Beyond power spectra
– Inflationary three-point functions –

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Plan of the talk

1. Some essential remarks on the evaluation of inflationary power spectra
2. Evaluation of the scalar bi-spectrum generated during inflation
3. Constraints from Planck on the scalar bi-spectrum
4. Are features in the power spectrum consistent with small non-Gaussianities?
5. Evaluating the other three-point functions
6. The squeezed limit and the consistency relations
7. Outlook

This talk is based on

- J. Martin and L. Sriramkumar, *The scalar bi-spectrum in the Starobinsky model: The equilateral case*, JCAP **1201**, 008 (2012).
- D. K. Hazra, L. Sriramkumar and J. Martin, *BINGO: A code for the efficient computation of the scalar bi-spectrum*, JCAP **1305**, 026 (2013).
- V. Sreenath, R. Tibrewala and L. Sriramkumar, *Numerical evaluation of the three-point scalar-tensor cross-correlations and the tensor bi-spectrum*, JCAP **1312**, 037 (2013).
- V. Sreenath and L. Sriramkumar, *Examining the consistency relations involving the three-point scalar and tensor correlations*, In preparation.

A few words on the conventions and notations

- We shall work in units such that $c = \hbar = 1$, and define the Planck mass to be $\mathrm{M}_{_{\mathrm{Pl}}} = (8\,\pi\,G)^{-1/2}$.
- As is often done, particularly in the context of inflation, we shall assume the background universe to be described by the spatially flat, Friedmann line-element.
- We shall denote differentiation with respect to the cosmic and the conformal times t and η by an overdot and an overprime, respectively.
- Moreover, N shall denote the number of e-folds.
- Further, as usual, a and $H = \dot{a}/a$ shall denote the scale factor and the Hubble parameter associated with the Friedmann universe.

The curvature perturbation and the governing equation

On quantization, the operator corresponding to the curvature perturbation $\mathcal{R}(\eta, \boldsymbol{x})$ can be expressed as[1]

$$\begin{aligned}\hat{\mathcal{R}}(\eta, \boldsymbol{x}) &= \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\,\pi)^{3/2}}\,\hat{\mathcal{R}}_{\boldsymbol{k}}(\eta)\,\mathrm{e}^{i\,\boldsymbol{k}\cdot\boldsymbol{x}}\\ &= \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\,\pi)^{3/2}}\,\left[\hat{a}_{\boldsymbol{k}}\,f_{\boldsymbol{k}}(\eta)\,\mathrm{e}^{i\,\boldsymbol{k}\cdot\boldsymbol{x}} + \hat{a}_{\boldsymbol{k}}^{\dagger}\,f_{\boldsymbol{k}}^{*}(\eta)\,\mathrm{e}^{-i\,\boldsymbol{k}\cdot\boldsymbol{x}}\right],\end{aligned}$$

where $\hat{a}_{\boldsymbol{k}}$ and $\hat{a}_{\boldsymbol{k}}^{\dagger}$ are the usual creation and annihilation operators that satisfy the standard commutation relations.

The modes $f_{\boldsymbol{k}}$ are governed by the differential equation

$$f_{\boldsymbol{k}}'' + 2\,\frac{z'}{z}\,f_{\boldsymbol{k}}' + k^2\,f_{\boldsymbol{k}} = 0,$$

where $z = a\,\mathrm{M_{_{Pl}}}\,\sqrt{2\,\epsilon_1}$, with $\epsilon_1 = -\,\mathrm{d}\ln H/\mathrm{d}N$ being the first slow roll parameter.

[1] See, for instance, L. Sriramkumar, Curr. Sci. **97**, 868 (2009).

The Bunch-Davies initial conditions

While studying the evolution of the curvature perturbation, it often proves to be more convenient to work in terms of the so-called Mukhanov-Sasaki variable $v_{\boldsymbol{k}}$, which is defined as $v_{\boldsymbol{k}} = z\, f_{\boldsymbol{k}}$. In terms of the variable $v_{\boldsymbol{k}}$, the above equation of motion for $f_{\boldsymbol{k}}$ reduces to the following simple form:

$$v_{\boldsymbol{k}}'' + \left(k^2 - \frac{z''}{z}\right) v_{\boldsymbol{k}} = 0.$$

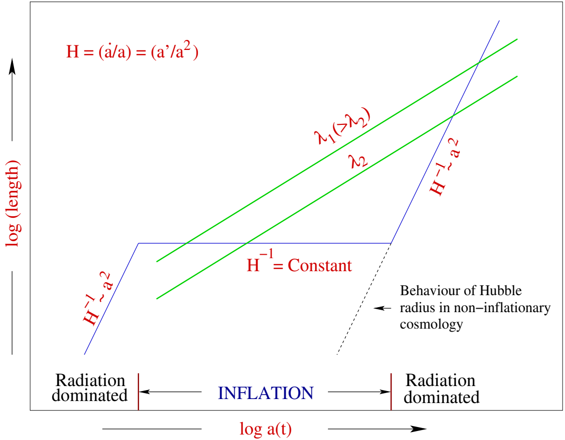
The initial conditions on the perturbations are imposed when the modes are well inside the Hubble radius during inflation.

Usually, it is the so-called Bunch-Davies initial conditions that are imposed on the modes, which amounts to demanding that the Mukhanov-Sasaki variable $v_{\boldsymbol{k}}$ reduces to following Minkowski-like positive frequency form in the sub-Hubble limit[2]:

$$\lim_{k/(a\,H)\to\infty} v_{\boldsymbol{k}} = \frac{1}{\sqrt{2\,k}}\, \mathrm{e}^{-i\,k\,\eta}.$$

[2] T. Bunch and P. C. W. Davies, Proc. Roy. Soc. Lond. A **360**, 117 (1978).

The behavior of modes during inflation



A schematic diagram illustrating the behavior of the physical wavelength $\lambda_{\rm P} \propto a$ (the green lines) and the Hubble radius H^{-1} (the blue line) during inflation and the radiation dominated epochs[3].

[3]See, for example, E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley Publishing Company, New York, 1990), Fig. 8.4.

