# QUANTUM FIELDS IN NON-TRIVIAL BACKGROUNDS 

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by

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## To my parents

For their patience and support

## Contents

Acknowledgements ..... viii
Declaration ..... ix
Abstract ..... X
1 Introduction and background ..... 1
1.1 Coordinate dependence of the particle concept: an example in flat spacetime ..... 5
1.1.1 Canonical quantization in Minkowski coordinates ..... 6
1.1.2 Canonical quantization in Rindler coordinates ..... 9
1.1.3 Bogolubov transformations ..... 12
1.2 Particle production in a curved spacetime: a simple example ..... 18
1.3 Concept of a detector ..... 21
1.3.1 The Unruh-DeWitt detector ..... 23
1.3.2 Inertial and uniformly accelerated Unruh-DeWitt detectors in flat spacetime ..... 27
1.3.3 Unruh-DeWitt detectors in Schwarzschild and de-Sitter spacetimes ..... 31
1.4 Pair production in a constant electric field background ..... 35
1.4.1 Quantization in the time dependent gauge: Bogolubov co- efficients ..... 37
1.4.2 Quantization in the space dependent gauge: tunneling prob- ability ..... 40
1.5 The effective Lagrangian approach ..... 44
1.5.1 Effective Lagrangian for a constant electromagnetic back- ground ..... 49
1.6 Backreaction on the classical background ..... 59
2 Finite time detectors ..... 63
2.1 Aspects of finite time detection ..... 66
2.2 Detector response with window functions ..... 77
2.2.1 Gaussian window function ..... 78
2.2.2 Window function with an exponential cut-off ..... 85
2.2.3 A rectangular window function (sum of two step functions) ..... 89
2.3 Discussion ..... 92
2.4 Limitations of the detector concept ..... 96
3 Quantum field theory in classical electromagnetic backgrounds ..... 99
3.1 Schwinger's proper time formalism for evaluating effective La- grangians ..... 100
3.2 Examining the validity of the tunneling interpretation ..... 104
3.2.1 Effective Lagrangian for a time independent magnetic field background ..... 109
3.2.2 Tunneling probability in a time independent magnetic field background ..... 112
3.2.3 Implications ..... 116
3.3 Limitations of the Klein approach ..... 122
3.4 Effective Lagrangian: a conjecture ..... 126
3.4.1 A time independent electromagnetic example ..... 129
3.4.2 Effective Lagrangian for a plane electromagnetic wave back- ground ..... 133
3.4.3 An example from gravity ..... 135
3.4.4 Discussion ..... 138
3.5 Some remarks on the Schwinger's formalism ..... 142
4 Limited validity of the semiclassical theory ..... 146
4.1 Friedmann universe with a massless scalar field: minisuperspace model ..... 149
4.2 Criterion for drawing the limits on the validity of the semiclassical theory ..... 153
$4.3 \Delta_{S C}$ for different quantum states of the scalar field mode ..... 156
4.3.1 For a vacuum state ..... 156
4.3.2 For a $n$-particle state ..... 157
4.3.3 For a coherent state ..... 158
4.4 $\Delta_{K F}$ for different quantum states of the scalar field mode ..... 160
4.4.1 For a vacuum state ..... 161
4.4.2 For a $n$-particle state ..... 163
4.4.3 For a coherent state ..... 164
4.5 Implications ..... 167
5 Analogues of quantum effects in classical field theory ..... 169
5.1 Power spectrum of a real, monochromatic wave in a uniformly ac- celerated frame ..... 171
5.1.1 Power spectrum of a scalar field mode ..... 171
5.1.2 Power spectrum of a plane electromagnetic wave ..... 177
5.2 Generalization to other field configurations ..... 179
5.3 Planckian ambience in Schwarzschild and de-Sitter spacetimes ..... 184
5.4 Discussion ..... 188
6 Conclusions and outlook ..... 192
A Contour integrals ..... 201
A. 1 Response of the inertial detector ..... 201
A. 2 Response of the accelerated detector ..... 205

## List of Tables

$2.1 \mathcal{F}_{\text {ine }}(\Omega, \epsilon, T)$ and $\mathcal{R}_{\text {ine }}(\Omega, \epsilon, T)$ in different limits ..... 93
4.1 $\Delta_{S C}$ in the limit of $\beta \rightarrow 0$ ..... 159
4.2 $\Delta_{S C}$ in the limit of $\beta \rightarrow \infty$ ..... 159
$4.3 \Delta_{K F}$ in the limit of $\beta \rightarrow 0$ ..... 166
4.4 $\Delta_{K F}$ in the limit of $\beta \rightarrow \infty$ ..... 166

## List of Figures

A. 1 Contour for $\mathcal{F}_{\text {ine1 }}(\Omega, T)$ ..... 202
A. 2 Contour for $\mathcal{F}_{\text {ine } 2 A}(\Omega, T)$ ..... 203
A. 3 Contour for $\mathcal{F}_{\text {ine } 2 B}(\Omega, T)$ ..... 204
A. 4 Contour for $\mathcal{F}_{\text {accin }}(\Omega, T)$ ..... 205
A. 5 Contour for $\mathcal{F}_{\text {acc } 2 n A}(\Omega, T)$ ..... 207
A. 6 Contour for $\mathcal{F}_{\text {acc2nB }}(\Omega, T)$ ..... 207

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## Declaration

CERTIFIED that the work incorporated in the thesis

## Quantum fields in non-trivial backgrounds

submitted by L. Sriramkumar was carried out by the candidate under my supervision. Such material as has been obtained from other sources has been duly acknowledged in the thesis.

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T. Padmanabhan
(Thesis supervisor)

## Abstract

Quantum field theory has been enormously successful as a theory describing the behavior of fields up to energy scales of the order of 100 GeV . Quantum electrodynamics, the earliest of the gauge theories, describes the interaction of the electromagnetic field with matter. Though, during the early stages of its formulation, the divergences that arise in the theory had seemed too big a hurdle to overcome, regularization and renormalization procedures have been developed to handle these divergences and the theory has come up with a large number of predictions. Several of these predictions, like, Lamb shift, anomalous magnetic moment of the electron, have been experimentally verified thereby firmly establishing the validity of quantum electrodynamics. The theory due to Salam and Weinberg has been able to successfully unify the electromagnetic and weak interactions into a single gauge theory. Also, the $W$ and the $Z$ bosons predicted by the theory have been observed experimentally thereby establishing the SalamWeinberg theory as the correct theory of weak interactions. Though, we are yet to have a theory that describes the strong interactions adequately, we have in hand a workable model in quantum chromodynamics. Efforts to describe all these three interactions by a unified gauge theory have also been successful.

The gravitational interaction has been the odd one out. All attempts to
provide a quantum framework for the gravitational field have so far proved to be unsuccessful. In the absence of a viable quantum theory of gravity, can one say anything at all about the influence of the gravitational field on quantum phenomena? In the early days of quantum theory, before the development of quantum electrodynamics, a picture of a classical electromagnetic field interacting with atomic and molecular systems was used to understand spectroscopic results. Such a semiclassical description yields some results that are in accordance with the full theory of quantum electrodynamics. One may therefore hope that a similar regime exists for gravity, a regime in which the gravitational field can be retained as a classical background, while the matter fields are quantized according to the conventional quantum field theory. Though, we are yet to have a quantum theory of gravity, there exist compelling reasons to believe that quantum gravitational effects will be important only at energy scales of the order of Planck energy ( $\sim 10^{19} \mathrm{GeV}$ ). There exists a domain of 17 orders of magnitude between the Planck energy and an energy scale of the order of 100 GeV , a domain in which the gravitational field can be assumed to behave classically and the matter fields can be assumed to have a quantum nature. Though, there exist other contesting theories to describe the classical gravitational field, experiments have pointed towards Einstein's general theory of relativity as the correct classical theory of gravity. Thus, adopting general relativity as a theory describing classical gravity, one is led to the subject of quantum field theory in curved spacetimes which has been an area of active research during the past couple of decades.

The conventional formulation of quantum field theory in Minkowski spacetime is invariant under the Poincare group, i.e. the theory is invariant only under linear coordinate transformations. Under non-linear coordinate transformations,
even in flat spacetime, quantum field theoretic concepts such as vacua, particles etc. do not, in general, seem to possess a covariant meaning. Similar problems are encountered when the evolution of quantum fields are studied in curved spacetimes. Further, in a curved spacetime the presence of the gravitational background can lead to production of particles corresponding to the quantum field. These particles that have been produced can also react back on the classical background. The metric, which is assumed here to be described by Einstein's equations, is a covariant concept. Therefore, if the backreaction of the quantum field on the gravitational background has to be studied meaningfully, a covariant description of the phenomenon of particle production is called for. This in turn requires an understanding of the concept of a particle in an arbitrary curved spacetime.

