k(\vec{x}, t)$ form an orthonormal set [1]

$$\int u_k^*(\vec{x}, t) i \overleftrightarrow{\partial}_0 u_{k'}(\vec{x}, t) d^3x = \delta^3(\vec{k} - \vec{k}'). \tag{1.22}$$

Using equation (1.22), we find the expressions

$$a(k) = \int d^3x [(2\pi)^3 2\omega_k]^{1/2} u_k^*(\vec{x}, t) i \overleftrightarrow{\partial}_0 \phi(\vec{x}, t), \tag{1.23}$$

and

$$a^\dagger(k) = \int d^3x' [(2\pi)^3 2\omega_k]^{1/2} \phi(\vec{x}', t) i \overleftrightarrow{\partial}_0 u_{k'}(\vec{x}', t), \tag{1.24}$$

From equations (1.19), (1.20), (1.23) and (1.24) we obtain the commutation relations [1]

$$\begin{aligned}[a(\hat{k}), \hat{a^\dagger}(k')] &= (2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{k'}), \\ [\hat{a}(k), \hat{a}(k')] &= 0, \\ [\hat{a}^\dagger(k), \hat{a}^\dagger(k')] &= 0.\end{aligned} \tag{1.25}$$

The annihilation and the creation operators $\hat{a}(k)$ and $\hat{a}^\dagger(k)$ respectively, plays a crucial role in the particle interpretation of the quantised field theory. We can construct the particle number operator $\hat{N}(k)$ from the above equations. It has the following form:

$$\hat{a^\dagger}(k)\hat{a}(k) = (2\pi)^3 2\omega_k \delta^3(0)\hat{N}(k). \tag{1.26}$$

The eigenstate of this operator forms the Fock basis

$$\hat{N}(k)|n(k)\rangle = n(k)|n(k)\rangle. \tag{1.27}$$

The commutation relations between the particle number operator and the creation and annihilation operator is as below:

$$\begin{aligned}[\hat{N}(k), \hat{a}^\dagger(k)] &= [\frac{\hat{a}^\dagger(k)\hat{a}(k)}{(2\pi)^3 2\omega_k \delta^3(0)}, \hat{a}^\dagger(k)] \\ &= \frac{1}{(2\pi)^3 2\omega_k \delta^3(0)}([\hat{a}^\dagger(k), \hat{a}^\dagger(k)]\hat{a}(k) + \hat{a}^\dagger(k)[\hat{a}(k), \hat{a}^\dagger(k)]) \\ &= \frac{1}{(2\pi)^3 2\omega_k \delta^3(0)}(2\pi)^3 2\omega_k \delta^3(0)\hat{a}^\dagger(k) \\ &= \hat{a}^\dagger(k),\end{aligned} \tag{1.28}$$

and,

$$\begin{aligned}[\hat{N}(k), \hat{a}(k)] &= [\frac{\hat{a}^\dagger(k)\hat{a}(k)}{(2\pi)^3 2\omega_k \delta^3(0)}, \hat{a}(k)] \\ &= \frac{1}{(2\pi)^3 2\omega_k \delta^3(0)}(\hat{a}^\dagger(k)[\hat{a}(k), \hat{a}(k)] + [\hat{a}^\dagger(k), \hat{a}(k)]\hat{a}(k)) \\ &= -\frac{1}{(2\pi)^3 2\omega_k \delta^3(0)}(2\pi)^3 2\omega_k \delta^3(0)\hat{a}(k) \\ &= -\hat{a}(k).\end{aligned} \tag{1.29}$$

Using equations (1.28) and (1.29), we find that,

$$\begin{aligned}\hat{N}(k)\hat{a}^\dagger(k)|n(k)\rangle &= \hat{a}^\dagger(k)\hat{N}(k)|n(k)\rangle + [\hat{N}(k), \hat{a}^\dagger(k)]|n(k)\rangle \\ &= \hat{a}^\dagger(k)\hat{N}(k)|n(k)\rangle + \hat{a}^\dagger(k)|n(k)\rangle \\ &= [n(k) + 1]\hat{a}^\dagger(k)|n(k)\rangle,\end{aligned} \tag{1.30}$$

and

$$\begin{aligned}\hat{N}(k)\hat{a}(k)|n(k)\rangle &= \hat{a}(k)\hat{N}(k)|n(k)\rangle + [\hat{N}(k), \hat{a}(k)]|n(k)\rangle \\ &= \hat{a}(k)\hat{N}(k)|n(k)\rangle - \hat{a}(k)|n(k)\rangle \\ &= [n(k) - 1]\hat{a}(k)|n(k)\rangle. \end{aligned} \tag{1.31}$$

From the above equations we can conclude that if the fock state $|n(k)\rangle$ has eigenvalue $n(k)$, that is,

$$\hat{N}(k)|n(k)\rangle = n(k)|n(k)\rangle,$$

then the state $\hat{a}^\dagger(k)|n(k)\rangle$ is an eigenstate of the operator $\hat{N}(k)$ with eigenvalue $[n(k)+1]$. Similarly, state $\hat{a}(k)|n(k)\rangle$ is an eigenstate of $N(k)$with eigenvalue $[n(k)-1]$. This justifies our defining $N(k)$ as the particle number operator or as the density operator.

This also shows us how suitable the name creation and annihilation operator is for $\hat{a}^\dagger(k)$ and $\hat{a}(k)$, respectively. From equation (1.30) we can see that the creation operator raises the particle number by one, that is, it creates a particle. Similarly, from equation (1.31) it is clear that the annihilation operator lowers the particle number by one, that is, it annihilates a particle.

