Accelerating universe and the nature of dark energy

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by

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CERTIFICATE

This is to certify that the project titled **Accelerating universe and the nature of dark energy** is a bona fide record of work done by **Tirthankar Banerjee** towards the partial fulfillment of the requirements of the Master of Science degree in Physics at the Indian Institute of Technology, Madras, Chennai 600036, India.

(L. Sriramkumar, Project supervisor)

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ABSTRACT

About a decade and a half ago, observations from a special class of supernovae revealed certain facts about our universe that came as a surprise to the larger physics community. Until then, it was commonly believed that, being dominated by matter, the universe was undergoing a phase of deceleration. But, the supernovae observations hinted otherwise, and they seem to predict that the universe is, in fact, currently accelerating. Since the initial developments, observations of supernovae at increasingly higher redshifts have corroborated the original results. Our main aim in this work will be to reanalyze some of the supernovae data and arrive at the by-now established conclusion that the universe is presently in a state of accelerated expansion. An accelerating universe, in turn, implies that the universe is dominated by dark energy. While a cosmological constant is found to fit the data quite well, we also attempt to understand the nature of dark energy by modeling it in terms of scalar fields.

Contents

Chapter 1

Introduction

Cosmology aims to answer questions which have intrigued humankind, such as the birth of the universe, its evolution and its probable end. Theoretically, Einstein's general relativity serves as the bedrock for gaining an understanding of the universe on the largest scales. The famous discovery of the redshifting galaxies by Hubble suggested that the universe is expanding. The Cosmic Microwave Background (CMB) points to the fact that the energy density of radiation during the current epoch is negligible when compared to that of ordinary, non-relativistic matter. In order to describe the universe, one often assumes that it is homogeneous and isotropic. This crucial assumption is confirmed by the observations of the distribution of galaxies by surveys such as the two degree Field survey (2dF) and the Sloan Digital Sky Survey (SDSS) [1, 2].

Had the universe been composed only of matter and radiation, the dominance of matter during the present epoch would lead to the deceleration of the universe due to the gravitational attraction. But the determination of distances to far away galaxies through observations of a special class of supernovae have revealed that the universe today is actually dominated by another type of matter which, in fact, leads to an acceleration of the universe. This additional component does not interact with matter or radiation, and hence is often referred to as the dark energy.

1.1 Standard candles, and the role of supernovae

Consider a source which emits light of a given wavelength, say, λ_{e}. By the time the light reaches us, the universe would have expanded. As we shall illustrate, a consequence of the expansion is the fact that the wavelengths of light from other galaxies are stretched, and what we actually observe is light of a different wavelength, λ_{o}. This phenomenon is known as redshift, which is characterized by the redshift parameter, $z = (\lambda_{\mathrm{o}} - \lambda_{\mathrm{e}})/\lambda_{\mathrm{e}}$. It so happens that the measurement of the brightness of far away galaxies as a function of the redshift proves to be as a very effective tool in understanding the composition and, hence, the dynamics of the universe.

One should appreciate the fact the redshift is an easily measurable quantity. While the redshift associated with a single spectra line will, evidently, be impossible to determine, the presence of a wide variety of elements (and at different levels of ionization) results in a pattern of a large number of lines in the spectra of galaxies. It is the complete pattern that facilities the determination of the redshift of galaxies.

In contrast, the distance to an astronomical object is tricky to arrive at. While methods such as trigonometric parallax are sturdy and straightforward, their utility turns out to be limited when the distances involved are rather great and, hence, the angles involved very small. It is in this context that the concept of a standard candle comes in useful (for an extended discussion on this topic, see, for instance, Ref. [3]). By a standard candle one refers to an astronomical object with a known intrinsic luminosity or brightness. Given the intrinsic luminosity, the observed brightness then allows us to estimate the distance to the object. A simple example can illustrate the point. If one is informed that a light bulb emits $100\ \mathrm{W}$ of energy, then measuring the flux at our location allows us to arrive at the distance to the bulb from us. Supernovae are exploding stars, which turn extraordinarily bright at their peak intensity that they often outshine their host galaxy. Certain types of supernovae—known as type Ia—exhibit such strong universal characteristics (which is related to the way they were formed) that their intrinsic, maximum brightness is very closely related to the pattern of the rise and fall of their intensities with time. The universal characteristic of these type Ia supernovae allow them to be used as standard candles. Moreover, their high peak brightness admit us to utilize them to determine distances over cosmological scales. Various observational missions such as the Supernova Cosmology Project (SCP) [4] and the Super Nova Legacy Survey (SNLS) [5] have assembled an impressive amount of data, which have led to interesting implications for our understanding of the universe.

1.2 The composition of the universe

The universe is found to be homogeneous and isotropic over suitably large length scales, roughly of the order of $70\ \mathrm{Mpc}$ [1, 2]. The evolution of such a smooth universe is determined by its matter content through the Einstein's equations of the general theory of relativity. In an evolving universe, the distances to far away galaxies determined through their intrinsic luminosities and observed fluxes depend on the evolution. Therefore, determining the distances to far objects as a function of time or, equivalently, redshift, allows us to arrive at the composition of the universe.

Evidently, the universe contains matter and radiation. The universe will currently be going through a decelerating phase if it had contained no other component of matter. We had mentioned above that the type Ia supernovae facilitates the determination of the distances to galaxies very far away. As we shall discuss, interestingly, the data point to the fact that the universe today is dominated by another component of matter which can be conveniently referred to as the cosmological constant or, more generally, as dark energy. Moreover, the dominance of such a component of matter implies that the universe is currently accelerating.

1.3 Scalar fields as models of dark energy

Cosmological constant is called so since, in contrast to other components, it energy density remains unvarying as the universe evolves. Phenomenologically, a cosmological constant fits the supernovae data quite well. However, even a component whose energy density varies mildly with time (such a type of matter is often referred to as the dark energy) leads to an equally good fit to the data.

The attractive aspect of dark energy is that it allows modeling in terms of scalar fields. We shall discuss as to how certain types of scalar fields, called quintessence, can act as a viable alternative to the cosmological constant.

1.4 Notations and conventions

We shall set $\hbar = c = 1$, but shall display G explicitly, and define the Planck mass to be $M_{_{\mathrm{Pl}}} = (8\,\pi\,G)^{-1/2}$.

We shall work in $(3 + 1)$-spacetime dimensions, and adopt the metric signature of $(+, -, -, -)$. The Greek and Latin indices shall denote the spacetime and the spatial coordinates, respectively,

Chapter 2

The Friedmann universe

In this chapter, we shall provide a quick review of the Friedmann model of the universe. We shall discuss the motivation behind working with this model. We shall first present the arguments which lead us to consider the Friedmann line element for describing our universe. We shall then analyze the geometry associated with the Friedmann metric.

2.1 The Friedmann-Robertson-Walker metric

Observations of the CMB suggest that the universe is isotropic to one partial in 10^5 [6]. This, in turn, implies that the universe was highly homogeneous at an early epoch when the radiation decoupled from matter. Even though the anisotropies are present, they do not come into the picture when the large scale structure of the universe needs to be discussed. By homogeneous, we mean that there exists no uneven distribution of matter in three dimensional space, and isotropic implies that there is no favored direction. Homogeneity and isotropy in turn imply that the metric describing the spatial part of the universe should be spherically symmetric.

Therefore, the spatial part of the line element can be expressed as

$$\mathrm{d}l^2 = e^{\lambda(r)}\,\mathrm{d}r^2 + r^2\,\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) = l_{ab}dx^a dx^b, \tag{2.1}$$

where the quantity $e^{\lambda(r)}$ cannot depend on θ and ϕ if spherical symmetry needs to be preserved. For a space of constant curvature, we can express the Riemann tensor as

$$R_{abcd} = \kappa(l_{ac}l_{bd} - l_{ad}l_{bc}) \tag{2.2}$$

for which the Ricci tensor is

$$R_{ab} = 2\kappa l_{ab} \tag{2.3}$$

On calculating the Ricci tensor associated with Eq. (2.1) and using Eq. (2.3), we arrive at the following equations

$$\frac{1}{r}\frac{\mathrm{d}\lambda}{\mathrm{d}r} = 2\kappa e^{\lambda} \tag{2.4}$$

and

$$1 + \left(\frac{r}{2}\right)\left(\frac{d\lambda}{dr}\right)e^{-\lambda} - e^{-\lambda} = 2\kappa r^2 \tag{2.5}$$

Hence integrating the above equations , we have,

$$e^{\lambda(r)} = \frac{1}{(1-\kappa r^2)} \tag{2.6}$$

When $r \neq 0$, we rescale r and make κ equal to 1 or -1. We will see in the next section that these values of κ have interesting properties and how κ affects the nature of the three space associated with the line element. Now the complete metric can be written as

$$\mathrm{d}s^2 = g_{00}\mathrm{d}t^2 + 2g_{0i}\mathrm{d}t\mathrm{d}x^i + g_{ij}\mathrm{d}x^i\,\mathrm{d}x^j. \tag{2.7}$$

Isotropy of the expansion implies that all g_{0i} must vanish. A non-zero g_{0i} is related to a three vector v_i which points along the direction specified by the index i. Moreover in a coordinate system determined by the fundamental observers (to whom the universe appears isotropic), the spacelike surfaces can be labeled by using the proper time of clocks carried by these observers. This choice of t implies $g_{00} = 1$.

