

Bouncing universes

L. Sriramkumar

Department of Physics, Indian Institute of Technology Madras, Chennai

Saha Institute of Nuclear Physics, Kolkata

July 25, 2016

Plan of the talk

- 1 Bouncing scenarios
- 2 Generation of scale invariant magnetic fields in bouncing universes
- 3 Duality and scale invariant magnetic fields
- 4 The tensor bi-spectrum in a matter bounce
- 5 Summary



This talk is based on...

- L. Sriramkumar, K. Atmjeet and R. K. Jain, *Generation of scale invariant magnetic fields in bouncing universes*, JCAP **1509**, 010 (2015), E-print: arXiv:1506.06475 [astro-ph.CO].
- D. Chowdhury, V. Sreenath and L. Sriramkumar, *The tensor bi-spectrum in a matter bounce*, JCAP **1511**, 002 (2015), E-print: arXiv:1506.06475 [astro-ph.CO],
- D. Chowdhury, L. Sriramkumar and R. K. Jain, *Duality and scale invariant magnetic fields from bouncing universes*, arXiv:1604.02143 [gr-qc].



A few words on the conventions and notations

- ◆ We shall work in units such that $c = \hbar = 1$, and define the Planck mass to be $M_{\text{Pl}} = (8\pi G)^{-1/2}$.



A few words on the conventions and notations

- ◆ We shall work in units such that $c = \hbar = 1$, and define the Planck mass to be $M_{\text{Pl}} = (8\pi G)^{-1/2}$.
- ◆ We shall assume the background universe to be described by the following spatially flat, Friedmann-Lemaître-Robertson-Walker (FLRW) line-element:

$$ds^2 = -dt^2 + a^2(t) d\mathbf{x}^2 = a^2(\eta) (-d\eta^2 + d\mathbf{x}^2),$$

where t is the cosmic time, $a(t)$ is the scale factor and $\eta = \int dt/a(t)$ denotes the conformal time coordinate.



A few words on the conventions and notations

- ◆ We shall work in units such that $c = \hbar = 1$, and define the Planck mass to be $M_{\text{Pl}} = (8\pi G)^{-1/2}$.
- ◆ We shall assume the background universe to be described by the following spatially flat, Friedmann-Lemaître-Robertson-Walker (FLRW) line-element:

$$ds^2 = -dt^2 + a^2(t) d\mathbf{x}^2 = a^2(\eta) (-d\eta^2 + d\mathbf{x}^2),$$

where t is the cosmic time, $a(t)$ is the scale factor and $\eta = \int dt/a(t)$ denotes the conformal time coordinate.

- ◆ We shall denote differentiation with respect to the cosmic and the conformal times t and η by an overdot and an overprime, respectively.



A few words on the conventions and notations

- ◆ We shall work in units such that $c = \hbar = 1$, and define the Planck mass to be $M_{\text{Pl}} = (8\pi G)^{-1/2}$.
- ◆ We shall assume the background universe to be described by the following spatially flat, Friedmann-Lemaître-Robertson-Walker (FLRW) line-element:

$$ds^2 = -dt^2 + a^2(t) d\mathbf{x}^2 = a^2(\eta) (-d\eta^2 + d\mathbf{x}^2),$$

where t is the cosmic time, $a(t)$ is the scale factor and $\eta = \int dt/a(t)$ denotes the conformal time coordinate.

- ◆ We shall denote differentiation with respect to the cosmic and the conformal times t and η by an overdot and an overprime, respectively.
- ◆ Further, as usual, $H = \dot{a}/a$ shall denote the Hubble parameter associated with the FLRW universe.



Bouncing scenarios: An alternative to inflation¹

- Bouncing models correspond to situations wherein the universe initially goes through a period of contraction until the scale factor reaches a certain minimum value before transiting to the expanding phase.

¹See, for instance, M. Novello and S. P. Bergliaffa, *Phys. Rep.* **463**, 127 (2008);
D. Battefeld and P. Peter, *Phys. Rep.* **571**, 1 (2015).



Bouncing scenarios: An alternative to inflation¹

- Bouncing models correspond to situations wherein the universe initially goes through a period of contraction until the scale factor reaches a certain minimum value before transiting to the expanding phase.
- They offer an alternative to inflation to overcome the horizon problem, as they permit well motivated, Minkowski-like initial conditions to be imposed on the perturbations at early times during the contracting phase.

¹See, for instance, M. Novello and S. P. Bergliaffa, *Phys. Rep.* **463**, 127 (2008);
D. Battefeld and P. Peter, *Phys. Rep.* **571**, 1 (2015).



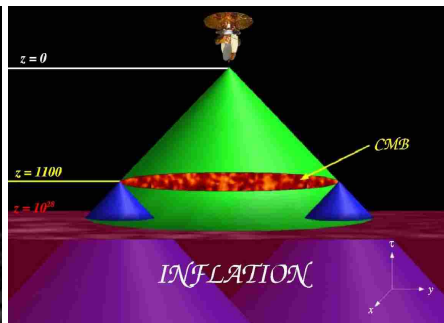
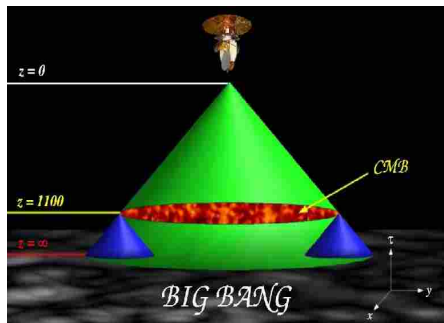
Bouncing scenarios: An alternative to inflation¹

- Bouncing models correspond to situations wherein the universe initially goes through a period of contraction until the scale factor reaches a certain minimum value before transiting to the expanding phase.
- They offer an alternative to inflation to overcome the horizon problem, as they permit well motivated, Minkowski-like initial conditions to be imposed on the perturbations at early times during the contracting phase.
- However, matter fields may have to violate the null energy condition near the bounce in order to give rise to such a scale factor. Also, there exist (genuine) concerns whether such an assumption about the scale factor is valid in a domain where general relativity is expected to fail and quantum gravitational effects are supposed to take over.

¹See, for instance, M. Novello and S. P. Bergliaffa, *Phys. Rep.* **463**, 127 (2008);
D. Battefeld and P. Peter, *Phys. Rep.* **571**, 1 (2015).



The resolution of the horizon problem in inflation

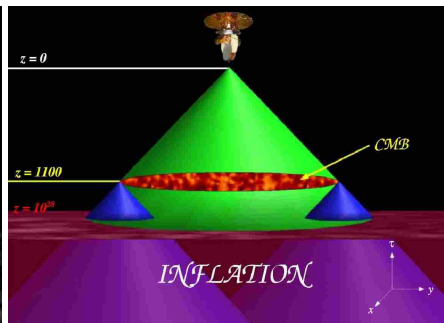
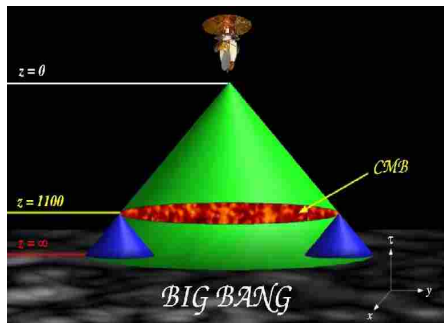


Left: The radiation from the CMB arriving at us from regions separated by more than the Hubble radius at the last scattering surface (which subtends an angle of about 1° today) could not have interacted before decoupling.

²Images from W. Kinney, astro-ph/0301448.



The resolution of the horizon problem in inflation



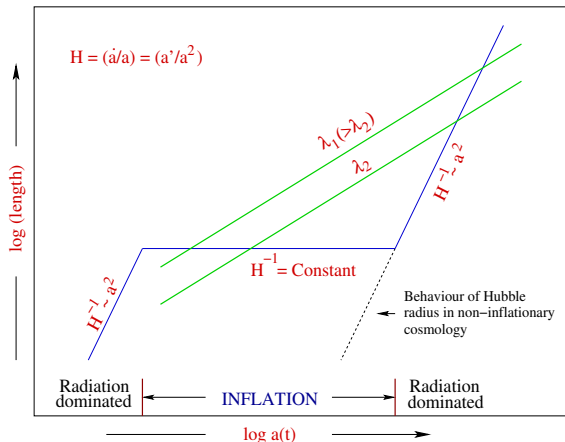
Left: The radiation from the CMB arriving at us from regions separated by more than the Hubble radius at the last scattering surface (which subtends an angle of about 1° today) could not have interacted before decoupling.

Right: An illustration of how an early and sufficiently long epoch of inflation helps in resolving the horizon problem².

²Images from W. Kinney, [astro-ph/0301448](https://arxiv.org/abs/astro-ph/0301448).



Bringing the modes inside the Hubble radius

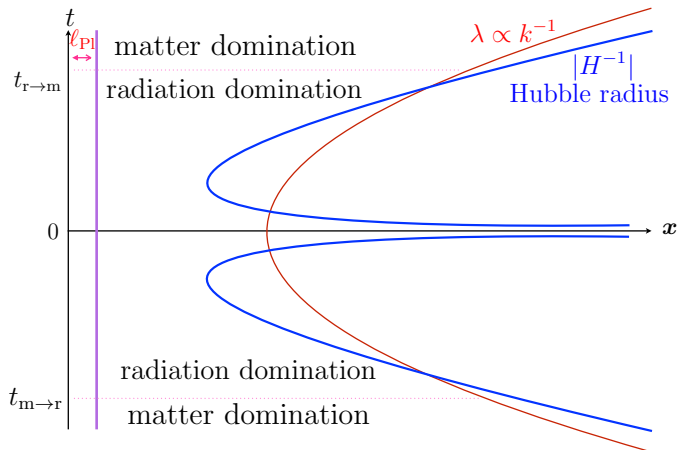


A schematic diagram illustrating the behavior of the physical wavelength $\lambda_p \propto a$ (the green lines) and the Hubble radius H^{-1} (the blue line) during inflation and the radiation dominated epochs³.

³See, for example, E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley Publishing Company, New York, 1990), Fig. 8.4.



Overcoming the horizon problem in bouncing models



The evolution of the physical wavelength and the Hubble radius in a typical bouncing scenario⁴.

