Observing the end of inflation

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Colloquium

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Tata Institute of Fundamental Research, Mumbai

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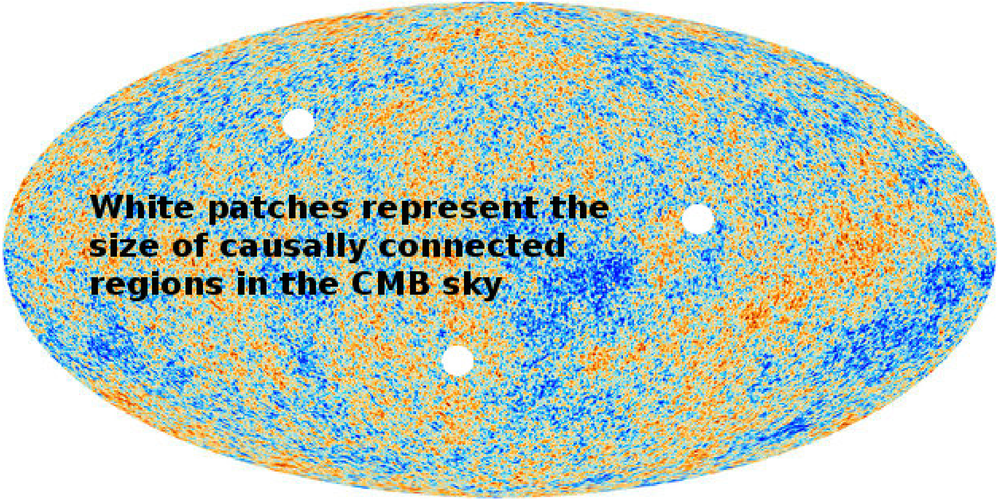
Plan of the talk

This talk is based on. . .

- A. Hoory, J. Martin, A. Paul and **L. Sriramkumar**, Primary gravitational waves at high frequencies I: Origin of suppression in the power spectrum, JCAP 05, 080 (2026) [arXiv:2512.03959 [astro-ph.CO]].
- A. Hoory, J. Martin, A. Paul and **L. Sriramkumar**, Primary gravitational waves at high frequencies II: Emergence of the exponential cut-off in the power spectrum, arXiv:2605.18286 [astro-ph.CO].

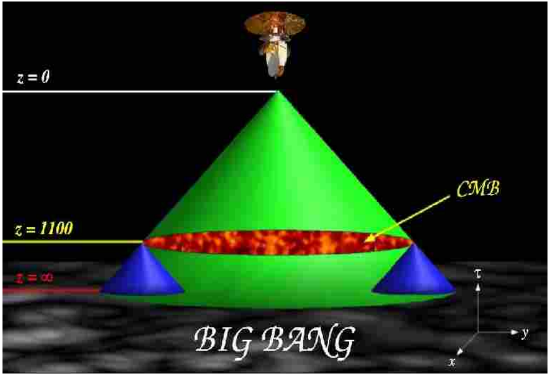
Plan of the talk

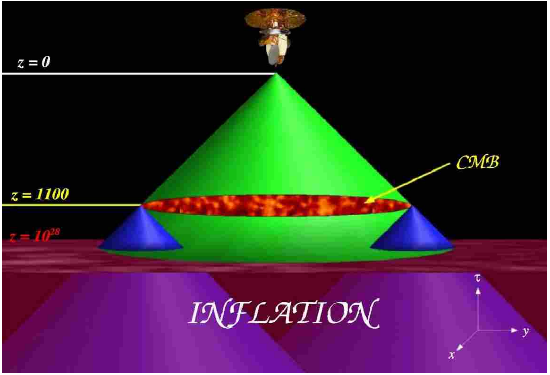
Horizon problem



The radiation from the cosmic microwave background (CMB) arriving at us from regions separated by more than the Hubble radius at the surface of last scattering, which subtends an angle of about 1° today, could not have interacted before decoupling.

Resolution of the horizon problem in the inflationary scenario

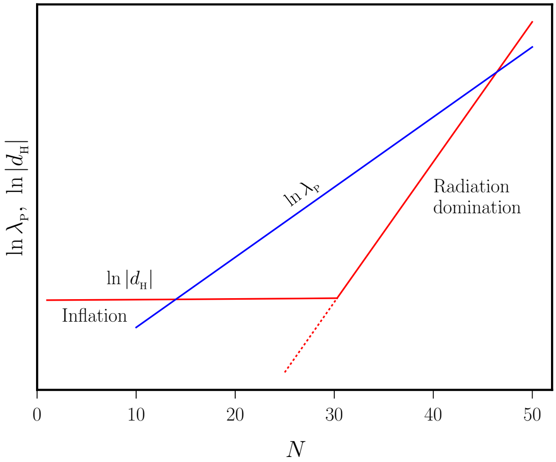




Another illustration of the horizon problem (on the left), and an illustration of its resolution (on the right) through an early and sufficiently long epoch of inflation[1].

[1] Images from W. Kinney, astro-ph/0301448.

Bringing the modes inside the Hubble radius

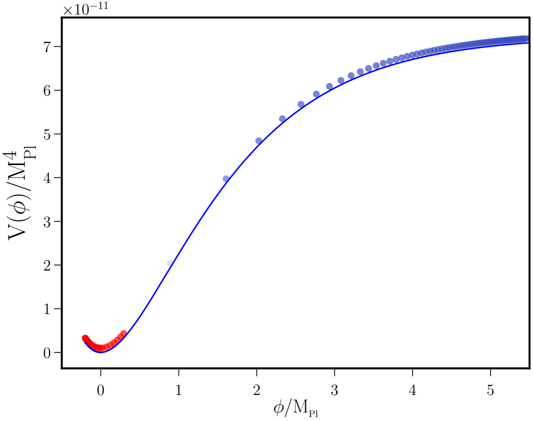


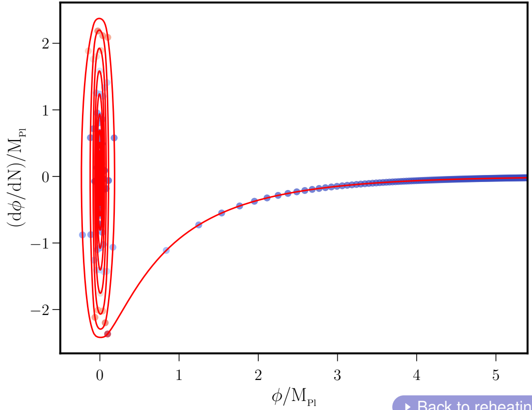
Evolution of comoving lengths

The physical wavelength $\lambda_{\rm P} \propto a$ (in blue) and the Hubble radius $d_{\rm H} = H^{-1}$ (in red) in the inflationary scenario[2]. The scale factor is expressed in terms of e-folds N as $a(N) \propto {\rm e}^N$.

[2] See, for example, E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley Publishing Company, New York, 1990), Fig. 8.4.

Inflationary attractor





Back to reheating

The evolution of the scalar field in the popular Starobinsky model, which leads to slow roll inflation, is indicated (as circles, in blue and red) at regular intervals of time (on the left). Illustration of the behavior of the scalar field in phase space (on the right)[3].

[3]Figure credit H. V. Ragavendra.

Origin of the primordial perturbations[4]

Scalar perturbations:

- The quantum fluctuations associated with the scalar fields that drive inflation are responsible for the primordial perturbations. The perturbations in the metric and matter are related through the Einstein's equations.
- The scalar perturbations leave the largest imprints on the CMB, and are primarily responsible for the large scale structure in the universe.

Tensor perturbations:

- The tensor perturbations, i.e. gravitational waves (GWs), can be generated from the quantum vacuum, even in the absence of sources. These are often referred to as primary GWs (PGWs).
- The GWs satisfy the wave equation in the given background spacetime.

[4] See, for instance, L. Sriramkumar, Curr. Sci. **97**, 868 (2009) [arXiv:0904.4584 [astro-ph.CO]].

Describing the primordial power spectra

While comparing with the observations, for convenience, one often uses the following power law, template for the primordial scalar and tensor power spectra:

$$\mathcal{P}_{\mathrm{S}}(k) = A_{\mathrm{S}} \left(\frac{k}{k_*}\right)^{n_{\mathrm{S}}-1}, \quad \mathcal{P}_{\mathrm{T}}(k) = A_{\mathrm{T}} \left(\frac{k}{k_*}\right)^{n_{\mathrm{T}}},$$

where $(A_{\mathrm{S}}, A_{\mathrm{T}})$ and $(n_{\mathrm{S}}, n_{\mathrm{T}})$ denote the (scalar, tensor) amplitudes and spectral indices, while k_* is the pivot scale at which these values are quoted.

The tensor-to-scalar ratio r is defined as

$$r(k) = \frac{\mathcal{P}_{\mathrm{T}}(k)}{\mathcal{P}_{\mathrm{S}}(k)}.$$

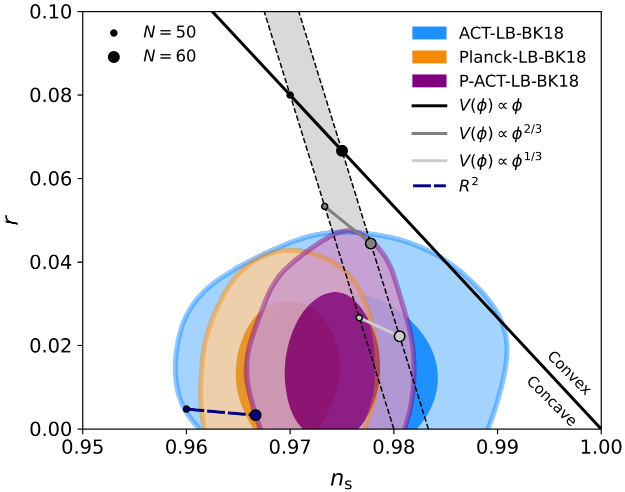
In slow roll inflation

$$n_{\mathrm{S}} \simeq 1 - 2\epsilon_1 - \epsilon_2, \quad n_{\mathrm{T}} \simeq -2\epsilon_1, \quad r \simeq 16\epsilon_1,$$

where $(\epsilon_1, \epsilon_2) \ll 1$ are the first two slow roll parameters[5].

[5]See, for instance, L. Sriramkumar, Curr. Sci. **97**, 868 (2009) [arXiv:0904.4584 [astro-ph.CO]].

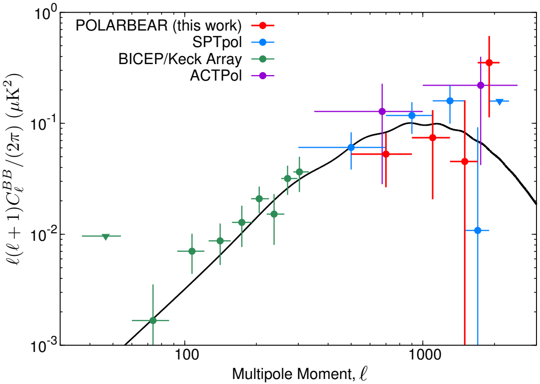
Performance of inflationary models in the $n_{\rm s}$-r plane

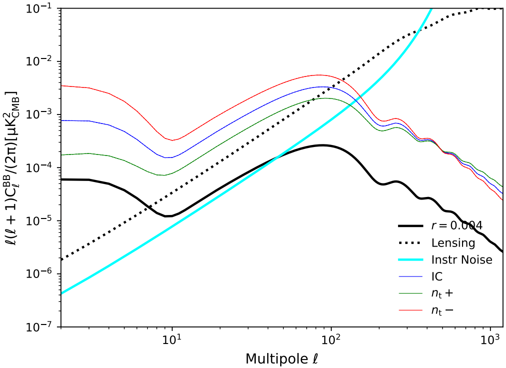


Latest constraints on $n_{\rm s}$ and r from ACT, in combination with other data sets, compared to the theoretical predictions of some of the popular inflationary models[6].

[6] ACT Collaboration (E. Calabrese *et al.*), arXiv:2503.14454 [astro-ph.CO].

Prospects for observing imprints of tensor perturbations on the CMB





Left: Constraints on the B-mode, angular power spectrum of the CMB from two years of POLARBEAR data[7].