So the metric that describes the spacetime of the universe is given by

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\,\left[\frac{\mathrm{d}r^2}{(1-\kappa r^2)} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\right] \tag{2.8}$$

the factor, $a(t)$ is known as the expansion factor or the scale factor. Note that a cannot be a function of any other coordinate other than t because the expansion is independent of position and direction. The above line-element given by Eq. (2.8) is known as the Friedmann line element.

In the following section, we shall try to understand the geometry of the spatial hypersurfaces of the Friedmann model. We shall consider the cases with $\kappa = 0$,1, -1 and understand the nature of the three space in each of these cases.

2.2 Understanding the geometry of the Friedmann universe

In order to understand the geometry of the Friedmann model, let us introduce a new coordinate, χ ([7], [8]), where

$$\chi = \int \frac{\mathrm{d}r}{\sqrt{(1-\kappa r^2)}}. \tag{2.9}$$

On integration, we obtain that

$$\chi = \sin^{-1} r, \quad r, \quad \sinh^{-1} r \tag{2.10}$$

for κ=1, 0, -1, respectively. The line element now, as a function of χ is given by

$$\mathrm{d}l^2 = a^2[\mathrm{d}\chi^2 + S_\kappa^2(\chi)\mathrm{d}\Omega_2^2] \tag{2.11}$$

where $\mathrm{d}\Omega_2$ denotes the infinitesimal solid angle and

$$S_\kappa(\chi) = \begin{cases} \sin\chi & \text{for} \quad \kappa = 1, \\ \chi & \text{for} \quad \kappa = 0, \\ \sinh\chi & \text{for} \quad \kappa = -1. \end{cases} \tag{2.12}$$

Let us first work with the simplest case, i.e. when $\kappa = 0$. This represents a flat three dimensional space defined by the relation

$$\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 = \text{constant}. \tag{2.13}$$

If we choose our coordinates such that

$$x_1 = a\,\chi\,\sin\theta\,\cos\phi, \quad x_2 = a\,\chi\,\sin\theta\,\sin\phi, \quad x_3 = a\,\chi\,\cos\theta \tag{2.14}$$

we obtain that $\mathrm{d}l^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2$. This Euclidean space is covered by the coordinate range $0 \leq \chi < \infty$, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. Hence, for $\kappa = 0$, we have the *spatially* flat universe.

Now let us consider the $\kappa = 1$ case. We have $r = \sin\chi$. In this case, Eq. (2.11) represents a three sphere which is embedded in a flat four dimensional Euclidean space described by, say, the line element

$$\mathrm{d}l^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \mathrm{d}x_4^2 \tag{2.15}$$

The definition of such a three sphere is given by

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = a^2, \tag{2.16}$$

where a is the radius of the sphere. Now, if we choose the four coordinates such that

$$x_1 = a\cos\chi, \quad x_2 = a\sin\chi\sin\theta\cos\phi, \quad x_3 = a\sin\chi\sin\theta\sin\phi, \quad x_4 = a\sin\chi\cos\theta, \tag{2.17}$$

then, evidently, we will the be able to satisfy the above constraint.

The metric given by Eq.(2.11) now can be written in terms of θ, ϕ and the new angle variable χ.

$$dl^2_{3-sphere} = a^2(t)\left[\, d\chi^2 + \sin^2\chi(\, d\theta^2 + \sin^2\theta\, d\phi^2)\right] \tag{2.18}$$

The coordinate range $0 \leq \chi \leq \pi$, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$ covers the entire three space and has a volume given by

$$V = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{\pi} d\chi\sqrt{g} = a^3 \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\; d\theta \int_0^{\pi} \sin^2\chi\; d\chi = 2\pi^2 a^3 \tag{2.19}$$

For $\chi = \text{constant}$, we have a two sphere with surface area $4\pi a^2\sin^2\chi$. The dependence of the surface area on χ is clear. It increases with χ till $\chi = \frac{\pi}{2}$, where it is maximum and decreases thereafter till χ reaches π. Hence the $\kappa = 1$ model is known as the *closed* model.

Let us now investigate the case when the curvature term κ has value -1. In this case, $r = \sinh\chi$; For $\kappa = -1$, Eq.(2.11) represents a hyperboloid which is embedded in a four dimensional Minkowski space. This hyperboloid is defined by the relation

$$x_1^2 - x_2^2 - x_3^2 - x_4^2 = a^2 \tag{2.20}$$

If we choose coordinates such that

$$x_1 = a\cosh\chi, \quad x_2 = a\sinh\chi\sin\theta\cos\phi, \quad x_3 = a\sinh\chi\sin\theta\sin\phi, \quad x_4 = a\sinh\chi\cos\theta \tag{2.21}$$

we will see that the above constraint (2.20) is satisfied.

We can then write the metric given by Eq.(2.11) in terms of θ, ϕ and the new variable χ.

$$\mathrm{d}l^2_{hyperboloid} = a^2(t)[\mathrm{d}\chi^2 + \sinh^2\chi(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)] \tag{2.22}$$

The coordinate range to encompass this hypersurface is given by $0 \leq \chi < \infty, 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi$. The surface area for a constant χ is given by $4\pi a^2\sinh^2\chi$ and it keeps on increasing with χ. Hence it is also referred to as the *open* model.

Chapter 3

Kinematics of the Friedmann universe

In this chapter, we shall discuss the kinematical properties of the Friedmann universe. We shall look at the evolution of the frequency of radiation and the momenta of material particles with the scale factor of the expanding universe. We define the redshift parameter z. We shall introduce the Hubble constant and see how it is related to the age of the universe. Next, we define the concept of luminosity distance in an expanding Friedmann universe.

3.1 Redshift

In order to understand the behavior of motion of photons or particles in the universe, we need to begin with the concept of geodesics. A geodesic on a surface is the shortest path between two points. For any spacetime,the geodesic equation is given by [9]

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = 0, \tag{3.1}$$

where

$$u^\alpha = \frac{dx^\alpha}{\mathrm{d}s} \tag{3.2}$$

The quantity Γ^a_{bc} denotes the Christoffel symbols associated with the metric tensor describing the spacetime.

For the Friedmann metric, we have from the zeroth component of the geodesic equation

$$\frac{\mathrm{d}^2 t}{\mathrm{d}s^2} + \left(\frac{a\dot{a}}{1-\kappa r^2}\right)\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 = 0 \tag{3.3}$$

Also for material particles, we have

$$u^\mu u_\mu = 1 \tag{3.4}$$

which comes from the fact $p_\mu p^\mu = m^2c^2$. On combining the last two equations, we arrive at the following result

$$\frac{\mathrm{d}^2 t}{\mathrm{d}s^2} + \left(\frac{\dot{a}}{a}\right)\left[\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 - 1\right] = 0, \tag{3.5}$$

Integrating Eq. (3.5) we get the relation

$$a\left[\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 - 1\right]^{1/2} = \text{constant} \tag{3.6}$$

Using Eqs. (3.6) and (3.4), we get

$$|p|^2 \propto a^{-2}, \tag{3.7}$$

where, $|p|$ denotes the amplitude of the three momentum of the particle. So, in the Friedmann universe, the momentum of a particle goes down as a^{-1}.

Let us now investigate the behavior of photons in the Friedmann universe. Since $\mathrm{d}s^2 = 0$ for light rays, they satisfy the relation $u^\mu u_\mu = 0$. Now in Eq.(3.3),if we replace the quantity $\mathrm{d}s$ by a parameter say , $\mathrm{d}\lambda$, where λ is the affine parameter [9] , we will see that the frequency of a photon satisfies the relation, $\omega \propto a^{-1}$.

Astronomically, the redshift parameter z is defined as

$$1 + z = \frac{\lambda_o}{\lambda_e} = \frac{\omega_e}{\omega_o} \tag{3.8}$$

Cosmologically, since the redshift occurs because of the expansion of the universe, we can write

$$1 + z = \frac{\omega_e}{\omega_o} = \frac{a(t_o)}{a(t_e)}, \tag{3.9}$$

where ω_e represents the frequency of light emitted at a certain epoch (t_e) and ω_o represents the frequency received at the present epoch (t_o). λ_e and λ_o are the corresponding associated wavelengths [9]. $a(t_o)$ and $a(t_e)$ refer to the scale factor of the universe today and at the time of emission, respectively. In other words, $z = 0$ corresponds to the present epoch.