⁴Figure from, D. Battefeld and P. Peter, *Phys. Rept.* **571**, 1 (2015).



Violation of the null energy condition

Recall that, according to the Friedmann equations

$$\dot{H} = -4\pi G (\rho + p).$$

In any bouncing scenario, the Hubble parameter is negative before the bounce, crosses zero at the bounce and is positive thereafter.



Violation of the null energy condition

Recall that, according to the Friedmann equations

$$\dot{H} = -4\pi G (\rho + p).$$

In any bouncing scenario, the Hubble parameter is negative before the bounce, crosses zero at the bounce and is positive thereafter.

Evidently, \dot{H} will be positive near the bounce, which implies that $(\rho + p)$ has to be negative in this domain. In other words, the null energy condition needs to be violated in order to achieve a bounce.



Classical bounces and sources

Consider for instance, bouncing models of the form

$$a(\eta) = a_0 \left(1 + \frac{\eta^2}{\eta_0^2} \right)^q = a_0 (1 + k_0^2 \eta^2)^q,$$

where a_0 is the value of the scale factor at the bounce (*i.e.* when $\eta = 0$), $\eta_0 = 1/k_0$ denotes the time scale of the duration of the bounce and $q > 0$. We shall assume that the scale k_0 associated with the bounce is of the order of the Planck scale M_{Pl} .



Classical bounces and sources

Consider for instance, bouncing models of the form

$$a(\eta) = a_0 \left(1 + \frac{\eta^2}{\eta_0^2} \right)^q = a_0 (1 + k_0^2 \eta^2)^q,$$

where a_0 is the value of the scale factor at the bounce (*i.e.* when $\eta = 0$), $\eta_0 = 1/k_0$ denotes the time scale of the duration of the bounce and $q > 0$. We shall assume that the scale k_0 associated with the bounce is of the order of the Planck scale M_{Pl} .

The above scale factor can be achieved with the help of two fluids with constant equation of state parameters $w_1 = (1 - q)/(3q)$ and $w_2 = (2 - q)/(3q)$. The energy densities of these fluids behave as $\rho_1 = M_1/a^{(2q+1)/q}$ and $\rho_2 = M_2/a^{2(1+q)/q}$, where $M_1 = 12 k_0^2 M_{\text{Pl}}^2 a_0^{1/q}$ and $M_2 = -M_1 a_0^{1/q}$.



Classical bounces and sources

Consider for instance, bouncing models of the form

$$a(\eta) = a_0 \left(1 + \frac{\eta^2}{\eta_0^2} \right)^q = a_0 (1 + k_0^2 \eta^2)^q,$$

where a_0 is the value of the scale factor at the bounce (*i.e.* when $\eta = 0$), $\eta_0 = 1/k_0$ denotes the time scale of the duration of the bounce and $q > 0$. We shall assume that the scale k_0 associated with the bounce is of the order of the Planck scale M_{Pl} .

The above scale factor can be achieved with the help of two fluids with constant equation of state parameters $w_1 = (1 - q)/(3q)$ and $w_2 = (2 - q)/(3q)$. The energy densities of these fluids behave as $\rho_1 = M_1/a^{(2q+1)/q}$ and $\rho_2 = M_2/a^{2(1+q)/q}$, where $M_1 = 12 k_0^2 M_{\text{Pl}}^2 a_0^{1/q}$ and $M_2 = -M_1 a_0^{1/q}$.

Note that, when $q = 1$, during very early times wherein $\eta \ll -\eta_0$, the scale factor behaves as in a matter dominated universe (*i.e.* $a \propto \eta^2$). Therefore, the $q = 1$ case is often referred to as the matter bounce scenario. In such a case, $\rho_1 = 12 k_0^2 M_{\text{Pl}}^2 a_0/a^3$ and $\rho_2 = -12 k_0^2 M_{\text{Pl}}^2 a_0^2/a^4$.



The non-minimal action and the equation of motion

We shall consider a case wherein the electromagnetic field is coupled non-minimally to a scalar field ϕ and is described by the action

$$S[\phi, A^\mu] = -\frac{1}{16\pi} \int d^4x \sqrt{-g} J^2(\phi) F_{\mu\nu} F^{\mu\nu},$$

where $F_{\mu\nu}$ denotes the electromagnetic field tensor which is given in terms of the vector potential A^μ as follows:

$$F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu} = A_{\nu,\mu} - A_{\mu,\nu}.$$



The non-minimal action and the equation of motion

We shall consider a case wherein the electromagnetic field is coupled non-minimally to a scalar field ϕ and is described by the action

$$S[\phi, A^\mu] = -\frac{1}{16\pi} \int d^4x \sqrt{-g} J^2(\phi) F_{\mu\nu} F^{\mu\nu},$$

where $F_{\mu\nu}$ denotes the electromagnetic field tensor which is given in terms of the vector potential A^μ as follows:

$$F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu} = A_{\nu,\mu} - A_{\mu,\nu}.$$

The scalar field ϕ could be, for instance, the primary matter field that is driving the background evolution and J is an arbitrary function of the field.



Quantization of the electromagnetic field

In a spatially flat, FLRW universe, we can choose to work in the Coulomb gauge wherein $A_0 = 0$ and $\partial_i A^i = 0$. In such a gauge, upon quantization, the vector potential \hat{A}_i can be Fourier decomposed as follows⁵:

$$\hat{A}_i(\eta, \mathbf{x}) = \sqrt{4\pi} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \sum_{\lambda=1}^2 \tilde{\epsilon}_{\lambda i}(\mathbf{k}) \left[\hat{a}_{\mathbf{k}}^\lambda \bar{A}_k(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a}_{\mathbf{k}}^{\lambda\dagger} \bar{A}_k^*(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}} \right],$$

where the modes \bar{A}_k satisfy the differential equation

$$\bar{A}_k'' + 2 \frac{J'}{J} \bar{A}_k' + k^2 \bar{A}_k = 0.$$

⁵See, for instance, J. Martin and J. Yokoyama, JCAP **0801**, 025 (2008);
K. Subramanian, Astron. Nachr. **331**, 110 (2010).



Quantization of the electromagnetic field

In a spatially flat, FLRW universe, we can choose to work in the Coulomb gauge wherein $A_0 = 0$ and $\partial_i A^i = 0$. In such a gauge, upon quantization, the vector potential \hat{A}_i can be Fourier decomposed as follows⁵:

$$\hat{A}_i(\eta, \mathbf{x}) = \sqrt{4\pi} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \sum_{\lambda=1}^2 \tilde{\epsilon}_{\lambda i}(\mathbf{k}) \left[\hat{a}_{\mathbf{k}}^\lambda \bar{A}_k(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a}_{\mathbf{k}}^{\lambda\dagger} \bar{A}_k^*(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}} \right],$$

where the modes \bar{A}_k satisfy the differential equation

$$\bar{A}_k'' + 2 \frac{J'}{J} \bar{A}_k' + k^2 \bar{A}_k = 0.$$

If we define a new variable $\mathcal{A}_k = J \bar{A}_k$, then the above equation simplifies to

$$\mathcal{A}_k'' + \left(k^2 - \frac{J''}{J} \right) \mathcal{A}_k = 0,$$

and one can impose the standard Bunch-Davies initial conditions on the modes \mathcal{A}_k at suitably early times.

⁵See, for instance, J. Martin and J. Yokoyama, JCAP **0801**, 025 (2008);
K. Subramanian, Astron. Nachr. **331**, 110 (2010).



Power spectra of the electric and magnetic fields

The energy densities associated with the electric and magnetic fields can be written in terms of the vector potential A_i and its time and spatial derivatives as follows:

$$\rho_E = \frac{J^2}{8\pi a^2} g^{ij} A'_i A'_j,$$

$$\rho_B = \frac{J^2}{16\pi} g^{ij} g^{lm} (\partial_j A_m - \partial_m A_j) (\partial_i A_l - \partial_l A_i),$$

where $g^{ij} = \delta^{ij}/a^2$ denotes the spatial components of the FLRW metric.



Power spectra of the electric and magnetic fields

The energy densities associated with the electric and magnetic fields can be written in terms of the vector potential A_i and its time and spatial derivatives as follows:

$$\rho_E = \frac{J^2}{8\pi a^2} g^{ij} A'_i A'_j,$$

$$\rho_B = \frac{J^2}{16\pi} g^{ij} g^{lm} (\partial_j A_m - \partial_m A_j) (\partial_i A_l - \partial_l A_i),$$

where $g^{ij} = \delta^{ij}/a^2$ denotes the spatial components of the FLRW metric.

The expectation values of the corresponding operators, *i.e.* $\hat{\rho}_E$ and $\hat{\rho}_B$, can be evaluated in the vacuum state annihilated by the operator $\hat{a}_{\mathbf{k}}^\lambda$.



Power spectra of the electric and magnetic fields

The energy densities associated with the electric and magnetic fields can be written in terms of the vector potential A_i and its time and spatial derivatives as follows:

$$\rho_E = \frac{J^2}{8\pi a^2} g^{ij} A'_i A'_j,$$

$$\rho_B = \frac{J^2}{16\pi} g^{ij} g^{lm} (\partial_j A_m - \partial_m A_j) (\partial_i A_l - \partial_l A_i),$$

where $g^{ij} = \delta^{ij}/a^2$ denotes the spatial components of the FLRW metric.

The expectation values of the corresponding operators, *i.e.* $\hat{\rho}_E$ and $\hat{\rho}_B$, can be evaluated in the vacuum state annihilated by the operator $\hat{a}_{\mathbf{k}}^\lambda$.