The phenomenon of particle production takes place in classical electromagnetic backgrounds too. We can possibly learn lessons for the gravitational case by studying the evolution of quantum fields in electromagnetic backgrounds. In fact, some of the conceptual problems that arise while studying quantum fields in curved spacetimes are encountered in electromagnetic backgrounds too. Just as a covariant formulation of the phenomenon of particle production is required for gravitational backgrounds, a gauge invariant description of the same phenomenon is called for in the case of electromagnetic backgrounds.

This thesis work is focussed towards improving our understanding of the phenomenon of particle production and also the backreaction of these particles that have been produced on the classical backgrounds.

A chapter wise summary of the thesis is given below.

In chapter 1, we introduce the basic terminology and the mathematical
framework that is used to study the evolution of quantum fields in classical gravitational and electromagnetic backgrounds. This chapter reviews some of the essential results that serve as a background for the chapters that follow. We begin this chapter by illustrating the coordinate dependence of the particle concept with the aid of a simple example in flat spacetime. We then present an example of a time dependent gravitational background in which the phenomenon of particle production takes place. Motivating the usefulness of the detector concept, we introduce the Unruh-DeWitt detector. We discuss the response of inertial and uniformly accelerated Unruh-DeWitt detectors in flat spacetime and also analyze the response of these detectors in Schwarzschild and de-Sitter spacetimes. Carrying out the canonical quantization of a complex scalar field in a constant electric field background, we illustrate how the tunneling interpretation is invoked to explain the phenomenon of particle production in time independent gauges. Introducing the effective Lagrangian approach, we show that invariant results can be obtained by this approach with the help of an electromagnetic example. Finally, we discuss as to how the backreaction of the quantum field on the classical background can be taken into account and introduce the semiclassical Einstein's equations.

Chapter 2 is devoted to the study of finite time response of Unruh-DeWitt detectors. We begin this chapter by motivating the need for a finite time detector. We then study the response inertial and uniformly accelerated Unruh-DeWitt detectors in flat spacetime when they are switched on smoothly as well as abruptly for a finite proper time interval. We identify the divergences that appear in the response functions of the detectors when they are switched on abruptly and point out the origin of these divergences. We conclude this chapter by pointing out the limitations of the detector concept.

In chapter 3, we study the evolution of a quantized complex scalar field in classical electromagnetic backgrounds. We begin this chapter by introducing Schwinger's proper time formalism to evaluate effective Lagrangians. We then examine the validity of the tunneling interpretation that is usually invoked in literature to explain the phenomenon of particle production in time independent gauges. With the aid of an example, we show that the tunneling interpretation can be inconsistent with the effective Lagrangian approach. The effective Lagrangian being a more reliable approach, we conclude that this lack of consistency between these two approaches calls into question the validity of the tunneling interpretation. We then discuss the limitations of the the Klein approach that is used to study particle production in time independent gauges.

Though the effective Lagrangian approach is more reliable, the evaluation of the effective Lagrangian even for a given classical background proves to be a rather difficult task. In chapter 3, we also propose a conjecture that can possibly help us guess the form of the the effective Lagrangian for an arbitrary background. We put forward the conjecture that the effective Lagrangian for a classical background will be zero if all the invariant scalars (involving the field and its derivatives) describing the background vanish identically. We verify this conjecture by explicitly evaluating the effective Lagrangian for some non-trivial electromagnetic and gravitational backgrounds. We conclude this chapter with a few remarks on the boundary condition that is implicitly assumed in the evaluation of effective Lagrangians using Schwinger's formalism.

In chapters 2 and 3 , we had neglected the backreaction of the quantum field on the classical background and had concentrated our efforts on obtaining an invariant description of the phenomenon of particle production. Once such
description is at hand the backreaction of the quantum field on the classical background can be taken into account. It is generally assumed that the backreaction of the quantum field on a gravitational background is given by the expectation value of the energy-momentum tensor of the quantum field. Since such a semiclassical theory is incapable of providing a preferred state for the quantum field by itself, the expectation value of the energy-momentum tensor has to be evaluated in a state specified by hand. This semiclassical theory can then be relied upon only if the fluctuations in the energy-momentum densities of the quantum field are small when compared to their expectation values. Using this as the criterion, in chapter 4, we analyze the validity of the semiclassical theory for a minisuperspace model of a massless scalar field in a Friedmann universe. We evaluate the magnitude of the fluctuations in the backreaction term for the states of the scalar field mode that correspond to the vacuum, $n$-particle and coherent states of the quantized scalar field. We find that the fluctuations in the backreaction term are small, even when a large amount of particles are being produced, only for coherent states with a large value for the parameter describing them. We therefore conclude that the semiclassical theory we have considered will be valid during all stages of evolution, only if the quantum fields are assumed to be in 'coherent' like states.

In quantum field theory, it is the coefficients of the positive frequency components of the normal modes of the quantum field that are identified to be annihilation operators. Therefore, the evolution of a quantum field is governed by the behavior of the normal modes of the equation of motion satisfied by it. But, even a classical field satisfies the same equations of motion as does a quantum field. If so, can some of the non-trivial effects that arise in quantum field theory
arise in classical field theory too? In chapter 5 , we show that this indeed can be the case. Fourier analyzing real plane waves modes of scalar and electromagnetic fields in flat spacetime with respect to the proper time of a uniformly accelerated observer, we find that the resulting power spectrum has a Planckian nature. We then outline as to how such a Planckian spectrum can also prove to be a feature of observers stationed at a constant radius in Schwarzschild and de-sitter spacetimes. We conclude this chapter by presenting a model of a detector which responds to the Fourier spectrum of the field with respect to its proper time thereby illustrating that it should, in principle, be possible to physically measure the power spectrum we have obtained.

Finally, in chapter 6, we present our conclusions and outlook.

This thesis is mainly based on the following publications.

- L. Sriramkumar and T. Padmanabhan, Finite-time response of inertial and uniformly accelerated Unruh-DeWitt detectors, Class. Quantum Grav. 13, 2061 (1996).
- L. Sriramkumar, Limits on the validity of the semiclassical theory-a minisuperspace example, IUCAA preprint 15/95, Accepted for publication in Int. J. Mod. Phys. D.
- L. Sriramkumar and T. Padmanabhan, Does a nonzero tunneling probability imply particle production in time independent classical electromagnetic backgrounds?, Phys. Rev. D 54, 7599 (1996).
- L. Sriramkumar, R. Mukund and T. Padmanabhan, Non-trivial classical backgrounds with vanishing quantum corrections, IUCAA preprint 42/96,

Accepted for publication in Phys. Rev. D.

- K. Srinivasan, L. Sriramkumar and T. Padmanabhan, Possible quantum interpretation of certain power spectra in classical field theory, IUCAA preprint 18/96, Submitted for publication.
- K. Srinivasan, L. Sriramkumar and T. Padmanabhan, Plane waves viewed from an accelerated frame: quantum physics in classical setting, Submitted for publication.


## Chapter 1

## Introduction and background

During the past couple of decades or so the subject of quantum field theory in curved spacetime has been an area of active research. The original motivation to study the behavior of quantum fields in classical gravitational backgrounds was the belief that such a study will provide useful clues for a quantum theory of gravity. Though a quantum theory of gravity still remains a distant dream, it will be fair to say that there has been very interesting discoveries in this area. Many important lessons have been learned from this effort, but it is difficult to list down direct clues to quantum gravity obtained from this study-if anything, the conceptual problems faced in this subject make quantum gravity look all the more puzzling.

The basic formalism of quantum field theory in flat spacetime can be generalized to a curved spacetime in a straightforward way (see, for e.g., any of the following textbooks $[1,2,3]$, or one of the following review articles $[4,5,6]$ ). A quantum field is described in a curved spacetime by the generally covariant version of flat spacetime Lagrangian, varying which one obtains the generally covariant field equations. Quantization of the field then proceeds by defining a set of canon-
ical commutation relations for the field operators. The evolution of the quantum field is governed by the behavior of the normal modes of the field equation in the spacetime of interest. Departure from flat spacetime field theory comes at the next level when one tries to construct the Fock basis and define particles corresponding to the quantum field. (Throughout this thesis, we shall restrict our analysis to free quantum fields, because our interest is the interaction of the quantum field with gravity or electromagnetism rather than its interaction with itself. Also, we shall assume here that the gravitational background is described by Einstein's equations.)