If we consider a massless spin 0 particle, then the field energy can be obtained by substituting equation (1.17) in equation (1.16)

$$\begin{aligned}\mathcal{H} &= \int \frac{d^3k}{(2\pi)^3 2k_0} \frac{k_0}{2} [\hat{a}^\dagger(k)\hat{a}(k) + \hat{a}(k)\hat{a}^\dagger(k)] \\ &= \int d^3k k_0 [\hat{N}(k) + \frac{1}{2}], \end{aligned} \tag{1.32}$$

where we have substituted $\omega_k = k_0$ since $m = 0$.
And, the field momentum is obtained by substituting equation (1.17) in equation (1.13)

$$\vec{P} = \int d^3k \vec{k} [\hat{N}(k) + \frac{1}{2}]. \tag{1.33}$$

Hence, $\hat{N}(k)$ is clearly the number of particles with momentum $\vec{k}$ and energy k_0.

Also,

$$[\hat{a}(k) \mid n(k)\rangle]^\dagger [\hat{a}(k) \mid n(k)\rangle] > 0, \tag{1.34}$$

as all Hilbert space states must be.
Therefore, evaluating the innerproduct in the above equation, we get,

$$\langle n(k) \mid \hat{a}^\dagger(k)\hat{a}(k) \mid n(k)\rangle = n(k)\langle n(k) \mid n(k)\rangle > 0. \tag{1.35}$$

So, the number operator eigenvalue $n(k)$ has to be non-negative, that is, positive or zero .

From equation (1.31) we can see that $\hat{a}(k)$ when acts on state $|n(k)\rangle$ reduces it by one. So continuous application of $\hat{a}(k)$ on $|n(k)\rangle$ will reduce it and can eventually make the state negative. But as seen from equation (1.35) $|n(k)\rangle$ has to be non negative. So, to prevent $|n(k)\rangle$ from becoming negative we define a ground state, $|0\rangle$, such that,

$$\hat{a}(k)|0\rangle = 0. \tag{1.36}$$

As a result,

$$\hat{N}(k)|0\rangle = \hat{a}^{\dagger}(k)\hat{a}(k)|0\rangle = 0. \tag{1.37}$$

The ground state, $|0\rangle$, also called the vaccum state, contains no particles. Also, the application of $\hat{a}^{\dagger}(k)$ now increases $n(k)$ in steps of 1 starting from 0. So $n(k)$ must be integral.This provides a complete justification for interpreting $\hat{N}(k)$ as the number operator, and also for the particle interpretation of the quantised theory.

Chapter 2

The Casimir Effect

One of the effects arising due to field quantisation is the Casimir effect. It is an experimentally verified prediction of quantum field theory. It is named after the Dutch physicist, Hendrik Casimir, who formulated the theory in 1948.

The Casimir effect is exhibited when two uncharged conducting plates are placed, a few nanometers apart, in vacuum. In a classical description, the lack of an external field means that there is no field between the plates. Therefore, no force should be measured between them. However, if this field is studied using the QED vacuum of quantum electrodynamics, it is seen that the plates do affect the virtual photons which constitute the field, and generate a net force, which is either an attraction or a repulsion depending on the specific arrangement of the two plates. Although Casimir effect can be expressed in terms of virtual particles interacting with the conducting plates, it is best described and more easily calculated in terms of the zero-point energy of a quantised field in the intervening space between the conducting plates. This force has been measured and is a striking example of an effect captured formally by field quantisation.

2.1 Vacuum energy divergences

Vacuum intutively means the ’absence of anything’ or ’an empty space’. Generally, the vacuum state is the state with the lowest possible energy. In case of classical field theory, the vacuum is a state where the field is absent, that is, $\phi(\vec{x}, t) = 0$. However, in quantum field theory the vacuum state is by no means a simple empty space. It contains virtual particles that come into existence and then annihilates in a timespan too short to observe. However, the vacuum state does not contain any physical particles.

Let us calculate the energy and momentum of a quantum field in the vacuum state. For this purpose, we will consider a n-dimensional spacetime in this section and the vectors are also n-dimensional.
The momentum of the field in this state is:[2]

$$\langle 0|\vec{P}|0\rangle = \int d^{n-1}x \langle 0|T^{0i}|0\rangle = 0. \tag{2.1}$$

So, the momentum of the field in the vacuum state is zero. This is as expected, since the lack of any external field would mean that the net flux is zero.

We also expect it to carry zero energy, as no field quanta are present. However, a simple calculation shows us that the energy is as follows:[2]

$$\langle 0|H|0\rangle = \int d^{n-1}x \langle 0|T^{00}|0\rangle = \sum_{\vec{k}} \frac{1}{2}\omega_{\vec{k}}. \tag{2.2}$$

Now, computing the summation we find that, [2]

$$\begin{aligned} \sum_{\vec{k}} \frac{1}{2}\omega_{\vec{k}} &= \frac{1}{2}(L/2\pi)^{n-1} \int \omega_{\vec{k}} d^{n-1}k \\ &= (L^2/4\pi)^{(n-1)/2} \frac{1}{\Gamma((n-1)/2)} \int_0^\infty (k^2+m^2)^{\frac{1}{2}} k^{n-2} dk, \end{aligned} \tag{2.3}$$

which diverges like k^n for large k. So, we can see that, the energy of the vacuum state of the quantum field is not only non zero, rather it is infinite.

The fact that equation (2.3) is divergent apparently indicates that the vacuum contains an infinite density of energy. The trouble comes from the $\frac{1}{2}\omega_{\vec{k}}$ zero-point energy associated with each simple harmonic oscillator mode of the scalar field. As $\omega_{\vec{k}}$ has no upper bound the zero-point energy can be arbitrarily large. This is a problem which will plague the subject of quantum fields in curved spacetime throughout. However, in flat spacetime, it is easily circumvented. Energy as such is not measurable in non-gravitational physics, so we can rescale or renormalize the zero point energy, even by an infinite amount, without affecting observable quantities. This may be acomplished by simply neglecting the

$$\frac{1}{2}\sum_{\vec{k}} \omega_{\vec{k}}$$

term in the expression for the Hamiltonian. Or, we can define a normal ordering operation, denoted by ::, in which one demands that wherever a product of creation and annihilation operator appears, it is understood that all annihilation operators stand to the right of the creation operators. Thus, the expression for the Hamiltonian becomes,

$$:H: \;=\; \sum_k a_k^\dagger a_k \omega_{\vec{k}}, \tag{2.4}$$

and the troublesome $\frac{1}{2}\omega_{\vec{k}}$ term has disappeared. [2]