The scalar power spectrum

The dimensionless scalar power spectrum $\mathcal{P}_{\rm S}(k)$ is defined in terms of the correlation function of the Fourier modes of the curvature perturbation $\hat{\mathcal{R}}_{\boldsymbol{k}}$ as follows:

$$\langle \hat{\mathcal{R}}_{\boldsymbol{k}}(\eta)\, \hat{\mathcal{R}}_{\boldsymbol{p}}(\eta)\rangle = \frac{(2\,\pi)^2}{2\,k^3}\, \mathcal{P}_{\rm S}(k)\, \delta^{(3)}\,(\boldsymbol{k}+\boldsymbol{p})\,.$$

In the Bunch-Davies vacuum, say, $|0\rangle$, which is defined as $\hat{a}_{\boldsymbol{k}}|0\rangle = 0\ \forall\ \boldsymbol{k}$, we can express the scalar power spectrum in terms of the quantities $f_{\boldsymbol{k}}$ and $v_{\boldsymbol{k}}$ as

$$\mathcal{P}_{\rm S}(k) = \frac{k^3}{2\,\pi^2}\, |f_{\boldsymbol{k}}|^2 = \frac{k^3}{2\,\pi^2}\, \left(\frac{|v_{\boldsymbol{k}}|}{z}\right)^2$$

and, analytically, the spectrum is evaluated in the super-Hubble limit, *i.e.* when $k/(a\,H) \to 0$.

As is well known, numerically, the Bunch-Davies initial conditions are imposed on the modes when they are *well inside the Hubble radius*, and the power spectrum is evaluated at suitably late times when the modes are *sufficiently outside*[4].

[4] See, for example, D. S. Salopek, J. R. Bond and J. M. Bardeen, Phys. Rev. D **40**, 1753 (1989); C. Ringeval, Lect. Notes Phys. **738**, 243 (2008).

Equations governing the tensor perturbations

Upon quantization, the tensor perturbations can be written in terms of the corresponding modes, say, $g_{\boldsymbol{k}}$, as follows:

$$\hat{\gamma}_{ij}(\eta, \boldsymbol{x}) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\,\pi)^{3/2}}\, \hat{\gamma}_{ij}^{\boldsymbol{k}}(\eta)\, \mathrm{e}^{i\,\boldsymbol{k}\cdot\boldsymbol{x}}$$

$$= \sum_s \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\,\pi)^{3/2}} \left(\hat{b}^s_{\boldsymbol{k}}\, \varepsilon^s_{ij}(\boldsymbol{k})\, g_k(\eta)\, \mathrm{e}^{i\,\boldsymbol{k}\cdot\boldsymbol{x}} + \hat{b}^{s\dagger}_{\mathbf{k}}\, \varepsilon^{s*}_{ij}(\boldsymbol{k})\, g^*_k(\eta)\, \mathrm{e}^{-i\,\boldsymbol{k}\cdot\boldsymbol{x}}\right),$$

where $\hat{b}^s_{\boldsymbol{k}}$ and $\hat{b}^{s\,\dagger}_{\boldsymbol{k}}$ are the usual creation and annihilation operators that satisfy the standard commutation relations, while $\varepsilon^s_{ij}(\boldsymbol{k})$ represents the transverse and traceless polarization tensor describing gravitational waves.

The modes $g_{\boldsymbol{k}}$ are governed by the differential equation

$$g_{\boldsymbol{k}}'' + 2\,\mathcal{H}\,g_{\boldsymbol{k}}' + k^2\,g_{\boldsymbol{k}} = 0$$

and, in terms of the variable $u_{\boldsymbol{k}} = \mathrm{M}_{_{\mathrm{Pl}}}\,a\,g_{\boldsymbol{k}}/\sqrt{2}$, the above equation reduces to

$$u_{\boldsymbol{k}}'' + \left(k^2 - \frac{a''}{a}\right)\,u_{\boldsymbol{k}} = 0.$$

The tensor power spectrum

The tensor power spectrum $\mathcal{P}_{_{\rm T}}(k)$ is defined through the relation

$$\langle\hat{\gamma}^{\boldsymbol{k}}_{m_1 n_1}\,\hat{\gamma}^{\boldsymbol{p}}_{m_2 n_2}\rangle = \frac{(2\,\pi)^2}{8\,k^3}\,\Pi^{\boldsymbol{k}}_{m_1 n_1, m_2 n_2}\,\mathcal{P}_{_{\rm T}}(k)\,\delta^3\,(\boldsymbol{k}+\boldsymbol{p})\,,$$

where

$$\Pi^{\boldsymbol{k}}_{m_1 n_1, m_2 n_2} = \sum_s\,\varepsilon^s_{m_1 n_1}(\boldsymbol{k})\,\varepsilon^{s*}_{m_2 n_2}(\boldsymbol{k}).$$

In terms of the quantities $g_{\boldsymbol{k}}$ and $u_{\boldsymbol{k}}$, the tensor power spectrum $\mathcal{P}_{_{\rm T}}(k)$ in the Bunch-Davies vacuum is given by

$$\mathcal{P}_{_{\rm T}}(k) = 4\,\frac{k^3}{2\,\pi^2}\,|g_{\boldsymbol{k}}|^2 = \frac{8}{{\rm M}_{_{\rm Pl}}^2}\,\frac{k^3}{2\,\pi^2}\,\left(\frac{|u_{\boldsymbol{k}}|}{a}\right)^2,$$

with the right hand side being evaluated, as in the scalar case, when the modes are sufficiently outside the Hubble radius[5].

[5] See, for example, L. Sriramkumar, Curr. Sci. **97**, 868 (2009).

The scalar bi-spectrum

The scalar bi-spectrum $\mathcal{B}_{\mathcal{R}\mathcal{R}\mathcal{R}}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ is related to the three-point correlation function of the Fourier modes of the curvature perturbation, evaluated towards the end of inflation, say, at the conformal time $\eta_{\rm e}$, as follows[6]:

$$\langle \hat{\mathcal{R}}_{\boldsymbol{k}_1}(\eta_{\rm e})\, \hat{\mathcal{R}}_{\boldsymbol{k}_2}(\eta_{\rm e})\, \hat{\mathcal{R}}_{\boldsymbol{k}_3}(\eta_{\rm e}) \rangle = (2\,\pi)^3\ \mathcal{B}_{\mathcal{R}\mathcal{R}\mathcal{R}}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)\ \delta^{(3)}\left(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3\right).$$

For convenience, we shall set

$$\mathcal{B}_{\mathcal{R}\mathcal{R}\mathcal{R}}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = (2\,\pi)^{-9/2}\ G_{\mathcal{R}\mathcal{R}\mathcal{R}}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3).$$

[6] D. Larson *et al.*, Astrophys. J. Suppl. **192**, 16 (2011);
E. Komatsu *et al.*, Astrophys. J. Suppl. **192**, 18 (2011).