Actually this departure arises even when one attempts to formulate quantum field theory in a noninertial coordinate system in flat spacetime [7, 8]. After all, there is no reason why field quantization should be carried out in the Minkowski coordinates alone. An accelerating observer, for example, will find it more natural to carry out the field quantization in a coordinate system obtained by a suitable transformation of the Minkowski coordinates. It then turns out, that the vacuum state defined in an inertial coordinate system and the vacuum state defined in a noninertial coordinate system can, in general, be different states $[9,10]$. Hence, the definition of a particle in the two coordinate systems can also be different. These features are encountered when the evolution of quantum fields is studied in a curved spacetime [4].

The hope of providing an operational definition of the concept of a particle in a curved spacetime led to the idea of a detector. The development of the idea of detectors have emphasized the observer dependence of the particle concept. It was shown that the response of detectors depends on the state of their motion, if the quantum field is assumed to be in a particular state, say, the Minkowski vacuum
state in flat spacetime. The monopole detector due to Unruh and DeWitt [11, 12], for instance, does not respond in the Minkowski vacuum state when in inertial motion in flat spacetime but responds when it is accelerating uniformly or when it is in rotational motion $[10,13]$. Similar features arise when the response of detectors are studied in curved spacetimes [14]. It has become clear that the conventional formulation of quantum field theory in flat spacetime is not invariant under non-linear coordinate transformations and in an arbitrary curved spacetime the very definition of a particle becomes dependent on the coordinate system chosen by an observer.

In a curved spacetime, even if we choose a particular coordinate system, a quantum field which was initially in the vacuum state may not remain in the vacuum state at a later time. One finds that the time variation of the classical gravitational background can lead to production of particles corresponding to the quantum field $[15,16,17,18,19,20,21,22,23,24,25,26]$. The other aspect of the study of quantum fields in a curved spacetime is the backreaction of the particles that have been produced on the classical gravitational background. It is generally believed that this backreaction should be described by Einstein's equations with the right hand side replaced by the expectation value of the energy-momentum tensor of the quantum field, evaluated in a given state $[27,28,29,30,31,32]$.

Phenomena such as particle production and vacuum polarization also arise in classical electromagnetic backgrounds [33, 34, 35, 36]. Some of the conceptual issues that arise in the study of quantum fields in classical gravitational backgrounds are encountered in electromagnetic backgrounds too [37, 38]. So, the evolution of quantum fields in electromagnetic backgrounds has been repeatedly studied in literature with the hope that such a study will teach us some useful
lessons to handle the gravitational case.

In this chapter, we introduce the basic mathematical formalism of the different approaches that are used to study the evolution of quantum fields in classical gravitational and electromagnetic backgrounds. We also review here some of the essential results that will serve as a background for the chapters that follow. Most of the our analysis in this chapter and the chapters that follow will be carried out for the case of a quantum scalar field, but our results will, in general, hold good for fields of higher spins too. This restriction will enable the results presented in this thesis to emerge with the minimum of mathematical complexity. Also, we shall set $\hbar=c=G=1$ in all our calculations.

This chapter is organized as follows. In section 1.1, we illustrate the coordinate dependence of the particle concept with the aid of a simple example in flat spacetime. In section 1.2, we present an example of a time dependent gravitational background in which the phenomenon of particle production takes place. After motivating the usefulness of the detector concept, we introduce the UnruhDeWitt detector in section 1.3. In the same section, we discuss the response of inertial and uniformly accelerated Unruh-DeWitt detectors in flat spacetime and also analyze the response of these detectors when they are stationed at a constant radius in Schwarzschild and de-Sitter spacetimes. Carrying out the normal mode analysis of a complex scalar field in a constant electric field background, in section 1.4, we illustrate how the tunneling interpretation is invoked to explain the phenomenon of particle production in time independent gauges. In section 1.5, we introduce the effective Lagrangian approach and show that invariant results can be obtained by this approach with the aid of the example of a constant electromagnetic background. In section 1.6, we discuss as to how the backreaction of
the quantum field on the classical background can possibly be taken into account and introduce the semiclassical Einstein's equations.

### 1.1 Coordinate dependence of the particle concept: an example in flat spacetime

In this section, we shall illustrate the coordinate dependence of the particle concept with the help of a simple example in flat spacetime. For the sake of mathematical simplicity, we shall mostly work here in $(1+1)$ dimensions.

The system we shall consider is a real, massless scalar field $\Phi$ described by the action

$$
\begin{equation*}
\mathcal{S}[\Phi]=\int d^{2} x \sqrt{-g} \mathcal{L}(\Phi)=\frac{1}{2} \int d^{2} x \sqrt{-g} g_{\mu \nu} \partial^{\mu} \Phi \partial^{\nu} \Phi \tag{1.1}
\end{equation*}
$$

where

$$
\begin{equation*}
\partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}} . \tag{1.2}
\end{equation*}
$$

The equation of motion for the scalar field $\Phi$ described by the action above is given by

$$
\begin{equation*}
\square \Phi \equiv \frac{1}{\sqrt{-g}} \partial_{\mu}\left(\sqrt{-g} g^{\mu \nu} \partial_{\nu}\right) \Phi=0 . \tag{1.3}
\end{equation*}
$$

With the help of the following conserved four current $j^{\mu}$

$$
\begin{equation*}
j_{\mu}=\left(\Phi^{*} \partial_{\mu} \Phi-\Phi \partial_{\mu} \Phi^{*}\right) \tag{1.4}
\end{equation*}
$$

we can define a scalar product for any two solutions $\Phi_{1}$ and $\Phi_{2}$ of the scalar field $\Phi$ as follows:

$$
\begin{equation*}
\left(\Phi_{1}, \Phi_{2}\right)=-i \int d \Sigma^{\mu} \sqrt{-g_{\Sigma}}\left(\Phi_{1} \partial_{\mu} \Phi_{2}^{*}-\Phi_{2}^{*} \partial_{\mu} \Phi_{1}\right) \tag{1.5}
\end{equation*}
$$

where $d \Sigma^{\mu}=n^{\mu} d \Sigma$, with $n^{\mu}$ being a future directed unit vector orthogonal to the spacelike hypersurface $\Sigma(d \Sigma$ is the volume element on $\Sigma)$ and the asterisk denotes complex conjugation.

### 1.1.1 Canonical quantization in Minkowski coordinates

In $(1+1)$ dimensions and in Minkowski coordinates $(t, x)$, flat spacetime is described by the line element

$$
\begin{equation*}
d s^{2}=d t^{2}-d x^{2} \tag{1.6}
\end{equation*}
$$

The equation of motion for the scalar field $\Phi$, viz. equation (1.3), corresponding to this metric is given by

$$
\begin{equation*}
\left(\partial_{t}^{2}-\partial_{x}^{2}\right) \Phi(t, x)=0 \tag{1.7}
\end{equation*}
$$

The solutions to this equation are plane waves i.e.

$$
\begin{equation*}
u_{k}(t, x) \propto \exp -i(\omega t-k x) \tag{1.8}
\end{equation*}
$$

where $\omega=|k|$ and $k$ can take values continuously in the range $-\infty$ and $\infty$. Since the flat spacetime metric is independent of the Minkowski time coordinate $t$, positive frequency modes can be defined with respect to the timelike Killing vector $(\partial / \partial t)$. That is, normal modes $u_{k}$ are defined to be positive frequency modes if they are eigenfunctions of the operator $(\partial / \partial t)$ :

$$
\begin{equation*}
\partial_{t} u_{k}(t, x)=-i \omega u_{k}(t, x) \quad \text { with } \quad \omega>0 . \tag{1.9}
\end{equation*}
$$

In the Minkowski coordinates we are considering here we can choose the hypersurface $d \Sigma^{\mu}$ in the scalar product (1.5) to be a constant $t$ surface. Then, if we choose

$$
\begin{equation*}
u_{k}(t, x)=\frac{1}{\sqrt{4 \pi \omega}} \exp -i(\omega t-k x) \tag{1.10}
\end{equation*}
$$

we find that the modes $u_{k}$ and their complex conjugates $u_{k}^{*}$ satisfy the following orthonormality relations

$$
\begin{equation*}
\left(u_{k}, u_{k^{\prime}}\right)=\delta_{D}\left(k-k^{\prime}\right) \quad ; \quad\left(u_{k}^{*}, u_{k^{\prime}}^{*}\right)=-\delta_{D}\left(k-k^{\prime}\right) \quad \text { and } \quad\left(u_{k}, u_{k^{\prime}}^{*}\right)=0 \tag{1.11}
\end{equation*}
$$

where $\delta_{D}(z)$ is the Dirac delta function of the corresponding argument.