2.2 Cylindrical two-dimensional spacetime

The simplest generalization of Minkowski space quantum field theory is the introduction of non-trivial topological structures in a locally flat spacetime. The easiest such generalization is the $R^1 \times S^1$ two-dimensional spacetime with closed spatial sections. [2]

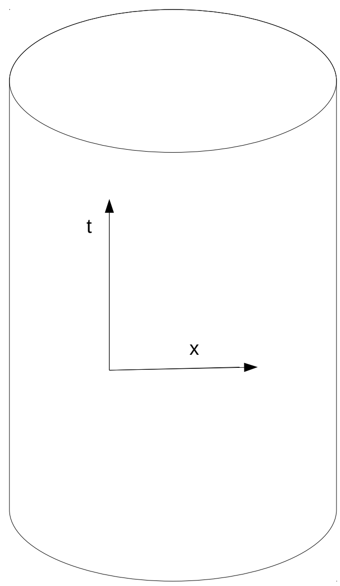

Fig. 1: Two-dimensional spacetime with compact spatial sections ($R^1 \times S^1$). The circumference of the cylinder is L.

This spacetime has the two-dimensional Minkowski space line element given by

$$ds^2 = c^2 dt^2 - dx^2. \tag{2.5}$$

The spatial points x and $x + L$ are identical, where L is the periodicity length.

In general, the field modes in n-dimension are given by

$$u_k(\vec{x},t) = [2\omega(2\pi)^{n-1}]^{-\frac{1}{2}} e^{i\vec{k}.\vec{x}-i\omega t}. \tag{2.6}$$

So, in this two dimensional spacetime, the field modes are:

$$u_k(x,t) = [2\omega(2\pi)]^{-\frac{1}{2}} e^{ikx-i\omega t}. \tag{2.7}$$

To account for the presence of closed spatial sections, we need to impose the boundary condition,

$$u_k(x,t) = u_k(x+nL,t). \tag{2.8}$$

Therefore, the restricted field modes, which is a discrete set, are as follows

$$u_k(x,t) = (2L\omega)^{-\frac{1}{2}} e^{i(kx-\omega t)}, \tag{2.9}$$

where $k = 2\pi n/L, n = 0, \pm 1, \pm 2, \pm 3, ...$

Let us restrict our attention to the massless case, that is, $\omega = |k|$. The modes labelled by positive values of n have the form $e^{[ik(x-t)]}$ and represent waves that move from left to right, while negative values of n give $e^{[ik(x+t)]}$, which represent waves moving from right to left.

As the field modes are forced into a discrete set, the field energy will be modified. So let us find the stress-tensor.

In general the stress-tensor operator, $T_{\mu\nu}$, is given by

$$\begin{aligned} T_{\alpha\beta} &= \phi(\vec{x},t)_{,\alpha}\phi(\vec{x},t)_{,\beta} - \eta_{\alpha\beta}\mathcal{L} \\ &= \phi(\vec{x},t)_{,\alpha}\phi(\vec{x},t)_{,\beta} - \frac{1}{2}\eta_{\alpha\beta}\eta^{\lambda\delta}\phi(\vec{x},t)_{,\lambda}\phi(\vec{x},t)_{,\delta} + \frac{1}{2}m^2\phi^2(\vec{x},t)\eta_{\alpha\beta}, \end{aligned} \tag{2.10}$$

where ',' denotes partial derivative. For the massless two-dimensional case, as considered here, the stress-tensor operator is:

$$T_{\alpha\beta} = \phi(x,t)_{,\alpha}\phi(x,t)_{,\beta} - \frac{1}{2}\eta_{\alpha\beta}\eta^{\lambda\delta}\phi(x,t)_{,\lambda}\phi(x,t)_{,\delta}. \tag{2.11}$$

So, in the present two-dimensional case, the Cartesian components of the stress-tensor operator becomes:

$$T_{tt} = T_{xx} = \frac{1}{2}\left(\frac{\partial\phi(x,t)}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi(x,t)}{\partial x}\right)^2, \tag{2.12}$$

$$T_{tx} = T_{xt} = \frac{\partial\phi(x,t)}{\partial t}\frac{\partial\phi(x,t)}{\partial x}. \tag{2.13}$$

Let us define $|0_L\rangle$ as the vacuum associated with the discrete modes. To find the energy in the vacuum state, we need to evaluate $\langle 0_L|T_{tt}|0_L\rangle$. Note that, the vacuum state associated with the discrete modes has the property:

$$|0_L\rangle \longrightarrow |0\rangle, \tag{2.14}$$

as $L \longrightarrow \infty$, where $|0\rangle$ is the usual Minkowski space vacuum. Now, from equation (2.9)

$$u_k(x,t) = (2L\omega)^{-\frac{1}{2}} e^{i(kx-\omega t)},$$

and, from equation (1.20),

$$\phi(x,t) = \sum_k \frac{1}{(2L|k|)^{1/2}}[\hat{a}_k u_k(x,t) + \hat{a}_k^\dagger u_k^*(x,t)], \tag{2.15}$$

Now, differentiating with respect to t, we obtain

$$\frac{\partial \phi(x,t)}{\partial t} = \sum_k \frac{1}{(2L|k|)^{1/2}}[\hat{a}_k u_k(-i\omega) + \hat{a}_k^\dagger u_k^*(i\omega)], \tag{2.16}$$

Differentiating with respect to x, we obtain

$$\frac{\partial \phi(x,t)}{\partial x} = \sum_k \frac{1}{(2L|k|)^{1/2}}[\hat{a}_k u_k(ik) + \hat{a}_k^\dagger u_k^*(-ik)], \tag{2.17}$$

Therefore, we find that

$$T_{tt} = T_{xx} = \sum_k \frac{|k|}{2L}[\hat{a}_k \hat{a}_k^\dagger + \hat{a}_k^\dagger \hat{a}_k]. \tag{2.18}$$

So,

$$\begin{aligned}
\langle 0_L|T_{tt}|0_L\rangle &= \sum_k \frac{|k|}{2L}\langle 0_L|\hat{a}_k \hat{a}_k^\dagger|0_L\rangle \\
&= \frac{1}{2L}\sum_k |k| \\
&= \frac{1}{2L}\sum_{n=-\infty}^{n=+\infty} \frac{2\pi n}{L} \\
&= \frac{2\pi}{L^2}\sum_{n=0}^{n=\infty} n,
\end{aligned} \tag{2.19}$$

which is clearly infinite. So, the vacuum state field energy diverges.