3.2 The luminosity distance in the Friedmann universe

For light rays moving radially in a Friedmann universe, we can write,

$$\frac{dt^2}{a^2} = \frac{dr^2}{(1 - \kappa r^2)} \tag{3.10}$$

As the universe expands during the emission time of the photon,t_e to the receiving time,t_o, we have,

$$\int_{t_e}^{t_o} \frac{1}{a}\, dt = \int_{r_{em}}^{0} \frac{1}{\sqrt{1-\kappa r^2}}\, dr \tag{3.11}$$

But using Eq. (3.9), we get

$$\frac{dt}{a} = \frac{dt}{dz}\frac{dz}{a}\frac{da}{da} = \frac{dz}{a_o} d_H(z) \tag{3.12}$$

with

$$d_H(z) = \frac{a}{\dot{a}} = \frac{1}{H(z)} \tag{3.13}$$

It may be noted that in a spatially flat universe (i.e. when $\kappa = 0$), we have

$$r_{em}(z) = \int_0^z \frac{d_H(z)}{a_o}\, dz \tag{3.14}$$

The luminosity of a source is the energy emitted by it per unit time. The flux received by an object at a certain distance is the energy received per unit area per unit time. So, if the distance between the source and the receiver is d_1, then, the flux, say, F is given by

$$F = \frac{L}{4\pi d_1^2}, \tag{3.15}$$

where L is the luminosity. Hence, the distance , $d_1 = \sqrt{\frac{L}{4\pi F}}$. We obtain the flux , F from observations. If we can know L, we will be able to determine the distance between the source and the receiver. But due to the evolution of the Friedmann universe, we have the frequency of light , ω decreasing as a^{-1}. Hence

$$\frac{\mathrm{d}E_{em}}{\mathrm{d}E_{rec}} = \left(\frac{a_o}{a}\right) \tag{3.16}$$

The subscripts *em* and *rec* refer to emitted and received energies, respectively. Again as dt is proportional to $(dE)^{-1}$, we have,

$$\frac{\mathrm{d}t_{em}}{\mathrm{d}t_{rec}} = \left(\frac{a_o}{a}\right) \tag{3.17}$$

Hence the luminosity,

$$L = \frac{\mathrm{d}E_{em}}{\mathrm{d}t_{em}} = \left(\frac{a_o}{a}\right)^2 \left(\frac{\mathrm{d}E_{rec}}{\mathrm{d}t_{rec}}\right) = \frac{1}{(1+z)^2}\left(\frac{\mathrm{d}E_{rec}}{\mathrm{d}t_{rec}}\right) \tag{3.18}$$

So the received flux is given by

$$F = \frac{(dE_{rec}/dt_{rec})}{4\pi a_o^2 r_{em}^2 (1+z)^2} \tag{3.19}$$

Comparing Eq. (3.15) with Eq.(3.19), we arrive at an expression for luminosity distance in Friedmann universe in terms of a and z,

$$d_{\mathrm{L}}(z) = a_o r_{em}(1+z), \tag{3.20}$$

where $a_o = a(t_o)$

Hence substituting with Eq.(3.14), we see that Eq.(3.20) gives the required expression for the luminosity distance in a spatially flat Friedmann universe.

3.3 The Hubble's law

For sufficiently nearby galaxies, observations show that there exists a linear relation between the redshift and the distance, a relation which is known as the Hubble's law [10]. For nearby galaxies, we can write

$$a(t_e) = a(t_o)\left[1 + \left(\frac{\dot{a}}{a}\right)_{t_o}(t_e - t_o)\right] \tag{3.21}$$

so that

$$\frac{a(t_e)}{a(t_o)} = 1 + H_0\,(t_e - t_o), \tag{3.22}$$

where H_0 is known as the Hubble constant. Now from Eq. (3.9) we can write

$$\frac{1}{1+z} = \frac{a(t_e)}{a(t_o)} \tag{3.23}$$

Expanding the left hand side of this equation up to first order and comparing with Eq.(3.22 we arrive at the relation

$$z = H_0\,(t_o - t_e) \tag{3.24}$$

Also, for a nearby galaxy, we can write,

$$r_{em} = \int_{t_e}^{t_o} \frac{\mathrm{d}t}{a(t)} \simeq \int_{t_e}^{t_o} \frac{\mathrm{d}t}{a_o} \simeq \frac{1}{a_o}\,(t_o - t_e) \tag{3.25}$$

Therefore, we have, in such a case,

$$d_{\mathrm{L}}(z) = a_o r_{em}(1+z) \simeq (t_o - t_e)\;(1+z)\; \simeq H_0^{-1}\,z, \tag{3.26}$$

where, in arriving at the final expression, we have retained terms only up to first order in z. So we see that up to first order in z the redshift velocity, $v = cz$ varies linearly with the proper distance, $a_o r_{em}$. Hence if we know the distance, $d_{\rm L}$ which can be measured by measuring the flux of objects of known luminosity, a plot of $d_{\rm L}(z)$ with z will help us to determine H_0. This quantity , H_0 determines the rate at which the universe is presently expanding.Observations(see, for instance [9]) have shown that the value of H_o is around $72\,\mathrm{km}\,\mathrm{sec}^{-1}\mathrm{Mpc}^{-1}$. It is clear that H_o has dimensions of time inverse.

The quantity, Hubble time is defined as,

$$t_H = \frac{1}{H_0} = 9.78h^{-1}Gyr, \tag{3.27}$$

where $h \approx 0.72$ and Gyr means 10^9 years.
The significance of t_H is that it gives us a rough estimate of the *age* of our universe.

We have thus arrived at an expression for the luminosity distance which will be used later for analyzing the supernovae data. In the next section we will study the dynamics of the Friedmann universe. We will look at the behavior of the scale factor, a when different energy densities dominate the universe. We will also plot contours for the age of universe as functions of different energy densities.

Chapter 4

Dynamics of the Friedmann universe

4.1 The Friedmann equations and the evolution of the various components of matter

Given a stress energy tensor, say, $T^{\mu\nu}$, the dynamics of a given spacetime is governed by the following Einstein's equations [8, 9]

$$G^{\mu\nu} = (8\pi G)T^{\mu\nu}, \tag{4.1}$$

where G is Newton's gravitational constant. The quantity $G_{\mu\nu}$ is known as the Einstein tensor which is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{4.2}$$

with R being the scalar curvature.

In case of a perfect fluid, the stress energy tensor is given by [8]

$$T^{\mu\nu} = (\rho + p)u^{\mu}u^{\nu} + p\eta^{\mu\nu} \tag{4.3}$$

ρ is the energy density associated with the fluid and p is the pressure and u^{μ} is the four velocity of the fluid.

The homogeneity and isotropy of the Friedmann universe implies that the stress energy tensor associated with any matter component should be diagonal and of the following form: $T^{\mu}_{\nu} = \mathrm{diag.}\ (\rho, -p, -p, -p)$. Upon making use of this stress energy tensor in the Einstein's equations (4.1), we arrive at the following Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} = \frac{8\pi G}{3}\rho, \tag{4.4}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} = -(8\pi G)p. \tag{4.5}$$

The above equations lead to

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{4.6}$$

The universe will have an accelerated expansion if $\ddot{a} > 0$. It is clear from Eq.(4.6) to achieve this, the energy density, ρ *must be less than 3p*.

The equation of state for a perfect fluid is defined by the relation

$$p = w\rho, \tag{4.7}$$

where, w is known as the equation of state parameter. The fact that the energy-momentum tensor has to be conserved leads to the equation

$$T^{\mu}_{\nu;\mu} = 0 \tag{4.8}$$

In a Friedmann universe, the time component of this equation leads to

$$\frac{d}{dt}(\rho a^3) = -p\frac{da^3}{dt} \tag{4.9}$$

or

$$\frac{d}{da}(\rho a^3) = -(3a^2 p) \tag{4.10}$$

For the case of a perfect fluid described by the equation of state (4.7), upon integrating the above differential, we obtain that

$$\rho \propto a^{-3(1+w)} \tag{4.11}$$

This result determines the evolution of the density of the different components of the universe. Now from the relations between energy densities and the scale factor we thus see
$\rho_m(t) \propto a(t)^{-3}$, $\rho_R(t) \propto a(t)^{-4}$, $\rho_\Lambda(t) = constant$,
where $\rho_m(t)$, $\rho_R(t)$, $\rho_\Lambda(t)$ are the energy densities of non relativistic matter, radiation and the cosmological constant , respectively. It is worth noting that when $w = -1$, ρ becomes independent of the scale factor, a. Such a component of matter, whose energy density remains a constant despite the expansion is known as the cosmological constant.

The Friedmann line element in terms of cosmic time, t is written as

$$ds^2 = dt^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{(1-\kappa r^2)} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right] \tag{4.12}$$

If we make a change of variable such that $t = \int a(\eta)\,\mathrm{d}\eta$, the above line element takes the following form

$$ds^2 = a^2(\eta)\left[d(\eta)^2 - \frac{\mathrm{d}r^2}{(1-\kappa r^2)} - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right] \tag{4.13}$$

η is called the *conformal time*.

In the first Friedmann equation, if we substitute $\kappa = 0$ and the relation between ρ and a, we arrive at the dependence of a on time for a universe comprising of a perfect fluid,

$$a(t) \propto t^{2/3(1+w)} \tag{4.14}$$

In terms of the conformal time, thus we have

$$a(\eta) \propto \eta^{(2/1+3w})} \tag{4.15}$$

Hence we see,

$$\eta = t^{(\frac{1+3w}{3(1+w)})} \tag{4.16}$$

4.2 The different epochs of the universe

The first Friedmann equation can be written as

$$\frac{\kappa}{a^2} = \left(\frac{\dot{a}}{a}\right)^2 \left[\left(\frac{8\pi G}{3H^2}\right)\rho - 1\right] \tag{4.17}$$

where $\frac{3H^2}{8\pi G} = \rho_c$ and $H = \frac{\dot{a}}{a}$ with ρ_c being a quantity that is known as the critical density. The density parameter, Ω is defined as

$$\Omega = \frac{\rho}{\rho_c} \tag{4.18}$$

So Eq.(4.17) becomes

$$\frac{\kappa}{a^2 H^2(t)} = \Omega(t) - 1 \tag{4.19}$$

At this stage one should note that for $\kappa = 0$, i.e., when the universe is spatially flat, Ω is always equal to 1.