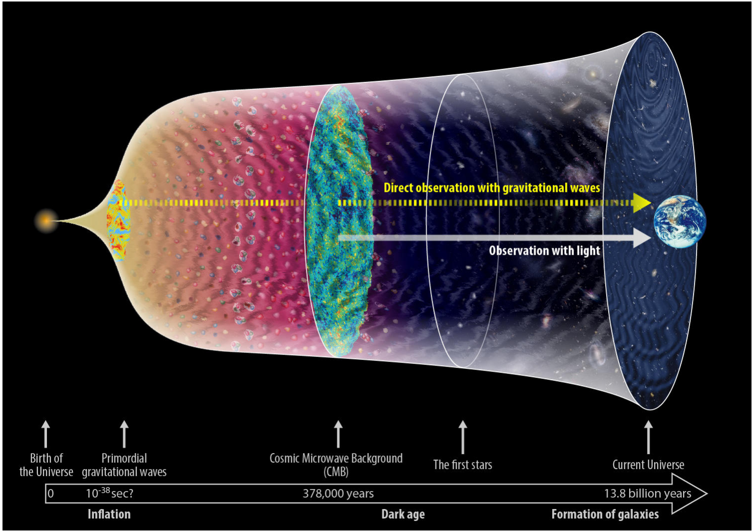
Right: Possibility of observing the B-mode angular power spectra of the CMB by LiteBIRD[8].

[7] POLARBEAR Collaboration (P. A. R. Ade *et al.*), Ap. J. **848**, 141, (2017).

[8] D. Paoletti, F. Finelli, J. Valiviita and M. Hazumi, Phys. Rev. D **106**, 083528 (2022).

Plan of the talk

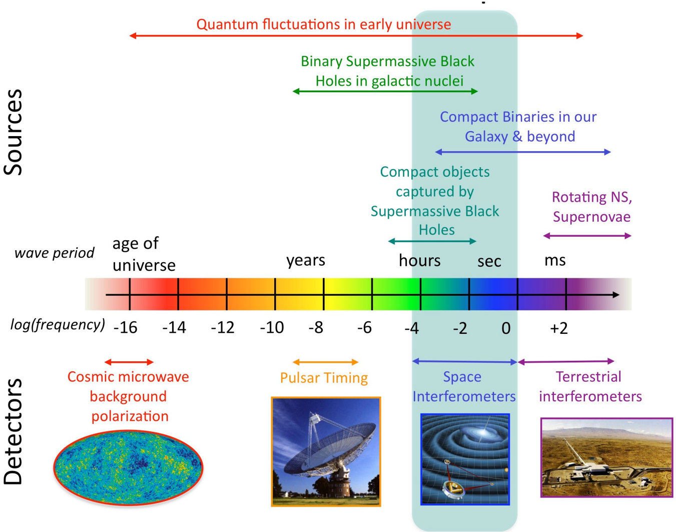
Probing the primordial universe through GWs



GWs provide a unique window to probe the primordial universe[9].

[9]Image from https://gwpo.nao.ac.jp/en/gallery/000061.html.

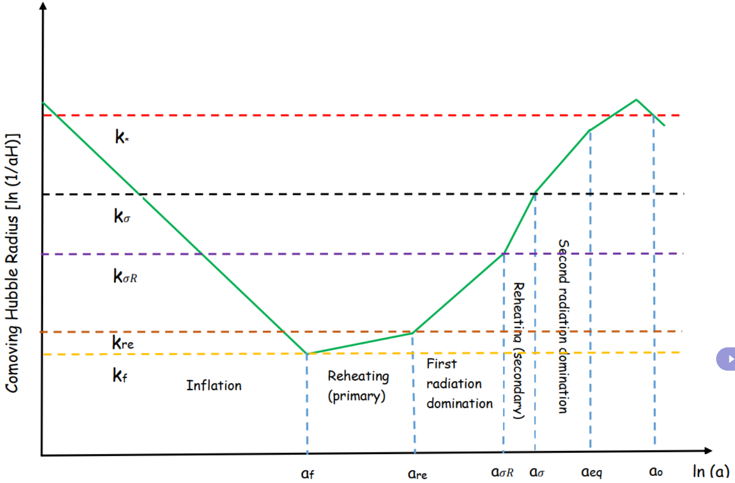
Spectral range of GWs



Different sources of GWs and corresponding detectors[10].

[10] J. B. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (Pearson Education, Delhi, 2003).

Behavior of comoving wave numbers and the Hubble radius



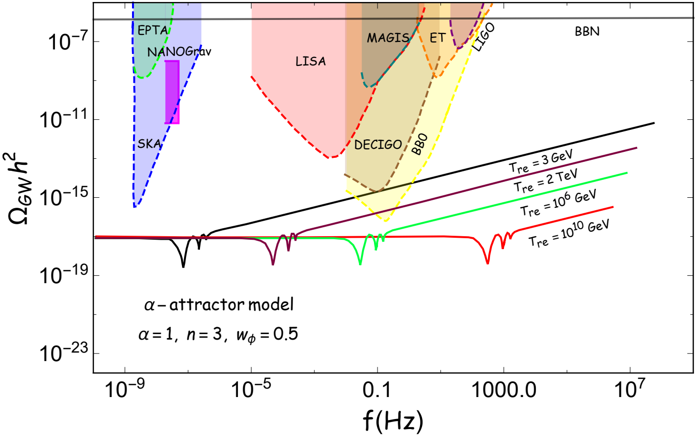
‣ Evolution of physical lengths

‣ Inflationary attractor

Behavior of the comoving wave number k (horizontal lines in different colors) and the comoving Hubble radius $d_{\rm H}/a = (a\,H)^{-1}$ (in green) across different epochs[11].

[11] Md. R. Haque, D. Maity, T. Paul and L. Sriramkumar, Phys. Rev. D **104**, 063513 (2021) [arXiv:2105.09242 [astro-ph.CO]].

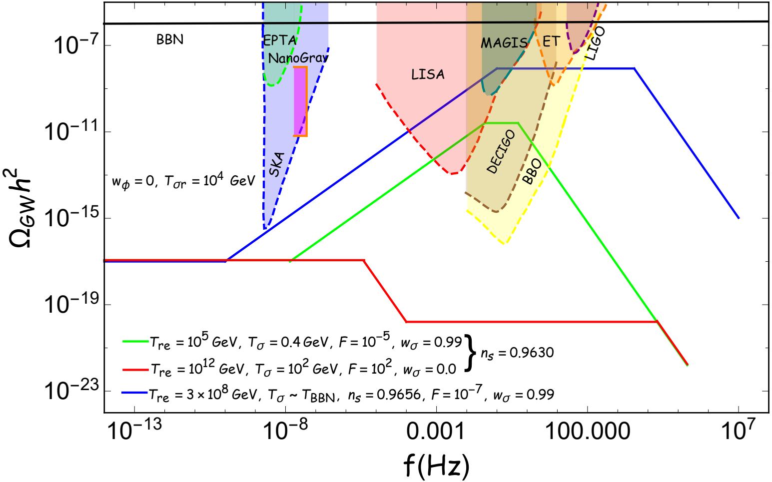
Effects on the spectral energy density of PGWs due to reheating



The dimensionless spectral energy density of PGWs today, viz. $\Omega_{\rm GW}$, is plotted over a wide range of frequency f, for different reheating temperatures (in red, green, brown and black)[12]. Note that $n_{\rm GW} = 2\,(3\,w_{\rm re} - 1)/(3\,w_{\rm re} + 1)$.

[12] K. Nakayama, S. Saito, Y. Suwa and J. Yokoyama, JCAP **0806**, 020 (2008) [arXiv:0804.1827 [astro-ph]];
Md. R. Haque, D. Maity, T. Paul and L. Sriramkumar, Phys. Rev. D **104**, 063513 (2021) [arXiv:2105.09242 [astro-ph.CO]].

Effects on PGWs due to a secondary phase of reheating



The dimensionless spectral energy density of PGWs observed today $\Omega_{\mathrm{GW}}(f)$ is plotted in scenarios involving a secondary phase of reheating[13].

[13] Md. R. Haque, D. Maity, T. Paul and L. Sriramkumar, Phys. Rev. D **104**, 063513 (2021) [arXiv:2105.09242 [astro-ph.CO]].

Plan of the talk

Essentials of PGWs

When the tensor perturbations, say, h_{ij}, are taken into account, the line-element describing the spatially flat, FLRW universe is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\,[\delta_{ij} + h_{ij}(t,\boldsymbol{x})]\,\mathrm{d}x^i\mathrm{d}x^j = a^2(\eta)\,\left\{-\mathrm{d}\eta^2 + [\delta_{ij} + h_{ij}(\eta,\boldsymbol{x})]\,\mathrm{d}x^i\mathrm{d}x^j\right\}.$$

Since the tensor perturbations are transverse and traceless, they satisfy the conditions

$$\delta^{ij}\partial_i h_{jl} = 0,\ \delta^{ij} h_{ij} = 0.$$

Also, when no sources with an anisotropic stress-energy tensor are present, the Einstein's equations at the first order in the perturbations lead to the following equation of motion for the PGWs:

$$h_{ij}'' + 2\mathcal{H}h_{ij}' - \partial^2 h_{ij} = 0,$$

where $\mathcal{H} = a'/a = aH$ is the conformal Hubble parameter, with $H = \dot{a}/a$ being the Hubble parameter, and $\partial^2 = \delta^{ij}\partial_i\partial_j$.

Quantization of PGWs

On quantization, the tensor perturbations h_{ij} can be decomposed in terms of the rescaled mode functions, say, $\mu_k(\eta)$, as follows:

$$\hat{h}_{ij}(\eta, \boldsymbol{x}) = \frac{\sqrt{2}}{M_{\mathrm{Pl}}\, a(\eta)} \sum_{\lambda=(+,\times)} \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}} \varepsilon_{ij}^{\lambda}(\hat{\boldsymbol{n}}) \left[\hat{a}_{\boldsymbol{k}}^{\lambda} \mu_k(\eta) \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \hat{a}_{\boldsymbol{k}}^{\lambda\dagger} \mu_k^{*}(\eta) \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right],$$

where $\boldsymbol{k} = k\hat{\boldsymbol{n}}$ with $\hat{\boldsymbol{n}}$ being a unit vector, and $\varepsilon_{ij}^{\lambda}(\hat{\boldsymbol{n}})$ represents the polarization tensor.

The annihilation and creation operators $\hat{a}_{\boldsymbol{k}}^{\lambda}$ and $\hat{a}_{\boldsymbol{k}}^{\lambda\dagger}$ obey the standard commutation relations, viz.

$$\left[\hat{a}_{\boldsymbol{k}}^{\lambda}, \hat{a}_{\boldsymbol{k}'}^{\lambda'}\right] = \left[\hat{a}_{\boldsymbol{k}}^{\lambda\dagger}, \hat{a}_{\boldsymbol{k}'}^{\lambda'\dagger}\right] = 0, \quad \left[\hat{a}_{\boldsymbol{k}}^{\lambda}, \hat{a}_{\boldsymbol{k}'}^{\lambda'\dagger}\right] = \delta^{(3)}\left(\boldsymbol{k} - \boldsymbol{k}'\right) \delta^{\lambda\lambda'}.$$

In the absence of sources with anisotropic stresses, the equation of motion for h_{ij} leads to the following differential equation for the rescaled mode function $\mu_k(\eta)$:

$$\mu_k'' + \left(k^2 - \frac{a''}{a}\right)\mu_k = 0.$$

PS of PGWs

The power spectrum (PS) of PGWs at the conformal time η, say, $\mathcal{P}_{_{\rm T}}(k,\eta)$, is defined through the relation

$$\delta^{im}\delta^{jn}\left\langle 0\left|\hat{h}_{mn}(\eta,\boldsymbol{x})\hat{h}_{ij}(\eta,\boldsymbol{x})\right|0\right\rangle = \int_0^\infty \frac{{\rm d}k}{k}\mathcal{P}_{_{\rm T}}(k,\eta),$$

where, as is usually done, we have assumed that the expectation value on the left hand side is evaluated in the quantum vacuum. The vacuum state $|0\rangle$ is defined as $\hat{a}_{\boldsymbol{k}}^{\lambda}|0\rangle = 0$ for all $\boldsymbol{k}$ and λ.

On utilizing the decomposition of the quantized tensor perturbations $\hat{h}_{ij}$ in terms of the rescaled mode functions $\mu_k(\eta)$, the PS of PGWs at any time can be expressed as follows:

$$\mathcal{P}_{_{\rm T}}(k,\eta) = \frac{8}{M_{_{\rm Pl}}^2}\frac{k^3}{2\pi^2}\frac{|\mu_k(\eta)|^2}{a^2(\eta)}.$$

Back to characteristic strain

PS of PGWs in de Sitter inflation

In de Sitter inflation $a(\eta) = -1/(H_{\mathrm{e}}\eta)$, where H_{e} is the constant Hubble parameter.