It can be shown that the spectral energy densities of the magnetic and electric fields are given by

$$\mathcal{P}_B(k) = \frac{d\langle 0|\hat{\rho}_B|0\rangle}{d\ln k} = \frac{J^2(\eta)}{2\pi^2} \frac{k^5}{a^4(\eta)} |\bar{A}_k(\eta)|^2,$$

$$\mathcal{P}_E(k) = \frac{d\langle 0|\hat{\rho}_E|0\rangle}{d\ln k} = \frac{J^2(\eta)}{2\pi^2} \frac{k^3}{a^4(\eta)} |\bar{A}'_k(\eta)|^2.$$



Power spectra in power law inflation

For power law inflation described by the scale factor $a(\eta) = a_1 (-\eta/\eta_1)^{\beta+1}$ and for coupling function of the form $J(\eta) = J_0 a^n(\eta)$, one can show that the power spectrum of the magnetic field is given by⁶

$$\mathcal{P}_B(k) = \mathcal{F}(m) H^4 (-k\eta)^{4+2m},$$

where $m = (\beta + 1) n = \alpha$ for $\alpha \leq 1/2$ and $m = 1 - \alpha$ for $\alpha \geq 1/2$, while

$$\mathcal{F}(m) = [(2\pi) 2^{2m+1} \Gamma^2(m + 1/2) \cos^2(\pi m)]^{-1}.$$

⁶See, for instance, J. Martin and J. Yokoyama, JCAP **0801**, 025 (2008);
K. Subramanian, Astron. Nachr. **331**, 110 (2010).



Power spectra in power law inflation

For power law inflation described by the scale factor $a(\eta) = a_1 (-\eta/\eta_1)^{\beta+1}$ and for coupling function of the form $J(\eta) = J_0 a^n(\eta)$, one can show that the power spectrum of the magnetic field is given by⁶

$$\mathcal{P}_B(k) = \mathcal{F}(m) H^4 (-k\eta)^{4+2m},$$

where $m = (\beta + 1)n = \alpha$ for $\alpha \leq 1/2$ and $m = 1 - \alpha$ for $\alpha \geq 1/2$, while

$$\mathcal{F}(m) = [(2\pi) 2^{2m+1} \Gamma^2(m + 1/2) \cos^2(\pi m)]^{-1}.$$

The corresponding spectrum for the electric field can be obtained to be

$$\mathcal{P}_E(k) = \frac{\mathcal{G}(m)}{2\pi^2} H^4 (-k\eta)^{4+2m},$$

where $m = 1 + \alpha$ if $\alpha \leq -1/2$ and $m = -\alpha$ for $\alpha \geq -1/2$, while

$$\mathcal{G}(m) = [(2\pi) 2^{2m+3} \Gamma^2(m + 3/2) \cos^2(\pi m)]^{-1}.$$

⁶See, for instance, J. Martin and J. Yokoyama, JCAP **0801**, 025 (2008);
K. Subramanian, Astron. Nachr. **331**, 110 (2010).



Power spectra in power law inflation

For power law inflation described by the scale factor $a(\eta) = a_1 (-\eta/\eta_1)^{\beta+1}$ and for coupling function of the form $J(\eta) = J_0 a^n(\eta)$, one can show that the power spectrum of the magnetic field is given by⁶

$$\mathcal{P}_B(k) = \mathcal{F}(m) H^4 (-k\eta)^{4+2m},$$

where $m = (\beta + 1)n = \alpha$ for $\alpha \leq 1/2$ and $m = 1 - \alpha$ for $\alpha \geq 1/2$, while

$$\mathcal{F}(m) = [(2\pi) 2^{2m+1} \Gamma^2(m + 1/2) \cos^2(\pi m)]^{-1}.$$

The corresponding spectrum for the electric field can be obtained to be

$$\mathcal{P}_E(k) = \frac{\mathcal{G}(m)}{2\pi^2} H^4 (-k\eta)^{4+2m},$$

where $m = 1 + \alpha$ if $\alpha \leq -1/2$ and $m = -\alpha$ for $\alpha \geq -1/2$, while

$$\mathcal{G}(m) = [(2\pi) 2^{2m+3} \Gamma^2(m + 3/2) \cos^2(\pi m)]^{-1}.$$

It is evident that $m = -2$ leads to a scale invariant spectrum for the magnetic field which corresponds to either $\alpha = 3$ or $\alpha = -2$.

⁶See, for instance, J. Martin and J. Yokoyama, JCAP **0801**, 025 (2008);
K. Subramanian, Astron. Nachr. **331**, 110 (2010).



Modeling the bounce and the non-minimal coupling

We shall model the bounce by assuming that the scale factor $a(\eta)$ behaves as follows:

$$a(\eta) = a_0 \left(1 + \frac{\eta^2}{\eta_0^2} \right)^q = a_0 (1 + k_0^2 \eta^2)^q,$$

which we had discussed earlier.



Modeling the bounce and the non-minimal coupling

We shall model the bounce by assuming that the scale factor $a(\eta)$ behaves as follows:

$$a(\eta) = a_0 \left(1 + \frac{\eta^2}{\eta_0^2} \right)^q = a_0 (1 + k_0^2 \eta^2)^q,$$

which we had discussed earlier.

Note that the above scale factor reduces to the simple power law form with $a(\eta) \propto \eta^{2q}$ at very early times (*i.e.* when $-\eta \gg \eta_0$).



Modeling the bounce and the non-minimal coupling

We shall model the bounce by assuming that the scale factor $a(\eta)$ behaves as follows:

$$a(\eta) = a_0 \left(1 + \frac{\eta^2}{\eta_0^2} \right)^q = a_0 (1 + k_0^2 \eta^2)^q,$$

which we had discussed earlier.

Note that the above scale factor reduces to the simple power law form with $a(\eta) \propto \eta^{2q}$ at very early times (*i.e.* when $-\eta \gg \eta_0$).

We shall assume that the coupling function can be expressed in terms of the scale factor as

$$J(\eta) = J_0 a^n(\eta).$$



E- \mathcal{N} -folds

The conventional e-fold N is defined $N = \log(a/a_0)$ so that $a(N) = a_0 \exp N$. However, the function e^N is a monotonically increasing function of N .

⁷L. Sriramkumar, K. Atmjeet and R. K. Jain, JCAP **1509**, 010 (2015).



E- \mathcal{N} -folds

The conventional e-fold N is defined $N = \log(a/a_0)$ so that $a(N) = a_0 \exp N$. However, the function e^N is a monotonically increasing function of N .

In a bouncing scenario, an obvious choice for the scale factor seems to be⁷

$$a(\mathcal{N}) = a_0 \exp(\mathcal{N}^2/2),$$

with \mathcal{N} being the new time variable that we shall consider for integrating the differential equation governing \bar{A}_k .

⁷L. Sriramkumar, K. Atmjeet and R. K. Jain, JCAP **1509**, 010 (2015).



E- \mathcal{N} -folds

The conventional e-fold N is defined $N = \log(a/a_0)$ so that $a(N) = a_0 \exp N$. However, the function e^N is a monotonically increasing function of N .

In a bouncing scenario, an obvious choice for the scale factor seems to be⁷

$$a(\mathcal{N}) = a_0 \exp(\mathcal{N}^2/2),$$

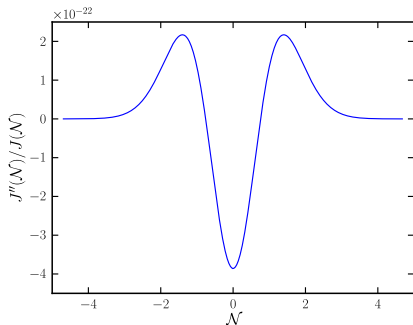
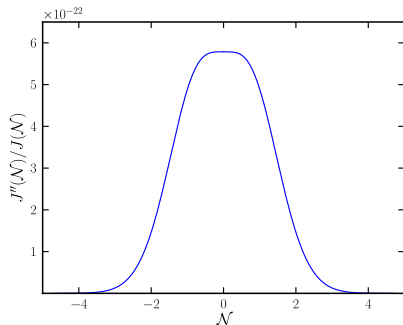
with \mathcal{N} being the new time variable that we shall consider for integrating the differential equation governing \bar{A}_k .

For want of a better name, we shall refer to the variable \mathcal{N} as e- \mathcal{N} -fold since the scale factor grows roughly by the amount $e^{\mathcal{N}}$ between \mathcal{N} and $(\mathcal{N} + 1)$.

⁷L. Sriramkumar, K. Atmjeet and R. K. Jain, JCAP **1509**, 010 (2015).



The behavior of J''/J



The behavior of the quantity J''/J has been plotted as a function of \mathcal{N} for $q = 1$ and $n = 3/2$ (on the left) and $n = -1$ (on the right). Note that the maximum value of J''/J is roughly of the order of k_0^2 .



Analytical solutions for the modes at early times

At very early times (*i.e.* for $-\eta \gg \eta_0$), the scale factor simplifies to the power law form $a(\eta) \propto \eta^{2q}$. During such times, the non-minimal coupling function J also behaves as $J(\eta) \propto \eta^\gamma$, where we have set $\gamma = 2nq$.



Analytical solutions for the modes at early times

At very early times (*i.e.* for $-\eta \gg \eta_0$), the scale factor simplifies to the power law form $a(\eta) \propto \eta^{2q}$. During such times, the non-minimal coupling function J also behaves as $J(\eta) \propto \eta^\gamma$, where we have set $\gamma = 2nq$.

In such a case, we have $J''/J \simeq \gamma(\gamma - 1)/\eta^2$ and it is easy to show that the solutions to the modes of the electromagnetic vector potential \mathcal{A}_k can be expressed as

$$\mathcal{A}_k(\eta) = \sqrt{-k\eta} \left[C_1(k) J_{\gamma-1/2}(-k\eta) + C_2(k) J_{-\gamma+1/2}(-k\eta) \right].$$

One finds that, for the Bunch-Davies initial conditions, $C_1(k)$ and $C_2(k)$ are given by

$$C_1(k) = \sqrt{\frac{\pi}{4k}} \frac{e^{-i\pi\gamma/2}}{\cos(\pi\gamma)} \quad \text{and} \quad C_2(k) = \sqrt{\frac{\pi}{4k}} \frac{e^{i\pi(\gamma+1)/2}}{\cos(\pi\gamma)}.$$



Analytical solutions for the modes at early times

At very early times (*i.e.* for $-\eta \gg \eta_0$), the scale factor simplifies to the power law form $a(\eta) \propto \eta^{2q}$. During such times, the non-minimal coupling function J also behaves as $J(\eta) \propto \eta^\gamma$, where we have set $\gamma = 2nq$.