For a flat universe, we have

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho. \tag{4.20}$$

But $\rho \propto a^{-3(1+w)}$. Hence using Eq. (4.20) we get

$$\dot{a} \propto a^{-\frac{1}{2}(1+3w)} \tag{4.21}$$

So for $w = -1$, $a(t) \propto e^{\lambda t}$ and for $w \neq -1$, $a(t) \propto t^{\frac{2}{3(1+w)}}$

From the equation of state, we know the following:

1. For non-relativistic matter,$w = 0$, hence $a(t) \propto t^{2/3}$.
2. For radiation, $w = \frac{1}{3}$, hence $a(t) \propto t^{1/2}$.

In a universe dominated by a perfect fluid, using Eq.(4.19) and Eq.(4.1), we then arrive at the following result

$$\frac{d\Omega}{d(\ln a)} = a\frac{d\Omega}{da} \tag{4.22}$$

We can see that the energy density of radiation decays faster than matter and the cosmological constant as the universe expands. Therefore, if we go back in time, the radiation energy density will start dominating. Hence the very early universe had to be radiation dominated, then it was matter dominated and the late-time universe is thus supposed to be dominated by the cosmological constant. So, now we can write

$$\frac{\rho_{_{\rm NR}}(t)}{\rho_{_{\rm NR}}(0)} = \left(\frac{a_0}{a}\right)^3 = (1+z)^3 \tag{4.23}$$

But, since

$$\Omega_{_{\rm NR}} = \frac{\rho_{_{\rm NR}}}{\rho_c(0)} \tag{4.24}$$

we can write

$$\rho_{_{\rm NR}}(t) = \Omega_{_{\rm NR}}\rho_c(0)\left(\frac{a_o}{a}\right)^3 \tag{4.25}$$

The total energy density of the Friedmann universe can be written as

$$\rho_{total}(t) = \rho_{_{\rm NR}}(t) + \rho_R(t) + \rho_\Lambda = \rho_c(0)\left[\Omega_{_{\rm NR}}\left(\frac{a_o}{a}\right)^3 + \Omega_R\left(\frac{a_o}{a}\right)^4 + \Omega_\Lambda\right] \tag{4.26}$$

4.3 Components of the universe

Plugging the value of ρ_{total} in the first Friedmann equation,viz. Eq.(4.5), we get

$$H^2 + \frac{\kappa}{a^2} = \frac{8\pi G}{3}\rho_c(0)\left[\Omega_{\rm NR}\left(\frac{a_o}{a}\right)^3 + \Omega_R\left(\frac{a_o}{a}\right)^4 + \Omega_\Lambda\right] \tag{4.27}$$

Using the definition of ρ_c and the redshift parameter, z we finally have

$$\left(\frac{H(z)}{H_0}\right)^2 = \left[\Omega_{\rm NR}(1+z)^3 + \Omega_R(1+z)^4 + \Omega_\Lambda - (\Omega - 1)(1+z)^2\right] \tag{4.28}$$

with

$$\Omega = \Omega_{\rm NR} + \Omega_R + \Omega_\Lambda \tag{4.29}$$

If we use the value of $H_0 = 72$ km $s^{-1}\mathrm{Mpc}^{-1}$, then we will get $\rho_c(0) \approx 10^{-26} kg/m^3$.

The Cosmic Microwave Background is considered as the most dominant contribution to the radiation energy density of the universe today [6]. The temperature, T corresponding to the CMB is obtained from the peak of the Planckian curve. This turns out to be , $T_o \approx 2.73$ K [6]. The radiation energy density follows Stefan's law,.i.e, $\rho_R = \sigma T^4$, where σ is Stefan's constant. So we can calculate Ω_R today since we know values of the critical density, ρ_c.

$$\Omega_R = \frac{\rho_R}{\rho_c} \tag{4.30}$$

Hence from Eq.(4.30), we get,

$$\Omega_R h^2 \approx 2.56 \times 10^{-5}, \tag{4.31}$$

where, $H_0 = 100h$ km$s^{-1}\mathrm{Mpc}^{-1}$. $h = \frac{H_o}{100}$

This is a very important result in our analysis. As we shall now show, this will help us to identify the redshift at which the energy densities of matter and radiation were equal. Let us call this redshift as z_{eq}.

At z_{eq}, $\rho_{\rm NR} = \rho_R$. Hence we can show

$$(1 + z_{eq}) = \left(\frac{\Omega_{\rm NR}}{\Omega_R}\right) \approx 3.9 \times 10^4(\Omega_{\rm NR} h^2) \tag{4.32}$$

We will now calculate the temperature of the radiation at this epoch.

The energy density of radiation, $\rho_R \propto a^{-4}$. Since radiation also follows Stefan's law, $\rho_R \propto T^4$. So using these two arguments we can say

$$T \propto a^{-1} \tag{4.33}$$

So we have

$$T_{eq} = T_o(1 + z_{eq}) \approx 9.04 \times (\Omega_{\mathrm{NR}} h^2)\, eV \tag{4.34}$$

So we find that the universe has cooled down considerably (to 2.73 K) from what it was during the equality of matter and radiation (around 10^4 K). Similarly we can also find the redshift when the non-relativistic matter density and the cosmological constant were equal. This turns out to be approximately 0.282.

We will now derive expressions for the age of the universe in terms of the parameters Ω_{NR}, Ω_R , Ω_Λ and h. Next we will plot contours for the age of the universe as functions of these parameters.

4.4 Age of the universe

From Eq. (4.28), we can write

$$\frac{da}{adt} = H_0\sqrt{\Omega_{\mathrm{NR}}(1+z)^3 + \Omega_R(1+z)^4 + \Omega_\Lambda - (\Omega - 1)(1+z)^2} \tag{4.35}$$

We can change the variable a to z by using the relation

$$da = -\frac{adz}{1+z} \tag{4.36}$$

Hence integrating Eq.(4.35) from $t = 0$ to the present time we get the expression,

$$t_{age} = \int_0^\infty \frac{dz}{100h(1+z)\sqrt{\Omega_{\mathrm{NR}}(1+z)^3 + \Omega_R(1+z)^4 + \Omega_\Lambda - (\Omega - 1)(1+z)^2}} \tag{4.37}$$

So we can evaluate the age of the universe for different functions of the parameters h and the densities. We are going to use $\Omega_R h^2 = 2.56 \times 10^{-5}$ and $\Omega = \Omega_{\mathrm{NR}} + \Omega_R + \Omega_\Lambda$.

The plots are obtained in the $\Omega_{\mathrm{NR}} - h$ plane. Each contour corresponds to 10^9 years.

In the next chapter, we will work with the supernovae data. We will see which energy component dominates the universe today. We will try to get constraints on the energy densities and the equation of state parameter, w.

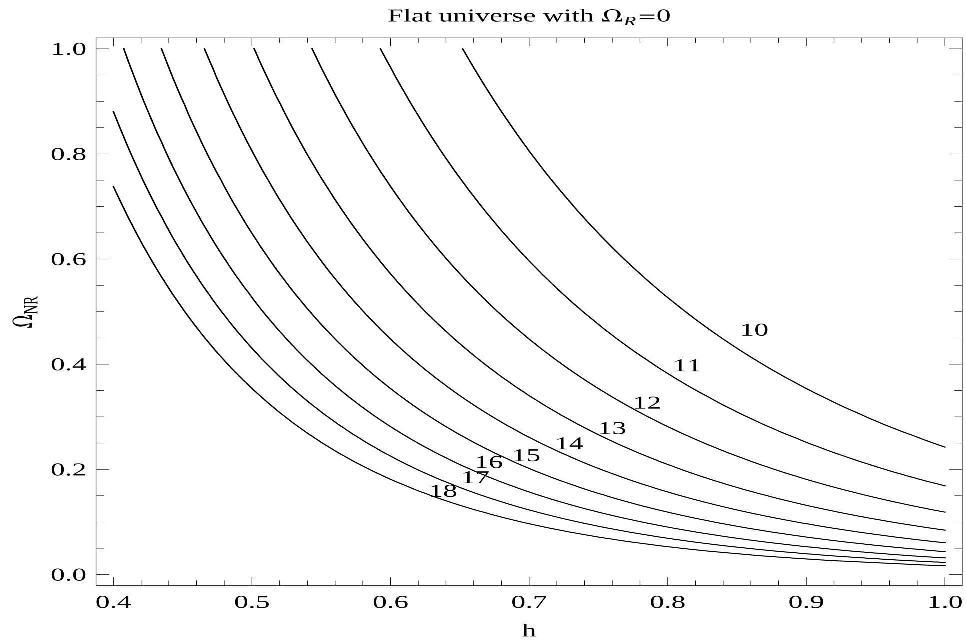

Figure 4.1: The age of the universe has been plotted as a function of the parameters Ω_{NR} and h. Given limits on the age of the universe, such as for instance, determined by the oldest observed object, the above contours allow us to arrive at constraints on the cosmological parameters. These contours correspond to a spatially flat universe with the radiation energy density set to zero.