The canonical quantization of the scalar field can be carried out by treating $\Phi$ as an operator and imposing the following equal time commutation relations

$$
\left.\begin{array}{l}
{\left[\Phi(t, x), \Phi\left(t, x^{\prime}\right)\right]=0}  \tag{1.12}\\
{\left[\Pi(t, x), \Pi\left(t, x^{\prime}\right)\right]=0} \\
{\left[\Phi(t, x), \Pi\left(t, x^{\prime}\right)\right]=i \delta_{D}\left(x-x^{\prime}\right),}
\end{array}\right\}
$$

where $\Pi$ is the canonically conjugate momentum corresponding to the scalar field defined as

$$
\begin{equation*}
\Pi=\frac{\partial \mathcal{L}}{\partial\left(\partial_{0} \Phi\right)}=\partial_{0} \Phi \tag{1.13}
\end{equation*}
$$

(In an arbitrary curved spacetime, the canonically conjugate variable $\Pi$ corresponding to the scalar field $\Phi$ is given by the relation

$$
\begin{equation*}
\Pi=\frac{\partial \mathcal{L}}{\partial\left(\partial_{0} \Phi\right)}=\sqrt{-g} g^{0 \mu} \partial_{\mu} \Phi \tag{1.14}
\end{equation*}
$$

In flat spacetime and in Minkowski coordinates $g^{00}=1$ and $g^{01}=0$. Therefore, in such a case, the above relation for $\Pi$ simplifies to equation (1.13).)

The normal modes (1.10) and their complex conjugates satisfying the relations (1.11) form a complete orthonormal basis so that the quantized scalar field can be expanded as follows:

$$
\begin{equation*}
\Phi(t, x)=\int_{-\infty}^{\infty} d k\left(\hat{a}_{k} u_{k}(t, x)+\hat{a}_{k}^{\dagger} u_{k}^{*}(t, x)\right), \tag{1.15}
\end{equation*}
$$

where $\hat{a}_{k}$ and $\hat{a}_{k}^{\dagger}$ are the annihilation and the creation operators for the mode $k$. (Note that in the decomposition above we have identified the coefficients of the positive frequency normal modes to be the annihilation operators.) The equal time commutation relations (1.12) then correspond to

$$
\left.\begin{array}{rl}
{\left[\hat{a}_{k}, \hat{a}_{k^{\prime}}\right]} & =0  \tag{1.16}\\
{\left[\hat{a}_{k}^{\dagger}, \hat{a}_{k^{\prime}}^{\dagger}\right]} & =0 \\
{\left[\hat{a}_{k}, \hat{a}_{k^{\prime}}^{\dagger}\right]} & =\delta_{D}\left(k-k^{\prime}\right) .
\end{array}\right\}
$$

The Minkowski vacuum state $\left|0_{M}\right\rangle$ is then defined to be the state that is annihilated by the annihilation operator $\hat{a}_{k}$, i.e.

$$
\begin{equation*}
\hat{a}_{k}\left|0_{M}\right\rangle=0, \quad \forall k . \tag{1.17}
\end{equation*}
$$

The many particle states can then be obtained by repeatedly operating the creation operator $\hat{a}_{k}^{\dagger}$ on the Minkowski vacuum state. For instance, the one particle state $\left|1_{k}\right\rangle$ can be obtained by operating the creation operator once on the Minkowski vacuum state as follows:

$$
\begin{equation*}
\left|1_{k}\right\rangle=\hat{a}_{k}^{\dagger}\left|0_{M}\right\rangle . \tag{1.18}
\end{equation*}
$$

The Fock space thus constructed from the Minkowski vacuum state is invariant under the action of the Poincaré group. The operator $\hat{N}_{k} \equiv\left(\hat{a}_{k}^{\dagger} \hat{a}_{k}\right)$ is called the number operator for the mode $k$ and its expectation value in a state $\left|n_{k}\right\rangle$ is equal to $n_{k}$, the number of quanta in the mode $k$. For instance, in the Minkowski vacuum state

$$
\begin{equation*}
\left\langle 0_{M}\right| \hat{N}_{k}\left|0_{M}\right\rangle=\left\langle 0_{M}\right| \hat{a}_{k}^{\dagger} \hat{a}_{k}\left|0_{M}\right\rangle=0, \quad \forall k \tag{1.19}
\end{equation*}
$$

### 1.1.2 Canonical quantization in Rindler coordinates

Now, consider the following non-linear transformations of the Minkowski coordinates $t$ and $x[39,40,41]$

$$
\begin{equation*}
t=g^{-1} e^{g \xi} \sinh (g \tau) \quad \text { and } \quad x=g^{-1} e^{g \xi} \cosh (g \tau), \tag{1.20}
\end{equation*}
$$

where $g$ is a constant. The new coordinates $\tau$ and $\xi$ which we shall refer to as the Rindler coordinates, cover only the wedge $x>|t|$ of the Minkowski $t-x$ plane. In terms of new coordinates $\tau$ and $\xi$, the flat spacetime line element (1.6) takes the following form:

$$
\begin{equation*}
d s^{2}=e^{2 g \xi}\left(d \tau^{2}-d \xi^{2}\right) \tag{1.21}
\end{equation*}
$$

From equation (1.20) it can be easily noted that

$$
\begin{equation*}
x^{2}-t^{2}=g^{-2} e^{2 g \xi} \quad \text { and } \quad \tanh (g \tau)=(t / x) . \tag{1.22}
\end{equation*}
$$

These relations then imply that curves of constant $\tau$ are straight lines passing through the origin, while curves of constant $\xi$ are hyperbolae in the Minkowski $t-x$ plane. Each of these hyperbolae is the spacetime trajectory of a uniformly accelerating observer having a proper acceleration $\left(g e^{-g \xi}\right)$.

This can be shown as follows (see, for instance, ref. [42], pp. 22-23). Consider an observer who is traveling with a uniform acceleration $\lambda$ along the $x$-axis. The equation of motion of such an observer is given by

$$
\begin{equation*}
\frac{d}{d t}\left(\frac{v}{\sqrt{1-v^{2}}}\right)=\lambda \tag{1.23}
\end{equation*}
$$

where $v=(d x / d t)$. This equation can be integrated with the result

$$
\begin{equation*}
v=\frac{d x}{d t}=\lambda t\left(1+\lambda^{2} t^{2}\right)^{-1 / 2}, \tag{1.24}
\end{equation*}
$$

where we have chosen the initial condition to be $v=0$ at $t=0$. Integrating this equation again and setting $x=\lambda^{-1}$ at $t=0$, we obtain that

$$
\begin{equation*}
x=\lambda^{-1}\left(1+\lambda^{2} t^{2}\right)^{1 / 2} \tag{1.25}
\end{equation*}
$$

The proper time $s$ as measured by a clock carried by the accelerated observer is related to the Minkowski time $t$ as follows:

$$
\begin{align*}
s(t) & =\int_{0}^{t} d t \sqrt{1-v^{2}} \\
& =\int_{0}^{t} \frac{d t}{\sqrt{1+\lambda^{2} t^{2}}} \\
& =\lambda^{-1} \operatorname{arcsinh}(\lambda t) . \tag{1.26}
\end{align*}
$$

Using this relation and equation (1.25), we can express the trajectory of the observer accelerating with a proper acceleration $\lambda$ as follows:

$$
\begin{equation*}
t=\lambda^{-1} \sinh (\lambda s) \quad \text { and } \quad x=\lambda^{-1} \cosh (\lambda s) . \tag{1.27}
\end{equation*}
$$

(If we now choose $\lambda=\left(g e^{-g \xi}\right)$ and $s=\left(e^{g \xi} \tau\right)$, we find that these relations reduce to the transformations (1.20).) This trajectory is a hyperbolae in the $t$ $x$ plane confirming our claim that the Rindler coordinates $(\tau, \xi)$ correspond to the coordinates of an observer accelerating uniformly along the spatial coordinate $x$. Note that different hyperbolae correspond to different uniform accelerations, with the acceleration decreasing as one moves out towards positive $x$. A uniformly accelerated observer traveling along one of these hyperbolae is called a Rindler observer. The null lines $x= \pm t$ are asymptotes of the hyperbolae and hence a Rindler observer never intersects these lines. These null lines therefore act as horizons for the uniformly accelerated observers.