In such a case, we have $J''/J \simeq \gamma(\gamma - 1)/\eta^2$ and it is easy to show that the solutions to the modes of the electromagnetic vector potential \mathcal{A}_k can be expressed as

$$\mathcal{A}_k(\eta) = \sqrt{-k\eta} \left[C_1(k) J_{\gamma-1/2}(-k\eta) + C_2(k) J_{-\gamma+1/2}(-k\eta) \right].$$

One finds that, for the Bunch-Davies initial conditions, $C_1(k)$ and $C_2(k)$ are given by

$$C_1(k) = \sqrt{\frac{\pi}{4k}} \frac{e^{-i\pi\gamma/2}}{\cos(\pi\gamma)} \quad \text{and} \quad C_2(k) = \sqrt{\frac{\pi}{4k}} \frac{e^{i\pi(\gamma+1)/2}}{\cos(\pi\gamma)}.$$

It can also be shown that

$$\mathcal{A}'_k(\eta) - \frac{J'}{J} \mathcal{A}_k(\eta) = k \sqrt{-k\eta} \left[C_1(k) J_{\gamma+1/2}(-k\eta) - C_2(k) J_{-\gamma-1/2}(-k\eta) \right]$$



Analytical solutions near the bounce

Note that, when $n > 0$, J''/J has a maximum at the bounce. In such a case, for $k \ll k_0$, $k^2 \ll J''/J$ around the bounce. Hence, upon ignoring the k^2 in the equation governing \bar{A}_k , we can integrate the equation to yield

$$\bar{A}'_k(\eta) \simeq \bar{A}'_k(\eta_*) \frac{J^2(\eta_*)}{J^2(\eta)},$$

where η_* is a time when $k^2 \ll J''/J$ before the bounce. The above equation can be integrated to arrive at

$$\bar{A}_k(\eta) \simeq \bar{A}_k(\eta_*) + \bar{A}'_k(\eta_*) \int_{\eta_*}^{\eta} d\eta \frac{J^2(\eta_*)}{J^2(\eta)} = \bar{A}_k(\eta_*) + \bar{A}'_k(\eta_*) a^{2n}(\eta_*) \int_{\eta_*}^{\eta} \frac{d\eta}{a^{2n}(\eta)},$$

where we have set the constant of integration to be $\bar{A}_k(\eta_*)$.



Analytical solutions near the bounce

Note that, when $n > 0$, J''/J has a maximum at the bounce. In such a case, for $k \ll k_0$, $k^2 \ll J''/J$ around the bounce. Hence, upon ignoring the k^2 in the equation governing \bar{A}_k , we can integrate the equation to yield

$$\bar{A}'_k(\eta) \simeq \bar{A}'_k(\eta_*) \frac{J^2(\eta_*)}{J^2(\eta)},$$

where η_* is a time when $k^2 \ll J''/J$ before the bounce. The above equation can be integrated to arrive at

$$\bar{A}_k(\eta) \simeq \bar{A}_k(\eta_*) + \bar{A}'_k(\eta_*) \int_{\eta_*}^{\eta} d\eta \frac{J^2(\eta_*)}{J^2(\eta)} = \bar{A}_k(\eta_*) + \bar{A}'_k(\eta_*) a^{2n}(\eta_*) \int_{\eta_*}^{\eta} \frac{d\eta}{a^{2n}(\eta)},$$

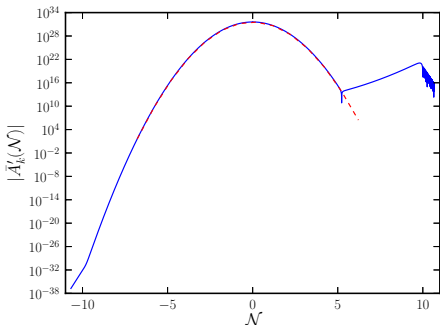
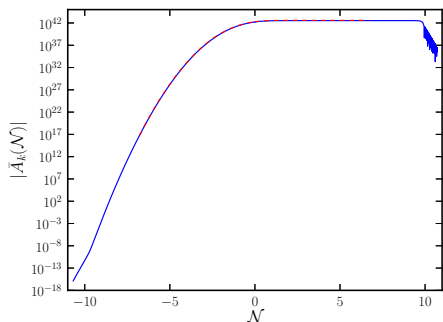
where we have set the constant of integration to be $\bar{A}_k(\eta_*)$.

When $\gamma = 3$, we can evaluate the above integral to obtain that

$$\begin{aligned} \bar{A}_k(\eta) \simeq & \bar{A}_k(\eta_*) + \bar{A}'_k(\eta_*) \frac{a^{2n}(\eta_*)}{a_0^{2n}} \frac{\eta_0}{8} \left\{ \frac{\eta}{\eta_0} \frac{5 + 3(\eta/\eta_0)^2}{[1 + (\eta/\eta_0)^2]^2} + 3 \tan^{-1} \left(\frac{\eta}{\eta_0} \right) \right. \\ & \left. - \frac{\eta_*}{\eta_0} \frac{5 + 3(\eta_*/\eta_0)^2}{[1 + (\eta_*/\eta_0)^2]^2} - 3 \tan^{-1} \left(\frac{\eta_*}{\eta_0} \right) \right\}. \end{aligned}$$



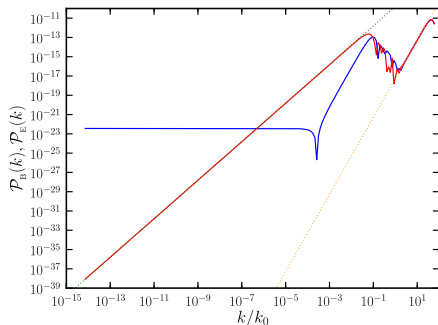
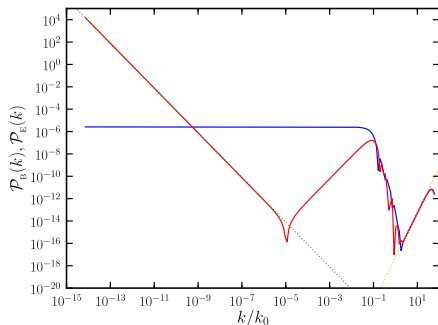
Comparison of the numerical and analytical results



The behavior of the absolute values of \bar{A}_k (on the left) and its derivative \bar{A}'_k (on the right) has been plotted for the mode $k = 10^{-10} k_0$ with $k_0/M_{\text{Pl}} = e^{-25} = 1.389 \times 10^{-11}$ for the case wherein $n = 3/2$, $q = 1$, $a_0 = 10^{-10}$ and $J_0 = 10^4$. The dashed red curves represent the analytical approximation around the bounce that can be arrived at for modes such that $k \ll k_0$.



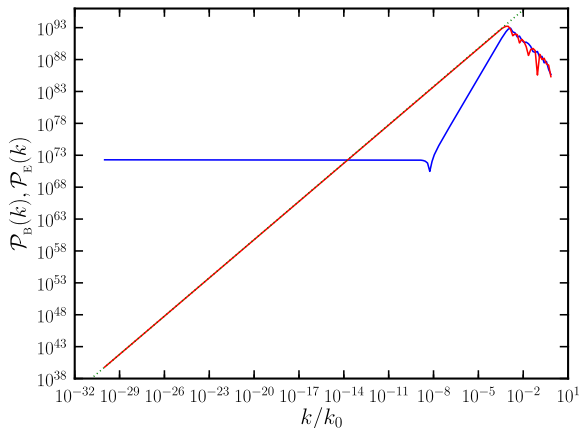
Power spectra of magnetic and electric fields



The power spectra of the magnetic (in blue) and the electric (in red) fields for the cases wherein $(q, n) = (1, 3/2)$ (corresponding to $\gamma = 3$, on the left) and $(q, n) = (1, -1)$ (corresponding to $\gamma = -2$, on the right). We have worked with the same values of η_0 , a_0 and J_0 as in the previous figure. The power spectra of the electric field are along expected lines, behaving as $k^{4-2\gamma} = k^{-2}$ when $\gamma = 3$ and $k^{6+2\gamma} = k^2$ when $\gamma = -2$ (indicated by the dotted green lines).



Spectrum of observable strengths

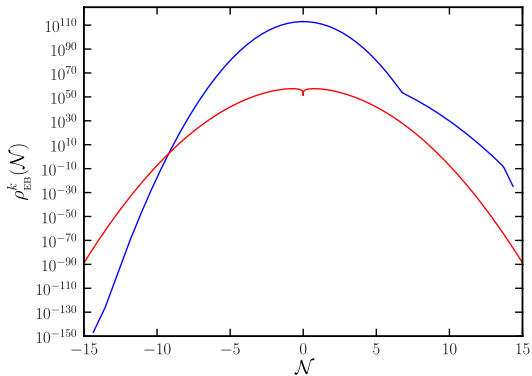


The power spectra with $q = 1$ and $n = -1$, corresponding to $\gamma = -2$ has been plotted for a wide range of wavenumbers. We have set $k_0/M_{\text{Pl}} = 1$, $a_0 = 4 \times 10^{-29}$ and $J_0 = 10^4$, which lead to magnetic fields in the early universe that correspond to observable strengths today⁸.

⁸L. Sriramkumar, K. Atmjeet and R. K. Jain, JCAP **1509**, 010 (2015).



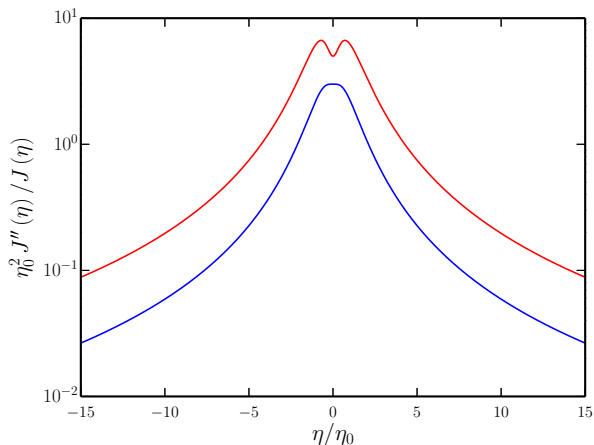
The issue of backreaction



The behavior of the energy density in the electric and magnetic fields for the mode $k = 10^{-20} k_0$ has been plotted (in blue) along with the energy density of the background (in red). We have worked with the same values of the various parameters as in the last figure.



The behavior of J''/J



The behavior of $\eta_0^2 J''/J$, which depends only on η/η_0 , has been plotted for $\gamma = 3$ (in blue) and $\gamma = 5$ (in red). The figure has been plotted over a very narrow range of η/η_0 in order to illustrate the presence of a single maximum for $\gamma = 3$ and two maxima and one minimum for $\gamma = 5$.