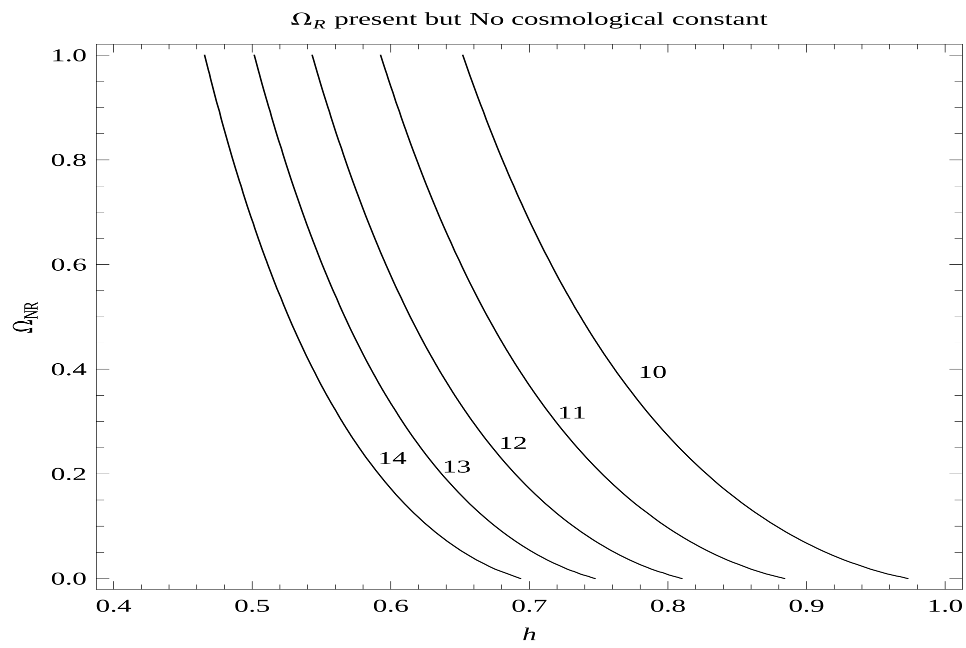

Figure 4.2: Contours for a non flat Friedmann universe comprising of no cosmological constant but both radiation and pressureless matter.

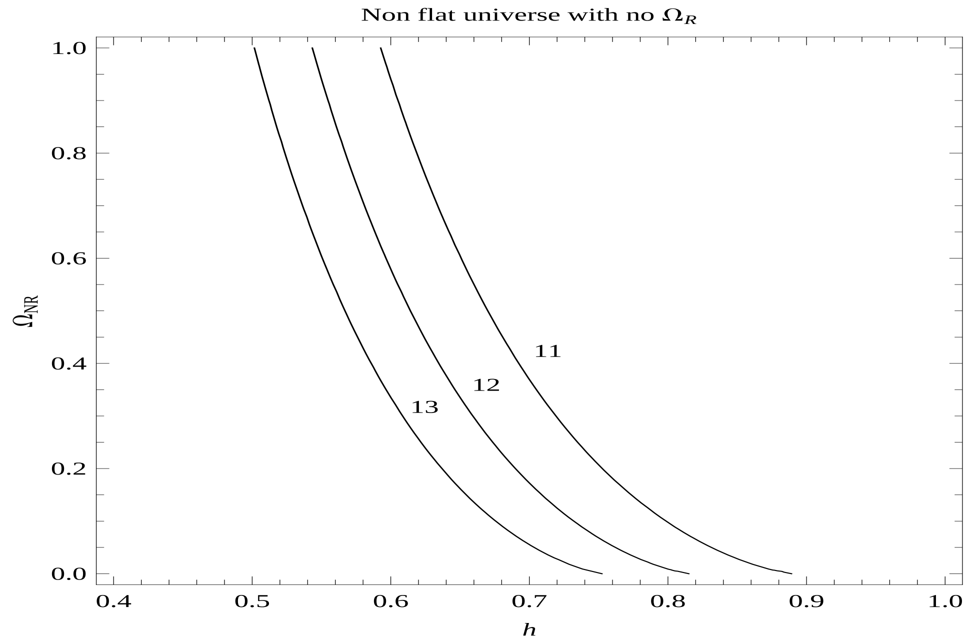

Figure 4.3: These contours represent a non flat universe with non zero cosmological constant and pressureless matter but no radiation.

Chapter 5

Distant supernovae and the accelerating universe

In this chapter, after a discussion on the classification of the various types of supernovae, we shall outline the reasons as to why a special class of the supernovae can act as a standard candle. Thereafter, we shall make use of the redshift-magnitude data available from SCP and SNLS to arrive at constraints on some of the cosmological parameters. Specifically, restricting ourselves to the case of the spatially flat Friedman model (i.e. the case wherein κ is zero), we shall vary the parameters $\Omega_{\rm NR}$ and w [cf. Eq. (4.11)] and utilize the standard chi-square minimization technique to arrive at the best fit values for these parameters from the data.

5.1 Classification of supernovae

Let us begin with a discussion on the classification of the various types of supernovae and the reasons behind choosing a certain type of supernovae as a standard candle.

During the final stages of stellar evolution, instabilities occur due to imbalances between sources of pressure and the gravitational forces in the core of stars. Such imbalances lead to an explosion if the mass of the star turns out to be much higher than the mass of the Sun. These explosions prove to be extremely powerful and their brightness remain fairly constant over an extended period of time (in fact, for tens of days). The term *supernova* was coined to describe these exploding stars. It was soon realized there exist two types of supernovae, which can be distinguished based on their spectra (for a detailed discussion on the the formation of stars and the evolution of some into supernovae, see Ref. [11])

The difference between the two types of supernovae is essentially the presence or absence of hydrogen in their spectra. Supernovae whose spectra do not contain prominent hydrogen lines are referred to as type I, and the remaining as type II. A careful infra-red analysis of the spectra later led to two further sub classes of the type I supernovae, which are called as types Ia and Ib. Type Ib supernovae are those which contain He I lines, while type Ia are classified as those supernovae which had neither He I nor Si II lines in their spectra. As we shall discuss in the following section, it is the type Ia supernovae which proves to be very useful as indicators of distances.

5.2 Type Ia supernovae as a standard candle

As we had discussed in the introductory chapter, a standard candle is an astronomical object whose intrinsic luminosity is largely independent of its location in the universe. Observationally, one measures the apparent luminosity. It is then clear that, if the intrinsic luminosity is known by other means, then the observed or the apparent luminosity will then allow us to estimate the distance to the source.

The intrinsic luminosities of the standard candles are arrived in two steps. Typically, a tight correlation between the apparent luminosity of such sources and another property (such as the period of variation in, say, variable stars) which is easily measurable is obtained. Then, if the distance to some of these objects are known through other unambiguous distance indicators (such as the simple trigonometric parallax), a relationship can then be constructed between the intrinsic luminosity of these sources and the simpler to measure property. It is this relation that then allows us to estimate the distance to the source.

It is in such a context that the bright and frequent type Ia supernovae come in extremely handy. Their frequency implies that we have a good opportunity to observe them, while their brightness allows us to resolve them even at very large distances. A type Ia supernova is easily identified by the absence of Si absorption feature at 6150 Å in their spectrum (which, in contrast, is present in type Ib). Further, they exhibit a characteristic light curve, viz. the rise and the fall of the brightness of the supernovae, which helps us determine their absolute luminosity (see Ref. [12]; in this context, also see Ref. [13]).

In the following section, using the SCP and the SNLS data and the concept of luminosity distance that we had defined earlier in Chap. 3, we shall numerically arrive at the constraints on the cosmological parameters.

5.3 Analysis of the SCP and the SNLS data

The SCP mission is more a decade old, and it was the first mission dedicated to the observations of the supernovae, aimed at determining the distances of far away galaxies and thereby understanding the corresponding implications for cosmology. SNLS is more recent and ongoing mission. It has been specifically designed for observing supernovae between redshifts of 0.3 and 1 and arriving at their precise characteristics. The main goal of the SNLS team has been to arrive at stronger constraints (than SCP could) on the cosmological parameters, in particular, on the equation of state of dark energy and also the value of the corresponding density parameter.

5.3.1 The redshift-magnitude relation

As we have mentioned, measurements of redshift of distant object is a straightforward exercise and poses no difficulties. The SCP and the SNLS teams actually measure the apparent magnitude m which is related to the absolute magnitude M and the luminosity distance $d_{\rm L}$ [cf. Eq. (3.20)] through the following relation (see, for example, Ref. [16]):

$$m(z) = M + 5\,\log_{10}\left[\frac{d_{\rm L}(z)}{1\ {\rm Mpc}}\right] + 25. \tag{5.1}$$

The absolute magnitude M is arrived at based on the light curves of the supernovae and the apparent magnitude m. According to the standard candle hypothesis, it is the intrinsic quantity M which is the same for all type Ia supernovae. Note that $d_{\rm L}(z)$ contains the quantity H_0. However, we shall choose to work with a quantity $Q(z)$ which is related to $d_{\rm L}(z)$ as follows:

$$Q(z) = H_0\, d_{\rm L}(z), \tag{5.2}$$

and it is important to mention here that the function $Q(z)$ is independent of the Hubble parameter H_0. In such a case, we have

$$\mu_{\rm B}(z) \equiv m(z) - M = 5\log_{10}[Q(z)] + 25 + 5\log_{10}\left[\frac{H_0^{-1}}{1\,{\rm Mpc}}\right]. \tag{5.3}$$

The data allow us to determine the quantity M to be $19.31 \pm 0.03 + 5\log_{10} h_{70}$ [18]. This in turn leads to

$$\mu_{\mathrm{B}}(z) = 43.15 + 5\log_{10}[Q(z)]. \tag{5.4}$$

SCP and SNLS provide us with μ_{B} and z for a collection of supernovae. Since the quantity $d_{\mathrm{L}}(z)$ depends on the cosmological parameters, our goal is to use the data to arrive at constraints on the parameters. Independent observations from the CMB point to the fact that our universe is spatially flat to a very good extent (see, for example, Ref. [6]). So, we shall assume that $\kappa = 0$ or, equivalently, $\Omega = 1$. Further, at late times such as today, the energy density in radiation can be completely ignored. These conditions leave us with only two cosmological parameters, viz. the non-relativistic matter density parameter (Ω_{NR}) and the equation of state parameter for dark energy (w) to be determined. We shall arrive at constraints on these parameters using the standard chi-squared technique of comparing theoretical models with the data.