The non-Gaussianity parameter $f_{\rm NL}$

The observationally relevant non-Gaussianity parameter $f_{\rm NL}$ is basically introduced through the relation[7]

$$\mathcal{R}(\eta, \boldsymbol{x}) = \mathcal{R}_{\rm G}(\eta, \boldsymbol{x}) - \frac{3\, f_{\rm NL}}{5}\, \left[\mathcal{R}_{\rm G}^2(\eta, \boldsymbol{x}) - \left\langle\mathcal{R}_{\rm G}^2(\eta, \boldsymbol{x})\right\rangle\right],$$

where $\mathcal{R}_{\rm G}$ denotes the Gaussian quantity, and the factor of $3/5$ arises due to the relation between the Bardeen potential and the curvature perturbation during the matter dominated epoch.

Utilizing the above relation and Wick's theorem, one can arrive at the three-point correlation function of the curvature perturbation in Fourier space in terms of the parameter $f_{\rm NL}$. It is found to be

$$\begin{aligned}\langle\hat{\mathcal{R}}_{\boldsymbol{k}_1}\, \hat{\mathcal{R}}_{\boldsymbol{k}_2}\, \hat{\mathcal{R}}_{\boldsymbol{k}_3}\rangle &= -\frac{3\, f_{\rm NL}}{10}\, (2\,\pi)^{5/2}\, \left(\frac{1}{k_1^3\, k_2^3\, k_3^3}\right)\, \delta^{(3)}(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3)\\ &\quad\times\, \left[k_1^3\, \mathcal{P}_{\rm S}(k_2)\, \mathcal{P}_{\rm S}(k_3) + \text{two permutations}\right].\end{aligned}$$

[7] E. Komatsu and D. N. Spergel, Phys. Rev. D **63**, 063002 (2001).

The relation between $f_{_{\rm NL}}$ and the scalar bi-spectrum

Upon making use of the above expression for the three-point function of the curvature perturbation and the definition of the scalar bi-spectrum, we can, in turn, arrive at the following relation[8]:

$$\begin{aligned} f_{_{\rm NL}}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) &= -\frac{10}{3}\,(2\,\pi)^{1/2}\,\left(k_1^3\,k_2^3\,k_3^3\right)\,\mathcal{B}_{\mathcal{R}\mathcal{R}\mathcal{R}}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)\\ &\quad\times\,\left[k_1^3\,\mathcal{P}_{_{\rm S}}(k_2)\,\mathcal{P}_{_{\rm S}}(k_3) + \text{two permutations}\right]^{-1}\\ &= -\frac{10}{3}\,\frac{1}{(2\,\pi)^4}\,\left(k_1^3\,k_2^3\,k_3^3\right)\,G_{\mathcal{R}\mathcal{R}\mathcal{R}}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)\\ &\quad\times\,\left[k_1^3\,\mathcal{P}_{_{\rm S}}(k_2)\,\mathcal{P}_{_{\rm S}}(k_3) + \text{two permutations}\right]^{-1}. \end{aligned}$$

[8] J. Martin and L. Sriramkumar, JCAP **1201**, 008 (2012).

The action at the cubic order

It can be shown that, the third order term in the action describing the curvature perturbation is given by[9]

$$\mathcal{S}^3_{\mathcal{R}\mathcal{R}\mathcal{R}}[\mathcal{R}] = \mathrm{M}_{_{\mathrm{Pl}}}^2 \int \mathrm{d}\eta \int \mathrm{d}^3\boldsymbol{x} \biggl[a^2\,\epsilon_1^2\,\mathcal{R}\,\mathcal{R}'^2 + a^2\,\epsilon_1^2\,\mathcal{R}\,(\partial\mathcal{R})^2 - 2\,a\,\epsilon_1\,\mathcal{R}'\,(\partial^i\mathcal{R})\,(\partial_i\chi) + \frac{a^2}{2}\,\epsilon_1\,\epsilon_2'\,\mathcal{R}^2\,\mathcal{R}' + \frac{\epsilon_1}{2}\,(\partial^i\mathcal{R})\,(\partial_i\chi)\,(\partial^2\chi) + \frac{\epsilon_1}{4}\,(\partial^2\mathcal{R})\,(\partial\chi)^2 + \mathcal{F}_1\left(\frac{\delta\mathcal{L}^2_{\mathcal{R}\mathcal{R}}}{\delta\mathcal{R}}\right)\biggr],$$

where $\mathcal{F}_1(\delta\mathcal{L}^2_{\mathcal{R}\mathcal{R}}/\delta\mathcal{R})$ denotes terms involving the variation of the second order action with respect to $\mathcal{R}$, while χ is related to the curvature perturbation $\mathcal{R}$ through the relation

$$\partial^2\chi = a\,\epsilon_1\,\mathcal{R}'.$$

[9] J. Maldacena, JHEP **0305**, 013 (2003);
D. Seery and J. E. Lidsey, JCAP **0506**, 003 (2005);
X. Chen, M.-x. Huang, S. Kachru and G. Shiu, JCAP **0701**, 002 (2007).

Evaluating the scalar bi-spectrum

At the leading order in the perturbations, one then finds that the scalar three-point correlation function in Fourier space is described by the integral[10]