Analytical solutions near the bounce for arbitrary γ

Recall that, near the bounce, when $n > 0$, for scales of cosmological interest such that $k \ll k_0$, we had obtained that

$$\bar{A}_k(\eta) \simeq \bar{A}_k(\eta_*) + \bar{A}'_k(\eta_*) \int_{\eta_*}^{\eta} d\tilde{\eta} \frac{J^2(\eta_*)}{J^2(\tilde{\eta})} = \bar{A}_k(\eta_*) + \bar{A}'_k(\eta_*) a^{2n}(\eta_*) \int_{\eta_*}^{\eta} \frac{d\tilde{\eta}}{a^{2n}(\tilde{\eta})},$$

where η_* is a time when $k^2 \ll J''/J$ before the bounce and we have set the constant of integration to be $\bar{A}_k(\eta_*)$.

⁹D. Chowdhury, L. Sriramkumar and R. K. Jain, arXiv:1604.02143 [gr-qc].



Analytical solutions near the bounce for arbitrary γ

Recall that, near the bounce, when $n > 0$, for scales of cosmological interest such that $k \ll k_0$, we had obtained that

$$\bar{A}_k(\eta) \simeq \bar{A}_k(\eta_*) + \bar{A}'_k(\eta_*) \int_{\eta_*}^{\eta} d\tilde{\eta} \frac{J^2(\eta_*)}{J^2(\tilde{\eta})} = \bar{A}_k(\eta_*) + \bar{A}'_k(\eta_*) a^{2n}(\eta_*) \int_{\eta_*}^{\eta} \frac{d\tilde{\eta}}{a^{2n}(\tilde{\eta})},$$

where η_* is a time when $k^2 \ll J''/J$ before the bounce and we have set the constant of integration to be $\bar{A}_k(\eta_*)$.

The above integral can, in fact, be carried out for an arbitrary γ to arrive at

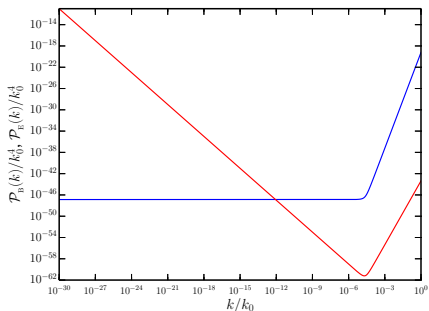
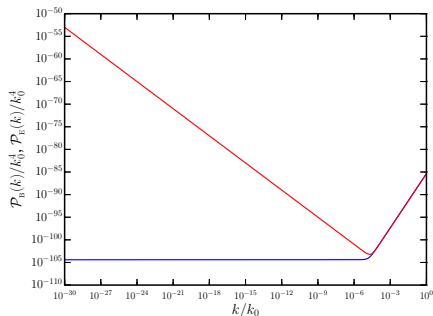
$$\begin{aligned} \bar{A}_k(\eta) &\simeq \bar{A}_k(\eta_*) + \bar{A}'_k(\eta_*) \frac{a^{2n}(\eta_*)}{a_0^{2n}} \\ &\times \left[\eta {}_2F_1\left(\frac{1}{2}, \gamma; \frac{3}{2}; -\frac{\eta^2}{\eta_0^2}\right) - \eta_* {}_2F_1\left(\frac{1}{2}, \gamma; \frac{3}{2}; -\frac{\eta_*^2}{\eta_0^2}\right) \right], \end{aligned}$$

where ${}_2F_1(a, b, c, z)$ denotes the hypergeometric function⁹.

⁹D. Chowdhury, L. Sriramkumar and R. K. Jain, arXiv:1604.02143 [gr-qc].



Power spectra before and after the bounce

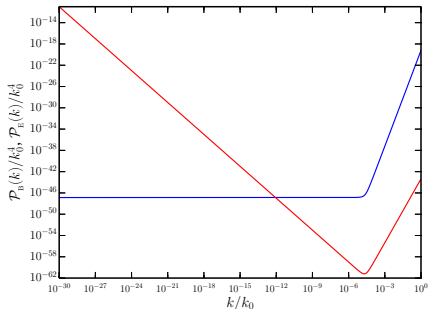
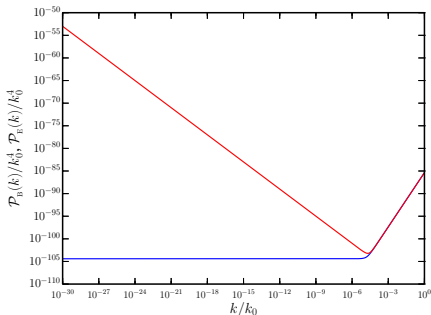


Left: The dimensionless power spectra of the magnetic (in blue) and electric (in red) fields, evaluated before the bounce at $\eta = -\alpha\eta_0$ have been plotted as a function of k/k_0 for $\gamma = 3$, $q = 1$, $a_0 = 8.73 \times 10^{10}$ and $\alpha = 10^5$.

¹⁰ D. Chowdhury, L. Sriramkumar and R. K. Jain, arXiv:1604.02143 [gr-qc].



Power spectra before and after the bounce



Left: The dimensionless power spectra of the magnetic (in blue) and electric (in red) fields, evaluated before the bounce at $\eta = -\alpha \eta_0$ have been plotted as a function of k/k_0 for $\gamma = 3$, $q = 1$, $a_0 = 8.73 \times 10^{10}$ and $\alpha = 10^5$.

Right: The corresponding power spectra evaluated after the bounce at $\eta = \beta \eta_0$, with $\beta = 10^2$. We should mention that the values of the parameters we have worked with lead to magnetic fields of observed strengths today corresponding to a few femto gauss¹⁰.

¹⁰D. Chowdhury, L. Sriramkumar and R. K. Jain, arXiv:1604.02143 [gr-qc].



Duality transformations

It is known that the solutions to the equations of motion governing the scalar and tensor perturbations are invariant under a certain transformation referred to as the duality transformation¹¹. For instance, it can be shown that the dual solution to the de Sitter case corresponds to the matter bounce. Both these cases lead to scale invariant spectra.

¹¹D. Wands, *Phys. Rev. D* **60**, 023507 (1999).



Duality transformations

It is known that the solutions to the equations of motion governing the scalar and tensor perturbations are invariant under a certain transformation referred to as the duality transformation¹¹. For instance, it can be shown that the dual solution to the de Sitter case corresponds to the matter bounce. Both these cases lead to scale invariant spectra.

In the case of electromagnetic fields of our interest here, given a coupling function J , its dual function, say, \tilde{J} , which leads to the same \tilde{J}''/\tilde{J} is found to be

$$J(\eta) \rightarrow \tilde{J}(\eta) = C J(\eta) \int_{\eta_*}^{\eta} \frac{d\bar{\eta}}{J^2(\bar{\eta})},$$

where C and η_* are constants. These constants can be suitably chosen to arrive at a physically reasonable form for \tilde{J}

¹¹D. Wands, *Phys. Rev. D* **60**, 023507 (1999).



Duality transformations

It is known that the solutions to the equations of motion governing the scalar and tensor perturbations are invariant under a certain transformation referred to as the duality transformation¹¹. For instance, it can be shown that the dual solution to the de Sitter case corresponds to the matter bounce. Both these cases lead to scale invariant spectra.

In the case of electromagnetic fields of our interest here, given a coupling function J , its dual function, say, \tilde{J} , which leads to the same \tilde{J}''/\tilde{J} is found to be

$$J(\eta) \rightarrow \tilde{J}(\eta) = C J(\eta) \int_{\eta_*}^{\eta} \frac{d\bar{\eta}}{J^2(\bar{\eta})},$$

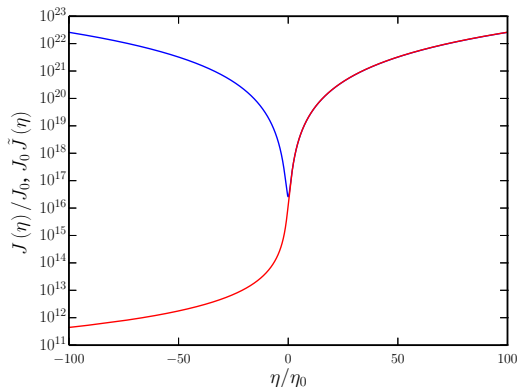
where C and η_* are constants. These constants can be suitably chosen to arrive at a physically reasonable form for \tilde{J}

It can be shown that the cases corresponding to $\gamma = 3$ and $\gamma = -2$ in the bouncing models which had led to scale invariant spectra are dual to each other.

¹¹D. Wands, *Phys. Rev. D* **60**, 023507 (1999).



A symmetric coupling function and its asymmetric dual



The coupling function J (in blue) and its dual \tilde{J} (in red) have been plotted as a function of η/η_0 for $\gamma = 3$ and $\eta_* \rightarrow -\infty$. Also, we have chosen the constant C to be $C/k_0 = 5.7 \times 10^{32}$ so that the dual function \tilde{J} matches the original coupling function J after the bounce¹².

¹²D. Chowdhury, L. Sriramkumar and R. K. Jain, arXiv:1604.02143 [gr-qc].



Equation governing the tensor perturbations

Upon quantization, the tensor perturbations can be written in terms of the corresponding modes, say, h_k , as follows:

$$\begin{aligned}\hat{\gamma}_{ij}(\eta, \mathbf{x}) &= \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \hat{\gamma}_{ij}^{\mathbf{k}}(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= \sum_s \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left(\hat{b}_{\mathbf{k}}^s \varepsilon_{ij}^s(\mathbf{k}) h_k(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{b}_{\mathbf{k}}^{s\dagger} \varepsilon_{ij}^{s*}(\mathbf{k}) h_k^*(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}} \right),\end{aligned}$$

where $\hat{b}_{\mathbf{k}}^s$ and $\hat{b}_{\mathbf{k}}^{s\dagger}$ are the usual creation and annihilation operators that satisfy the standard commutation relations, while $\varepsilon_{ij}^s(\mathbf{k})$ represents the transverse and traceless polarization tensor describing gravitational waves.