Clearly, the metric (1.21) is conformally related to the metric (1.6) (for a discussion on conformal transformations, see, for e.g., ref. [1], section 3.1). In
the $(1+1)$ dimensional case we are considering here the action (1.1) is invariant under conformal transformations. Therefore, the equation of motion (1.3) for the massless scalar field $\Phi$ in terms of the new coordinates $\tau$ and $\xi$ reduces to

$$
\begin{equation*}
\left(\partial_{\tau}^{2}-\partial_{\xi}^{2}\right) \Phi(\tau, \xi)=0 \tag{1.28}
\end{equation*}
$$

The solutions to this equation, as it was in the case of Minkowski coordinates, are just plane waves, i.e.

$$
\begin{equation*}
v_{l}(\tau, \xi) \propto \exp -i(\nu \tau-l \xi) \tag{1.29}
\end{equation*}
$$

where $\nu=|l|$ and $l$ can take values continuously between $-\infty$ and $\infty$. Since the metric (1.21) is independent of the Rindler time coordinate $\tau$, the positive frequency modes for the new coordinates can be defined with respect to the timelike Killing vector field $(\partial / \partial \tau)$ as follows:

$$
\begin{equation*}
\partial_{\tau} v_{l}(\tau, \xi)=-i \nu v_{l}(\tau, \xi) \quad \text { with } \quad \nu>0 . \tag{1.30}
\end{equation*}
$$

In the Rindler case, the hypersurface $d \Sigma^{\mu}$ in the scalar product (1.5) can be chosen to be a constant $\tau$ hypersurface. Then, the normalized modes are given by

$$
\begin{equation*}
v_{l}(\tau, \xi)=\frac{1}{\sqrt{4 \pi \nu}} \exp -i(\nu \tau-l \xi) \tag{1.31}
\end{equation*}
$$

These modes and their complex conjugates $v_{l}^{*}$ satisfy the following set of orthonormality relations

$$
\begin{equation*}
\left(v_{l}, v_{l^{\prime}}\right)=\delta_{D}\left(l-l^{\prime}\right) \quad ; \quad\left(v_{l}^{*}, v_{l^{\prime}}^{*}\right)=-\delta_{D}\left(l-l^{\prime}\right) \quad \text { and } \quad\left(v_{l}, v_{l^{\prime}}^{*}\right)=0 . \tag{1.32}
\end{equation*}
$$

Just as in the Minkowski case, the quantized scalar field can now be expanded in terms of modes (1.31) and their complex conjugates as follows:

$$
\begin{equation*}
\Phi(\tau, \xi)=\int_{-\infty}^{\infty} d l\left(b_{l} v_{l}(\tau, \xi)+\hat{b}_{l}^{\dagger} v_{l}^{*}(\tau, \xi)\right) \tag{1.33}
\end{equation*}
$$

where $\hat{b}_{l}$ and $\hat{b}_{l}^{\dagger}$ are the creation and the annihilation operators corresponding to the conformal Rindler mode $l$. The operators $\hat{b}_{l}$ and $\hat{b}_{l}^{\dagger}$ follow the same commutation relations as the Minkowski operators $\hat{a}_{k}$ and $\hat{a}_{k}^{\dagger}$. The vacuum state corresponding to the new Rindler coordinates can then be defined as

$$
\begin{equation*}
\hat{b}_{l}\left|0_{R}\right\rangle=0, \quad \forall l . \tag{1.34}
\end{equation*}
$$

### 1.1.3 Bogolubov transformations

In the last two subsections, we have carried out the canonical quantization of the scalar field $\Phi$ in flat spacetime in two different coordinate systems which are related by a non-linear coordinate transformation. We find that the scalar field $\Phi$ can be decomposed in these two coordinate systems in terms of two complete, orthonormal set of modes $u_{k}$ and $v_{l}$. These two decompositions lead to two vacuum states $\left|0_{M}\right\rangle$ and $\left|0_{R}\right\rangle$ and their associated Fock space. Are these two quantization equivalent?

As both sets of the normal modes $u_{k}$ and $v_{l}$ are complete, one set of modes can be expanded in terms of the other as follows:

$$
\begin{equation*}
v_{l}[\tau(t, x), \xi(t, x)]=\int_{-\infty}^{\infty} d k\left(\alpha(l, k) u_{k}(t, x)+\beta(l, k) u_{k}^{*}(t, x)\right) . \tag{1.35}
\end{equation*}
$$

Conversely

$$
\begin{equation*}
u_{k}[t(\tau, \xi), x(\tau, \xi)]=\int_{-\infty}^{\infty} d l\left(\alpha^{*}(l, k) v_{l}(\tau, \xi)-\beta(l, k) v_{l}^{*}(\tau, \xi)\right) . \tag{1.36}
\end{equation*}
$$

These relations are known as the Bogolubov transformations [43, 44, 45, 46]. The quantities $\alpha(l, k)$ and $\beta(l, k)$ are called the Bogolubov coefficients. Using equation (1.35) and the orthonormality relations (1.11), the Bogolubov coefficients
can be expressed as

$$
\begin{equation*}
\alpha(l, k)=\left(v_{l}, u_{k}\right) \quad \text { and } \quad \beta(l, k)=-\left(v_{l}, u_{k}^{*}\right) . \tag{1.37}
\end{equation*}
$$

Making use of the orthonormality conditions (1.11) and (1.32) of the normal modes $u_{k}$ and $v_{l}$, it can be shown that

$$
\begin{equation*}
\hat{a}_{k}=\int_{-\infty}^{\infty} d l\left(\alpha(l, k) \hat{b}_{l}+\beta^{*}(l, k) \hat{b}_{l}^{\dagger}\right) \tag{1.38}
\end{equation*}
$$

and

$$
\begin{equation*}
\hat{b}_{l}=\int_{-\infty}^{\infty} d k\left(\alpha^{*}(l, k) \hat{a}_{k}-\beta^{*}(l, k) \hat{a}_{k}^{\dagger}\right) . \tag{1.39}
\end{equation*}
$$

The Bogolubov transformations also possess the following properties

$$
\begin{align*}
\left.\int_{-\infty}^{\infty} d k(\alpha(l, k)) \alpha^{*}\left(l^{\prime}, k\right)-\beta(l, k) \beta^{*}\left(l^{\prime}, k\right)\right) & =\delta_{D}\left(l-l^{\prime}\right),  \tag{1.40}\\
\left.\int_{-\infty}^{\infty} d k(\alpha(l, k)) \beta\left(l^{\prime}, k\right)-\beta(l, k) \alpha\left(l^{\prime}, k\right)\right) & =0 . \tag{1.41}
\end{align*}
$$

It follows immediately from equations (1.38) and (1.39) that the two Fock spaces constructed out of the modes of the Minkowski and the Rindler coordinates will prove to be different if the Bogolubov coefficient $\beta$ is nonzero. For example, if $\beta$ proves to be nonzero then it can be easily seen from equation (1.39) that the Minkowski vacuum $\left|0_{M}\right\rangle$ will not be annihilated by the Rindler annihilation operator $\hat{b}_{l}$. This indeed happens to be the case. The Bogolubov coefficients between the Minkowski and the Rindler modes can be evaluated with the aid of equation (1.37). If we choose to evaluate the scalar products in equation (1.37) on the $\tau=0$ hypersurface, we find that the Bogolubov coefficients relating the Minkowski and the Rindler modes are described by the following integrals:

$$
\begin{align*}
& \alpha(l, k)=\frac{1}{4 \pi \sqrt{\omega \nu}} \int_{-\infty}^{\infty} d \xi\left(\omega e^{g \xi}+\nu\right) e^{i l \xi} \exp -i\left(k g^{-1} e^{g \xi}\right), \\
& \beta(l, k)=\frac{1}{4 \pi \sqrt{\omega \nu}} \int_{-\infty}^{\infty} d \xi\left(\omega e^{g \xi}-\nu\right) e^{i l \xi} \exp i\left(k g^{-1} e^{g \xi}\right) . \tag{1.42}
\end{align*}
$$

Changing the integration variable to $z=e^{g \xi}$, we find that these integrals reduce to

$$
\begin{align*}
& \alpha(l, k)=\frac{g^{-1}}{4 \pi \sqrt{\omega \nu}} \int_{0}^{\infty} \frac{d z}{z}(\omega z+\nu) z^{i l g^{-1}} e^{-i k z g^{-1}}  \tag{1.43}\\
& \beta(l, k)=\frac{g^{-1}}{4 \pi \sqrt{\omega \nu}} \int_{0}^{\infty} \frac{d z}{z}(\omega z-\nu) z^{i l g^{-1}} e^{i k z g^{-1}} \tag{1.44}
\end{align*}
$$