Hence, we see that the energy of the $R^1 \times S^1$ spacetime becomes infinite if we consider all the frequency modes from 0 to ∞. The compactified spatial sections can modify the long wavelength modes, but the ultraviolet region still diverges.

A similar situation arises in Minkowski spacetime. However, in case of the Minkowski spacetime while calculating the vacuum energy, the ultraviolet divergence is removed by normal ordering with respect to the creation and annihilation operators of the Fock space associated with the modes in equation (2.9). In the case of a general state $|\psi\rangle$ in this Fock space, applying normal ordering we get,

$$\langle\psi| : T_{\alpha\beta} : |\psi\rangle = \langle\psi|T_{\alpha\beta}|\psi\rangle - \langle 0|T_{\alpha\beta}|0\rangle, \tag{2.20}$$

so

$$\langle 0| : T_{\alpha\beta} : |0\rangle = \langle 0|T_{\alpha\beta}|0\rangle - \langle 0|T_{\alpha\beta}|0\rangle \tag{2.21}$$

$$= 0. \tag{2.22}$$

Considering Minkowski space as the covering space of $R^1 \times S^1$, $|0_L\rangle$ can be considered as a state in the above Fock space. Hence, the divergence can be removed by applying equation (2.20) as follows,

$$\begin{aligned}\langle 0_L| : T_{tt} : |0_L\rangle &= \langle 0_L|T_{tt}|0_L\rangle - \langle 0|T_{tt}|0\rangle \\ &= \langle 0_L|T_{tt}|0_L\rangle - \lim_{L' \longrightarrow \infty} \langle 0_{L'}|T_{tt}|0_{L'}\rangle. \end{aligned} \tag{2.23}$$

Since both terms on the right-hand side of equation (2.23) are individually divergent, they cannot be subtracted without careful analysis. So, let us follow a simpler procedure here. We introduce a cut-off factor $e^{-\alpha|k|}$ into the divergent sums of the type of equation (2.19), and let $\alpha \longrightarrow 0$ at the end of the calculation. [2]

After introducing the cut-off factor, the sum in equation (2.19) becomes finite and can be readily performed :

$$\langle 0_L|T_{tt}|0_L\rangle = \frac{2\pi}{L^2}\sum_{n=0}^{\infty} n e^{-\alpha|k|} = \frac{2\pi}{L^2}\sum_{n=0}^{\infty} n e^{-\frac{2\pi\alpha n}{L}}. \tag{2.24}$$

For ease of calculation, let us consider :

$$\frac{2\pi\alpha}{L} = \beta \tag{2.25}$$

Therefore, the summation in equation (2.24) becomes,

$$\begin{aligned}\sum_{n=0}^{\infty} n e^{-\frac{2\pi\alpha n}{L}} = \sum_{n=0}^{\infty} n e^{-n\beta} &= -\frac{\partial}{\partial\beta}\sum_{n=0}^{\infty} e^{-n\beta} \\ &= \frac{1}{\beta^2} - \frac{1}{12} + \frac{\beta^2}{64} + \mathcal{O}(\beta^4).\end{aligned} \tag{2.26}$$

Therefore, substituting for β we get,

$$\langle 0_L|T_{tt}|0_L\rangle = \frac{1}{2\pi\alpha^2} - \frac{\pi}{6L^2} + \mathcal{O}(\alpha^4). \tag{2.27}$$

A similar expression is obtained for $\langle 0_{L'}|T_{tt}|0_{L'}\rangle$ as well. Taking the limit,

$$\lim_{L'\longrightarrow\infty}\langle 0_{L'}|T_{tt}|0_{L'}\rangle = \frac{1}{2\pi\alpha^2}. \tag{2.28}$$

Substituting equation (2.27) and equation (2.28) in equation (2.23) and taking $\alpha \longrightarrow 0$, we find

$$\langle 0_L|: T_{tt} :|0_L\rangle = -\frac{\pi}{6L^2}. \tag{2.29}$$

Since $\langle 0_L|: T_{tt} :|0_L\rangle = \langle 0_L|: T_{xx} :|0_L\rangle$, we can write

$$\langle 0_L|: T_{xx} :|0_L\rangle = -\frac{\pi}{6L^2}. \tag{2.30}$$

Thus, we see that, although $\langle T_{\alpha\beta}\rangle$ diverges when evaluated for both states $|0\rangle$ and $|0_L\rangle$, the difference between the two results is finite. If we require that $\langle 0|: T_{\alpha\beta} :|0\rangle = 0$, then the state $|0_L\rangle$ contains a finite, negative energy density

$$\rho = \langle 0_L|: T_{tt} :|0_L\rangle = -\frac{\pi}{6L^2}, \tag{2.31}$$

and pressure

$$p = \langle 0_L|: T_{xx} :|0_L\rangle = -\frac{\pi}{6L^2}. \tag{2.32}$$

Thus, the cloud of negative vacuum energy is distributed uniformly throughout the $R^1 \times S^1$ universe with total energy $-\frac{\pi}{6L}$.

2.3 Parallel plates in (3+1) dimensions

So far we have seen a simple case in (1+1) dimensions, that is, one spatial and one time dimension. Let us now consider a more realistic case in (3+1) dimensions. To illustrate this, let us consider two large, parallel, perfectly conducting uncharged plates. Let the plates be squares of size L and let them be placed at a distance a from each other, with $a \ll L$.