Equation governing the tensor perturbations

Upon quantization, the tensor perturbations can be written in terms of the corresponding modes, say, h_k , as follows:

$$\begin{aligned}\hat{\gamma}_{ij}(\eta, \mathbf{x}) &= \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \hat{\gamma}_{ij}^{\mathbf{k}}(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= \sum_s \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left(\hat{b}_{\mathbf{k}}^s \varepsilon_{ij}^s(\mathbf{k}) h_k(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{b}_{\mathbf{k}}^{s\dagger} \varepsilon_{ij}^{s*}(\mathbf{k}) h_k^*(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}} \right),\end{aligned}$$

where $\hat{b}_{\mathbf{k}}^s$ and $\hat{b}_{\mathbf{k}}^{s\dagger}$ are the usual creation and annihilation operators that satisfy the standard commutation relations, while $\varepsilon_{ij}^s(\mathbf{k})$ represents the transverse and traceless polarization tensor describing gravitational waves.

The modes h_k are governed by the differential equation

$$h_k'' + 2\mathcal{H} h_k' + k^2 h_k = 0$$

where $\mathcal{H} = a'/a$ and, in terms of the variable $u_k = M_{\text{Pl}} a h_k / \sqrt{2}$, the above equation reduces to

$$u_k'' + \left(k^2 - \frac{a''}{a} \right) u_k = 0.$$



The tensor power spectrum: Definition

The tensor power spectrum $\mathcal{P}_T(k)$ is defined through the relation

$$\langle \hat{\gamma}_{m_1 n_1}^{\mathbf{k}} \hat{\gamma}_{m_2 n_2}^{\mathbf{p}} \rangle = \frac{(2\pi)^2}{8k^3} \Pi_{m_1 n_1, m_2 n_2}^{\mathbf{k}} \mathcal{P}_T(k) \delta^3(\mathbf{k} + \mathbf{p}),$$

where

$$\Pi_{m_1 n_1, m_2 n_2}^{\mathbf{k}} = \sum_s \varepsilon_{m_1 n_1}^s(\mathbf{k}) \varepsilon_{m_2 n_2}^{s*}(\mathbf{k}).$$

¹³See, for example, L. Sriramkumar, *Curr. Sci.* **97**, 868 (2009).



The tensor power spectrum: Definition

The tensor power spectrum $\mathcal{P}_T(k)$ is defined through the relation

$$\langle \hat{\gamma}_{m_1 n_1}^{\mathbf{k}} \hat{\gamma}_{m_2 n_2}^{\mathbf{p}} \rangle = \frac{(2\pi)^2}{8k^3} \Pi_{m_1 n_1, m_2 n_2}^{\mathbf{k}} \mathcal{P}_T(k) \delta^3(\mathbf{k} + \mathbf{p}),$$

where

$$\Pi_{m_1 n_1, m_2 n_2}^{\mathbf{k}} = \sum_s \varepsilon_{m_1 n_1}^s(\mathbf{k}) \varepsilon_{m_2 n_2}^{s*}(\mathbf{k}).$$

In terms of the quantities h_k and u_k , the tensor power spectrum $\mathcal{P}_T(k)$ in the Bunch-Davies vacuum is given by

$$\mathcal{P}_T(k) = 4 \frac{k^3}{2\pi^2} |h_k|^2 = \frac{8}{M_{\text{Pl}}^2} \frac{k^3}{2\pi^2} \left(\frac{|u_k|}{a} \right)^2,$$

with the right hand side being evaluated at suitably late times¹³.

¹³See, for example, L. Sriramkumar, *Curr. Sci.* **97**, 868 (2009).



The matter bounce

We shall assume that the scale factor describing the bouncing scenario is given in terms of the conformal time coordinate η by the relation

$$a(\eta) = a_0 \left(1 + \eta^2/\eta_0^2\right) = a_0 \left(1 + k_0^2 \eta^2\right).$$

As we had discussed earlier, at very early times, *viz.* when $\eta \ll -\eta_0$, the scale factor behaves as in a matter dominated epoch¹⁴.

¹⁴See, for example, R. Brandenberger, [arXiv:1206.4196](https://arxiv.org/abs/1206.4196).



The matter bounce

We shall assume that the scale factor describing the bouncing scenario is given in terms of the conformal time coordinate η by the relation

$$a(\eta) = a_0 \left(1 + \eta^2/\eta_0^2\right) = a_0 \left(1 + k_0^2 \eta^2\right).$$

As we had discussed earlier, at very early times, *viz.* when $\eta \ll -\eta_0$, the scale factor behaves as in a matter dominated epoch¹⁴.

The quantity a''/a corresponding to the above scale factor is given by

$$\frac{a''}{a} = \frac{2 k_0^2}{1 + k_0^2 \eta^2},$$

which has essentially a Lorentzian profile.

¹⁴See, for example, R. Brandenberger, [arXiv:1206.4196](https://arxiv.org/abs/1206.4196).



The tensor modes in the first domain

Let us divide the period before the bounce into two domains, with the first domain be determined by the condition $-\infty < \eta < -\alpha \eta_0$, where α is a relatively large number, which we shall set to be, say, 10^5 .



The tensor modes in the first domain

Let us divide the period before the bounce into two domains, with the first domain be determined by the condition $-\infty < \eta < -\alpha \eta_0$, where α is a relatively large number, which we shall set to be, say, 10^5 .

In the first domain, we can assume that the scale factor behaves as $a(\eta) \simeq a_0 k_0^2 \eta^2$, so that $a''/a = 2/\eta^2$. Since the condition $k^2 = a''/a$ corresponds to, say, $\eta_k = -\sqrt{2}/k$, the initial conditions can be imposed when $\eta \ll \eta_k$.



The tensor modes in the first domain

Let us divide the period before the bounce into two domains, with the first domain be determined by the condition $-\infty < \eta < -\alpha \eta_0$, where α is a relatively large number, which we shall set to be, say, 10^5 .

In the first domain, we can assume that the scale factor behaves as $a(\eta) \simeq a_0 k_0^2 \eta^2$, so that $a''/a = 2/\eta^2$. Since the condition $k^2 = a''/a$ corresponds to, say, $\eta_k = -\sqrt{2}/k$, the initial conditions can be imposed when $\eta \ll \eta_k$.

The modes h_k can be easily obtained in such a case and the positive frequency modes that correspond to the vacuum state at early times are given by

$$h_k(\eta) = \frac{\sqrt{2}}{M_{\text{Pl}}} \frac{1}{\sqrt{2k}} \frac{1}{a_0 k_0^2 \eta^2} \left(1 - \frac{i}{k\eta} \right) e^{-ik\eta}.$$



The modes in the second domain

Let us now consider the behavior of the modes in the domain $-\alpha\eta_0 < \eta < 0$. Since we are interested in scales much smaller than k_0 , we shall assume that $\eta_k \ll -\alpha\eta_0$, which corresponds to $k \ll k_0/\alpha$.



The modes in the second domain

Let us now consider the behavior of the modes in the domain $-\alpha \eta_0 < \eta < 0$. Since we are interested in scales much smaller than k_0 , we shall assume that $\eta_k \ll -\alpha \eta_0$, which corresponds to $k \ll k_0/\alpha$.

In such a case, upon ignoring the k^2 term, the equation governing h_k can be immediately integrated to yield

$$h_k(\eta) \simeq h_k(\eta_*) + h'_k(\eta_*) a^2(\eta_*) \int_{\eta_*}^{\eta} \frac{d\tilde{\eta}}{a^2(\tilde{\eta})},$$

where η_* is a suitably chosen time and the scale factor $a(\eta)$ is given by the complete expression.



The modes in the second domain

Let us now consider the behavior of the modes in the domain $-\alpha \eta_0 < \eta < 0$. Since we are interested in scales much smaller than k_0 , we shall assume that $\eta_k \ll -\alpha \eta_0$, which corresponds to $k \ll k_0/\alpha$.

In such a case, upon ignoring the k^2 term, the equation governing h_k can be immediately integrated to yield

$$h_k(\eta) \simeq h_k(\eta_*) + h'_k(\eta_*) a^2(\eta_*) \int_{\eta_*}^{\eta} \frac{d\tilde{\eta}}{a^2(\tilde{\eta})},$$

where η_* is a suitably chosen time and the scale factor $a(\eta)$ is given by the complete expression.

If we choose $\eta_* = -\alpha \eta_0$, we can make use of the solution in the first domain to obtain the following solution in the second domain:

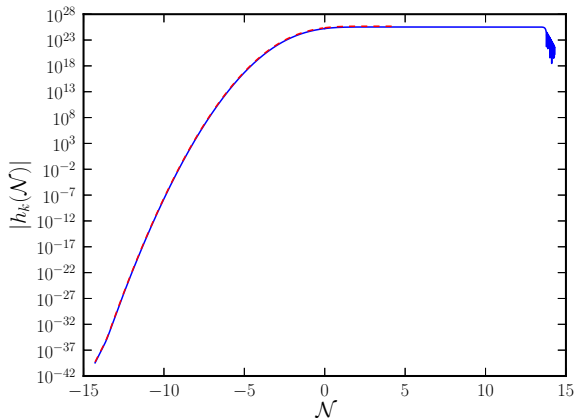
$$h_k = A_k + B_k f(k_0 \eta),$$

where

$$f(k_0 \eta) = \frac{k_0 \eta}{1 + k_0^2 \eta^2} + \tan^{-1}(k_0 \eta).$$



Evolution of the tensor modes across the bounce



A comparison of the numerical results (in blue) with the analytical results (in red) for the amplitude of the tensor mode $|h_k|$ corresponding to the wavenumber $k/k_0 = 10^{-20}$. We have set $a_0 = 10^5$ and, for plotting the analytical results we have chosen $\alpha = 10^5$.