Carrying out these integrals by rotating the contour to the imaginary axis, we obtain that

$$
\begin{align*}
& \alpha(l, k)=\left(\frac{g^{-1}}{4 \pi k \sqrt{\omega \nu}}\right)(\omega l+k \nu)\left(k g^{-1}\right)^{-i l g^{-1}} \\
& \times \Gamma\left(i l g^{-1}\right) e^{\pi l / 2 g},  \tag{1.45}\\
& \beta(l, k)=-\alpha(l, k) e^{-\pi l / g}, \tag{1.46}
\end{align*}
$$

where $\Gamma(z)$ is the Gamma function. In fact, the expectation value of the Rindler number operator in the Minkowski vacuum state proves to be a thermal spectrum [7, 8, 11]. That is

$$
\begin{align*}
\left\langle 0_{M}\right| \hat{N}_{l}\left|0_{M}\right\rangle & =\left\langle 0_{M}\right| \hat{b}_{l}^{\dagger} \hat{b}_{l}\left|0_{M}\right\rangle \\
& =\int_{-\infty}^{\infty} d k|\beta(l, k)|^{2} \\
& =\int_{0}^{\infty} \frac{d k}{2 \pi k}\left(\frac{g^{-1}}{\exp \left(2 \pi \nu g^{-1}\right)-1}\right) \tag{1.47}
\end{align*}
$$

(The logarithmic divergence in the above integral is a feature of massless scalar fields in $(1+1)$ dimensions.) Therefore, quantization in the Minkowski and the Rindler coordinates are inequivalent.

As we have mentioned at the beginning of this section, our discussion above has been presented in $(1+1)$ dimensions so as to keep the mathematics simple. We shall now briefly outline as to how the Bogolubov coefficient $\beta$ between Minkowski and Rindler coordinates proves to be nonzero in $(3+1)$ dimensions too.

In flat spacetime and in Minkowski coordinates, the normalized modes of a real, massless scalar field in $(3+1)$ dimensions are given by

$$
\begin{equation*}
u_{\mathbf{k}}(t, \mathbf{x})=\frac{1}{\sqrt{(2 \pi)^{3} 2 \omega}} \exp -i(\omega t-\mathbf{k} \cdot \mathbf{x}) \tag{1.48}
\end{equation*}
$$

where $\omega=|\mathbf{k}|$. The quantized scalar field can be decomposed in terms of these modes and their complex conjugates $u_{\mathrm{k}}^{*}$ as follows:

$$
\begin{equation*}
\Phi(t, \mathbf{x})=\int d^{3} \mathbf{k}\left(\hat{a}_{\mathbf{k}} u_{\mathbf{k}}(t, \mathbf{x})+\hat{a}_{\mathbf{k}}^{\dagger} u_{\mathbf{k}}^{*}(t, \mathbf{x})\right) . \tag{1.49}
\end{equation*}
$$

In $(3+1)$ dimensions, the coordinate transformations (1.20) lead to the following Rindler metric:

$$
\begin{equation*}
d s^{2}=e^{2 g \xi}\left(d \tau^{2}-d \xi^{2}\right)-d y^{2}-d z^{2} \tag{1.50}
\end{equation*}
$$

where we have assumed that the $y$ and the $z$-coordinates remain unchanged. The normalized modes of a massless scalar field in these Rindler coordinates are given by (see, for e.g., ref. [10])

$$
\begin{equation*}
v_{\nu 1_{\perp}}\left(\tau, \xi, \mathbf{x}_{\perp}\right)=\left(\frac{\sinh \left(\pi \nu g^{-1}\right)}{4 \pi^{4} g}\right)^{1 / 2} e^{-i \nu \tau} e^{i 1_{\perp} \cdot \mathbf{x}_{\perp}} K_{i \nu g g^{-1}}\left(l_{\perp} g^{-1} e^{g \xi}\right), \tag{1.51}
\end{equation*}
$$

where $\mathbf{1}_{\perp} \equiv\left(l_{y}, l_{z}\right), \mathbf{x}_{\perp} \equiv(y, z), l_{\perp}=\left|\mathbf{1}_{\perp}\right|$ and $K_{i \nu g^{-1}}$ is the Macdonald function, a Bessel function of imaginary order and argument. These modes and their complex conjugates $v_{\nu 1_{\perp}}^{*}$ form a complete orthonormal basis. Therefore, the quantized scalar field can be decomposed in terms of these normal modes as follows:

$$
\begin{equation*}
\Phi\left(\tau, \xi, \mathbf{x}_{\perp}\right)=\int_{0}^{\infty} d \nu \int d^{2} \mathbf{1}_{\perp}\left(b_{\nu 1_{\perp}} v_{\nu \mathbf{1}_{\perp}}\left(\tau, \xi, \mathbf{x}_{\perp}\right)+b_{\nu 1_{\perp}}^{\dagger} v_{\nu 1_{\perp}}^{*}\left(\tau, \xi, \mathbf{x}_{\perp}\right)\right) . \tag{1.52}
\end{equation*}
$$

The Bogolubov coefficients between the modes (1.48) and (1.52) can evaluated using the scalar product (1.37). Evaluating the scalar product on the $\tau=0$ hypersurface, we obtain that (see, for instance, ref. [9])

$$
\alpha\left(\nu, \mathbf{l}_{\perp}, \mathbf{k}\right)=\left\{2 \pi \omega g^{-1}\left(1-\exp -\left(2 \pi \nu g^{-1}\right)\right)\right\}^{-1 / 2}
$$

$$
\begin{align*}
& \times\left(\frac{w-k_{x}}{l_{\perp}}\right)^{i \nu g^{-1}} \delta_{D}\left(\mathbf{k}_{\perp}-\mathbf{l}_{\perp}\right) \\
& \beta\left(\nu, \mathbf{l}_{\perp}, \mathbf{k}\right)=-\alpha\left(\nu, \mathbf{l}_{\perp}, \mathbf{k}\right) e^{-\pi \nu g^{-1}} \tag{1.53}
\end{align*}
$$

This result then shows that inequivalent quantization in Minkowski and Rindler coordinates is a feature that arises in $(3+1)$ dimensions too. Using the above expression for the Bogolubov coefficient $\beta$, it can be easily shown that the expectation value of the Rindler number operator in the Minkowski vacuum state is a thermal spectrum with a temperature $T=(g / 2 \pi)$, just as it was in the $(1+1)$ dimensional case.

The conventional formulation of quantum field theory in flat spacetime is invariant only the Poincaré group. The Poincaré group is basically a set of linear coordinate transformations. Our discussion above illustrates the fact that under non-linear coordinate transformations, even in flat spacetime, concepts such as vacuum, particles etc. can, in general, prove to be coordinate dependent. In a curved spacetime, the Poincaré group is no longer a symmetry group of the spacetime. Therefore, inequivalent quantization in different coordinates describing the same gravitational background can be expected to arise in curved spacetimes too.

The results regarding the Bogolubov transformations we have presented above are not restricted to flat spacetime alone but apply to complete, orthonormal sets of solutions in curved spacetimes too. Consider a curved spacetime in which more than one timelike Killing vector is available. We can define positive frequency normal modes with respect to these different timelike Killing vectors. If the Bogolubov coefficient $\beta$ proves to be nonzero between any two of these normal modes, then inequivalent quantization, as illustrated in the flat spacetime
example above, will arise and there is bound to be an ambiguity in the definition of particles. Further, in a generic spacetime, in which the metric is explicitly time dependent, a timelike Killing vector may not be available at all. In such a situation, positive frequency modes cannot be defined unambiguously. Thus, we may be faced with either a lack of uniqueness in the particle definition, or it may not be possible to define particles at all [4].

In certain limited cases, however, the particle concept is useful and one can obtain interesting results. Consider a spacetime which is static in the asymptotic past and in the asymptotic future. Then, timelike Killing vector fields are available in the asymptotic domains, but they need not be the same vector. We can define a vacuum state, in the past and a (possibly different) vacuum state in the future, even though a vacuum state cannot be defined in the intermediate times (due to the absence of a Killing vector field). If the quantum field was initially in the vacuum state defined in the asymptotic past, then at late times it will appear as if particles are present in that state. This result is interpreted as production of particles corresponding to the quantum field by the changing geometry of spacetime. The emission of Hawking radiation from a star undergoing gravitational collapse is a famous example of particle creation in a time dependent gravitational background [47, 48, 49, 50, 51, 52, 53, 54].

The phenomenon of particle production is clearly different from the one concerning the presence of the Rindler quanta in the Minkowski vacuum. The latter arises because there is more than one way of defining positive frequency modes in a given spacetime, even though the spacetime itself is static. On the other hand, particles are created in a time dependent metric because the natural definition of positive frequency modes are different at two different times [55, 56].