Now, let us consider the modes inside the volume $L^2 a$. Since, the component k_z is perpendicular to the plate, this component will be quantised, so

$$k_z = \frac{n\pi}{a},$$

where, $n = 0, 1, 2, \cdots$.
The other two components, k_x and k_y, which are parallel to the plate are continuous. Also, we should remember that there are two polarization states in general. If, however, k_z vanishes then only one mode survives.

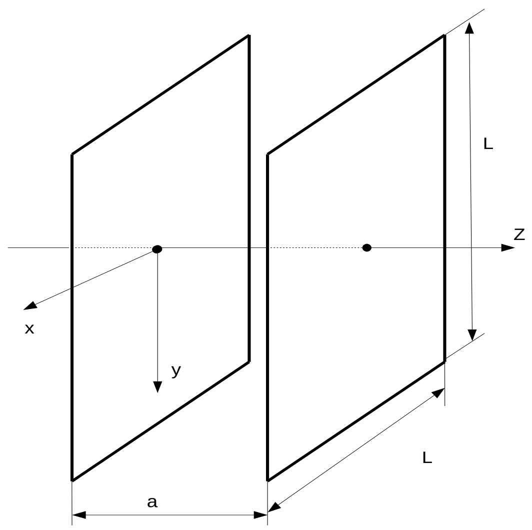

Fig. 2: Two parallel plates placed in vacuum a few micrometre apart.

Therefore the zero point energy of the configuration is [3]

$$\begin{aligned} E &= \sum_k \frac{1}{2}\hbar\omega_k &(2.33)\\ &= \frac{\hbar c}{2}\sum_k |\vec{k}_k| \\ &= \frac{\hbar c}{2}\int L^2 \frac{d^2 k_\parallel}{(2\pi)^2}\left[|\vec{k}_\parallel| + 2\sum_{n=1}^{\infty}\left(k_\parallel^2 + \frac{n^2\pi^2}{a^2}\right)^{1/2}\right]. &(2.34)\end{aligned}$$

This expression is infinite. However, we need to subtract the free value which contributes to this same volume a quantity,

$$E_0 = \frac{\hbar c}{2} \int \frac{L^2 d^2 k_\parallel}{(2\pi)^2} \int_{-\infty}^{+\infty} \frac{a dk_z}{2\pi} 2\sqrt{k_\parallel^2 + k_z^2}. \tag{2.35}$$

Changing the integration variable from k_z to n,

$$\begin{aligned} E_0 &= \frac{\hbar c}{2} \int \frac{L^2 d^2 k_\parallel}{(2\pi)^2} \int_0^{+\infty} 2\frac{a}{2\pi} d\left(\frac{n\pi}{a}\right) 2\sqrt{k_\parallel^2 + \left(\frac{n^2\pi^2}{a^2}\right)} \\ &= \frac{\hbar c}{2} \int \frac{L^2 d^2 k_\parallel}{(2\pi)^2} \int_0^{+\infty} 2dn\sqrt{k_\parallel^2 + \left(\frac{n^2\pi^2}{a^2}\right)}. \end{aligned} \tag{2.36}$$

Therefore, energy per unit surface is the difference between the zero point energy of the configuration and the free value of the energy divided by the area of the plates, that is,

$$\mathcal{E} = \frac{E - E_0}{L^2} \tag{2.37}$$

$$\begin{aligned} &= \frac{\hbar c}{2\pi} \int_0^\infty \frac{d^2 k_\parallel}{(2\pi)^2} \left(\left[|\vec{k}_\parallel| + 2\sum_{n=1}^{\infty} \left(k_\parallel^2 + \frac{n^2\pi^2}{a^2}\right)^{1/2} \right] - \int_0^\infty 2dn\sqrt{k_\parallel^2 + \frac{n^2\pi^2}{a^2}} \right) \\ &= \frac{\hbar c}{2} \int_0^\infty \frac{2\pi k dk}{(2\pi)^2} \left(k + 2\sum_{n=1}^{\infty} \sqrt{k^2 + \frac{n^2\pi^2}{a^2}} - \int_0^\infty 2dn\sqrt{k^2 + \frac{n^2\pi^2}{a^2}} \right) \\ &= \frac{\hbar c}{2\pi} \int_0^\infty k dk \left(\frac{k}{2} + \sum_{n=1}^{\infty} \sqrt{k^2 + \frac{n^2\pi^2}{a^2}} - \int_0^\infty dn\sqrt{k^2 + \frac{n^2\pi^2}{a^2}} \right). \end{aligned} \tag{2.38}$$

This quantity is apparently still not defined due to ultraviolet (large k) divergences. However, for wavelength shorter than the atomic size it is unrealistic to use a perfect conductor approximation. Therefore, let us introduce in the above integral a smooth cut off function $f(k)$ equal to unity for $k \lesssim k_m$ and vanishing for $k \gg k_m$, where k_m is of the order of the inverse atomic size. Let us set $u = a^2k^2/\pi^2$, then $du = 2\frac{a^2}{\pi^2} k dk$. Therefore,

$$\begin{aligned} \mathcal{E} &= \frac{\hbar c}{2\pi} \int_0^\infty \frac{\pi^2}{2a^2} du \left(\frac{1}{2}\sqrt{\frac{\pi^2 u}{a^2}} + \sum_{n=1}^{\infty} \sqrt{\frac{\pi^2 u^2}{a^2} + \frac{n^2\pi^2}{a^2}} - \int_0^\infty dn\sqrt{\frac{\pi^2 u^2}{a^2} + \frac{n^2\pi^2}{a^2}} \right) f(k) \\ &= \frac{\hbar c\pi}{4a^2} \int_0^\infty du \left(\frac{1}{2}\sqrt{u}\sqrt{\frac{\pi^2}{a^2}} + \sum_{n=1}^{\infty} \frac{\pi}{a}\sqrt{u + n^2} - \int_0^\infty dn \frac{\pi}{a}\sqrt{u + n^2} \right) f(k) \\ &= \frac{\hbar c\pi^2}{4a^3} \int_0^\infty du \left[\frac{\sqrt{u}}{2} + \sum_{n=1}^{\infty} \sqrt{u + n^2} - \int_0^\infty dn\sqrt{u + n^2} \right] f(k) \end{aligned}$$

$$= \frac{\hbar c \pi^2}{4a^3} \int_0^\infty du \left[\frac{\sqrt{u}}{2} f\left(\frac{\pi}{a}\sqrt{u}\right) + \sum_{n=1}^{\infty} \sqrt{u+n^2} f\left(\frac{\pi}{a}\sqrt{u^2+n^2}\right) - \int_0^\infty dn \sqrt{u+n^2} f\left(\frac{\pi}{a}\sqrt{u+n^2}\right)\right]. \quad (2.39)$$