The third domain and the tensor power spectrum

The quantities A_k and B_k are given by

$$A_k = \frac{\sqrt{2}}{M_{\text{Pl}}} \frac{1}{\sqrt{2k}} \frac{1}{a_0 \alpha^2} \left(1 + \frac{i k_0}{\alpha k}\right) e^{i \alpha k/k_0} + B_k f(\alpha),$$

$$B_k = \frac{\sqrt{2}}{M_{\text{Pl}}} \frac{1}{\sqrt{2k}} \frac{1}{2 a_0 \alpha^2} (1 + \alpha^2)^2 \left(\frac{3 i k_0}{\alpha^2 k} + \frac{3}{\alpha} - \frac{i k}{k_0}\right) e^{i \alpha k/k_0}.$$



The third domain and the tensor power spectrum

The quantities A_k and B_k are given by

$$A_k = \frac{\sqrt{2}}{M_{\text{Pl}}} \frac{1}{\sqrt{2k}} \frac{1}{a_0 \alpha^2} \left(1 + \frac{i k_0}{\alpha k}\right) e^{i \alpha k/k_0} + B_k f(\alpha),$$

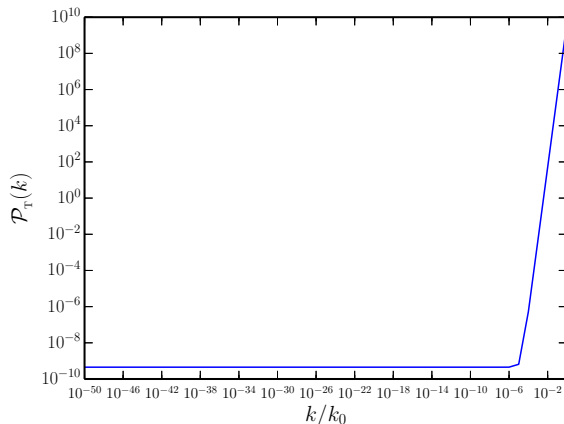
$$B_k = \frac{\sqrt{2}}{M_{\text{Pl}}} \frac{1}{\sqrt{2k}} \frac{1}{2 a_0 \alpha^2} (1 + \alpha^2)^2 \left(\frac{3 i k_0}{\alpha^2 k} + \frac{3}{\alpha} - \frac{i k}{k_0}\right) e^{i \alpha k/k_0}.$$

If we evaluate the tensor power spectrum after the bounce at $\eta = \beta \eta_0$, we find that it can be expressed as

$$\mathcal{P}_{\text{T}}(k) = 4 \frac{k^3}{2 \pi^2} |A_k + B_k f(\beta)|^2.$$



The tensor power spectrum



The behavior of the tensor power spectrum has been plotted as a function of k/k_0 for a wide range of wavenumbers. In plotting this figure, we have set $k_0/M_{\text{Pl}} = 1$, $a_0 = 10^5$, $\alpha = 10^5$ and $\beta = 10^2$. Note that the power spectrum is scale invariant for $k \ll k_0/\alpha$.



Tensor bi-spectrum and non-Gaussianity parameter

The tensor bi-spectrum, evaluated at the conformal time, say, η_e , is defined as

$$\langle \hat{\gamma}_{m_1 n_1}^{\mathbf{k}_1}(\eta_e) \hat{\gamma}_{m_2 n_2}^{\mathbf{k}_2}(\eta_e) \hat{\gamma}_{m_3 n_3}^{\mathbf{k}_3}(\eta_e) \rangle = (2\pi)^3 \mathcal{B}_{\gamma\gamma\gamma}^{m_1 n_1 m_2 n_2 m_3 n_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \\ \times \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)$$

and, for convenience, we shall set

$$\mathcal{B}_{\gamma\gamma\gamma}^{m_1 n_1 m_2 n_2 m_3 n_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = (2\pi)^{-9/2} G_{\gamma\gamma\gamma}^{m_1 n_1 m_2 n_2 m_3 n_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3).$$

¹⁵V. Sreenath, R. Tibrewala and L. Sriramkumar, JCAP **1312**, 037 (2013).



Tensor bi-spectrum and non-Gaussianity parameter

The tensor bi-spectrum, evaluated at the conformal time, say, η_e , is defined as

$$\langle \hat{\gamma}_{m_1 n_1}^{\mathbf{k}_1}(\eta_e) \hat{\gamma}_{m_2 n_2}^{\mathbf{k}_2}(\eta_e) \hat{\gamma}_{m_3 n_3}^{\mathbf{k}_3}(\eta_e) \rangle = (2\pi)^3 \mathcal{B}_{\gamma\gamma\gamma}^{m_1 n_1 m_2 n_2 m_3 n_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \times \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)$$

and, for convenience, we shall set

$$\mathcal{B}_{\gamma\gamma\gamma}^{m_1 n_1 m_2 n_2 m_3 n_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = (2\pi)^{-9/2} G_{\gamma\gamma\gamma}^{m_1 n_1 m_2 n_2 m_3 n_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3).$$

As in the scalar case, one can define a dimensionless non-Gaussianity parameter to characterize the amplitude of the tensor bi-spectrum as follows¹⁵:

$$h_{\text{NL}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = - \left(\frac{4}{2\pi^2} \right)^2 [k_1^3 k_2^3 k_3^3 G_{\gamma\gamma\gamma}^{m_1 n_1 m_2 n_2 m_3 n_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)] \times \left[\Pi_{m_1 n_1, m_2 n_2}^{\mathbf{k}_1} \Pi_{m_3 n_3, \bar{m}\bar{n}}^{\mathbf{k}_2} k_3^3 \mathcal{P}_{\text{T}}(k_1) \mathcal{P}_{\text{T}}(k_2) + \text{five permutations} \right]^{-1}.$$

¹⁵V. Sreenath, R. Tibrewala and L. Sriramkumar, JCAP **1312**, 037 (2013).



The third order action and the tensor bi-spectrum

The third order action that leads to the tensor bi-spectrum is given by¹⁶

$$S_{\gamma\gamma\gamma}^3[\gamma_{ij}] = \frac{M_{\text{Pl}}^2}{2} \int d\eta \int d^3\mathbf{x} \left[\frac{a^2}{2} \gamma_{lj} \gamma_{im} \partial_l \partial_m \gamma_{ij} - \frac{a^2}{4} \gamma_{ij} \gamma_{lm} \partial_l \partial_m \gamma_{ij} \right].$$

¹⁶ J. Maldacena, JHEP **0305**, 013 (2003).



The third order action and the tensor bi-spectrum

The third order action that leads to the tensor bi-spectrum is given by¹⁶

$$S_{\gamma\gamma\gamma}^3[\gamma_{ij}] = \frac{M_{\text{Pl}}^2}{2} \int d\eta \int d^3\mathbf{x} \left[\frac{a^2}{2} \gamma_{lj} \gamma_{im} \partial_l \partial_m \gamma_{ij} - \frac{a^2}{4} \gamma_{ij} \gamma_{lm} \partial_l \partial_m \gamma_{ij} \right].$$

The tensor bi-spectrum calculated in the perturbative vacuum using the Maldacena formalism, can be written in terms of the modes $h_{\mathbf{k}}$ as follows:

$$\begin{aligned} G_{\gamma\gamma\gamma}^{m_1 n_1 m_2 n_2 m_3 n_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) &= M_{\text{Pl}}^2 \left[\left(\Pi_{m_1 n_1, ij}^{\mathbf{k}_1} \Pi_{m_2 n_2, im}^{\mathbf{k}_2} \Pi_{m_3 n_3, lj}^{\mathbf{k}_3} - \frac{1}{2} \Pi_{m_1 n_1, ij}^{\mathbf{k}_1} \Pi_{m_2 n_2, ml}^{\mathbf{k}_2} \Pi_{m_3 n_3, ij}^{\mathbf{k}_3} \right) k_{1m} k_{1l} \right. \\ &\quad \left. + \text{five permutations} \right] \\ &\quad \times [h_{k_1}(\eta_e) h_{k_2}(\eta_e) h_{k_3}(\eta_e) \mathcal{G}_{\gamma\gamma\gamma}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) + \text{complex conjugate}], \end{aligned}$$

where $\mathcal{G}_{\gamma\gamma\gamma}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ is described by the integral

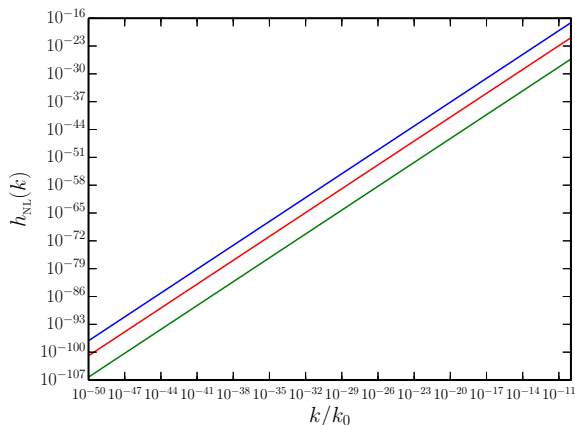
$$\mathcal{G}_{\gamma\gamma\gamma}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = -\frac{i}{4} \int_{\eta_i}^{\eta_e} d\eta a^2 h_{k_1}^* h_{k_2}^* h_{k_3}^*,$$

with η_i denoting the time when the initial conditions are imposed on the perturbations.

¹⁶J. Maldacena, JHEP 0305, 013 (2003).



The contributions due to the three domains



The contributions to the non-Gaussianity parameter h_{NL} in the equilateral limit from the first (in green), the second (in red) and the third (in blue) domains have been plotted as a function of k/k_0 for $k \ll k_0/\alpha$. Clearly, the third domain gives rise to the maximum contribution to h_{NL} .



The effect of the long wavelength tensor modes

Since the amplitude of a long wavelength mode freezes on super-Hubble scales during inflation, such modes can be treated as a background as far as the smaller wavelength modes are concerned. Let us denote the constant amplitude of the long wavelength tensor mode as γ_{ij}^B .

¹⁷S. Kundu, JCAP **1404**, 016 (2014).



The effect of the long wavelength tensor modes

Since the amplitude of a long wavelength mode freezes on super-Hubble scales during inflation, such modes can be treated as a background as far as the smaller wavelength modes are concerned. Let us denote the constant amplitude of the long wavelength tensor mode as γ_{ij}^B .

In the presence of such a long wavelength mode, the background FLRW metric can be written as

$$ds^2 = -dt^2 + a^2(t) [e^{\gamma^B}]_{ij} d\mathbf{x}^i d\mathbf{x}^j,$$

i.e. the spatial coordinates are modified according to a spatial transformation of the form $\mathbf{x}' = \Lambda \mathbf{x}$, where $\Lambda_{ij} = [e^{\gamma^B/2}]_{ij}$.

¹⁷ S. Kundu, JCAP **1404**, 016 (2014).