Before we go on to discuss the results of the comparison, let us say a few essential words on the χ^2 technique.

5.3.2 The χ^2 technique and the best fit values

The χ^2 technique is a tool to compare theory with experimental or observational data (in this context, see, for instance, the standard textbook [17]). The less the value of the quantity χ^2, the better is considered the fit of the theoretical model to the observations. The quantity χ^2 is defined as

$$\chi^2 = \sum_i \left[\frac{y_i - f(x_i)}{\sigma_i}\right]^2, \tag{5.5}$$

where y_i is the value of a quantity as obtained from observations or experiments, while $f(x_i)$ is the corresponding theoretical prediction for the quantity. The quantity σ_i denotes the error associated with the measurement that yields the value y_i. Moreover, $f(x_i)$ contains the parameters for the model. The best fit values of the theoretical parameters are considered to be those values which correspond to the smallest χ^2.

5.3.3 The best fit values for the cosmological parameters

We had mentioned that we shall vary the non-relativistic matter density parameter ($\Omega_{\rm NR}$) and the equation of state parameter for dark energy (w) to arrive at the best fit values. In Figs. 5.1 and 5.2 below, after fixing w to be minus -1 (which corresponds to a cosmological constant), we have plotted χ^2 as a function of $\Omega_{\rm NR}$. Clearly, χ^2 exhibits a minimum in both the cases. We find that the minimal value of χ^2 corresponds to $\Omega_{\rm NR} \simeq 0.25$ for the SCP data and $\Omega_{\rm NR} \simeq 0.28$ for the SNLS data. In Fig. 5.3 below, we have plotted the theoretical curves

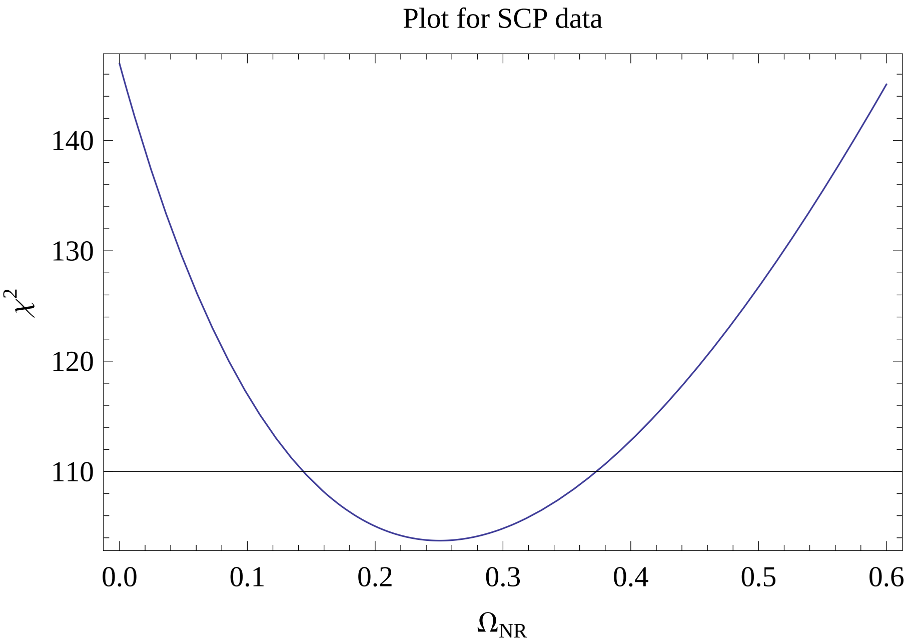

Figure 5.1: The quantity χ^2 has been plotted as a function of $\Omega_{\rm NR}$ with $w = -1$ for the SCP data. The minimum value for χ^2 is obtained for $\Omega_{\rm NR} \simeq 0.25$.

and the SNLS data or different values of $\Omega_{\rm NR}$ with w fixed to be -1. It is evident from the figure that an $\Omega_{\rm NR} \simeq 0.25$ fits the data better than the her two cases.

Relaxing the condition on w, in Fig. 5.4 below, we have plotted χ^2 as a function of $\Omega_{\rm NR}$ as well as w for the SNLS data. In such a case, we find that the minimal value of χ^2 occurs when $\Omega_{\rm NR} \simeq 0.25$ and $w \simeq -0.93$. In Fig. 5.5 below, we have plotted the theoretical curves for the luminosity distance and the SNLS data when $w \simeq -0.93$ for the three cases of $\Omega_{\rm NR}$ that we had plotted earlier in Fig. 5.3.

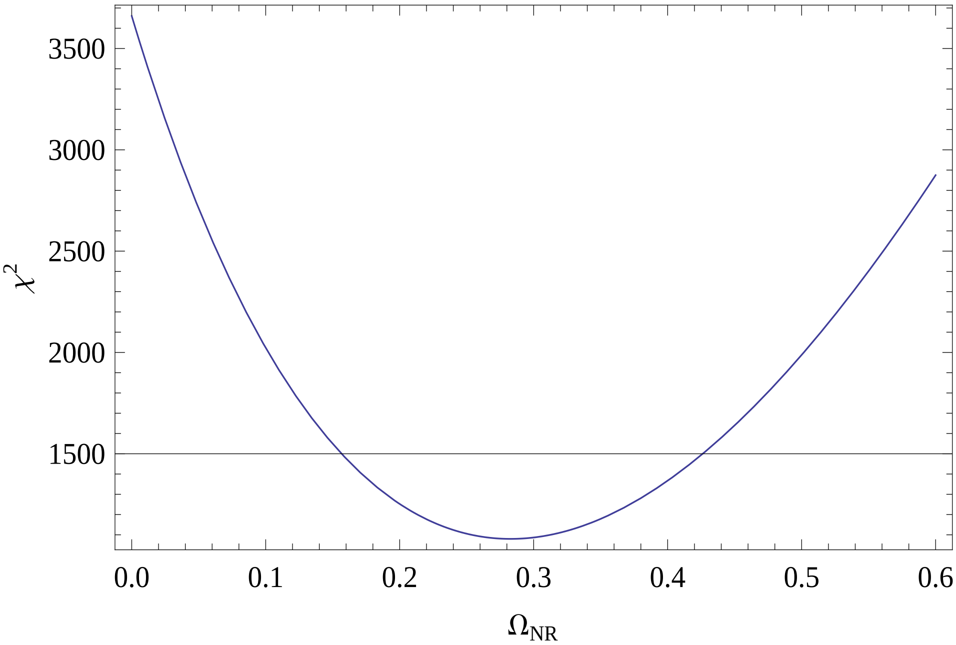

Figure 5.2: The quantity χ^2 plotted as a function of $\Omega_{_{\rm NR}}$ for the SNLS data with w set to -1. The minimum of χ^2 is found to be around $\Omega_{_{\rm NR}} \simeq 0.28$.

5.3.4 Implications

It is clear from the above discussion the introduction of a non zero cosmological constant provides (corresponding to $\Omega_\Lambda \simeq 0.75$) leads to an improved fit to the data. It is the fact that the universe is in fact dominated by the cosmological constant which leads to the inference that the universe is presently in an accelerating phase.

From the last section, it becomes quite clear from Fig. (5.5) that the presence of both matter and dark energy provides a much better fit to the data than the cases when only one component is present. We find that the value of the equation of state parameter, w turns out to be less than $-\frac{1}{3}$. So comparing with Eq.(4.6), we can see that the data indeed corresponds to an accelerating universe.

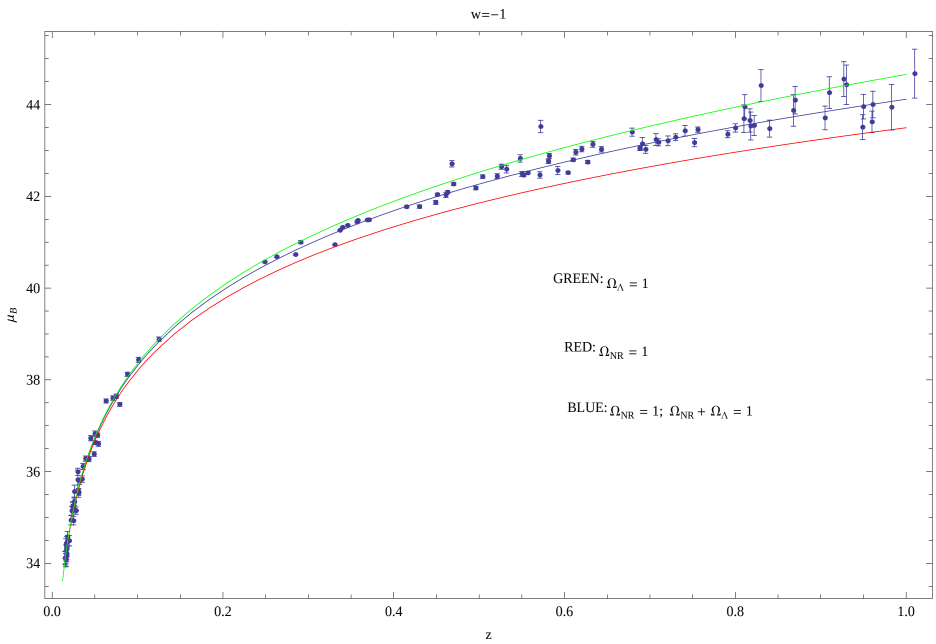

Figure 5.3: A plot of the theoretical curves as well as the SNLS data for the luminosity distance. The black dots with error bars denote the data. The green, the blue and the red curves are the theoretical predictions corresponding to $\Omega_{\rm NR}$ of zero, 0.28 and unity, with w set to be -1. It is visually evident that the blue theoretical curve corresponding to $\Omega_{\rm NR} \simeq 0.28$ fits the data better than the two curves.