In such a background, the rescaled mode function $\mu_k(\eta)$ is given by

$$\mu_k(\eta) = -A_k\sqrt{\frac{2}{\pi}}\left(1 - \frac{i}{k\eta}\right)\mathrm{e}^{-ik\eta} = \sqrt{\frac{1}{2k}}\left(1 - \frac{i}{k\eta}\right)\mathrm{e}^{-ik(\eta-\eta_{\mathrm{i}})}.$$

The PS of PGWs at the end of inflation at the conformal time η_{e} can be evaluated to be

$$\mathcal{P}_{_{\mathrm{S}}}(k,\eta_{\mathrm{e}}) = \frac{2H_{\mathrm{e}}^2}{\pi^2 M_{_{\mathrm{Pl}}}^2}\left(1 + \frac{k^2}{k_{\mathrm{e}}^2}\right),$$

where $k_{\mathrm{e}} = a_{\mathrm{e}}H_{\mathrm{e}} = -1/\eta_{\mathrm{e}}$ is the wave number that leaves the Hubble radius at the end of inflation. Evidently, a_{e} is the scale factor at η_{e}.

Note that, over small scales such that $k \gtrsim k_{\mathrm{e}}$, the PS behaves as

$$\mathcal{P}_{_{\mathrm{S}}}(k,\eta_{\mathrm{e}}) \simeq \frac{2H_{\mathrm{e}}^2}{\pi^2 M_{_{\mathrm{Pl}}}^2} y_{\mathrm{e}}^2,$$

where we have set $y_{\mathrm{e}} = k/k_{\mathrm{e}}$.

Transition from inflation to the epoch of radiation domination

Recall that, during de Sitter inflation and the epoch of radiation domination, the scale factors are given by

$$a(\eta) = -\frac{1}{H_{\mathrm{e}}\eta}, \quad a(\eta) = a_{\mathrm{r}}(\eta - \eta_{\mathrm{r}}).$$

The corresponding conformal Hubble parameters can be immediately evaluated to be

$$\mathcal{H} = -\frac{1}{\eta}, \quad \mathcal{H} = \frac{1}{(\eta - \eta_{\mathrm{r}})}.$$

Then, the continuity of $\mathcal{H}$ and the scale factor imply that

$$\eta_{\mathrm{r}} = 2\eta_{\mathrm{e}}, \quad a_{\mathrm{r}} = \frac{1}{H_{\mathrm{e}}\eta_{\mathrm{e}}^2},$$

where, recall that, $\eta_{\mathrm{e}} < 0$ is the conformal time at the end of inflation.

Mode functions during radiation domination

During the epoch of radiation domination, the general solution to the rescaled mode function $\mu_k(\eta)$ can be written as

$$\mu_k(\eta) = \alpha_k^{\mathrm{r}} m_k(\eta) + \beta_k^{\mathrm{r}} n_k(\eta),$$

where $(\alpha_k^{\mathrm{r}}, \beta_k^{\mathrm{r}})$ are the so-called Bogoliubov coefficients.

Since $a''/a = 0$ during radiation domination, the mode functions $m_k(\eta)$ and $n_k(\eta)$ have the following simple form:

$$m_k(\eta) = i\sqrt{\frac{2}{\pi}}\mathrm{e}^{-ix} = n_k^*(\eta),$$

where $x = k(\eta - \eta_{\mathrm{r}})$.

PS of PGWs during radiation domination

The resulting PS of PGWs during the epoch of radiation domination is given by

$$\mathcal{P}_{_{\mathrm{T}}}(k,\eta) = \frac{8k^3}{\pi^3 M_{_{\mathrm{Pl}}}^2 a^2}\left[|\alpha_k^{\mathrm{r}}|^2 + |\beta_k^{\mathrm{r}}|^2 - 2\mathrm{Re}(\alpha_k^{\mathrm{r}}\beta_k^{\mathrm{r}*})\cos(2x) - 2\mathrm{Im}(\alpha_k^{\mathrm{r}}\beta_k^{\mathrm{r}*})\sin(2x)\right].$$

The coefficients $(\alpha_k^{\mathrm{r}}, \beta_k^{\mathrm{r}})$ always satisfy the normalization condition

$$|\alpha_k^{\mathrm{r}}|^2 - |\beta_k^{\mathrm{r}}|^2 = |A_k|^2.$$

Therefore, the PS can be expressed as

$$\mathcal{P}_{_{\mathrm{T}}}(k,\eta) = \frac{2}{\pi^2}\left(\frac{H_{\mathrm{e}}}{M_{_{\mathrm{Pl}}}}\right)^2\left(\frac{a_{\mathrm{e}}}{a}\right)^2 \bar{y}_{\mathrm{e}}^2\Biggl[1 + 2\left|\frac{\beta_k^{\mathrm{r}}}{A_k}\right|^2 - 2\,\mathrm{Re}\left(\frac{\alpha_k^{\mathrm{r}}\beta_k^{\mathrm{r}*}}{|A_k|^2}\right)\cos(2x)$$
$$- 2\,\mathrm{Im}\left(\frac{\alpha_k^{\mathrm{r}}\beta_k^{\mathrm{r}*}}{|A_k|^2}\right)\sin(2x)\Biggr],$$

where $\bar{y}_{\mathrm{e}} = k/k_{\mathrm{e}}$ and $k_{\mathrm{e}} = a_{\mathrm{e}}H_{\mathrm{e}}$.

Bogoliubov coefficients and the explicit form of the PS

In the case of an instantaneous transition from de Sitter inflation to the epoch of radiation domination, the Bogoliubov coefficients $(\alpha_k^{\mathrm{r}}, \beta_k^{\mathrm{r}})$ can be evaluated to be Back to the linear case

$$\alpha_k^{\mathrm{r}} = -\frac{i A_k}{2 y_{\mathrm{e}}^2}\left(1 - 2 i y_{\mathrm{e}} - 2 y_{\mathrm{e}}^2\right) \mathrm{e}^{2 i y_{\mathrm{e}}}, \quad \beta_k^{\mathrm{r}} = -\frac{i A_k}{2 y_{\mathrm{e}}^2}.$$

On substituting these coefficients in the expression for the PS of PGWs during radiation domination, we obtain that

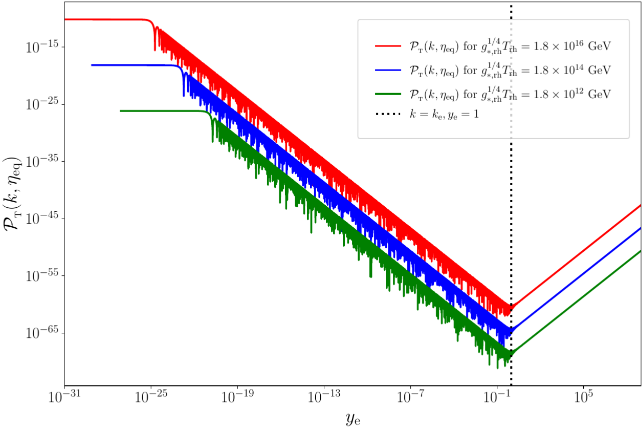
$$\mathcal{P}_{_{\mathrm{T}}}(k,\eta) = \frac{2}{\pi^2}\left(\frac{H_{\mathrm{e}}}{M_{_{\mathrm{Pl}}}}\right)^2 \left(\frac{a_{\mathrm{e}}}{a}\right)^2 \frac{1}{y_{\mathrm{e}}^2}\left\{y_{\mathrm{e}}^4 + \frac{1}{2} - \frac{1}{2}\left(1 - 2 y_{\mathrm{e}}^2\right)\cos\left[2(x - y_{\mathrm{e}})\right] + y_{\mathrm{e}} \sin\left[2(x - y_{\mathrm{e}})\right]\right\},$$

where, recall that, $x = k(\eta - \eta_{\mathrm{r}}) = a\bar{y}_{\mathrm{e}}/a_{\mathrm{e}} = a y_{\mathrm{e}}/a_{\mathrm{e}}$.

Over wave numbers that are in the sub-Hubble regime at the end of inflation, i.e. $y_{\mathrm{e}} \gg 1$ or $k \gg k_{\mathrm{e}}$, we find that the PS behaves as

$$\mathcal{P}_{_{\mathrm{T}}}(k,\eta) \simeq \frac{2}{\pi^2}\left(\frac{H_{\mathrm{e}}}{M_{_{\mathrm{Pl}}}}\right)^2 \left(\frac{a_{\mathrm{e}}}{a}\right)^2 y_{\mathrm{e}}^2.$$

PS of PGWs at the time of radiation-matter equality



The PS of PGWs $\mathcal{P}_{\rm T}(k,\eta)$, evaluated at the time of radiation-matter equality $\eta_{\rm eq}$, has been plotted for the case of instantaneous transition from de Sitter inflation to the epoch of radiation domination. For $g_{*,\rm rh}^{1/4}T_{\rm re}=1.8\times 10^{16}\,{\rm GeV}$, $r\simeq 0.034$. We have plotted the PS for $g_{*,\rm rh}^{1/4}T_{\rm re}=1.8\times(10^{16},10^{14},10^{12})\,{\rm GeV}$ (in red, blue and green).

Transition from inflation to the epoch of matter domination

During matter domination, the scale factor can be expressed as

$$a(\eta) = a_{\mathrm{m}}\,(\eta - \eta_{\mathrm{m}})^2,$$

where a_{m} and η_{m} are constants.

On matching the above scale factor and its time derivative at η_{eq} with the values from the earlier epoch of radiation domination, we obtain that

$$a_{\mathrm{m}} = \frac{a_{\mathrm{r}}}{2\,(\eta_{\mathrm{eq}} - \eta_{\mathrm{m}})}, \quad \eta_{\mathrm{m}} = -\eta_{\mathrm{eq}} + 2\,\eta_{\mathrm{r}} = -\eta_{\mathrm{eq}} + 4\,\eta_{\mathrm{e}}.$$

Mode functions and the Bogoliubov coefficients

During matter domination, the general solution for $\mu_k(\eta)$ can be expressed as

$$\mu_k(\eta) = \alpha_k^{\mathrm{m}}\, p_k(\eta) + \beta_k^{\mathrm{m}}\, q_k(\eta),$$

where the functions $p_k(\eta)$ and $q_k(\eta)$ are given by

$$p_k(\eta) = \sqrt{\frac{2}{\pi}}\,\mathrm{e}^{-iz}\left(-1 + \frac{i}{z}\right) = q_k^*$$

with $z = k(\eta - \eta_{\mathrm{m}})$ and, as earlier, $|\alpha_k^{\mathrm{m}}|^2 - |\beta_k^{\mathrm{m}}|^2 = |A_k|^2$.

Using the forms of the functions (m_k, n_k) and (p_k, q_k), we can obtain $(\alpha_k^{\mathrm{m}}, \beta_k^{\mathrm{m}})$ to be

$$\alpha_k^{\mathrm{m}} = \frac{\mathrm{e}^{ix_{\mathrm{eq}}}}{2}\left[\alpha_k^{\mathrm{r}}\left(-2i + \frac{1}{x_{\mathrm{eq}}} + \frac{i}{4x_{\mathrm{eq}}^2}\right) - \frac{i\beta_k^{\mathrm{r}}}{4x_{\mathrm{eq}}^2}\mathrm{e}^{2ix_{\mathrm{eq}}}\right],$$

$$\beta_k^{\mathrm{m}} = \frac{\mathrm{e}^{-ix_{\mathrm{eq}}}}{2}\left[\frac{i\alpha_k^{\mathrm{r}}}{4x_{\mathrm{eq}}^2}\mathrm{e}^{-2ix_{\mathrm{eq}}} + \beta_k^{\mathrm{r}}\left(2i + \frac{1}{x_{\mathrm{eq}}} - \frac{i}{4x_{\mathrm{eq}}^2}\right)\right],$$

where $x_{\mathrm{eq}} = k(\eta_{\mathrm{eq}} - \eta_{\mathrm{r}})$.

PS of PGWs during matter domination

The PS of PGWs during matter domination can be expressed as

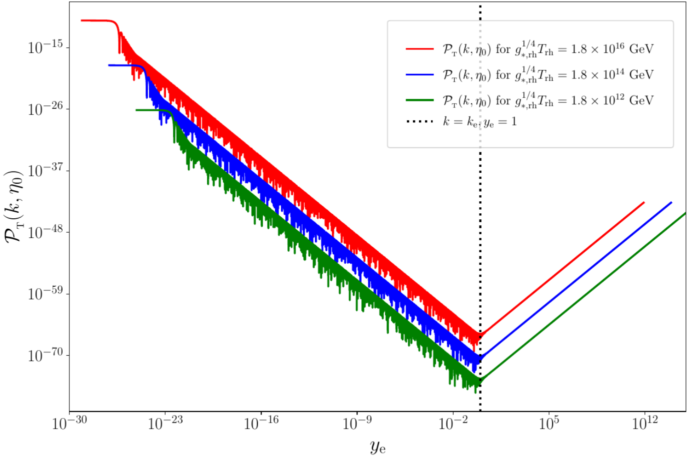
$$\mathcal{P}_{\mathrm{T}}(k,\eta) = \frac{2}{\pi^2}\left(\frac{H_{\mathrm{e}}}{M_{\mathrm{Pl}}}\right)^2\left(\frac{a_{\mathrm{e}}}{a}\right)^2\bar{y}_{\mathrm{e}}^2\Biggl\{\left(1+2\left|\frac{\beta_k^{\mathrm{m}}}{A_k}\right|^2\right)\left(1+\frac{1}{z^2}\right) + 2\left(1-\frac{1}{z^2}\right)\left[\mathrm{Re}\left(\frac{\alpha_k^{\mathrm{m}}\beta_k^{\mathrm{m}*}}{|A_k|^2}\right)\cos(2z) + \mathrm{Im}\left(\frac{\alpha_k^{\mathrm{m}}\beta_k^{\mathrm{m}*}}{|A_k|^2}\right)\sin(2z)\right] - \frac{4}{z}\left[\mathrm{Re}\left(\frac{\alpha_k^{\mathrm{m}}\beta_k^{\mathrm{m}*}}{|A_k|^2}\right)\sin(2z) - \mathrm{Im}\left(\frac{\alpha_k^{\mathrm{m}}\beta_k^{\mathrm{m}*}}{|A_k|^2}\right)\cos(2z)\right]\Biggr\},$$

where, recall that, $\bar{y}_{\mathrm{e}} = k/(a_{\mathrm{e}}H_{\mathrm{e}})$.