The effect of the long wavelength tensor modes

Since the amplitude of a long wavelength mode freezes on super-Hubble scales during inflation, such modes can be treated as a background as far as the smaller wavelength modes are concerned. Let us denote the constant amplitude of the long wavelength tensor mode as γ_{ij}^B .

In the presence of such a long wavelength mode, the background FLRW metric can be written as

$$ds^2 = -dt^2 + a^2(t) [e^{\gamma^B}]_{ij} d\mathbf{x}^i d\mathbf{x}^j,$$

i.e. the spatial coordinates are modified according to a spatial transformation of the form $\mathbf{x}' = \Lambda \mathbf{x}$, where $\Lambda_{ij} = [e^{\gamma^B/2}]_{ij}$.

Under such a spatial transformation, the small wavelength tensor perturbation transforms as¹⁷

$$\gamma_{ij}^{\mathbf{k}} \rightarrow \det(\Lambda^{-1}) \gamma_{ij}^{\Lambda^{-1} \mathbf{k}},$$

where $\det(\Lambda^{-1}) = 1$.

¹⁷S. Kundu, JCAP **1404**, 016 (2014).



The behavior of the two and three-point functions

On using the above results, one finds that the tensor two-point function in the presence of a long wavelength mode denoted by, say, the wavenumber k , can be written as

$$\begin{aligned} \langle \hat{\gamma}_{m_1 n_1}^{\mathbf{k}_1} \hat{\gamma}_{m_2 n_2}^{\mathbf{k}_2} \rangle_k &= \frac{(2\pi)^2}{2k_1^3} \frac{\Pi_{m_1 n_1, m_2 n_2}^{\mathbf{k}_1}}{4} \mathcal{P}_T(k_1) \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2) \\ &\times \left[1 - \left(\frac{n_T - 3}{2} \right) \gamma_{ij}^B \hat{n}_{1i} \hat{n}_{1j} \right], \end{aligned}$$

where $\hat{n}_{1i} = k_{1i}/k_1$.

¹⁸V. Sreenath and L. Sriramkumar, JCAP **1410**, 021 (2014).



The behavior of the two and three-point functions

On using the above results, one finds that the tensor two-point function in the presence of a long wavelength mode denoted by, say, the wavenumber k , can be written as

$$\begin{aligned} \langle \hat{\gamma}_{m_1 n_1}^{\mathbf{k}_1} \hat{\gamma}_{m_2 n_2}^{\mathbf{k}_2} \rangle_k &= \frac{(2\pi)^2}{2k_1^3} \frac{\Pi_{m_1 n_1, m_2 n_2}^{\mathbf{k}_1}}{4} \mathcal{P}_T(k_1) \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2) \\ &\times \left[1 - \left(\frac{n_T - 3}{2} \right) \gamma_{ij}^B \hat{n}_{1i} \hat{n}_{1j} \right], \end{aligned}$$

where $\hat{n}_{1i} = k_{1i}/k_1$.

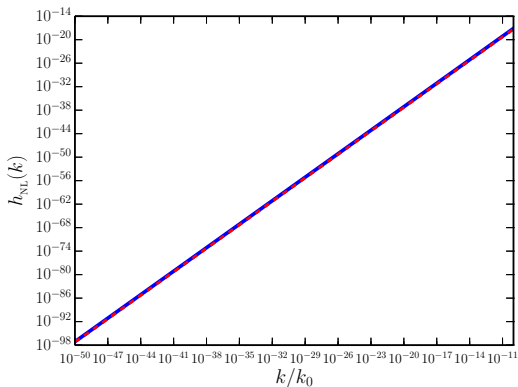
One can also show that, in the presence of a long wavelength mode, the tensor bi-spectrum can be written as¹⁸

$$\begin{aligned} \langle \hat{\gamma}_{m_1 n_1}^{\mathbf{k}_1} \hat{\gamma}_{m_2 n_2}^{\mathbf{k}_2} \hat{\gamma}_{m_3 n_3}^{\mathbf{k}_3} \rangle_{k_3} &= -\frac{(2\pi)^{5/2}}{4k_1^3 k_3^3} \left(\frac{n_T - 3}{32} \right) \mathcal{P}_T(k_1) \mathcal{P}_T(k_3) \\ &\times \Pi_{m_1 n_1, m_2 n_2}^{\mathbf{k}_1} \Pi_{m_3 n_3, ij}^{\mathbf{k}_3} \hat{n}_{1i} \hat{n}_{1j} \delta^3(\mathbf{k}_1 + \mathbf{k}_2). \end{aligned}$$

¹⁸V. Sreenath and L. Sriramkumar, JCAP **1410**, 021 (2014).



The complete contribution to h_{NL}



The behavior of h_{NL} in the equilateral (in blue) and the squeezed (in red) limits plotted as a function of k/k_0 for $k \ll k_0/\alpha$. The resulting h_{NL} is considerably small when compared to the values that arise in de Sitter inflation wherein $3/8 \lesssim h_{\text{NL}} \lesssim 1/2$. Moreover, we find that h_{NL} behaves as k^2 in the equilateral and the squeezed limits, with similar amplitudes¹⁹.

¹⁹D. Chowdhury, V. Sreenath and L. Sriramkumar, JCAP **1511**, 002 (2015).



Summary

- Scale invariant magnetic fields of observable strengths can be generated in a class of bouncing models. However, as in the inflationary context, they are also plagued by the problem of backreaction.



Summary

- Scale invariant magnetic fields of observable strengths can be generated in a class of bouncing models. However, as in the inflationary context, they are also plagued by the problem of backreaction.
- As in the case of the scalar and tensor perturbations, the power spectrum of the magnetic field remains invariant under a two parameter family of transformations (called the duality transformations) of the non-minimal coupling function.



Summary

- Scale invariant magnetic fields of observable strengths can be generated in a class of bouncing models. However, as in the inflationary context, they are also plagued by the problem of backreaction.
- As in the case of the scalar and tensor perturbations, the power spectrum of the magnetic field remains invariant under a two parameter family of transformations (called the duality transformations) of the non-minimal coupling function.
- In a matter bounce which leads to a scale invariant tensor power spectrum as de Sitter inflation does, the amplitude of the tensor bi-spectrum proves to be considerably smaller. Moreover, due to the rapid growth of the amplitude of the tensor modes as one approaches the bounce, the consistency relation governing the tensor bi-spectrum is violated in these scenarios.



Issues confronting bouncing models

- The growth of the perturbations as one approaches the bounce during the contracting phase causes serious concerns about the validity of linear perturbation theory near the bounce. Is it, for instance, sufficient if the perturbations remain small in specific gauges? Is a divergent curvature perturbation acceptable? These are issues of considerable importance and they need to be addressed satisfactorily.

²⁰ L. E. Allen and D. Wands, *Phys. Rev.* **70**, 063515 (2004).

²¹ Y-F. Cai, R. Brandenberger and X. Zhang, *Phys. Letts. B* **703**, 25 (2011).

²² J. Quintin, Z. Sherkatghanad, Y-F. Cai and R. Brandenberger, *Phys. Rev. D* **92**, 062532 (2015).



Issues confronting bouncing models

- The growth of the perturbations as one approaches the bounce during the contracting phase causes serious concerns about the validity of linear perturbation theory near the bounce. Is it, for instance, sufficient if the perturbations remain small in specific gauges? Is a divergent curvature perturbation acceptable? These are issues of considerable importance and they need to be addressed satisfactorily.
- Analysis in the cases of a few specific examples seem to suggest that bouncing models lead to a large tensor-to-scalar ratio that is inconsistent with the observations²⁰. Is it possible to construct models with lower tensor amplitudes?

²⁰ L. E. Allen and D. Wands, *Phys. Rev.* **70**, 063515 (2004).

²¹ Y-F. Cai, R. Brandenberger and X. Zhang, *Phys. Letts. B* **703**, 25 (2011).

²² J. Quintin, Z. Sherkatghanad, Y-F. Cai and R. Brandenberger, *Phys. Rev. D* **92**, 062532 (2015).



Issues confronting bouncing models

- The growth of the perturbations as one approaches the bounce during the contracting phase causes serious concerns about the validity of linear perturbation theory near the bounce. Is it, for instance, sufficient if the perturbations remain small in specific gauges? Is a divergent curvature perturbation acceptable? These are issues of considerable importance and they need to be addressed satisfactorily.
- Analysis in the cases of a few specific examples seem to suggest that bouncing models lead to a large tensor-to-scalar ratio that is inconsistent with the observations²⁰. Is it possible to construct models with lower tensor amplitudes?
- After the bounce, the universe needs to transit to a radiation dominated epoch. How can this be achieved? Does this process affect the evolution of the large scale perturbations²¹?

²⁰ L. E. Allen and D. Wands, *Phys. Rev.* **70**, 063515 (2004).

²¹ Y-F. Cai, R. Brandenberger and X. Zhang, *Phys. Letts. B* **703**, 25 (2011).

²² J. Quintin, Z. Sherkatghanad, Y-F. Cai and R. Brandenberger, *Phys. Rev. D* **92**, 062532 (2015).



Issues confronting bouncing models

- The growth of the perturbations as one approaches the bounce during the contracting phase causes serious concerns about the validity of linear perturbation theory near the bounce. Is it, for instance, sufficient if the perturbations remain small in specific gauges? Is a divergent curvature perturbation acceptable? These are issues of considerable importance and they need to be addressed satisfactorily.
- Analysis in the cases of a few specific examples seem to suggest that bouncing models lead to a large tensor-to-scalar ratio that is inconsistent with the observations²⁰. Is it possible to construct models with lower tensor amplitudes?
- After the bounce, the universe needs to transit to a radiation dominated epoch. How can this be achieved? Does this process affect the evolution of the large scale perturbations²¹?
- Does the growth of perturbations near the bounce naturally lead to large levels of non-Gaussianities in bouncing models²²?

²⁰ L. E. Allen and D. Wands, *Phys. Rev.* **70**, 063515 (2004).

²¹ Y-F. Cai, R. Brandenberger and X. Zhang, *Phys. Letts. B* **703**, 25 (2011).

²² J. Quintin, Z. Sherkatghanad, Y-F. Cai and R. Brandenberger, *Phys. Rev. D* **92**, 062532 (2015).



Thank you for your attention