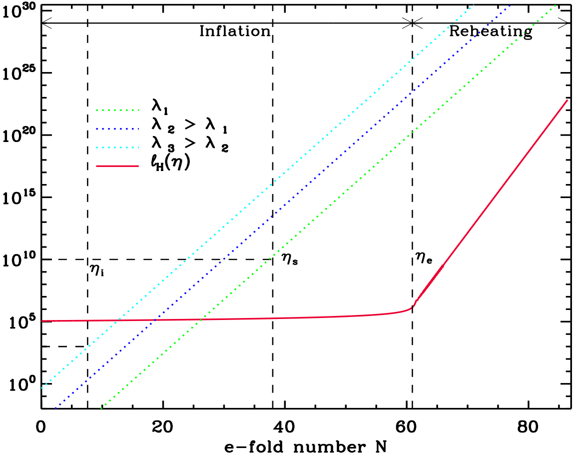
$$\langle\hat{\mathcal{R}}_{\boldsymbol{k}_1}(\eta_{\rm e})\,\hat{\mathcal{R}}_{\boldsymbol{k}_2}(\eta_{\rm e})\,\hat{\mathcal{R}}_{\boldsymbol{k}_3}(\eta_{\rm e})\rangle = -i\int_{\eta_{\rm i}}^{\eta_{\rm e}} {\rm d}\eta\; a(\eta)\;\left\langle\left[\hat{\mathcal{R}}_{\boldsymbol{k}_1}(\eta_{\rm e})\,\hat{\mathcal{R}}_{\boldsymbol{k}_2}(\eta_{\rm e})\,\hat{\mathcal{R}}_{\boldsymbol{k}_3}(\eta_{\rm e}),\hat{H}_{\rm I}(\eta)\right]\right\rangle,$$

where $\hat{H}_{\rm I}$ is the Hamiltonian corresponding to the above third order action, while $\eta_{\rm i}$ denotes a sufficiently early time when the initial conditions are imposed on the modes, and $\eta_{\rm e}$ denotes a very late time, say, close to when inflation ends.

Note that, while the square brackets imply the commutation of the operators, the angular brackets denote the fact that the correlations are evaluated in the given initial state, say, the Bunch-Davies vacuum.

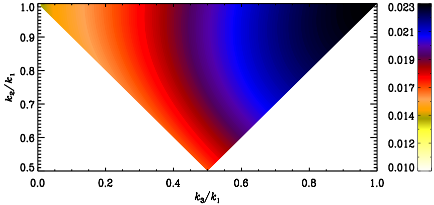
[10] See, for example, D. Seery and J. E. Lidsey, JCAP **0506**, 003 (2005); X. Chen, Adv. Astron. **2010**, 638979 (2010).

The various times of interest



The exact behavior of the physical wavelengths and the Hubble radius plotted as a function of the number of e-folds in the case of the archetypical quadratic potential, which allows us to illustrate the various times of our interest, *viz.* $\eta_{\rm i}$, $\eta_{\rm s}$ and $\eta_{\rm e}$.

Results from BINGO



A comparison of the analytical results (on the left) for the non-Gaussianity parameter $f_{\rm NL}$ with the numerical results (on the right) from the BIspectra and Non-Gaussianity Operator (BINGO) code for a generic triangular configuration of the wavevectors in the case of the standard quadratic potential[11]. The maximum difference between the numerical and the analytic results is found to be about 5%.

[11] D. K. Hazra, L. Sriramkumar and J. Martin, JCAP **05**, 026 (2013).

Template bispectra

For comparison with the observations, the scalar bi-spectrum is often expressed as follows[12]:

$$G_{\mathcal{R}\mathcal{R}\mathcal{R}}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = f_{\rm NL}^{\rm loc}\, G_{\mathcal{R}\mathcal{R}\mathcal{R}}^{\rm loc}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) + f_{\rm NL}^{\rm eq}\, G_{\mathcal{R}\mathcal{R}\mathcal{R}}^{\rm eq}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) + f_{\rm NL}^{\rm orth}\, G_{\mathcal{R}\mathcal{R}\mathcal{R}}^{\rm orth}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3),$$

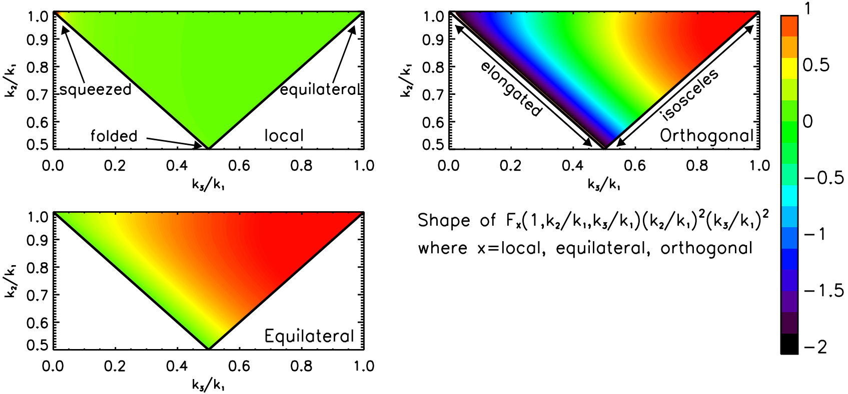
where $f_{\rm NL}^{\rm loc}$, $f_{\rm NL}^{\rm eq}$ and $f_{\rm NL}^{\rm orth}$ are free parameters that are to be estimated, and the local, the equilateral, and the orthogonal template bi-spectra are given by:

$$\begin{aligned}
G_{\mathcal{R}\mathcal{R}\mathcal{R}}^{\rm loc}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) &= \frac{6}{5}\left[\frac{(2\,\pi^2)^2}{k_1^3\,k_2^3\,k_3^3}\right]\left(k_1^3\,\mathcal{P}_{\rm S}(k_2)\,\mathcal{P}_{\rm S}(k_3) + \text{two permutations}\right),\\
G_{\mathcal{R}\mathcal{R}\mathcal{R}}^{\rm eq}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) &= \frac{3}{5}\left[\frac{(2\,\pi^2)^2}{k_1^3\,k_2^3\,k_3^3}\right)\left(6\,k_2\,k_3^2\,\mathcal{P}_{\rm S}(k_1)\,\mathcal{P}_{\rm S}^{2/3}(k_2)\,\mathcal{P}_{\rm S}^{1/3}(k_3) - 3\,k_3^3\,\mathcal{P}_{\rm S}(k_1)\,\mathcal{P}_{\rm S}(k_2)\right.\\
&\quad \left. -2\,k_1\,k_2\,k_3\,\mathcal{P}_{\rm S}^{2/3}(k_1)\,\mathcal{P}_{\rm S}^{2/3}(k_2)\,\mathcal{P}_{\rm S}^{2/3}(k_3) + \text{five permutations}\right),\\
G_{\mathcal{R}\mathcal{R}\mathcal{R}}^{\rm orth}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) &= \frac{3}{5}\left[\frac{(2\,\pi^2)^2}{k_1^3\,k_2^3\,k_3^3}\right]\left(18\,k_2\,k_3^2\,\mathcal{P}_{\rm S}(k_1)\,\mathcal{P}_{\rm S}^{2/3}(k_2)\,\mathcal{P}_{\rm S}^{1/3}(k_3) - 9\,k_3^3\,\mathcal{P}_{\rm S}(k_1)\,\mathcal{P}_{\rm S}(k_2)\right.\\
&\quad \left. -8\,k_1\,k_2\,k_3\,\mathcal{P}_{\rm S}^{2/3}(k_1)\,\mathcal{P}_{\rm S}^{2/3}(k_2)\,\mathcal{P}_{\rm S}^{2/3}(k_3) + \text{five permutations}\right).
\end{aligned}$$

The basis $(f_{\rm NL}^{\rm loc}, f_{\rm NL}^{\rm eq}, f_{\rm NL}^{\rm orth})$ for the scalar three-point function is considered to be large enough to encompass a range of interesting models.