In the limits $x_{\mathrm{eq}} \gg 1$ and $y_{\mathrm{e}} \gg 1$ or, equivalently, when $k \gg k_{\mathrm{e}}$, we obtain that

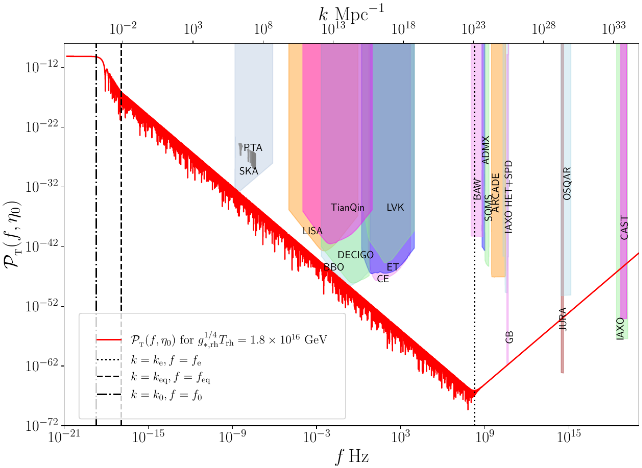
$$\mathcal{P}_{\mathrm{T}}(k,\eta) \simeq \frac{2}{\pi^2}\left(\frac{H_{\mathrm{e}}}{M_{\mathrm{Pl}}}\right)^2\left(\frac{a_{\mathrm{e}}}{a}\right)^2\left[y_{\mathrm{e}}^2 + \cos\left(2z - 2y_{\mathrm{e}} - 2x_{\mathrm{eq}}\right)\right].$$

PS of PGWs during late stages of matter domination I



The PS of PGWs $\mathcal{P}_{\mathrm{T}}(k,\eta)$ evaluated today, i.e. at the conformal time η_0, has been plotted for the case of instantaneous transitions from de Sitter inflation to the epochs of radiation and matter domination. We have plotted the PS for the three values of $g_{*,\mathrm{rh}}^{1/4} T_{\mathrm{re}}$ we had plotted in the previous figure.

PS of PGWs during late stages of matter domination II



The PS of PGWs $\mathcal{P}_{\mathrm{T}}(k, \eta)$ evaluated today, i.e. at the conformal time η_0, has been plotted assuming $g_{*,\mathrm{rh}}^{1/4} T_{\mathrm{re}} = 1.8 \times 10^{16}\,\mathrm{GeV}$ (corresponding to $r \simeq 0.034$) for the case of instantaneous transitions from de Sitter inflation to the epochs of radiation and matter domination.

Plan of the talk

Response of a detector to classical, stochastic GWs

At a given spatial location, say, $\boldsymbol{x}=0$, the signal, say, $\mathcal{S}(t)$, observed by a GW detector can be expressed as[14]

$$\mathcal{S}(t)=D^{ij}h_{ij}(t),$$

where D^{ij} depends on the properties of the detector and is known as the detector tensor. Then, the mean-squared displacement of the detector can be expressed as

$$\left\langle\mathcal{S}^2(t)\right\rangle=D^{ij}D^{mn}\left\langle h_{ij}(t)h_{mn}(t)\right\rangle,$$

where the angular brackets denote averages over the random variable h_{ij}. On using the in the above expression, we obtain that

$$\left\langle\mathcal{S}^2(t)\right\rangle=\frac{F}{4\pi}\int_0^\infty \mathrm{d}f\,S_h(f)=\frac{F}{16\pi}\int_0^\infty\frac{\mathrm{d}f}{f}\,\mathcal{P}_{_\mathrm{T}}(f,t),$$

where the quantity F is a number that depends on the detector, and is given by

$$F=\sum_{\lambda=(+,\times)}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}\theta\,\sin\theta\,\left[D^{ij}\varepsilon_{ij}^{\lambda}(\hat{\boldsymbol{n}})\right]^2.$$

[14] M. Maggiore, Phys. Rept. **331** 283 (2000) [gr-qc/9909001].

Response of a detector to quantum GWs

If the GWs are of quantum origin, we can define the mean-squared displacement as

$$\left\langle \hat{\mathcal{S}}^2(\eta) \right\rangle = D^{ij}\, D^{mn} \left\langle \hat{h}_{ij}(\eta, \boldsymbol{x}) \hat{h}_{mn}(\eta, \boldsymbol{x}) \right\rangle,$$

where the averages are to be evaluated in the quantum vacuum.

Using the decomposition of the PGWs, it is straightforward to show that, in the quantum vacuum, the mean-squared displacement can be expressed as

$$\left\langle \hat{\mathcal{S}}^2(\eta) \right\rangle = \frac{2F}{M_{_{\mathrm{Pl}}}^2\, a^2} \int_0^\infty \frac{\mathrm{d}k}{(2\pi)^3} k^2 |\mu_k|^2 = \frac{F}{16\pi} \int_0^\infty \frac{\mathrm{d}k}{k} \mathcal{P}_{_{\mathrm{T}}}(k, \eta),$$

which is the same as the expression in the case of stochastic and isotropic, classical GWs.

WKB ansatz

In general, a suitably rescaled Fourier mode function of the field in a FLRW universe, say, χ_k, will satisfy an equation of motion of the following form:

$$\chi_k''(\eta) + \left[\omega_k^2(\eta) + \sigma^2(\eta)\right] \chi_k(\eta) = 0,$$

where $\omega_k^2(\eta)$ and $\sigma^2(\eta)$ are functions of time.

To solve such an equation, let us assume the WKB ansatz

$$\chi_k^{\mathrm{ad}}(\eta) = \frac{1}{\sqrt{2W_k(\eta)}} \exp\left[-i \int_{\eta_{\mathrm{i}}}^{\eta} \mathrm{d}\tilde{\eta}\, W_k(\tilde{\eta})\right],$$

where η_{i} is some initial time.

On substituting this ansatz into the above equation of motion, we obtain that

$$W_k^2(\eta) = \omega_k^2(\eta) + \frac{\sigma^2(\eta)}{T^2} - \frac{1}{2T^2}\left[\frac{W_k''}{W_k} - \frac{3}{2}\left(\frac{W_k'}{W_k}\right)^2\right],$$

where we have introduced powers of $1/T$ to facilitate the adiabatic expansion.

Adiabatic expansion I

We can now expand $W_k(\eta)$ in adiabatic order, i.e. we can write[15]

$$W_k(\eta) = \sum_{n=0}^{\infty} \frac{W_k^{(n)}(\eta)}{T^n},$$

where the index (n) refers to the adiabatic order of the expansion.

We shall assume that $\omega_k(\eta)$ and $\sigma^2(\eta)$ are of adiabatic orders zero and two, respectively.

On separating the terms by the adiabatic order (i.e. equating the coefficients of T^{-n}), we obtain that

$$W_k^{(0)} = \omega_k, \quad W_k^{(1)} = 0, \quad W_k^{(2)} = \frac{\sigma^2}{2\omega_k} + \frac{3\omega_k'^2}{8\omega_k^3} - \frac{\omega_k''}{4\omega_k^2}.$$

[15] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982);
L. E. Parker and D. Toms, *Quantum Field Theory in Curved Spacetime: Quantized Field and Gravity* (Cambridge University Press, Cambridge, England, 2009).

Adiabatic expansion II

Note that, the adiabatic solution leads to

$$|\chi_k^{\rm ad}|^2 = \frac{1}{2W_k}.$$

When T is eventually set to unity, we have

$$W_k^{-1} \equiv (W_k^{-1})^{(0)} + (W_k^{-1})^{(2)} + \cdots,$$

with the terms at the different adiabatic orders $(W_k^{-1})^{(n)}$ defined as

$$(W_k^{-1})^{(0)} = \frac{1}{\omega_k}, \quad (W_k^{-1})^{(2)} = -\frac{W_k^{(2)}}{\omega_k^2}.$$

Regularization of the PS of PGWs

In the case of the rescaled mode function $\mu_k(\eta)$ that characterizes the PGWs we have

$$\omega_k = k, \quad \sigma^2(\eta) = -\frac{a''}{a}.$$

To regularize the PS of PGWs, we need to calculate the quantity $|\mu_k|^2_{\mathrm{ad}}$ up to the second adiabatic order.

We have

$$W_k^{(0)} = k, \quad W_k^{(2)} = -\frac{1}{2k}\frac{a''}{a}$$

and we obtain the contribution to the PS of PGWs due to the adiabatic solution up to the second order to be

$$\mathcal{P}_{\mathrm{T}}^{\mathrm{ad}}(k,\eta) = \frac{8}{M_{\mathrm{Pl}}^2}\frac{k^3}{2\pi^2}\frac{|\mu_k^{\mathrm{ad}}|^2}{a^2} = \frac{8}{M_{\mathrm{Pl}}^2}\frac{k^3}{2\pi^2 a^2}\frac{1}{2W_k}$$
$$= \frac{8}{M_{\mathrm{Pl}}^2}\frac{k^3}{2\pi^2 a^2}\frac{1}{2W_k^{(0)}}\left(1-\frac{W_k^{(2)}}{W_k^{(0)}}\right) = \frac{8}{M_{\mathrm{Pl}}^2}\frac{k^3}{2\pi^2 a^2}\frac{1}{2k}\left(1+\frac{a''}{2k^2 a}\right).$$

Note that the final expression is valid for arbitrary scale factor.

Regularized PS of PGWs during radiation domination

In the post-inflationary universe, we can express the adiabatic contribution $\mathcal{P}_{\mathrm{T}}^{\mathrm{ad}}(k,\eta)$ as follows:

$$\mathcal{P}_{\mathrm{T}}^{\mathrm{ad}}(k,\eta) = \frac{2}{\pi^2}\left(\frac{H_{\mathrm{e}}}{M_{\mathrm{Pl}}}\right)^2 \left(\frac{a_{\mathrm{e}}}{a}\right)^2 \bar{y}_{\mathrm{e}}^2 \left(1+\frac{1}{2k^2}\frac{a''}{a}\right).$$

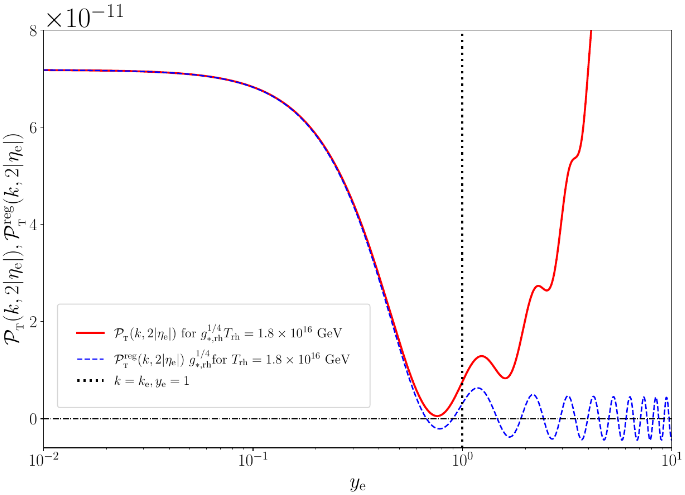
Upon subtracting this contribution from the PS of PGWs, in the case of an instantaneous transition from de Sitter inflation to radiation domination (wherein $\bar{y}_{\mathrm{e}} = y_{\mathrm{e}}$), we obtain that

$$\mathcal{P}_{\mathrm{T}}^{\mathrm{reg}}(k,\eta) = \frac{2}{\pi^2}\left(\frac{H_{\mathrm{e}}}{M_{\mathrm{Pl}}}\right)^2 \left(\frac{a_{\mathrm{e}}}{a}\right)^2 \frac{1}{2y_{\mathrm{e}}^2}\left\{1-\left(1-2y_{\mathrm{e}}^2\right)\cos\left[2(x-y_{\mathrm{e}})\right]+2y_{\mathrm{e}}\sin\left[2(x-y_{\mathrm{e}})\right]\right\}.$$

For wave numbers that are in the sub-Hubble domain at the end of inflation, i.e. when $y_{\mathrm{e}} \gg 1$, we find that

$$\mathcal{P}_{\mathrm{T}}^{\mathrm{reg}}(k,\eta) \simeq \frac{2}{\pi^2}\left(\frac{H_{\mathrm{e}}}{M_{\mathrm{Pl}}}\right)^2 \left(\frac{a_{\mathrm{e}}}{a}\right)^2 \cos\left[2(x-y_{\mathrm{e}})\right].$$

Regularized PS of PGWs during early stages of radiation domination



The actual and the regularized PS of PGWs, viz. $\mathcal{P}_{\rm T}(k, \eta)$ and $\mathcal{P}_{\rm T}^{\rm reg}(k, \eta)$, evaluated during the early stages of the radiation-dominated epoch at $\eta = -2\eta_{\rm e} = 2|\eta_{\rm e}|$, have been plotted (in red and blue) for the case of instantaneous transition from de Sitter inflation.