Let us define,

$$F(n) = \int_0^\infty du \sqrt{u+n^2} f\left(\frac{\pi}{a}\sqrt{u+n^2}\right). \quad (2.40)$$

Therefore, energy per unit surface is

$$\begin{aligned} \mathcal{E} &= \hbar c \frac{\pi^2}{4a^3}\left[\frac{1}{2}F(0) + \sum_{n=1}^{\infty} F(n) - \int_0^\infty dn F(n)\right] \\ &= \hbar c \frac{\pi^2}{4a^3}\left[\frac{1}{2}F(0) + F(1) + F(2) + \cdots - \int_0^\infty dn F(n)\right]. \end{aligned} \quad (2.41)$$

The interchange of the sum and integral was justified due to the absolute convergence in the presence of the cutoff function. As $n \longrightarrow \infty$, $F(n) \longrightarrow 0$. So, we use the Euler-MacLaurin formula to compute the difference between the sum and integral occuring in the above bracket: [3]

$$\frac{1}{2}F(0) + F(1) + F(2) + \cdots - \int_0^\infty dn F(n) = -\frac{1}{2!}B_2 F^{'}(0) - \frac{1}{4!}B_4 F^{'''}(0) + \cdots. \quad (2.42)$$

The Bernoulli numbers B_ν are defined through the series

$$\frac{y}{e^y - 1} = \sum_{\nu=0}^{\infty} B_\nu \frac{y^\nu}{\nu!}, \quad (2.43)$$

with $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $\cdots$. We have,

$$F(n) = \int_{n^2}^\infty du \sqrt{u} f\left(\frac{\pi\sqrt{u}}{a}\right). \quad (2.44)$$

Therefore,

$$F^{'}(n) = -2n^2 f\left(\frac{\pi n}{a}\right). \quad (2.45)$$

We assume that $f(0) = 1$, while its derivatives vanish at the origin, so that $F^{'}(0) = 0$, $F^{'''}(0) = -4$, and higher derivatives of F are equal to zero. Therefore, all reference to the cutoff has disappeared from the final result. Hence, energy per unit surface is

$$\mathcal{E} = \frac{\hbar c \pi^2}{a^3}\frac{B_4}{4!} = -\frac{\pi^2}{720}\frac{\hbar c}{a^3}. \quad (2.46)$$

So, the force per unit area $\mathcal{F}$ is,

$$\mathcal{F} = -\frac{\pi^2}{240}\frac{\hbar c}{a^4} = -\frac{0.013}{(a_{\mu m})^4}\text{dyn/cm}^2, \tag{2.47}$$

where $a_{\mu m}$ mean that the distance between the two plates is measured in units of micrometre. The negative sign implies that the force is attractive in nature.

As we can see, the strength of the force varies inversely with the fourth power of the distance between the plates. So the strength of the force falls off rapidly as the distance increases. Hence it is measurable when the distance between the plates is extremely small, say, only a few micrometers. In such range, the force becomes so strong that it becomes the dominant force between the two uncharged conductors.

To get an estimate of this force let us do an order of magnitude calculation. Let the distance between the two square plates be 1 μm while the other two dimensions be about 1 cm.
With $\hbar = 1.055 \times 10^{-27}\text{gcm}^2\text{s}^{-1}$, we get the force per unit area:

$$\begin{aligned}\mathcal{F} &= -\frac{(3.142)^2 \times 1.055 \times 10^{-27} \times 3 \times 10^{10}}{240 \times (1 \times 10^{-4})^4}\text{dyne/cm}^2 \\ &= -0.013\text{dyne/cm}^2.\end{aligned}$$

Therefore, if two 1 square centimetre conducting plates are placed in vacuum, 1 micrometre apart, then the force acting between them is about 0.013 dyne and the nature of the force is attractive. Though this force is pretty small, but it is measurable. In fact, measurement of this tiny force has been done in the year 1996 by Steven Lamoreaux. His results were in agreement with the theory to within the experimental uncertainty of 5%. [6]

So to conclude, the Casimir effect is a manifestation of quantum vacuum fluctuation. According to quantum field theory, the vacuum consists of fluctuating electromagnetic waves of all possible wavelengths which imbue it with a vast amount of energy, normally invisible to us. When two conducting plates are placed parallel to each other in vacuum, then between the two plates only those unseen electromagnetic waves whose wavelengths fit a whole number of times into the gap contribute to the vacuum energy. As the gap between the plates is narrowed, say to a few micrometers, fewer waves can contribute to the vacuum energy and so the energy density between the plates falls below the energy density of the surrounding space. This results into a tiny force between the plates, trying to pull the plates together. The force that exists between the two plates is the Casimir force. This force has been experimentally measured and thus provides proof of existence of the quantum vacuum.