[12] C. L. Bennett *et al.*, arXiv:1212.5225v1 [astro-ph.CO].

Illustration of the template bi-spectra



An illustration of the three template basis bi-spectra, *viz.* the local (top left), the equilateral (bottom) and the orthogonal (top right) forms for a generic triangular configuration of the wavevectors[13].

[13] E. Komatsu, Class. Quantum Grav. **27**, 124010 (2010).

Constraints on $f_{\rm NL}$

The constraints on the non-Gaussianity parameters from the recent Planck data are as follows[14]:

$$\begin{aligned} f_{\rm NL}^{\rm loc} &= 2.7 \pm 5.8, \\ f_{\rm NL}^{\rm eq} &= -42 \pm 75, \\ f_{\rm NL}^{\rm orth} &= -25 \pm 39. \end{aligned}$$

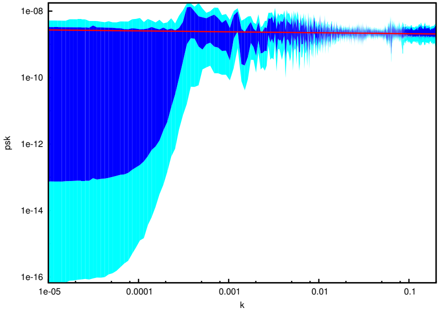
It should be stressed here that these are constraints on the primordial values.

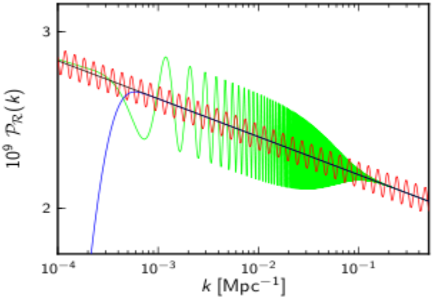
Also, the constraints on each of the $f_{\rm NL}$ parameters have been arrived at assuming that the other two parameters are zero.

We should also add that these constraints become less stringent if the primordial spectra are assumed to contain features.

[14] P. A. R. Ade *et al.*, arXiv:1303.5084 [astro-ph.CO].

Does the primordial power spectrum contain features?





Left: Reconstructed primordial spectra, obtained upon assuming the concordant background ΛCDM model. Recovered spectra improve the fit to the WMAP nine-year data by $\Delta\chi^2_{\rm eff} \simeq 300$, with respect to the best fit power law spectrum[15].

Right: Three different spectra with features that lead to an improved fit (of $\Delta\chi^2_{\rm eff} \simeq 10$) to the Planck data[16].

[15] D. K. Hazra, A. Shafieloo and T. Souradeep, JCAP **1307**, 031 (2013).

[16] P. A. R. Ade *et al.*, arXiv:1303.5082 [astro-ph.CO].

Inflationary models permitting deviations from slow roll

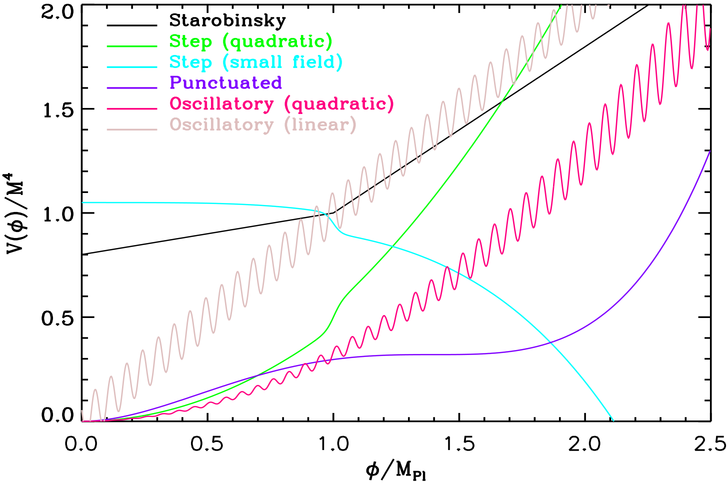
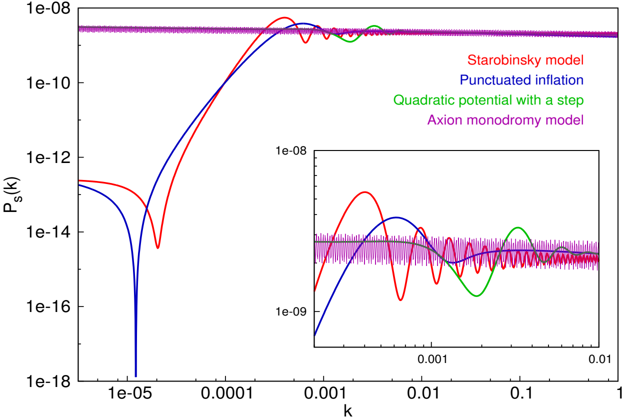


Illustration of potentials that admit departures from slow roll[17].

[17] J. Martin and L. Sriramkumar, JCAP 1201, 008 (2012);
D. K. Hazra, L. Sriramkumar and J. Martin, JCAP 1305, 026 (2013).

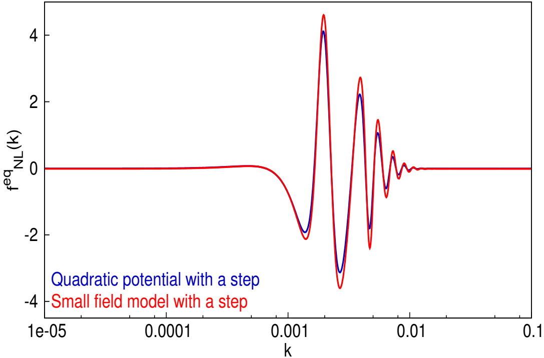
Spectra leading to an improved fit to the WMAP data



The scalar power spectra in the different inflationary models that lead to a better fit to the CMB data than the conventional power law spectrum[18].