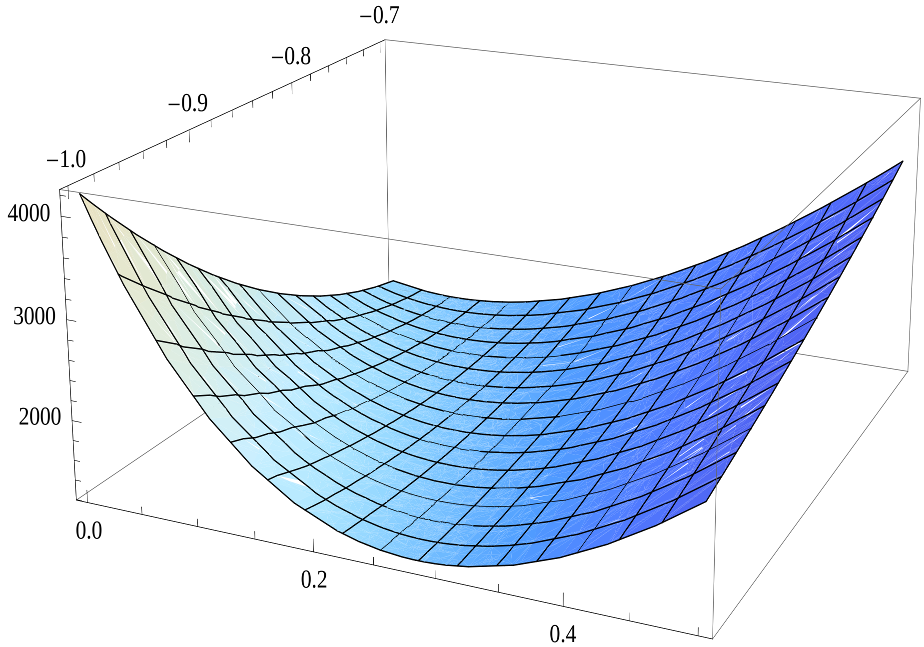

Figure 5.4: The quantity χ^2 for the SNLS data has been plotted as a function of $\Omega_{\rm NR}$ and w. While we have varied w from -1 to -0.7, $\Omega_{\rm NR}$ has been allowed to vary from 0 to 0.5. The minimum value of χ^2 in the $\Omega_{\rm NR}$-w plane is found to be located at $\Omega_{\rm NR} \simeq 0.25$ and $w \simeq -0.93$.

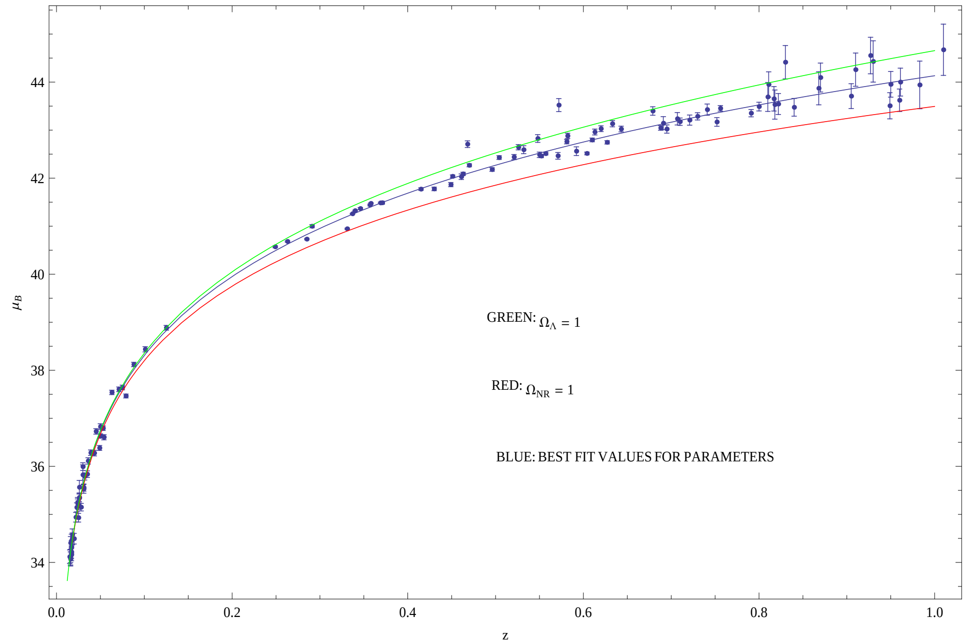

Figure 5.5: The theoretical for the luminosity distance plotted for the three different values of $\Omega_{\rm NR}$ that we had plotted earlier in Fig. 5.3. Note that, however, we have set w to be -0.93 in arriving at these curves. The blue curve, which corresponds to $\Omega_{\rm NR} \simeq 0.25$, clearly fits the data quite well.

Chapter 6

The nature of dark energy

As we have seen in the preceding chapters, the universe has three main components, namely, the non-relativistic matter density, Ω_{NR}, the radiation energy density, Ω_R and the term that was put by hand for better fitting of the data, the cosmological constant, Ω_Λ. The presence of Ω_Λ has a striking significance. This is what sources the acceleration of the universe

We now focus on the reasons behind the acceleration. We try to understand whether the dark energy is actually a constant or varies with time. Is it growing or decaying? A dark energy density that varies with time is also known as *quintessence*(for reference, see [20]).

Now we will try to arrive at a model for dark energy that drives the late-time acceleration of the universe. We will work with the scalar field model. The motivation behind this, as we will see comes from the facts, that firstly it is *easy* to arrive at conditions for the accelerating state of the universe using scalar fields. Secondly, for suitable choices of potentials, we will see that this model satisfies conditions for all the previous epochs the universe went through,i.e., the radiation and matter dominated epochs.

6.1 Scalar fields in the Friedmann universe

A *field* is characterized by infinite degrees of freedom. It is the carrier of interaction between particles. Locally, a field exists everywhere at every instant of time. A scalar field is one which is characterized by magnitude only,i.e., it does not depend upon directions.
If we do a coordinate transformation such that x^μ goes to x'^μ, then the scalar will satisfy the relation $\phi'(x'^\mu) = \phi(x^\mu)$

The action describing the scalar field comprises of two parts, the kinetic energy and the potential energy. In a flat spacetime, the quantity d^4x acts as the invariant infinitesimal 4-volume. But when the spacetime is curved, the actual invariant quantity is $\sqrt{-g}d^4x$, with g being the determinant of the metric tensor describing the spacetime.Moreover, the equation of motion for the scalar field should not contain more than second order time derivatives of the fundamental variable (ϕ) . These arguments motivate us to write down the action for the scalar field in the following manner.

$$S = \int \left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]\sqrt{-g}\,d^4x, \tag{6.1}$$

where $V(\phi)$ describes the potential energy associated with the scalar field ϕ.

The equation of motion for the field can be arrived at by varying Eq.(6.1) with respect to ϕ, which gives

$$\frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}g^{\mu\nu}\partial_\mu)\phi + \frac{\partial V}{\partial\phi} = 0 \tag{6.2}$$

So for the Friedmann metric, we arrive at the equation of motion for the scalar field which is given by

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = 0 \tag{6.3}$$

$V(\phi)$ in Eq.(6.3) denotes derivative w.r.t ϕ. We can also arrive at the expression for the Stress-Energy tensor, $T_{\mu\nu}$ associated with the scalar field by varying Eq.(6.1) with respect to the metric tensor $g_{\mu\nu}$. This will give the following relation:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi + V(\phi)g_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\partial^k\phi\partial_k\phi \tag{6.4}$$

We now formulate the components of $T_{\mu\nu}$, namely, the energy density, ρ and the pressure, p for the Friedmann universe. Since the universe is homogeneous and isotropic , we can say that the scalar field, ϕ doesn't have spatial dependence but that it only depends on time. Using this condition and Eq.(4.3) and (6.4), we get the relations

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{6.5}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{6.6}$$

These are the two most important equations which we will be frequently using in the subsequent sections. Substituting these values in Eq.(4.6), we arrive at the condition for the accelerating universe,viz.,

$$\dot{\phi}^2 < V(\phi) \tag{6.7}$$

In other words the universe dominated by a scalar field will accelerate when the potential energy dominates over the kinetic energy. We shall also discuss the equation of state parameter, w for the scalar field, ϕ.

$$w = \frac{p}{\rho} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \tag{6.8}$$

Clearly from Eq.(6.8), when $\dot{\phi}^2 << V(\phi)$, we find that $w \approx -1$. Thus when the potential energy term dominates the dynamics of the field much more than the kinetic term, the scalar field seems to behave as the *cosmological constant*.