Chapter 3

Particle Production

In this section we study particle production in a constant electric field background. Let us consider two oppositely charged plates placed in vacuum. If σ is the surface charge density on the plates and d is the separation distance between them, then, a constant electric field

$$E = \frac{\sigma}{\epsilon_0}$$

is produced in the region between them and is directed from the positively charged plate towards the negatively charged plate. Here ϵ_0 is the permittivity of free space.
Since potential is

$$V = -\int_d^0 \vec{E}.d\vec{r},$$

the electromagnetic potential developed in the empty space between the two plates is

$$V = \frac{\sigma d}{\epsilon_0}.$$

The energy between the two plates can be obtained from the potential as,

$$\mathcal{E} = QV = \sigma AV = \frac{\sigma^2 Ad}{\epsilon_0},$$

where A is the surface area of the plates.
Hence, either by increasing the charge density on the plates or by maintaining a greater distance between the plates or by increasing the surface area of the plates, we can increase the energy of the field in the space between them. If this energy becomes equal to, say, $2m_ec^2$, where m_ec^2 is the rest mass energy of an electron, then a virtual electron-positron pair is created in the vacuum. If the electric field energy is much greater than $2m_ec^2$, then

the virtual electron-positron pair produced will gain kinetic energy from the field and travel towards the positive and negative plates, respectively. So the particle-antiparticle pair no longer remain as virtual particles. This is how particle production can take place in vacuum.

3.1 Bogolubov Transformation

We know, the field $\phi(\vec{x}, t)$ can be expressed as in equation (1.20),

$$\phi(\vec{x}, t) = \sum_i [\hat{a}_i u_i(\vec{x}, t) + \hat{a}_i^\dagger u_i^*(\vec{x}, t)]. \tag{3.1}$$

We can also consider a second complete orthonormal set of modes $\bar{u}_j(\vec{x}, t)$. The field $\phi(\vec{x}, t)$ may, then, be expanded in this set as,

$$\phi(\vec{x}, t) = \sum_j [\hat{\bar{a}}_j \bar{u}_j(\vec{x}, t) + \hat{\bar{a}}_j^\dagger \bar{u}_j^*(\vec{x}, t)]. \tag{3.2}$$

Corresponding to this decomposition of $\phi(\vec{x}, t)$ we define a new vacuum state $|\bar{0}\rangle$

$$\hat{\bar{a}}_j |\bar{0}\rangle = 0, \forall j \tag{3.3}$$

and a new Fock space.

As both sets are complete, the new modes $\bar{u}_j(\vec{x}, t)$ can be expanded in terms of the modes $u_i(\vec{x}, t)$ in the following manner,

$$\bar{u}_j(\vec{x}, t) = \sum_i (\alpha_{ji} u_i(\vec{x}, t) + \beta_{ji} u_i^*(\vec{x}, t)). \tag{3.4}$$

Conversely,

$$u_i(\vec{x}, t) = \sum_j (\alpha_{ji}^* \bar{u}_j(\vec{x}, t) - \beta_{ji} \bar{u}_j^*(\vec{x}, t)). \tag{3.5}$$

These relations are known as Bogolubov transformations and the matrices α_{ij}, β_{ij} are called Bogolubov coefficients. [2]

The Bogolubov coefficients can be evaluated as:

$$\alpha_{ij} = (\bar{u}_i, u_j) \tag{3.6}$$

and

$$\beta_{ij} = -(\bar{u}_i, u_j^*). \tag{3.7}$$

3.2 Pair Production in a constant electric field background

In this section, we will study the evolution of a quantum field in a constant electromagnetic background by the method of normal mode analysis. [4]

We will consider a system, which consists of a complex scalar field $\phi(\vec{x}, t)$, interacting with the electromagnetic field having the vector potential A^μ. The action describing the system is,

$$\begin{aligned} S[\phi(\vec{x},t), A^\mu] &= \int d^4x \mathcal{L}(\phi(\vec{x},t), A^\mu) \\ &= \int d^4x [(\partial_\mu \phi(\vec{x},t) + iqA_\mu \phi(\vec{x},t))(\partial_\mu \phi^*(\vec{x},t) - iqA_\mu \phi^*(\vec{x},t)) \\ &\qquad - m^2 \phi(\vec{x},t)\phi^*(\vec{x},t) - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}], \end{aligned} \tag{3.8}$$

where q and m are the charge and mass associated with a single quantum of the complex scalar field and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

We will assume that the electromagnetic field behaves classically, and the complex scalar field is a quantum field. Therefore, A^μ is just a c-number and $\phi(\vec{x}, t)$ is an operator valued distribution.

Varying the action, with respect to the complex scalar field $\phi(\vec{x}, t)$, we obtain the Klein-Gordon equation:

$$(\partial_\mu \phi(\vec{x},t) + iqA_\mu)(\partial^\mu \phi(\vec{x},t) + iqA^\mu) + m^2 \phi(\vec{x},t) = 0. \tag{3.9}$$

The electromagnetic background we will consider in this section is a constant electric field described by the field vector

$$\vec{E} = E\hat{x}, \tag{3.10}$$

where, E is a constant and $\hat{x}$ is the unit vector along the positive x-axis. We will describe this electromagnetic background using the time dependent vector potential

$$A_1^\mu = (0, -Et, 0, 0). \tag{3.11}$$

3.3 Bogolubov Coefficients

We will begin by quantising the complex scalar field $\phi(\vec{x}, t)$ in the time dependent gauge A_1^μ. Then, we will obtain a complete set of orthonormal solutions which can be identified as positive and negative frequency solutions in the asymptotic past, that is, at $t \longrightarrow -\infty$. We will identify as positive frequency modes those solutions which have a decreasing phase. In a similar manner, we can obtain the positive and negative frequency modes in the asymptotic future, that is, as $t \longrightarrow \infty$. [4]

Since the vector potential, A_1^μ, is time dependent, a mode which is purely positive frequency in the infinite past will evolve into a combination of positive and negative frequency modes in the infinite future. This phenomenon can be interpreted as particle production. [4]

Substituting the vector potential in equation (3.11) in the Klein-Gordon equation, equation (3.9), we obtain that,

$$\begin{aligned}(\partial_\mu \partial^\mu + 2iqA_\mu \partial^\mu - q^2 A_\mu A^\mu + m^2)\phi = 0 \\ (\partial_t^2 - \bigtriangledown^2 - 2iqEt\partial_x + (qEt)^2 + m^2)\phi = 0.\end{aligned} \tag{3.12}$$