### 1.2 Particle production in a curved spacetime: a simple example

We shall now discuss a simple model of particle creation in a spacetime that is Minkowskian in the asymptotic past and asymptotic future but is non-static in between. (The example we present here was investigated originally in ref. [57]). The spacetime is a two dimensional Friedmann-Robertson-Walker universe described by the line element

$$
\begin{equation*}
d s^{2}=d t^{2}-a^{2}(t) d x^{2} \tag{1.54}
\end{equation*}
$$

where the spatial sections expand or contract uniformly as described by the scale factor $a(t)$. Introducing a new time parameter (the so called conformal time) defined as

$$
\begin{equation*}
\eta=\int \frac{d t}{a(t)} \tag{1.55}
\end{equation*}
$$

the metric (1.54) can be rewritten in terms of the conformal time $\eta$ as follows:

$$
\begin{align*}
d s^{2} & =a^{2}(\eta)\left(d \eta^{2}-d x^{2}\right) \\
& =C(\eta)\left(d \eta^{2}-d x^{2}\right) \tag{1.56}
\end{align*}
$$

where we have defined the conformal scale factor as: $C(\eta)=a^{2}(\eta)$. This form of the line element is manifestly conformal to the flat spacetime line element in Minkowski coordinates. Suppose that

$$
\begin{equation*}
C(\eta)=A+B \tanh (\rho \eta) \tag{1.57}
\end{equation*}
$$

where $A, B$ and $\rho$ are constants, then in the asymptotic past and the asymptotic future the spacetime becomes Minkowskian since

$$
\begin{equation*}
C(\eta) \rightarrow A \pm B, \quad \text { as } \quad \eta \rightarrow \pm \infty \tag{1.58}
\end{equation*}
$$

Consider a massive, real scalar field described by the action

$$
\begin{equation*}
\mathcal{S}[\Phi]=\int d^{2} x \sqrt{-g} \mathcal{L}(\Phi)=\frac{1}{2} \int d^{2} x \sqrt{-g}\left(g_{\mu \nu} \partial^{\mu} \Phi \partial^{\nu} \Phi-m^{2} \Phi^{2}\right) \tag{1.59}
\end{equation*}
$$

Varying this action with respect to the scalar field $\Phi$, we obtain the equation of motion satisfied by the scalar field to be

$$
\begin{equation*}
\left(\square+m^{2}\right) \Phi \equiv\left(\frac{1}{\sqrt{-g}} \partial_{\mu}\left(\sqrt{-g} g^{\mu \nu} \partial_{\nu}\right)+m^{2}\right) \Phi=0 \tag{1.60}
\end{equation*}
$$

Substituting the metric (1.56) in this equation, we obtain that

$$
\begin{equation*}
\left(\partial_{\eta}^{2}-\partial_{x}^{2}+m^{2} C(\eta)\right) \Phi(\eta, x)=0 \tag{1.61}
\end{equation*}
$$

If we decompose the modes of the scalar field $\Phi$ as

$$
\begin{equation*}
u_{k}(\eta, x) \propto \chi_{k}(\eta) e^{i k x} \tag{1.62}
\end{equation*}
$$

we find that the function $\chi_{k}(\eta)$ satisfies the following differential equation

$$
\begin{equation*}
\frac{d^{2} \chi_{k}}{d \eta^{2}}+\left(k^{2}+m^{2} C(\eta)\right) \chi_{k}=0 . \tag{1.63}
\end{equation*}
$$

For the case of $C(\eta)$ given by (1.57), this differential equation can be solved in terms of hypergeometric functions [58]. The normalized modes which behave as positive frequency Minkowski modes in the asymptotic past (i.e. as $\eta, t \rightarrow-\infty$ ) are

$$
\begin{align*}
& u_{k}^{i n}(\eta, x)= \frac{1}{\sqrt{4 \pi \omega_{i n}}} \exp i\left\{k x-\omega_{+} \eta-\left(\omega_{-} / \rho\right) \ln [2 \cosh (\rho \eta)]\right\} \\
& \times F\left(1+\left(i \omega_{-} / \rho\right), i \omega_{-} / \rho, 1-\left(i \omega_{i n} / \rho\right),[1+\tanh (\rho \eta)] / 2\right) \\
& \xrightarrow{\eta \rightarrow-\infty} \frac{1}{\sqrt{4 \pi \omega_{i n}}} \exp -i\left(\omega_{i n} \eta-k x\right), \tag{1.64}
\end{align*}
$$

where

$$
\left.\begin{array}{l}
\omega_{\text {in }}=\left(k^{2}+m^{2}(A-B)\right)^{1 / 2}  \tag{1.65}\\
\omega_{\text {out }}=\left(k^{2}+m^{2}(A+B)\right)^{1 / 2} \\
\omega_{ \pm}=\frac{1}{2}\left(\omega_{\text {out }} \pm \omega_{\text {in }}\right)
\end{array}\right\}
$$

On the other hand, the modes which behave like positive frequency Minkowski modes in the asymptotic future (i.e. as $\eta, t \rightarrow \infty$ ) are found to be

$$
\begin{align*}
& u_{k}^{\text {out }}(\eta, x)= \frac{1}{\sqrt{4 \pi \omega_{\text {out }}}} \exp i\left\{k x-\omega_{+} \eta-\left(\omega_{-} / \rho\right) \ln [2 \cosh (\rho \eta)]\right\} \\
& \times F\left(1+\left(i \omega_{-} / \rho\right), i \omega_{-} / \rho, 1+\left(i \omega_{\text {out }} / \rho\right),[1-\tanh (\rho \eta)] / 2\right) \\
& \xrightarrow{\eta \rightarrow \infty} \frac{1}{\sqrt{4 \pi \omega_{\text {out }}}} \exp -i\left(\omega_{\text {out }} \eta-k x\right) . \tag{1.66}
\end{align*}
$$

Clearly, $u_{k}^{i n}$ and $u_{k}^{o u t}$ are not equal which means that the Bogolubov coefficient $\beta$ relating these two modes must be non-vanishing. To see this explicitly we can use the linear transformation properties of hypergeometric functions (see, for instance, ref. [59], p. 559) to write $u_{k}^{i n}$ in terms of $u_{k}^{o u t}$ as

$$
\begin{equation*}
u_{k}^{i n}(\eta, x)=\alpha(k) u_{k}^{\text {out }}(\eta, x)+\beta(k) u_{-k}^{\text {out* }}(\eta, x), \tag{1.67}
\end{equation*}
$$

where

$$
\begin{align*}
& \alpha(k)=\left(\frac{\omega_{\text {out }}}{\omega_{\text {in }}}\right)^{1 / 2}\left(\frac{\Gamma\left[1-\left(i \omega_{\text {in }} / \rho\right)\right] \Gamma\left(-i \omega_{\text {out }} / \rho\right)}{\Gamma\left(-i \omega_{+} / \rho\right) \Gamma\left[1-\left(i \omega_{+} / \rho\right)\right]}\right)  \tag{1.68}\\
& \beta(k)=\left(\frac{\omega_{\text {out }}}{\omega_{\text {in }}}\right)^{1 / 2}\left(\frac{\Gamma\left[1-\left(i \omega_{\text {in }} / \rho\right)\right] \Gamma\left(i \omega_{\text {out }} / \rho\right)}{\Gamma\left(i \omega_{-} / \rho\right) \Gamma\left[1+\left(i \omega_{-} / \rho\right)\right]}\right) \tag{1.69}
\end{align*}
$$

and $\Gamma(z)$ represents the Gamma function. Comparision of equation (1.67) with equation (1.35) reveals that the Bogolubov coefficients are given by

$$
\begin{equation*}
\alpha\left(k, k^{\prime}\right)=\alpha(k) \delta_{D}\left(k-k^{\prime}\right) \quad \text { and } \quad \beta\left(k, k^{\prime}\right)=\beta(k) \delta_{D}\left(k+k^{\prime}\right) \tag{1.70}
\end{equation*}
$$

From these two equations one obtains

$$
\begin{align*}
|\alpha(k)|^{2} & =\left(\frac{\sinh ^{2}\left(\pi \omega_{+} / \rho\right)}{\sinh \left(\pi \omega_{\text {in }} / \rho\right) \sinh \left(\pi \omega_{\text {out }} / \rho\right)}\right)  \tag{1.71}\\
|\beta(k)|^{2} & =\left(\frac{\sinh ^{2}\left(\pi \omega_{-} / \rho\right)}{\sinh \left(\pi \omega_{\text {in }} / \rho\right) \sinh \left(\pi \omega_{\text {out }} / \rho\right)}\right) \tag{1.72}
\end{align*}
$$

from which the normalization condition

$$
\begin{equation*}
|\alpha(k)|^{2}-|\beta(k)|^{2}=1, \tag{1.73}
\end{equation*}
$$

follows immediately.
Consider the case when the field is assumed to be in the $i n$-vacuum $\left|0_{i n}\right\rangle$ as defined by a Minkowski observer as $\eta, t \rightarrow-\infty$. As the spacetime expands and reaches the asymptotic future, i.e. as $\eta, t \rightarrow \infty$, the field is still in the state $\left|0_{i n}\right\rangle$ (we are working in the Heisenberg picture). However, the Minkowski observer in the out-region defines a different state $\left|0_{\text {out }}\right\rangle$ as the vacuum state and finds that the state $\left|0_{i n}\right\rangle$ is populated with $|\beta(k)|^{2}$ (given by equation (1.72)) number of particles, as defined by her. It is in this sense, we say that particle production has taken place. However it is not meaningful to ask whether or not these particles were created during expansion, because the particle concept is not well defined in the intermediate times.