In the next section we will work with the Inverse Power Law(I.P.L.) model, where the potential has an inverse power law dependence on ϕ. We will see how this potential satisfies conditions for both early-time and late-time universe. We will try to arrive at constraints on the parameters of this model which can satisfy conditions for late-time scalar field driven acceleration.

6.2 Quintessence

Potentials of the form ϕ^2 and ϕ^4 also lead to accelerated models of the universe. But these oscillatory models imply that the acceleration stops after some finite interval of time(because of the presence of the *friction* term in Eq.(6.3)). Although these models successfully explain the inflationary universe, they can't be used to describe the late time acceleration because unlike inflation, the late time acceleration is not supposed to be halted by any source of energy. So we need a model for the dark energy that satisfies conditions of both early and late time universe.

In the I.P.L. model, the potential energy density of the scalar field, ϕ takes the following form

$$V(\phi) = V_o(\phi)^{-\alpha}, \tag{6.9}$$

where $V_o = m^{4+\alpha}$ (where m has dimensions of mass) and $\alpha > 0$.This model, originally introduced by B.Ratra and P.J.E. Peebles was motivated from the ideas of SUSY and QCD [21]. We will now investigate the dynamics of the universe under the action of such a potential.

Let us first look into the radiation dominated epoch. We have seen in this case, the scale factor, $a(t) \propto t^{\frac{1}{2}}$. Hence,$H = \frac{1}{2t}$. The I.P.L. potential when substituted in Eq.(6.3) , gives

$$\ddot{\phi} + \frac{3}{2t}\dot{\phi} - \alpha\, m^{4+\alpha}\phi^{-\alpha-1} = 0 \tag{6.10}$$

We thus have a solution for ϕ given by

$$\phi = \left[\frac{\alpha(2+\alpha)^2 m^{4+\alpha} t^2}{6+\alpha}\right]^{\frac{1}{2+\alpha}} \tag{6.11}$$

Let us now perturb the field by a small amount, $\delta\phi$. We can see that this perturbation satisfies the relation

$$\delta\ddot{\phi} + \frac{3}{2t}\delta\dot{\phi} + \alpha(\alpha+1)\phi^{-\alpha-1}\delta\phi = 0 \tag{6.12}$$

Substituting the expression obtained for ϕ from Eq.(6.11) in Eq.(6.12) we get

$$\ddot{\phi} + \frac{3}{2t}\delta\dot{\phi} + \left[\frac{(\alpha+6)(\alpha+1)}{(2+\alpha)^2 t^2}\right]\delta\phi = 0 \tag{6.13}$$

Solving the above equation for the perturbation we arrive at the following result

$$\delta\phi \propto t^{\gamma}, \tag{6.14}$$

where

$$\gamma = -\frac{1}{4} \pm \sqrt{\frac{1}{16} - \frac{(\alpha+6)(\alpha+1)}{(2+\alpha)^2}} \tag{6.15}$$

In Eq.(6.15), γ turns imaginary for any $\alpha > 0$. So we see that the perturbations decay as $t^{-\frac{1}{4}}$ with increasing time.Thus Eq.(6.11) is called the tracker solution. It means that any other solution that comes close to ϕ will eventually approach it as t increases.Thus if we assume that ϕ obeys the tracker solution today, then we may say that this solution is insensitive to the initial conditions. This has the advantage that then there are only two parameters which we need to work with, namely m and α.

From Eq.(6.11) it is evident that both $\dot{\phi}^2$ and $V(\phi)$ go as $t^{-\frac{2\alpha}{2+\alpha}}$.Hence we find that ρ_R dominates over ρ_ϕ in the early universe since $\rho_R \propto t^{-2}$. So we see that the I.P.L. model satisfies conditions for the early universe.

In the matter dominated epoch, $\rho_{\rm NR} \propto t^{-2}$ and $\rho_R \propto t^{-\frac{8}{3}}$.But the tracker solution has the same behavior as in the radiation dominated epoch and so do $V(\phi)$ and $\dot{\phi}^2$. So it is clear that at late times ρ_ϕ will dominate over the other components of the universe. Let us now investigate the dynamics at very late times when the universe becomes dominated by the scalar field.

At very late times, Eq.(4.5) can be re-written in the form

$$H^2 = \frac{8\pi G}{3}\rho_\phi \tag{6.16}$$

Thus in the scalar field dominated epoch, the equation of motion becomes

$$\ddot{\phi} + \sqrt{24\pi G\rho_\phi}\dot{\phi} - \alpha m^{4+\alpha}\phi^{-\alpha-1} = 0 \tag{6.17}$$

We have already shown the condition for accelerating universe in Eq.(6.7). Using this we can say that at very late times, $\rho_\phi \approx V(\phi)$. We can also see this from the fact that the friction term in the equation of motion will eventually slow the growth of ϕ and hence $V(\phi)$ will start dominating over the kinetic term. The inertial term, $\ddot{\phi}$ will also be small compared to the other terms in the equation of motion with increasing time. We will show that these approximations are indeed satisfied by the model. In absence of the inertial and kinetic terms, Eq.(6.17) becomes

$$\sqrt{24\pi G m^{4+\alpha}\phi^{-\alpha}}\dot{\phi} - \alpha m^{4+\alpha}\phi^{-\alpha-1} = 0 \tag{6.18}$$

On solving for ϕ we get

$$\phi(t_l) = m\left[\frac{\alpha(2+\frac{\alpha}{2})t}{\sqrt{24\pi G}}\right]^{\frac{1}{2+\frac{\alpha}{2}}} \tag{6.19}$$

Let us now check the validity of the approximations used to get this solution.From Eq.(6.19), we see $\dot{\phi}^2 \propto t^{-\frac{2+\alpha}{2+\frac{\alpha}{2}}}$ and $V(\phi) \propto t^{-\frac{\alpha}{2+\frac{\alpha}{2}}}$. Thus at late times, $V(\phi) >> \dot{\phi}^2$.

To check the approximation related to the inertial term, we see $\ddot{\phi} \propto t^{-\frac{3+\alpha}{2+\frac{\alpha}{2}}}$ and $V_\phi \propto t^{\frac{-1-\alpha}{2+\frac{\alpha}{2}}}$. So clearly $\ddot{\phi} << V_\phi$ at late times.So we see that the solution obtained from Eq.(6.18) is valid for the accelerated universe at very late times.The late time solution, $\phi(t_l)$ is the asymptotic limit for the tracker solution,$\phi(t)$ (given by Eq.(6.11)) as $t \to \infty$. We have thus shown that the I.P.L. potential satisfies conditions for all the epochs the universe has gone through as well as for the late-time. It is now interesting to note the following cases.

From Eq. (6.9), we can say

$$\frac{\mathrm{d}V}{\mathrm{d}\phi} = \frac{-\alpha}{\phi} V(\phi) \tag{6.20}$$

and

$$\frac{\mathrm{d}^2V}{\mathrm{d}\phi^2} = \frac{\alpha(1+\alpha)}{\phi^2} V(\phi) \tag{6.21}$$

Thus we see that smaller values of α will lead to more flat potentials which will in turn lead to a more slowly varying behavior of the scalar field, ϕ. Hence from Eq. (6.8) we can say that smaller values of alpha will lead to values of w_ϕ closer to -1.

6.3 Relation between w_ϕ and α for different epochs

We have described the behavior of the scalar field and the associated quantities as functions of the cosmic time, t. It is preferable to express these behaviors as functions of the scale factor, a as we shall see that the relations between a and t for different epochs and the expression for evolution of the density of the quintessence field, ρ_ϕ can be used to express w_ϕ as a function of α in the respective epochs. This can help us to arrive at constraints on the value of the exponent parameter, α if constraints on w are available from observations.

Let us investigate the radiation dominated epoch. We have seen in Eq. (6.11) that $\phi \propto t^{2/2+\alpha}$. So $\rho \propto t^{\frac{-2\alpha}{2+\alpha}}$. But we also know that in the radiation dominated epoch, $a(t) \propto t^{1/2}$. Hence in the radiation dominated epoch, we can write

$$\rho \propto a^{\frac{-4\alpha}{\alpha+2}} \tag{6.22}$$

Therefore using the above expression and Eq. (4.11), we get the following relation in a radiation dominated universe

$$w_\phi = \frac{\alpha - 6}{3(\alpha+2)} \tag{6.23}$$

Similarly we can show that for the pressureless matter dominated universe, we will have the following relation

$$w_\phi = -\frac{2}{\alpha+2} \tag{6.24}$$

In general, the relation between w_ϕ and α is given by

$$w_\phi = \frac{w_B\alpha - 2}{\alpha+2}, \tag{6.25}$$

where w_B represents the equation of state parameter for the fluid component which dominates the background of the universe [22]. It is quite clear that when $w_B = -1$, we also have $w_\phi = -1$.

Chapter 7

Summary

We have seen that the introduction of an additional quantity as a component of our universe, namely the dark energy, was mandatory to explain the supernovae type 1a data. We have arrived at numerical values for the parameters $\Omega_{\rm NR}$ and w. We have shown using the data that the condition $\rho + 3p < 0$ is satisfied for our universe at the current epoch. This led us to conclude that we are living in an accelerating universe. The data which we have worked with has been used from [18] for SNLS and [4] for SCP.

In the second part of the project, we have tried to arrive at a model for dark energy. We have analytically arrived at the conclusion that the inverse power law scalar field potential satisfies conditions of both early-time and late-time universe.We have seen how the exponent, α can be constrained using constraints on w. We have also tried to understand the evolution of dark energy in the expanding universe.

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