[18] R. K. Jain, P. Chingangbam, J.-O. Gong, L. Sriramkumar and T. Souradeep, JCAP **0901**, 009 (2009); D. K. Hazra, M. Aich, R. K. Jain, L. Sriramkumar and T. Souradeep, JCAP **1010**, 008 (2010); M. Aich, D. K. Hazra, L. Sriramkumar and T. Souradeep, Phys. Rev. D **87**, 083526 (2013).

$f_{\rm NL}^{\rm loc}$ in models with a step

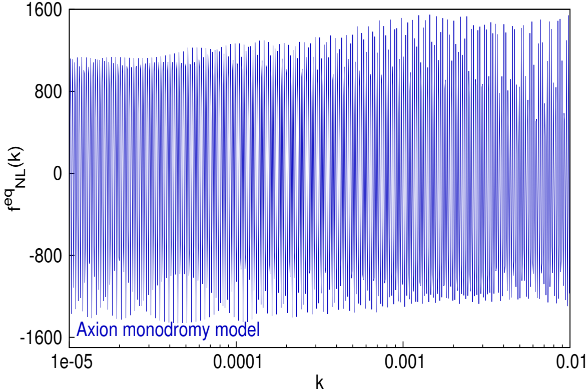


The non-Gaussianity parameter $f_{\rm NL}^{\rm loc}$ evaluated in the equilateral limit when a step has been introduced in the conventional chaotic inflationary model[19] involving the quadratic potential (in blue). The $f_{\rm NL}^{\rm loc}$ that arises in a small field model with a step[20] has also been illustrated (in red).

[19] X. Chen, R. Easther and E. A. Lim, JCAP **0706**, 023 (2007); JCAP **0804**, 010 (2008); P. Adshead, W. Hu, C. Dvorkin and H. V. Peiris, Phys. Rev. D **84**, 043519 (2011); P. Adshead, C. Dvorkin, W. Hu and E. A. Lim, Phys. Rev. D **85**, 023531 (2012).

[20] D. K. Hazra, L. Sriramkumar and J. Martin, JCAP **05**, 026 (2013).

$f_{\rm NL}^{\rm loc}$ in the axion monodromy model



The non-Gaussianity parameter $f_{\rm NL}^{\rm loc}$ evaluated in the equilateral limit in the axion monodromy model[21]. The modulations in the potential give rise to a certain resonant behavior[22], leading to a large $f_{\rm NL}^{\rm loc}$.

[21] D. K. Hazra, L. Sriramkumar and J. Martin, JCAP **05**, 026 (2013).

[22] S. Hannestad, T. Haugbolle, P. R. Jarnhus and M. S. Sloth, JCAP **1006**, 001 (2010); R. Flauger and E. Pajer, JCAP **1101**, 017 (2011).

The cross-correlations and the tensor bi-spectrum

The cross-correlations involving two scalars and a tensor and a scalar and two tensors are defined as

$$\langle \hat{\mathcal{R}}_{\boldsymbol{k}_1}(\eta_{\rm e})\, \hat{\mathcal{R}}_{\boldsymbol{k}_2}(\eta_{\rm e})\, \hat{\gamma}^{\boldsymbol{k}_3}_{m_3 n_3}(\eta_{\rm e})\,\rangle = (2\,\pi)^3\, \mathcal{B}^{m_3 n_3}_{\mathcal{R}\mathcal{R}\gamma}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)\, \delta^{(3)}\left(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3\right),$$

$$\langle \hat{\mathcal{R}}_{\boldsymbol{k}_1}(\eta_{\rm e})\, \hat{\gamma}^{\boldsymbol{k}_2}_{m_2 n_2}(\eta_{\rm e})\, \hat{\gamma}^{\boldsymbol{k}_3}_{m_3 n_3}(\eta_{\rm e})\rangle = (2\,\pi)^3\, \mathcal{B}^{m_2 n_2 m_3 n_3}_{\mathcal{R}\gamma\gamma}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) \times \delta^{(3)}\left(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3\right),$$

while the tensor bi-spectrum is given by

$$\langle \hat{\gamma}^{\boldsymbol{k}_1}_{m_1 n_1}(\eta_{\rm e})\, \hat{\gamma}^{\boldsymbol{k}_2}_{m_2 n_2}(\eta_{\rm e})\, \hat{\gamma}^{\boldsymbol{k}_3}_{m_3 n_3}(\eta_{\rm e})\rangle = (2\,\pi)^3\, \mathcal{B}^{m_1 n_1 m_2 n_2 m_3 n_3}_{\gamma\gamma\gamma}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) \times \delta^{(3)}\left(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3\right).$$

As in the pure scalar case, we shall set

$$\mathcal{B}_{\rm ABC}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = (2\,\pi)^{-9/2}\, G_{\rm ABC}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3),$$

where each of $(\mathrm{A}, \mathrm{B}, \mathrm{C})$ can be either a $\mathcal{R}$ or a γ.

The corresponding non-Gaussianity parameters

As in the scalar case, one can define dimensionless non-Gaussianity parameters to characterize the scalar-scalar-tensor and the scalar-tensor-tensor cross-correlations and the tensor bi-spectrum, respectively, as follows[23]:

$$C_{\rm NL}^{\mathcal{R}}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = -\frac{4}{(2\,\pi^2)^2}\, \left[k_1^3\, k_2^3\, k_3^3\; G_{\mathcal{R}\mathcal{R}\gamma}^{m_3 n_3}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)\right]$$

$$\times \left(\Pi_{m_3 n_3, \bar{m}\bar{n}}^{\boldsymbol{k}_3}\right)^{-1} \left\{\left[k_1^3\, \mathcal{P}_{_{\rm S}}(k_2) + k_2^3\, \mathcal{P}_{_{\rm S}}(k_1)\right]\, \mathcal{P}_{_{\rm T}}(k_3)\right\}^{-1},$$

$$C_{\rm NL}^{\gamma}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = -\frac{4}{(2\,\pi^2)^2}\, \left[k_1^3\, k_2^3\, k_3^3\; G_{\mathcal{R}\gamma\gamma}^{m_2 n_2 m_3 n_3}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)\right]$$