Regularized PS of PGWs during matter domination

During the matter domination, the contribution to the PS of PGWs at the second adiabatic order can be written as

$$\mathcal{P}_{\mathrm{T}}^{\mathrm{ad}}(k,\eta)=\frac{2}{\pi^2}\left(\frac{H_{\mathrm{e}}}{M_{\mathrm{Pl}}}\right)^2\left(\frac{a_{\mathrm{e}}}{a}\right)^2\bar{y}_{\mathrm{e}}^2\left(1+\frac{1}{z^2}\right).$$

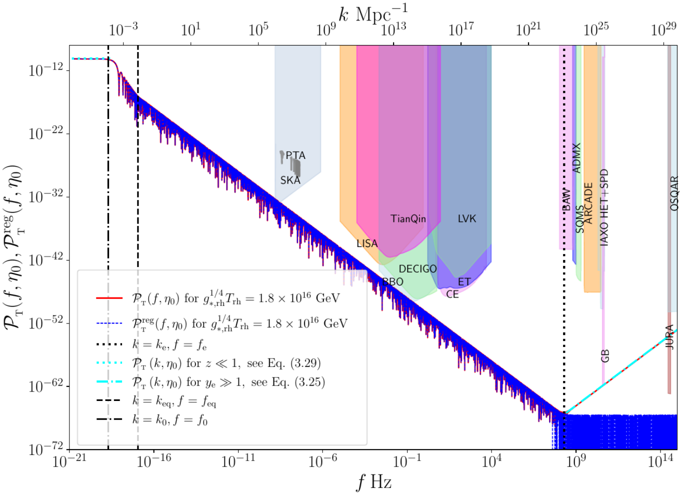
On subtracting this contribution from the PS of PGWs, in the limit $x_{\mathrm{eq}}\gg 1$, we obtain the regularized PS during the epoch of matter domination (after de Sitter inflation) to be

$$\mathcal{P}_{\mathrm{T}}^{\mathrm{reg}}(k,\eta)=\frac{2}{\pi^2}\left(\frac{H_{\mathrm{e}}}{M_{\mathrm{Pl}}}\right)^2\left(\frac{a_{\mathrm{e}}}{a}\right)^2\frac{1}{y_{\mathrm{e}}^2}\left[\frac{1}{2}-\frac{1}{2}\left(1-2y_{\mathrm{e}}^2\right)\cos\left(2z-2y_{\mathrm{e}}-2x_{\mathrm{eq}}\right)\right.$$
$$\left.+\,y_{\mathrm{e}}\sin\left(2z-2y_{\mathrm{e}}-2x_{\mathrm{eq}}\right)\right].$$

In the limit $y_{\mathrm{e}}\gg 1$, this regularized PS simplifies to the form

$$\mathcal{P}_{\mathrm{T}}^{\mathrm{reg}}(k,\eta)\simeq\frac{2}{\pi^2}\left(\frac{H_{\mathrm{e}}}{M_{\mathrm{Pl}}}\right)^2\left(\frac{a_{\mathrm{e}}}{a}\right)^2\cos\left(2z-2y_{\mathrm{e}}-2x_{\mathrm{eq}}\right).$$

Regularized PS of PGWs during late stages of matter domination



The actual and the regularized PS of PGWs, viz. $\mathcal{P}_{\rm T}(k,\eta)$ and $\mathcal{P}_{\rm T}^{\rm reg}(k,\eta)$, evaluated today at the conformal time η_0, have been plotted (in red and blue) for the case of instantaneous transitions from de Sitter inflation to the epochs of radiation and matter domination.

Plan of the talk

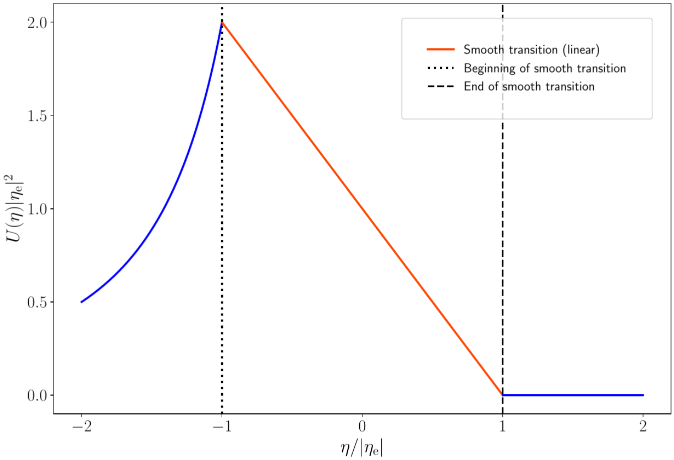
Smoothing the 'effective potential' with a linear function I

If we interpolate between the inflation and the epoch of radiation domination using a linear function for $U(\eta) = a''/a$, we obtain that

$$U(\eta) = \begin{cases} \frac{q\,(q-1)}{\eta^2}, & \text{for } \eta \le \eta_{\rm e}, \\ -\frac{q\,(q-1)}{\eta_{\rm e}^2\,\Delta\eta}\,\left[\eta - (\eta_{\rm e} + \Delta\eta)\right], & \text{for } \eta_{\rm e} \le \eta \le \eta_{\rm e} + \Delta\eta, \\ 0, & \text{for } \eta \ge \eta_{\rm e} + \Delta\eta. \end{cases}$$

It is straightforward to check that the quantity $U(\eta)$ is continuous at both $\eta_{\rm e}$ and $\eta_{\rm e} + \Delta\eta$.

Smoothing the 'effective potential' with a linear function II



The behavior of the quantity $U(\eta) = a''/a$ has been plotted in the case of transition from de Sitter inflation to the epoch of radiation domination, when $U(\eta)$ has been smoothed with a linear function.

Scale factor during the smooth linear transition

During the transition, the scale factor a satisfies the equation

$$a'' + \frac{q(q-1)}{\eta_{\mathrm{e}}^2\,\Delta\eta}\left[\eta - (\eta_{\mathrm{e}} + \Delta\eta)\right] a = 0.$$

The general solution to this equation can be expressed in terms of the Airy functions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ as follows:

$$a(\eta) = a_{\mathrm{e}}\left\{A_1 \mathrm{Ai}\left[\rho(\eta)\right] + A_2 \mathrm{Bi}\left[\rho(\eta)\right]\right\},$$

where $\rho(\eta) = \alpha U(\eta)$ and $\alpha = |\mathrm{d}U/\mathrm{d}\eta|^{-2/3}$.

The constants a_r and η_r describing the scale factor during radiation domination can be determined to be

$$\frac{\eta_{\mathrm{r}}}{|\eta_{\mathrm{e}}|} = \frac{\Delta\eta}{|\eta_{\mathrm{e}}|} - 1 - \left[q(q-1)\right]^{-1/3}\left(\frac{\Delta\eta}{|\eta_{\mathrm{e}}|}\right)^{1/3} \frac{\Gamma(1/3)}{3^{1/3}\Gamma(2/3)}\left(\frac{A_1 + \sqrt{3}A_2}{A_1 - \sqrt{3}A_2}\right),$$

$$a_{\mathrm{r}} = \frac{\left[q(q-1)\right]^{1/3}}{3^{1/3}\Gamma(1/3)|\eta_{\mathrm{e}}|}\left(\frac{\Delta\eta}{|\eta_{\mathrm{e}}|}\right)^{-1/3}\left(A_1 - \sqrt{3}A_2\right) a_{\mathrm{e}}.$$

Mode functions during the smoother linear transition

The general solution for the rescaled mode function can be expressed as

$$\mu_k(\eta) = B_1 i_k(\eta) + B_2 j_k(\eta) = B_1 \mathrm{Ai}[\tau(\eta)] + B_2 \mathrm{Bi}[\tau(\eta)],$$

where $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are the Airy functions, and the function $\tau(\eta)$ is given by

$$\tau(\eta) = -\alpha \left[k^2 - U(\eta)\right] = -\alpha k^2 + \rho(\eta).$$

Bogoliubov coefficients during radiation domination on small scales

We find that, at the leading order, the coefficients $(\alpha_k^{\mathrm{r}}, \beta_k^{\mathrm{r}})$ are given by

$$\alpha_k^{\mathrm{r}} \simeq -i A_k \mathrm{e}^{i[y_{\mathrm{e}}+(q-1)\pi/2]-i y_{\mathrm{e}} Y}\, \mathrm{e}^{i(\zeta_{\mathrm{e}}-\zeta_{\mathrm{tr}})},$$

$$\beta_k^{\mathrm{r}} \simeq \frac{q(q-1)A_k}{4y_{\mathrm{e}}^3} \mathrm{e}^{i[y_{\mathrm{e}}+\pi(q-1)\pi/2]+i y_{\mathrm{e}} Y}\, \mathrm{e}^{-i(\zeta_{\mathrm{e}}-\zeta_{\mathrm{tr}})} \left[1-\mathrm{e}^{i(\zeta_{\mathrm{e}}-\zeta_{\mathrm{tr}})} \frac{iX}{q(q-1)} \sin\left(\zeta_{\mathrm{e}}-\zeta_{\mathrm{tr}}\right)\right],$$

where the quantities ζ_{e} and ζ_{tr} are given by

$$\zeta_{\mathrm{tr}} = \frac{2}{3}\frac{y_{\mathrm{e}}^3}{X}, \quad \zeta_{\mathrm{e}} = \frac{2}{3}\frac{y_{\mathrm{e}}^3}{X}\left[1-\frac{q(q-1)}{y_{\mathrm{e}}^2}\right]^{3/2},$$

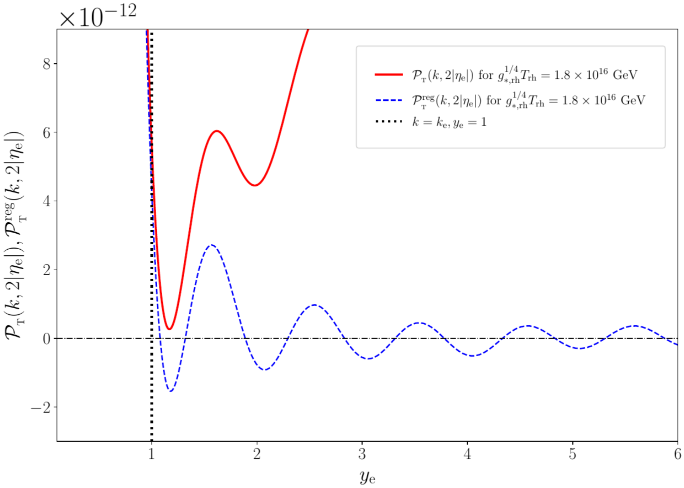
with X being

$$X = \frac{q\,(q-1)\,|\eta_{\mathrm{e}}|}{\Delta\eta}.$$

Note that, in contrast to the case of the instantaneous transition wherein β_k^{r}/A_k behaved as y_{e}^{-2}, it now behaves as y_{e}^{-3}.