The mode function for the scalar field $\phi(\vec{x}, t)$ can be decomposed as $u_k(t, \vec{x}) \propto f_k(t) e^{(i\vec{k}.\vec{x})}$, [4] where $\vec{k} \equiv (k_x, k_y, k_z) = (k_x, k_\perp)$ The function $f_k(t)$ satisfies equation (3.12) :

$$\begin{aligned}\frac{d^2 f_k}{dt^2} + [(k_x^2 + k_\perp^2) + 2qEtk_x + (qEt)^2 + m^2]f_k = 0 \\ \frac{d^2 f_k}{dt^2} + [m^2 + k_\perp^2 + (k_x + qEt)^2]f_k = 0.\end{aligned} \tag{3.13}$$

For ease of solving, we introduce new variables,

$$\tau = \sqrt{qE t} + (k_x/\sqrt{qE}) \tag{3.14}$$

$$\lambda = (k_\perp^2 + m^2)/qE \tag{3.15}$$

$$\nu = -(1 - i\lambda)/2. \tag{3.16}$$

Substituting these new variables, we get,

$$\frac{d^2 f_k}{d\tau^2} + (\tau^2 + \lambda)f_k = 0. \tag{3.17}$$

The solution of a differential equation which has the above form are the parabolic cylinder function. So, the solutions of equation (3.17) are

$$D_{\nu^*}((1+i)\tau), D_\nu((1-i)\tau), D_{\nu^*}(-(1+i)\tau), D_\nu(-(1-i)\tau), \tag{3.18}$$

where $D_\nu(z)$ is the parabolic cylinder function. [5]

From the asymptotic properties of the parabolic cylinder functions, we find that as $\tau \longrightarrow -\infty$

$$D_\nu(-(1-i)\tau) \longrightarrow (\sqrt{2}|\tau|)^\nu e^{-i\pi\nu/4} \exp i(\tau^2/2) \tag{3.19}$$

and

$$D_{\nu^*}(-(1+i)\tau) \longrightarrow (\sqrt{2}|\tau|)^{\nu^*} e^{-i\pi\nu^*/4} \exp -i(\tau^2/2). \tag{3.20}$$

Whereas, as $\tau \longrightarrow \infty$

$$D_\nu((1-i)\tau) \longrightarrow (\sqrt{2}|\tau|)^\nu e^{-i\pi\nu/4} \exp i(\tau^2/2) \tag{3.21}$$

and

$$D_{\nu^*}((1+i)\tau) \longrightarrow (\sqrt{2}|\tau|)^{\nu^*} e^{-i\pi\nu^*/4} \exp -i(\tau^2/2). \tag{3.22}$$

Since, the positive frequency mode should have a decreasing phase in the $\tau \longrightarrow -\infty$ limit, it is clear from the asymptotic forms of the parabolic cylinder functions that $D_\nu(-(1-i)\tau)$ is the positive frequency mode as $\tau \longrightarrow -\infty$. From the same argument, we can say that $D_{\nu^*}((1+i)\tau)$ is the positive frequency mode in the limit $\tau \longrightarrow \infty$.

Evolving $D_\nu(-(1-i)\tau)$ to $\tau \longrightarrow \infty$, we find that, [5]

$$D_\nu(-(1-i)\tau) = -\left(\frac{\sqrt{2\pi}}{\Gamma(-\nu)}\right) e^{i\pi(\nu-1)/2} D_{\nu^*}((1+i)\tau) + e^{i\pi\nu} D_\nu((1-i)\tau), \tag{3.23}$$

where $\Gamma(-\nu)$ is the Gamma function.

Therefore, The Bogolubov coefficients are:

$$\alpha(k) = -\left(\frac{\sqrt{2\pi}}{\Gamma(-\nu)}\right) e^{i\pi(\nu-1)/2} = \left(\frac{\sqrt{2\pi} e^{-(\lambda-i)\pi/4}}{\Gamma[(1-i\lambda)/2]}\right) \tag{3.24}$$

and

$$\beta(k) = e^{i\pi\nu} = e^{-(\lambda+i)\pi/2}. \tag{3.25}$$

Also,

$$|\alpha(k)|^2 = 1 + \exp -(\pi\lambda) \tag{3.26}$$

and

$$|\beta(k)|^2 = \exp -(\pi\lambda); \tag{3.27}$$

Therefore, from equation (3.26) and (3.27) we can clearly see that

$$|\alpha(k)|^2 - |\beta(k)|^2 = 1. \tag{3.28}$$

These results imply that the number of particles corresponding to the quantum scalar field produced by the electric field background is

$$|\beta(k)|^2 = \exp -(\pi(m^2 + k_\perp^2)/qE). \tag{3.29}$$

So from equation (3.29) we can see that

$$\beta(k) \propto e^{-(1/E)}.$$

So, when the electric field is zero, the number of particles produced is zero. This is as expected.

Chapter 4

Summary

Let us now briefly summarize the content of this report. We started with the interpretation of the Klein-Gordon equation as a single particle equation, and understood the difficulties that arises due to such an interpretation. We then quantised the field $\phi(\vec{x}, t)$ and reinterpreted the Klein-Gordon equation as a field equation. This was followed by the study of Casimir effect. To keep matters simple, we first studied Casimir effect by considering a cylinder in the two-dimensional Minkowski spacetime. From this we saw that the magnitude of the energy density is proportional to the second power of the circumference of the cylinder. This study was followed by a more realistic case in (3+1) dimensions. Here, we find that the magnitude of the force is inversely proportional to the fourth power of the distance between the two plates. This helps us appreciate the distance scale between the two plates required to observe the Casimir effect. In both cases, the force is observed to be attractive in nature. We then studied pair production in a constant electric field background, described by a time dependent gauge. We begin by quantising the scalar field $\phi(\vec{x}, t)$ in a time dependent gauge. We then use special functions and their properties to calculate the Bogolubov coefficients. From the value of these coefficients we can read off the number of particles produced by the quantum scalar field in a constant electromagnetic background described by the time dependent gauge.

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