$$\times \left\{\mathcal{P}_{_{\rm S}}(k_1)\, \left[\Pi_{m_2 n_2, m_3 n_3}^{\boldsymbol{k}_2}\, k_3^3\, \mathcal{P}_{_{\rm T}}(k_2) + \Pi_{m_3 n_3, m_2 n_2}^{\boldsymbol{k}_3}\, k_2^3\, \mathcal{P}_{_{\rm T}}(k_3)\right]\right\}^{-1},$$

$$h_{_{\rm NL}}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = -\left(\frac{4}{2\,\pi^2}\right)^2\, \left[k_1^3\, k_2^3\, k_3^3\; G_{\gamma\gamma\gamma}^{m_1 n_1 m_2 n_2 m_3 n_3}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)\right]$$

$$\times \left[\Pi_{m_1 n_1, m_2 n_2}^{\boldsymbol{k}_1}\, \Pi_{m_3 n_3, \bar{m}\bar{n}}^{\boldsymbol{k}_2}\, k_3^3\, \mathcal{P}_{_{\rm T}}(k_1)\, \mathcal{P}_{_{\rm T}}(k_2) + \text{five permutations}\right]^{-1}.$$

[23] V. Sreenath, R. Tibrewala and L. Sriramkumar, JCAP 1312, 037 (2013).

The actions governing the three-point functions

The actions that lead to the correlations involving two scalars and one tensor, one scalar and two tensors and three tensors are given by[24]

$$S^3_{\mathcal{R}\mathcal{R}\gamma}[\mathcal{R}, \gamma_{ij}] = \mathrm{M}_{_{\mathrm{Pl}}}^2 \int \mathrm{d}\eta \int \mathrm{d}^3\boldsymbol{x} \,\biggl[a^2\,\epsilon_1\,\gamma_{ij}\,\partial_i\mathcal{R}\,\partial_j\mathcal{R} + \frac{1}{4}\,\partial^2\gamma_{ij}\,\partial_i\chi\,\partial_j\chi$$
$$+ \frac{a\,\epsilon_1}{2}\,\gamma_{ij}'\,\partial_i\mathcal{R}\,\partial_j\chi + \mathcal{F}^2_{ij}(\mathcal{R})\,\frac{\delta\mathcal{L}^2_{\gamma\gamma}}{\delta\gamma_{ij}} + \mathcal{F}^3(\mathcal{R},\gamma_{ij})\,\frac{\delta\mathcal{L}^2_{\mathcal{R}\mathcal{R}}}{\delta\mathcal{R}}\biggr],$$

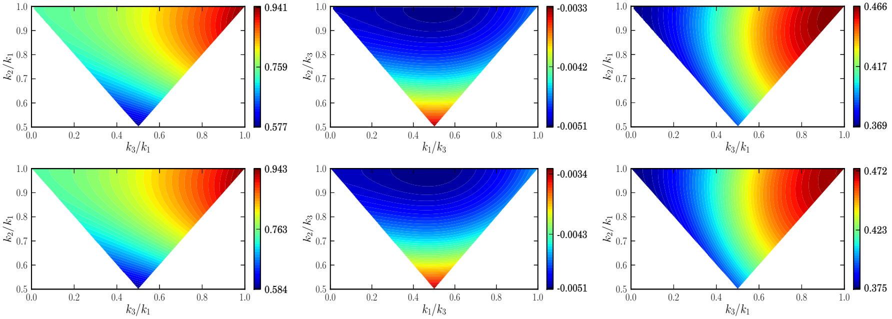
$$S^3_{\mathcal{R}\gamma\gamma}[\mathcal{R}, \gamma_{ij}] = \frac{\mathrm{M}_{_{\mathrm{Pl}}}^2}{4} \int \mathrm{d}\eta \int \mathrm{d}^3\boldsymbol{x} \,\biggl[\frac{a^2\,\epsilon_1}{2}\,\mathcal{R}\,\gamma_{ij}'\,\gamma_{ij}' + \frac{a^2\,\epsilon_1}{2}\,\mathcal{R}\,\partial_l\gamma_{ij}\,\partial_l\gamma_{ij}$$
$$- a\,\gamma_{ij}'\,\partial_l\gamma_{ij}\,\partial_l\chi + \mathcal{F}^4_{ij}(\mathcal{R},\gamma_{mn})\,\frac{\delta\mathcal{L}^2_{\gamma\gamma}}{\delta\gamma_{ij}}\biggr],$$

$$S^3_{\gamma\gamma\gamma}[\gamma_{ij}] = \frac{\mathrm{M}_{_{\mathrm{Pl}}}^2}{2} \int \mathrm{d}\eta \int \mathrm{d}^3\boldsymbol{x} \,\biggl[\frac{a^2}{2}\,\gamma_{lj}\,\gamma_{im}\,\partial_l\partial_m\gamma_{ij} - \frac{a^2}{4}\,\gamma_{ij}\,\gamma_{lm}\,\partial_l\partial_m\gamma_{ij}\biggr].$$

The quantities $\mathcal{L}^2_{\mathcal{R}\mathcal{R}}$ and $\mathcal{L}^2_{\gamma\gamma}$ are the second order Lagrangian densities comprising of two scalars and tensors which lead to the equations of motion.

[24] J. Maldacena, JHEP **0305**, 013 (2003).

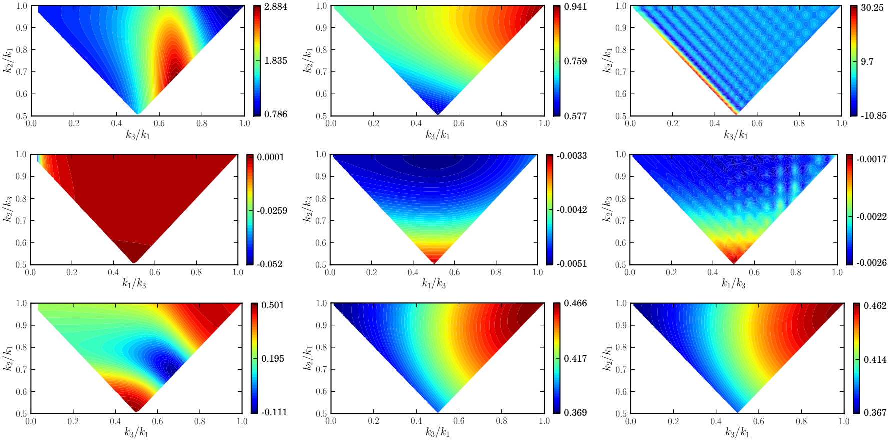
Comparison for an arbitrary triangular configuration



A comparison of the analytical results (at the bottom) for the non-Gaussianity parameters $C_{\rm NL}^{\mathcal{R}}$ (on the left), $C_{\rm NL}^{\gamma}$ (in the middle) and $h_{\rm NL}$ (on the right) with the numerical results (on top) for a generic triangular configuration of the wavevectors in the case of the standard quadratic potential[25]. As in the case of the scalar bi-spectrum, the maximum difference between the numerical and the analytic results is about 5%.