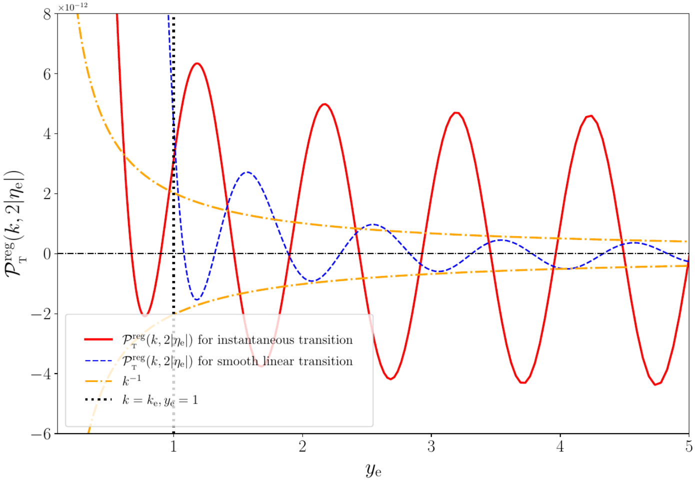
Instantaneous case

Regularized PS of PGWs after the smoother transition



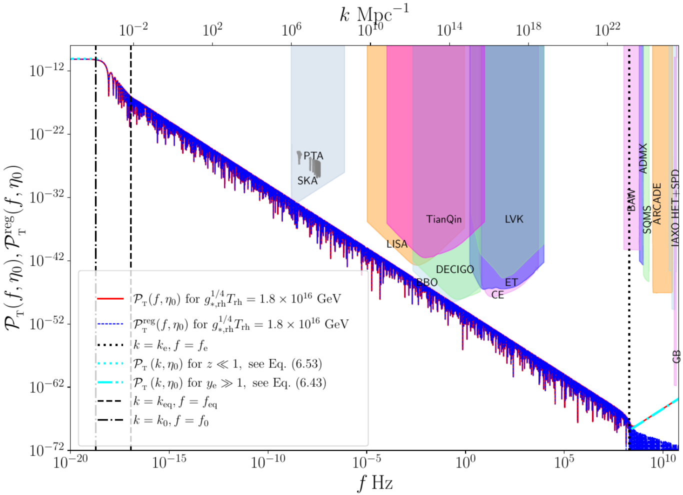
The actual and the regularized PS of PGWs, viz. $\mathcal{P}_{\mathrm{T}}(k, \eta)$ and $\mathcal{P}_{\mathrm{T}}^{\mathrm{reg}}(k, \eta)$, evaluated during the early stages of the epoch of radiation domination at the conformal time $\eta = 2|\eta_{\mathrm{e}}|$, have been plotted (in red and blue) in the case of the smoother linear transition.

Comparison of the PS after instantaneous and smoother transitions



The regularized PS of PGWs $\mathcal{P}_{\mathrm{T}}^{\mathrm{reg}}(k, \eta)$, evaluated during the early stages of the epoch of radiation domination at the conformal time $\eta = 2|\eta_{\mathrm{e}}|$, have been plotted (in red and blue) in the cases of instantaneous and smoother transitions from inflation to radiation domination.

Regularized PS of PGWs today, after the smoother transition



The actual and the regularized PS of PGWs, viz. $\mathcal{P}_{\mathrm{T}}(k,\eta)$ and $\mathcal{P}_{\mathrm{T}}^{\mathrm{reg}}(k,\eta)$, evaluated today (at the conformal time η_0), have been plotted (in red and blue), assuming a smooth linear transition from de Sitter inflation to the epoch of radiation domination.

Plan of the talk

Solution to the mode functions in the Born approximation

Let $\mu_k^0(\eta)$ be the solution for μ_k when $U(\eta)=0$.

For $k > k_{\mathrm{e}}$, at the leading order in the Born approximation, we can write the solution to $\mu_k(\eta)$ as

$$\mu_k(\eta) \simeq \mu_k^0(\eta) + \int_{-\infty}^{\infty} \mathrm{d}\bar{\eta}\, G(\eta,\bar{\eta})\, U(\bar{\eta})\, \mu_k^0(\bar{\eta}),$$

where $G(\eta,\bar{\eta})$ is the Green's function which satisfies the following differential equation:

$$\frac{\mathrm{d}^2 G(\eta,\bar{\eta})}{\mathrm{d}\eta^2} + k^2 G(\eta,\bar{\eta}) = \delta^{(1)}(\eta-\bar{\eta}).$$

The Green's function $G(\eta,\bar{\eta})$ can be easily determined to be

$$G(\eta,\bar{\eta}) = \theta(\eta-\bar{\eta})\frac{1}{k}\sin[k(\eta-\bar{\eta})].$$

For the Bunch-Davies initial conditions, we can express the solution to $\mu_k(\eta)$ as

$$\mu_k(\eta) \simeq -A_k\sqrt{\frac{2}{\pi}}\left\{\left[1+\frac{i}{2k}\int_{-\infty}^{\eta}\mathrm{d}\bar{\eta}\, U(\bar{\eta})\right]\mathrm{e}^{-ik\eta} - \left[\frac{i}{2k}\int_{-\infty}^{\eta}\mathrm{d}\bar{\eta}\, U(\bar{\eta})\,\mathrm{e}^{-2ik\bar{\eta}}\right]\mathrm{e}^{ik\eta}\right\}.$$

Bogoliubov coefficients in the Born approximation

Recall that, during the epoch of radiation domination, the general solution to $\mu_k(\eta)$ can be written as

$$\mu_k^{\rm r}(\eta) = i\sqrt{\frac{2}{\pi}}\left[\alpha_k^{\rm r}(\eta){\rm e}^{-ik(\eta-\eta_{\rm r})} - \beta_k^{\rm r}(\eta){\rm e}^{ik(\eta-\eta_{\rm r})}\right].$$

On comparing this solution with the solution evaluated in the Born approximation, we obtain that

$$\alpha_k^{\rm r}(\eta) = i A_k\left[1+\frac{i}{2k}\int_{-\infty}^{\eta}{\rm d}\bar{\eta}\, U(\bar{\eta})\right]{\rm e}^{-ik\eta_{\rm r}},$$

$$\beta_k^{\rm r}(\eta) = -\frac{A_k}{2k}\left[\int_{-\infty}^{\eta}{\rm d}\bar{\eta}\, U(\bar{\eta}){\rm e}^{-2ik\bar{\eta}}\right]{\rm e}^{ik\eta_{\rm r}}.$$

Bogoliubov coefficients in the case of the instantaneous transition

In the case of the instantaneous transition, we have

$$U(\eta) = \begin{cases} 2/\eta^2, & \text{for } \eta \leq \eta_{\mathrm{e}}, \\ 0, & \text{for } \eta > \eta_{\mathrm{e}}. \end{cases}$$

In such a case, the Bogoliubov coefficients are given by

$$\alpha_k^{\mathrm{r(it)}} = i A_k \left(1 + \frac{i}{k}\int_{-\infty}^{\eta_{\mathrm{e}}} \frac{\mathrm{d}\bar{\eta}}{\bar{\eta}^2}\right) \mathrm{e}^{-2ik\eta_{\mathrm{e}}} = i A_k \left(1 - \frac{i}{k\eta_{\mathrm{e}}}\right) \mathrm{e}^{-2ik\eta_{\mathrm{e}}} = i A_k \left(1 + \frac{i}{y_{\mathrm{e}}}\right) \mathrm{e}^{2iy_{\mathrm{e}}},$$

$$\beta_k^{\mathrm{r(it)}} = -\frac{A_k}{k}\left(\int_{-\infty}^{\eta_{\mathrm{e}}} \frac{\mathrm{d}\bar{\eta}}{\bar{\eta}^2}\,\mathrm{e}^{-2ik\bar{\eta}}\right) \mathrm{e}^{2ik\eta_{\mathrm{e}}} = A_k \left[2\pi - \frac{1}{y_{\mathrm{e}}}\mathrm{e}^{2iy_{\mathrm{e}}} + 2i\mathrm{Ei}(2iy_{\mathrm{e}})\right] \mathrm{e}^{-2iy_{\mathrm{e}}},$$

where $\mathrm{Ei}(x)$ is the exponential integral function. On using asymptotic the expansion for $\mathrm{Ei}(x)$, for $k \gg k_{\mathrm{e}}$, up to order y_{e}^{-7}, we obtain β_k^{it} to be

$$\beta_k^{\mathrm{r(it)}} \simeq A_k \left(-\frac{i}{2y_{\mathrm{e}}^2} - \frac{1}{2y_{\mathrm{e}}^3} + \frac{3i}{4y_{\mathrm{e}}^4} + \frac{3}{2y_{\mathrm{e}}^5} - \frac{15i}{4y_{\mathrm{e}}^6} - \frac{45}{4y_{\mathrm{e}}^7}\right).$$

Bogoliubov coefficients in the linear case

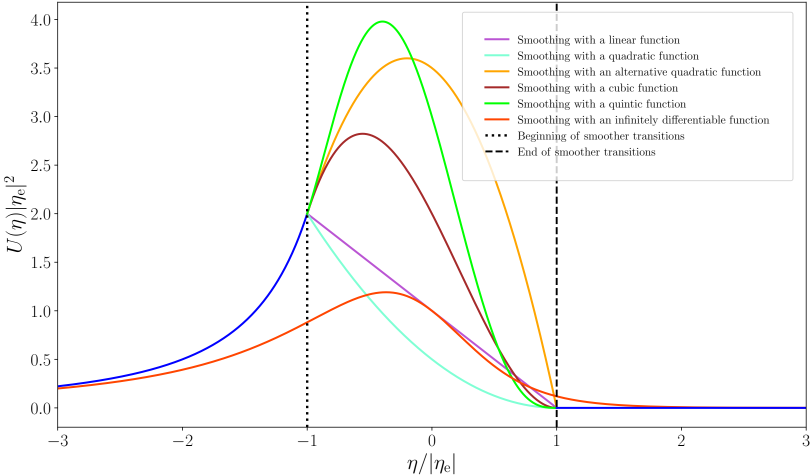
For the following $U(\eta)$:

$$U(\eta) = \begin{cases} \frac{2}{\eta^2}, & \text{for } \eta \leq \eta_{\mathrm{e}}, \\ -\frac{2}{\eta_{\mathrm{e}}^2 \Delta\eta_{\mathrm{e}}}\left[\eta - (\eta_{\mathrm{e}} + \Delta\eta_{\mathrm{e}})\right], & \text{for } \eta_{\mathrm{e}} \leq \eta \leq \eta_{\mathrm{e}} + \Delta\eta_{\mathrm{e}}, \\ 0, & \text{for } \eta \geq \eta_{\mathrm{e}} + \Delta\eta_{\mathrm{e}}, \end{cases}$$

we can evaluate the Bogoliubov coefficient β_k^{r} to be

$$\beta_k^{\mathrm{r}} = \frac{A_k}{y_{\mathrm{e}}} \mathrm{e}^{i y_{\mathrm{e}}(2+\eta_{\mathrm{r}}/|\eta_{\mathrm{e}}|)} \left[\sum_{n=2}^{\infty} \frac{n!}{(2 i y_{\mathrm{e}})^n} + \frac{1}{4 y_{\mathrm{e}}^2 k_{\mathrm{e}} \Delta\eta_{\mathrm{e}}} \left(\mathrm{e}^{-2 i y_{\mathrm{e}} k_{\mathrm{e}} \Delta\eta_{\mathrm{e}}} - 1\right)\right].$$

Behavior of the 'effective potential'



Behavior of the 'effective potential' $U(\eta) = a''/a$ in the cases of transition from de Sitter inflation to the epoch of radiation domination, when the quantity is smoothed with the aid of higher and higher order polynomials.

An example of completely smooth transitions

An 'effective potential' that describes a completely smooth transition from de Sitter inflation to the epoch of matter domination through the epoch of radiation domination is as follows:

$$U(\eta) = \frac{1}{\eta^2 + (\gamma_{\rm e}\eta_{\rm e})^2}\left[1 - \tanh\left(\frac{\eta - \eta_{\rm e}}{\Delta\eta_{\rm e}}\right)\right] + \frac{1}{(\eta - \eta_{\rm m})^2 + (\gamma_{\rm eq}\eta_{\rm eq})^2}\left[1 + \tanh\left(\frac{\eta - \eta_{\rm eq}}{\Delta\eta_{\rm eq}}\right)\right],$$

where $\Delta\eta_{\rm e}$ and $\Delta\eta_{\rm eq}$ are time intervals of the order of $|\eta_{\rm e}|$ and $\eta_{\rm eq}$, respectively.