[25] V. Sreenath, R. Tibrewala and L. Sriramkumar, JCAP **1312**, 037 (2013).

Three-point functions for models with features



Density plots of the non-Gaussianity parameters $C_{\rm NL}^{\mathcal{R}}$ (on top), $C_{\rm NL}^{\gamma}$ (in the middle) and $h_{\rm NL}$ (at the bottom) evaluated numerically for an arbitrary triangular configuration of the wavenumbers for the case of the punctuated inflationary scenario (on the left), the quadratic potential with the step (in the middle) and the axion monodromy model (on the right).

The consistency relation for scalars

In the so-called squeezed limit of the scalar bi-spectrum, *i.e.* when $\boldsymbol{k}_1 = -\boldsymbol{k}_2$ and $\boldsymbol{k}_3 \to 0$, it can be shown that the non-Gaussianity parameter $f_{_{\rm NL}}$ can be expressed as[26]

$$f_{_{\rm NL}}(k) = \frac{5}{12}\,(n_{_{\rm S}} - 1)\,,$$

where $n_{_{\rm S}}$ is the scalar spectral index defined as

$$n_{_{\rm S}} = 1 + \frac{{\rm d}\ln \mathcal{P}_{_{\rm S}}(k)}{{\rm d}\ln k}.$$

The above expression is often referred to as the consistency relation[27].

[26] J. Maldacena, JHEP **0305**, 013 (2003).

[27] P. Creminelli and M. Zaldarriaga, JCAP **0410**, 006 (2004).

Consistency relations involving scalars and tensors

In the squeezed limit, it can be shown that one can arrive at the following consistency relations for the non-Gaussainity parameters describing the other three-point functions[28]:

$$\begin{aligned} C_{_{\rm NL}}^{\mathcal{R}}(k) &= \frac{n_{_{\rm S}}}{4} - 1, \\ C_{_{\rm NL}}^{\gamma}(k) &= \frac{n_{_{\rm T}}}{2}, \\ h_{_{\rm NL}}(k) &= \frac{1}{2}\,(3 - n_{_{\rm T}})\,, \end{aligned}$$

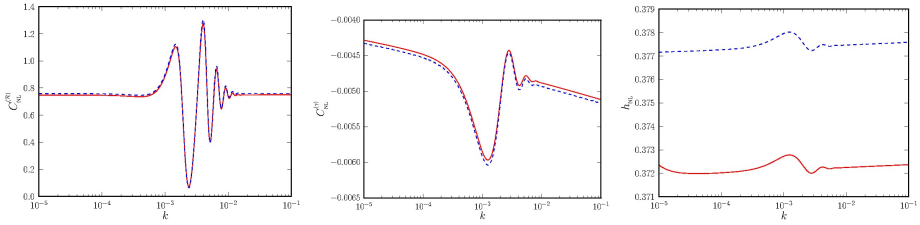
where, for simplicity, we have ignored quantities involving $\Pi^{\boldsymbol{k}}_{m_1 n_1, m_2 n_2}$, while $n_{_{\rm T}}$ is the tensor spectral index defined as

$$n_{_{\rm T}} = \frac{{\rm d}\ln \mathcal{P}_{_{\rm T}}(k)}{{\rm d}\ln k}.$$

Note that, while writing down the consistency relation for $C_{_{\rm NL}}^{\mathcal{R}}$, we have taken the tensor mode to be the squeezed mode. Similarly, in the case of $C_{_{\rm NL}}^{\gamma}$, we have considered the scalar mode to be the squeezed mode.

[28] D. Jeong and M. Kamionkowski, Phys. Rev. Lett. **108**, 251301 (2012);
L. Dai, D. Jeong and M. Kamionkowski, Phys. Rev. D **87**, 103006 (2013); Phys. Rev. D **88**, 043507 (2013);
S. Kundu, arXiv:1311.1575 [astro-ph.CO].

Examining the consistency relation away from slow roll



The quantities $C_{\rm NL}^{\mathcal{R}}$ (on the left), $C_{\rm NL}^{\gamma}$ (in the middle) and $h_{\rm NL}$ (on the right) plotted as a function of the wavenumber in the case of the quadratic potential with a step[29]. The red lines represent the numerical results obtained from the three-point functions, while the blue dashed lines denote those arrived at numerically using the consistency relations. We find the match between the two to be better than 2%.

[29] V. Sreenath and L. Sriramkumar, In preparation.

Outlook

- The strong constraints on the non-Gaussianity parameter f_{NL} from Planck suggests that inflationary and post-inflationary scenarios that lead to rather large non-Gaussianities are very likely to be ruled out by the data.
- In contrast, various analyses seem to point to the fact that the scalar power spectrum may contain features[30]. The possibility of such features can provide a strong handle on constraining inflationary models.
- Else, one may need to carry out a systematic search involving the scalar and the tensor power spectra[31], the scalar and the tensor bi-spectra and the cross-correlations to arrive at a small subset of viable inflationary models. The recent observations of the imprints of the tensor modes on the CMB by BICEP2[32] suggest that it may be possible to arrive at observational constraints on the non-Gaussainity parameters involving tensors in the not-too-distant future.
- The consistency relations seem to be a powerful tool in this regard, as establishing them observationally would unambiguously point to inflation driven by a single scalar field.

[30] P. A. R. Ade *et al.*, arXiv:1303.5082 [astro-ph.CO].

[31] In this context, see, J. Martin, C. Ringeval and V. Vennin, arXiv:1303.3787 [astro-ph.CO].

[32] P. A. R. Ade *et al.*, arXiv:1403.3985 [astro-ph.CO].

Thank you for your attention