In such a case, $\beta_k^{\rm m}$ can be evaluated to be

$$\begin{aligned}\beta_k^{\rm m} = \frac{\pi A_k}{k}\Bigg(&\frac{i}{2\gamma_{\rm e}\eta_{\rm e}}\left[1 - \tanh\left(\frac{-\eta_{\rm e} + i\gamma_{\rm e}\eta_{\rm e}}{\Delta\eta_{\rm e}}\right)\right]{\rm e}^{2k\gamma_{\rm e}\eta_{\rm e}}\\ &- \frac{1}{2i\gamma_{\rm eq}\eta_{\rm eq}}\left[1 + \tanh\left(\frac{\eta_{\rm m} - \eta_{\rm eq} - i\gamma_{\rm eq}\eta_{\rm eq}}{\Delta\eta_{\rm eq}}\right)\right]{\rm e}^{-2ik\eta_{\rm m}}\,{\rm e}^{-2k\gamma_{\rm eq}\eta_{\rm eq}}\\ &+ \sum_{n=0}^{\infty}\Bigg\{\frac{\Delta\eta_{\rm e}}{[\eta_{\rm e} - i(2n+1)(\pi/2)\Delta\eta_{\rm e}]^2 + (\gamma_{\rm e}\eta_{\rm e})^2}{\rm e}^{-2ik\eta_{\rm e}}\,{\rm e}^{-(2n+1)\pi k\Delta\eta_{\rm e}}\\ &- \frac{\Delta\eta_{\rm eq}}{[\eta_{\rm eq} - \eta_{\rm m} - i(2n+1)(\pi/2)\Delta\eta_{\rm eq}]^2 + (\gamma_{\rm eq}\eta_{\rm eq})^2}{\rm e}^{-2ik\eta_{\rm eq}}\,{\rm e}^{-(2n+1)\pi k\Delta\eta_{\rm eq}}\Bigg\}\Bigg){\rm e}^{ik\eta_{\rm m}}.\end{aligned}$$

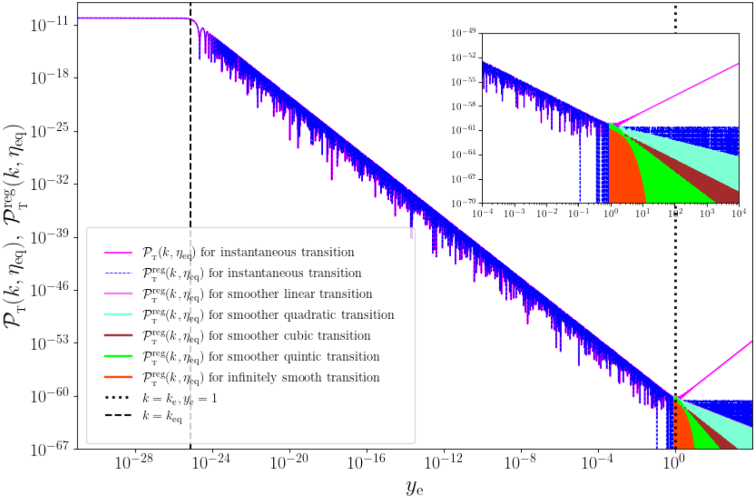
Bogoliubov coefficients and PS for increasingly smooth transitions

Behavior	Instantaneous	Linear	Quadratic	Cubic	Quintic	Completely continuous
Continuity up to	$\mathcal{H}$	U	U	U'	U''	All orders
Discontinuity from	U	U'	U' (one end)	U''	U'''	None
β_k/A_k	k^{-2}	k^{-3}	k^{-3}	k^{-4}	k^{-5}	$k^{-1}\,\mathrm{e}^{-\gamma k/k_\mathrm{e}}$
$\mathcal{P}_\mathrm{T}^\mathrm{reg}(k,\eta)$	k^{0}	k^{-1}	k^{-1}	k^{-2}	k^{-2}	$k\,\mathrm{e}^{-\gamma k/k_\mathrm{e}}$

The behavior of the Bogoliubov coefficients and the PS for increasingly smooth transitions over wave numbers that never leave the Hubble radius, i.e. over $k > k_\mathrm{e}$.

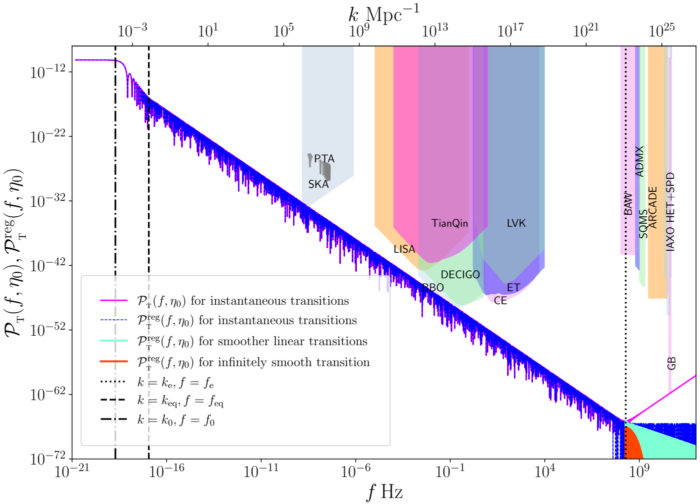
Note that completely continuous transitions lead to an exponential cut-off over $k > k_\mathrm{e}$.

Actual and regularized PS after smoother transitions I



The actual and the regularized PS of PGWs, i.e. $\mathcal{P}_{\mathrm{T}}(y_{\mathrm{e}},\eta)$ and $\mathcal{P}_{\mathrm{T}}^{\mathrm{reg}}(y_{\mathrm{e}},\eta)$, evaluated at the time of radiation-matter equality η_{eq}, after increasingly smoother and smoother transitions from de Sitter inflation to the epoch of radiation domination.

Actual and regularized PS after smoother transitions II



The actual and the regularized PS of PGWs, i.e. $\mathcal{P}_{\mathrm{T}}(y_{\mathrm{e}},\eta)$ and $\mathcal{P}_{\mathrm{T}}^{\mathrm{reg}}(y_{\mathrm{e}},\eta)$, evaluated today, after increasingly smoother transitions from de Sitter inflation to the epochs of radiation and matter domination.

Plan of the talk

Summary

- The regularization of the PS of PGWs is essential to avoid the quadratic rise in the PS on small scales. While regularization truncates the rise over $k \gtrsim k_{\mathrm{e}}$, it does not affect the standard predictions over $k \lesssim k_{\mathrm{e}}$.
- We find that, in the case of instantaneous transition from de Sitter inflation to radiation domination, the regularized PS of PGWs oscillates about zero with a constant amplitude in the domain $k \gtrsim k_{\mathrm{e}}$.
- We showed that, over $k \gtrsim k_{\mathrm{e}}$, there occurs a sharper and sharper suppression in the regularized PS of PGWs as the transition from inflation to radiation domination is smoothed further and further.
- We illustrated that, in the case of completely smooth transitions, the regularized PS of PGWs exhibits an exponential suppression over $k \gtrsim k_{\mathrm{e}}$.
- The observation of the exponential drop in the PS of PGWs can, in principle, help us determine the energy scale as well as the time of the end of inflation.

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Thank you for your attention

Properties of the polarization tensor

The quantity $\varepsilon_{ij}^{\lambda}(\hat{\boldsymbol{n}})$ is real and it represents the polarization tensor, with the index λ denoting the polarizations $+$ and $\times$ of the GWs.

The polarization tensors $\varepsilon_{ij}^{+}(\hat{\boldsymbol{n}})$ and $\varepsilon_{ij}^{\times}(\hat{\boldsymbol{n}})$ can be expressed in terms of the set of orthonormal unit vectors $(\hat{\boldsymbol{e}}_1(\hat{\boldsymbol{n}}), \hat{\boldsymbol{e}}_2(\hat{\boldsymbol{n}}), \hat{\boldsymbol{n}})$ in the following manner[16]:

$$\varepsilon_{ij}^{+}(\hat{\boldsymbol{n}}) = \hat{\boldsymbol{e}}_{1i}(\hat{\boldsymbol{n}})\hat{\boldsymbol{e}}_{1j}(\hat{\boldsymbol{n}}) - \hat{\boldsymbol{e}}_{2i}(\hat{\boldsymbol{n}})\hat{\boldsymbol{e}}_{2j}(\hat{\boldsymbol{n}}), \quad \varepsilon_{ij}^{\times}(\hat{\boldsymbol{n}}) = \hat{\boldsymbol{e}}_{1i}(\boldsymbol{k})\hat{\boldsymbol{e}}_{2j}(\hat{\boldsymbol{n}}) + \hat{\boldsymbol{e}}_{2i}(\boldsymbol{k})\hat{\boldsymbol{e}}_{1j}(\hat{\boldsymbol{n}}).$$

The polarization tensor $\varepsilon_{ij}^{\lambda}(\hat{\boldsymbol{n}})$ obeys the transverse and traceless conditions, viz.

$$\delta^{ij}\, k_i \varepsilon_{jl}^{\lambda}(\hat{\boldsymbol{n}}) = 0, \quad \delta^{ij}\varepsilon_{ij}^{\lambda}(\hat{\boldsymbol{n}}) = 0.$$

Also, the orthonormal nature of the unit vectors $\hat{\boldsymbol{e}}_1(\hat{\boldsymbol{n}})$ and $\hat{\boldsymbol{e}}_2(\hat{\boldsymbol{n}})$ lead to the normalization condition

$$\delta^{im}\delta^{jn}\varepsilon_{mn}^{\lambda}(\hat{\boldsymbol{n}})\varepsilon_{ij}^{\lambda'}(\hat{\boldsymbol{n}}) = 2\,\delta^{\lambda\lambda'}.$$

[16] M. Maggiore, Phys. Rept. **331**, 283 (2000) [arXiv:gr-qc/9909001].

Generation of PGWs during inflation

During inflation, the scale factor is typically of the form $a(\eta) = \ell_0(-\eta)^q$, where $q < -1$ in power-law inflation and $q = -1 - \epsilon_1$ (with $\epsilon_1 \ll 1$) in slow-roll inflation.

In the inflationary scenario, it is common to impose the so-called Bunch-Davies initial conditions on the perturbations when the modes are in the sub-Hubble domain during the early stages.

In such a case, the solution for the rescaled mode function $\mu_k(\eta)$ can be written as

$$\mu_k(\eta) = A_k f_k(y),$$

where $y \equiv -k\eta > 0$, and the function $f_k(y)$ is given by

$$f_k(\eta) = y^{1/2}\,\mathrm{H}_\mu^{(1)}(y),$$

with $\mu = 1/2 - q$. The constant A_k is given by

$$A_k = \sqrt{\frac{\pi}{4k}}\mathrm{e}^{i\pi\mu/2+i\pi/4+ik\eta_\mathrm{i}},$$

where $k\eta_\mathrm{i}$ is an arbitrary phase factor.

Mode functions in a generic post-inflationary background

In a background described by an equation-of-state parameter, say, w, the scale factor describing the spatially flat, FLRW universe can be written as

$$a(\eta) = a_w\,(\eta - \eta_w)^{2/(1+3w)}\,.$$

For such an $a(\eta)$, we have

$$\frac{a''}{a} = \frac{2(1-3w)}{(1+3w)^2(\eta-\eta_w)^2}.$$

In such a background, the general solution to the rescaled mode function $\mu_k(\eta)$ can be written as

$$\mu_k(\eta) = \alpha_k^w m_k(\eta) + \beta_k^w n_k(\eta),$$

where (α_k^w, β_k^w) are the so-called Bogoliubov coefficients. The functions $m_k(\eta)$ and $n_k(\eta)$ are given by

$$m_k(\eta) = x^{1/2}\mathrm{H}_{\nu(w)}^{(2)}(x), \quad n_k(\eta) = x^{1/2}\mathrm{H}_{\nu(w)}^{(1)}(x) = m_k^*(\eta),$$

where $x = k(\eta-\eta_w)$ and $\nu(w) = 3(1-w)/[2(1+3w)]$, while $\mathrm{H}_\nu^{(1)}(x)$ and $\mathrm{H}_\nu^{(2)}(x)$ are the Hankel functions of types one and two, respectively, and of order ν.

Values of the parameters involved

In the standard ΛCDM model of cosmology, it can be easily shown that

$$\left(\frac{H_{\mathrm{e}}}{M_{\mathrm{Pl}}}\right)^2 \left(\frac{a_{\mathrm{e}}}{a_{\mathrm{eq}}}\right)^2 = \frac{\pi}{\sqrt{90}} \left(\frac{H_0/h}{M_{\mathrm{Pl}}}\right) \left(\frac{g_{s,\mathrm{rh}}}{g_{s,\mathrm{eq}}}\right)^{-2/3} \left(\frac{g_{*,\mathrm{rh}}}{g_{*,\mathrm{eq}}}\right)^{1/2} \left(\frac{g_{*,\mathrm{rh}}^{1/4}\, T_{\mathrm{rh}}}{M_{\mathrm{Pl}}}\right)^2 \frac{\left(\Omega_{\mathrm{c}} h^2 + \Omega_{\mathrm{b}} h^2\right)^2}{\left(\Omega_\gamma h^2 + \Omega_\nu h^2\right)^{3/2}}$$

$$\simeq 3.725 \times 10^{-75} \left(\frac{g_{*,\mathrm{rh}}^{1/4}\, T_{\mathrm{rh}}}{10^9\ \mathrm{GeV}}\right)^2,$$

where a_{eq} denotes the scale factor at the time of radiation-matter equality.

Also, the ratio $a_{\mathrm{eq}}/a_{\mathrm{e}}$ can be expressed as

$$\frac{a_{\mathrm{eq}}}{a_{\mathrm{e}}} = \left(\frac{\pi^2}{90}\right)^{1/4} \left(\frac{H_0/h}{M_{\mathrm{Pl}}}\right)^{-1/2} \left(\frac{g_{s,\mathrm{rh}}}{g_{s,\mathrm{eq}}}\right)^{1/3} \left(\frac{g_{*,\mathrm{eq}}}{g_{*,\mathrm{rh}}}\right)^{1/4} \left(\frac{g_{*,\mathrm{rh}}^{1/4}\, T_{\mathrm{rh}}}{M_{\mathrm{Pl}}}\right) \frac{\left(\Omega_\gamma h^2 + \Omega_\nu h^2\right)^{3/4}}{\left(\Omega_{\mathrm{c}} h^2 + \Omega_{\mathrm{b}} h^2\right)}$$

$$\simeq 9.188 \times 10^{17} \left(\frac{g_{*,\mathrm{rh}}^{1/4}\, T_{\mathrm{rh}}}{10^9\ \mathrm{GeV}}\right).$$

Spectral density of stochastic GWs

Let us decompose the stochastic GWs at a given spatial point, say, $\boldsymbol{x} = 0$, as follows:

$$h_{ij}(t) = \sum_{\lambda=(+,\times)} \int_{-\infty}^{\infty} \mathrm{d}f \int \mathrm{d}\hat{\boldsymbol{n}}\, \varepsilon_{ij}^{\lambda}(\hat{\boldsymbol{n}})\tilde{h}_f^{\lambda}(\hat{\boldsymbol{n}})\mathrm{e}^{-2\pi i f t},$$

where $\varepsilon_{ij}^{\lambda}(\hat{\boldsymbol{n}})$ is the polarization tensor. Since $h_{ij}(t)$ is real, we have $\tilde{h}_f^{\lambda *}(\hat{\boldsymbol{n}}) = \tilde{h}_{-f}^{\lambda}(\hat{\boldsymbol{n}})$.

If we assume that the GW background is stochastic, isotropic, unpolarized and stationary, then the quantity $\tilde{h}_f^{\lambda}(\hat{\boldsymbol{n}})$ satisfies the condition[17]

$$\left\langle \tilde{h}_f^{\lambda}(\hat{\boldsymbol{n}})\tilde{h}_{f'}^{\lambda' *}(\hat{\boldsymbol{n}}') \right\rangle = \delta^{(1)}(f - f')\frac{1}{4\pi}\delta^{(2)}(\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}')\delta_{\lambda\lambda'}\frac{1}{2}S_h(f),$$

where the angular brackets on the left-hand side denote averaging over the random variables, and the quantity $S_h(f)$ is referred to as the spectral density of GWs.

[17] M. Maggiore, Phys. Rept. **331**, 283 (2000) [arXiv:gr-qc/9909001];
J. D. Romano and N. J. Cornish, Living Rev. Rel. **20**, 2 (2017) [arXiv:1608.06889 [gr-qc]].

Characteristic strain and the PS of stochastic GWs

If we make use of the relation $S_h(f) = S_h(-f)$, we obtain that

$$\delta^{im}\delta^{jn}\langle h_{mn}(t)h_{ij}(t)\rangle = 2\int_{-\infty}^{\infty} \mathrm{d}f\, S_h(f) = 4\int_0^{\infty} \mathrm{d}(\ln f)\, f\, S_h(f).$$

The so-called dimensionless characteristic strain $h_\mathrm{c}(f)$, which is a measure of the amplitude of GWs as a function of frequency, is defined through the relation

$$\delta^{im}\delta^{jn}\langle h_{mn}(t)h_{ij}(t)\rangle = 2\int_0^{\infty} \mathrm{d}(\ln f)\, h_\mathrm{c}^2(f)$$

and, from the above two equations, we obtain that

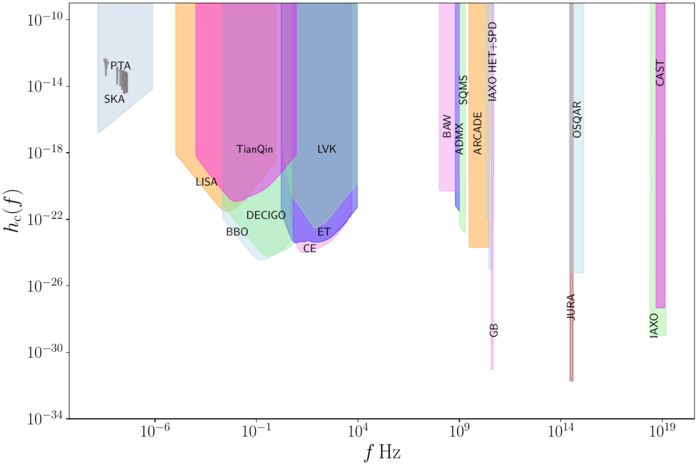
$$h_\mathrm{c}^2(f) = 2\, f\, S_h(f).$$

We can then arrive at the following relation between the PS of PGWs today, viz. $\mathcal{P}_{_\mathrm{T}}(f, \eta_0)$, the spectral density $S_h(f)$ and the characteristic strain $h_\mathrm{c}(f)$:

Definition of the PS

$$\mathcal{P}_{_\mathrm{T}}(f, \eta_0) = 4\, f\, S_h(f) = 2 h_\mathrm{c}^2(f).$$

From characteristic strain to the PS



The sensitivity curves for the dimensionless characteristic strain $h_c(f)$ have been illustrated for the different GW observatories[18].

[18] C. J. Moore, R. H. Cole and C. P. L.'Berry, Class. Quant. Grav. **32** 015014 (2015) [arXiv:1408.0740 [gr-qc]]; G. Franciolini, A. Maharana and F. Muia, Phys. Rev. D **106** 103520 [arXiv:2205.02153 [astro-ph.CO]].

Adiabatic contribution in real space I

In real space, the contribution to the two-point correlation function of the tensor perturbations due to the second adiabatic order in Fourier space can be expressed as

$$\delta^{im}\delta^{jn}\left\langle 0\left|\hat{h}^{ij}(\eta,\boldsymbol{x})\hat{h}_{ij}(\eta,\boldsymbol{x}')\right|0\right\rangle^{\mathrm{ad}} = \int_0^\infty \frac{\mathrm{d}k}{k}\,\frac{\sin(k|\boldsymbol{x}-\boldsymbol{x}'|)}{k|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathcal{P}_{\mathrm{T}}^{\mathrm{ad}}(k,\eta)$$
$$= \frac{2}{\pi^2 M_{\mathrm{Pl}}^2 a^2}\left[\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|^2}\int_0^\infty \mathrm{d}\kappa\,\sin\kappa + \frac{a''}{2a}\int_0^\infty \frac{\mathrm{d}\kappa}{\kappa^2}\,\sin\kappa\right],$$

where $\kappa = k|\boldsymbol{x}-\boldsymbol{x}'|$.

On carrying out the integrals, we obtain the adiabatic contribution at the second order to the two-point function in real space to be

$$\delta^{im}\delta^{jn}\left\langle 0\left|\hat{h}_{mn}(\eta,\boldsymbol{x})\hat{h}_{ij}(\eta,\boldsymbol{x}')\right|0\right\rangle^{\mathrm{ad}} = \frac{2}{\pi^2 M_{\mathrm{Pl}}^2 a^2}\left\{\frac{1}{|\boldsymbol{x}-\boldsymbol{x}|^2} + \frac{a''}{2a}\left[\frac{\sin\left(k_{\mathrm{IR}}|\boldsymbol{x}-\boldsymbol{x}'|\right)}{k_{\mathrm{IR}}|\boldsymbol{x}-\boldsymbol{x}'|} - \mathrm{Ci}(k_{\mathrm{IR}}|\boldsymbol{x}-\boldsymbol{x}'|)\right]\right\},$$

where k_{IR} is an infra-red cut-off and $\mathrm{Ci}(z)$ is the cosine integral function.

Adiabatic contribution in real space II

On Taylor expanding for small $|\boldsymbol{x}-\boldsymbol{x}'|$, we obtain that

$$\delta^{im}\delta^{jn}\left\langle 0\left|\hat{h}_{mn}(\eta,\boldsymbol{x})\hat{h}_{ij}(\eta,\boldsymbol{x}')\right|0\right\rangle^{\mathrm{ad}} = \frac{2}{\pi^2 M_{_{\mathrm{Pl}}}^2}\frac{1}{|\boldsymbol{x}_{\mathrm{p}}-\boldsymbol{x}'_{\mathrm{p}}|^2}\biggl\{1+\frac{a''}{2a^3}|\boldsymbol{x}_{\mathrm{p}}-\boldsymbol{x}'_{\mathrm{p}}|^2\biggl[1-\gamma_{_{\mathrm{E}}} - \ln\left(k_{_{\mathrm{IR}}}|\boldsymbol{x}-\boldsymbol{x}'|\right)+\mathcal{O}\left(k_{_{\mathrm{IR}}}^2|\boldsymbol{x}-\boldsymbol{x}'|^2\right)\biggr]\biggr\},$$

where $\boldsymbol{x}_{\mathrm{p}}=a\boldsymbol{x}$ denotes the physical distance and $\gamma_{_{\mathrm{E}}}$ is the Euler's constant.

PS of PGWs as a derived quantity

We find that the PS of PGWs can be expressed as

$$\mathcal{P}_{_{\mathrm{T}}}(k,\eta) = \frac{2k^2}{\pi}\int_0^\infty \mathrm{d}r\, r\sin(kr)\,\delta^{im}\delta^{jn}\left\langle 0\left|\hat{h}_{mn}(\eta,\boldsymbol{x})\hat{h}_{ij}(\eta,\boldsymbol{x}')\right|0\right\rangle$$

and one can easily verify that the regularized PS can similarly be written as

$$\mathcal{P}_{_{\mathrm{T}}}^{\mathrm{reg}}(k,\eta) = \frac{2k^2}{\pi}\int_0^\infty \mathrm{d}r\, r\sin(kr)$$
$$\times\,\delta^{im}\delta^{jn}\left[\left\langle 0\left|\hat{h}_{mn}(\eta,\boldsymbol{x})\hat{h}_{ij}(\eta,\boldsymbol{x}')\right|0\right\rangle - \left\langle 0\left|\hat{h}_{mn}(\eta,\boldsymbol{x})\hat{h}_{ij}(\eta,\boldsymbol{x}')\right|0\right\rangle^{\mathrm{ad}}\right].$$

From such a point of view, clearly, the fact that $\mathcal{P}_{_{\mathrm{T}}}^{\mathrm{reg}}(k,\eta)$ can turn negative should not appear as problematic.

Regularized PS of PGWs today, after the smoother transition II

In the limit $y_{\mathrm{e}} \gg 1$ the PS of PGWs during matter domination can be expressed as

$$\begin{aligned}\mathcal{P}_{\mathrm{T}}(k,\eta) = \frac{2}{\pi^2}\left(\frac{H_{\mathrm{e}}}{M_{\mathrm{Pl}}}\right)^2\left(\frac{a_{\mathrm{e}}}{a}\right)^2 y_{\mathrm{e}}^2 \Bigg\{ & 1 - \frac{1}{4x_{\mathrm{eq}}^2}\cos(4x_{\mathrm{eq}} - 2z) \\ & - \frac{1}{y_{\mathrm{e}}^3}\sin\left[2\left(x_{\mathrm{eq}} - Y\,y_{\mathrm{e}} - z + \zeta_{\mathrm{e}} - \zeta_{\mathrm{tr}}\right)\right] \\ & - \frac{X}{2y_{\mathrm{e}}^3}\cos\left(2x_{\mathrm{eq}} - 2Y\,y_{\mathrm{e}} - 2z + \zeta_{\mathrm{e}} - \zeta_{\mathrm{tr}}\right)\sin\left(\zeta_{\mathrm{e}} - \zeta_{\mathrm{tr}}\right)\Bigg\},\end{aligned}$$

where Y is given by

$$Y = -\left[q(q-1)\right]^{-1/3}\left(\frac{\Delta\eta}{|\eta_{\mathrm{e}}|}\right)^{1/3}\frac{\Gamma(1/3)}{3^{1/3}\Gamma(2/3)}\left(\frac{A_1 + \sqrt{3}A_2}{A_1 - \sqrt{3}A_2}\right).$$

The first term in the expression above will be present when we regularize the PS. We find that the second term becomes dominant in the regularized PS over $k \gtrsim k_{\mathrm{e}}^3/k_{\mathrm{eq}}^2$.