### 1.3 Concept of a detector

In a general curved spacetime, the particle concept is ambiguous. When formal methods, such as the canonical quantization procedure, lead to coordinate dependence of the particle concept, we can ask whether there exists an operational prescription of defining a particle which can help us resolve this ambiguity. One
such prescription would be to study the behavior of a measuring apparatus which interacts with the quantum field and can possibly respond to the particle content of the quantum field. After all, particles are what particle detectors are designed to detect [60]. The response of a particle detector in motion on a certain trajectory in the spacetime of our interest should then reflect the particle content of the quantum field in that spacetime. These motivations for an operational definition of the particle concept led to the idea of a detector.

In classical physics, if one wants to measure the strength of a field, say an electric field, it could be done by placing a charge in the field and by measuring the response of the charge, viz. its acceleration. Alternatively, one can measure the energy gained by a harmonically bound charge kept in an external electric field, and the energy gained by the charge will be proportional to the power spectrum of the field, evaluated at the frequency of the oscillator. The simplest analogue of this detection process in quantum mechanics would be an atom kept in an external quantized electric field and the rate of transition of the atom to the excited levels will then reflect the expectation value of the field.

Therefore, by a detector we have in mind a mathematical model involving a point like object which can be described by a classical worldline, but which nevertheless possesses internal degrees of freedom having a quantum description provided by energy levels. Such model detectors can essentially be described by the interaction Lagrangian for the coupling between the internal degrees of freedom of the detector and the quantum field. The worldline of the detector is assumed to be prescribed a priori; it is not considered to be a part of the dynamics. The detector is usually set in its ground state and the probability that as a result of its interaction with the quantum field, it will eventually be found in an excited state
is examined. Also, to qualify as a realistic detector, the detector, when it is on an inertial trajectory in flat spacetime, is not expected to respond in the Minkowski vacuum.

The response of such a detector in an arbitrary spacetime would, in general, depend on the following three elements: (i) the nature of the coupling between the detector and the field, (ii) the motion of the detector and (iii) the state of the quantum field. The simplest of the different possible detectors is the detector due to Unruh and DeWitt [11, 12]. In the following three subsections, we introduce the Unruh-DeWitt detector and analyze its response in flat as well as curved spacetimes.

### 1.3.1 The Unruh-DeWitt detector

The Unruh-DeWitt detector consists of an idealized point particle with internal energy levels labeled by the energy $E$ and coupled to the quantum field by a monopole interaction. Suppose the Unruh-DeWitt detector moves along the worldline described by the functions $x^{\mu}(\tau)$, where $\tau$ is the proper time as measured by the clock in the detector's frame. The interaction of the Unruh-DeWitt detector with a scalar field $\Phi$ is described by the interaction Lagrangian

$$
\begin{equation*}
\mathcal{L}_{i n t}(x(\tau))=c m(\tau) \Phi[x(\tau)], \tag{1.74}
\end{equation*}
$$

where $c$ is a small coupling constant and $m(\tau)$ is the detector's monopole operator.

Consider a Unruh-DeWitt detector that is assumed to be in its ground state $\left|E_{0}\right\rangle$ and is set in motion on an arbitrary trajectory in a particular spacetime. This detector, in general, will not remain in its ground state but will undergo a transition to an excited state $|E\rangle$ due to its interaction with the scalar field. The
amplitude for its transition to the excited state $|E\rangle$ will be given by

$$
\begin{equation*}
\mathcal{A}\left(E, E_{0}\right)=\langle E| \otimes\left\langle\Psi_{f}\right| T\left\{\exp i c\left(\int_{-\infty}^{\infty} d \tau m(\tau) \Phi[x(\tau)]\right)\right\}\left|\Psi_{i}\right\rangle \otimes\left|E_{0}\right\rangle \tag{1.75}
\end{equation*}
$$

where $T$ is the time ordering operator, $\left|\Psi_{i}\right\rangle$ is the initial state of the quantum field and $\left|\Psi_{f}\right\rangle$ is the state of the quantum field after its interaction with the detector. If we assume that the coupling constant $c$ is very small, then the transition amplitude can be approximated by the first order perturbation theory as follows:

$$
\begin{equation*}
\mathcal{A}\left(E, E_{0}\right)=i c\langle E| \otimes\left\langle\Psi_{f}\right|\left\{\int_{-\infty}^{\infty} d \tau m(\tau) \Phi[x(\tau)]\right\}\left|\Psi_{i}\right\rangle \otimes\left|E_{0}\right\rangle . \tag{1.76}
\end{equation*}
$$

If the time evolution of $m(\tau)$ is assumed to be

$$
\begin{equation*}
m(\tau)=e^{i H_{0} \tau} m(0) e^{-i H_{0} \tau} \tag{1.77}
\end{equation*}
$$

where $H_{0}$ is the Hamiltonian of the detector so that $H_{0}|E\rangle=E|E\rangle$ and $H_{0}\left|E_{0}\right\rangle=$ $E_{0}\left|E_{0}\right\rangle$, then the transition amplitude is given by

$$
\begin{equation*}
\mathcal{A}(\Omega)=\mathcal{M} \int_{-\infty}^{\infty} d \tau e^{i \Omega \tau}\left\langle\Psi_{f}\right| \Phi[x(\tau)]\left|\Psi_{i}\right\rangle \tag{1.78}
\end{equation*}
$$

where $\Omega=\left(E-E_{0}\right)$ and

$$
\begin{equation*}
\mathcal{M}=i c\langle E| m(0)\left|E_{0}\right\rangle \tag{1.79}
\end{equation*}
$$

We shall now examine whether the Unruh-DeWitt satisfies the demand we had made earlier for a detector to be realistic, viz. that the detector should not respond in the Minkowski vacuum state when it is on an inertial trajectory in flat spacetime. Earlier, in subsection 1.1.3, we had seen that, in flat spacetime and in $(3+1)$ dimensions, the quantized scalar field $\Phi$ can be decomposed in terms of the Minkowski normal modes as follows (cf. equations (1.48) and (1.49)):

$$
\begin{equation*}
\Phi(t, \mathbf{x})=\int \frac{d^{3} \mathbf{k}}{\sqrt{(2 \pi)^{3} 2 \omega}}\left(\hat{a}_{\mathbf{k}} e^{-i(\omega t-\mathbf{k} \cdot \mathbf{x})}+\hat{a}_{\mathbf{k}}^{\dagger} e^{i(\omega t-\mathbf{k} \cdot \mathbf{x})}\right) \tag{1.80}
\end{equation*}
$$

where $\omega=|\mathbf{k}|$. If we now assume that the initial state $\left|\Psi_{i}\right\rangle$ of the quantum field is the Minkowski vacuum state $\left|0_{M}\right\rangle$, then it is clear from the expression for the transition amplitude (1.78) that transitions can take place only to the one-particle state of the quantized scalar field, i.e. for $\left|\Psi_{f}\right\rangle=\left|1_{\mathbf{k}}\right\rangle$. Then

$$
\begin{equation*}
\left\langle\Psi_{f}\right| \Phi[x(\tau)]\left|0_{M}\right\rangle=\int \frac{d^{3} \mathbf{k}}{\sqrt{(2 \pi)^{3} 2 \omega}} \exp -i(\omega t-\mathbf{k} \cdot \mathbf{x}) . \tag{1.81}
\end{equation*}
$$

We must now take into account the fact that x is not an independent variable but is determined by the detector's trajectory. Let us assume that the detector follows an inertial world line, i.e.

$$
\begin{equation*}
\mathbf{x}(\tau)=\mathbf{x}_{0}+\mathbf{v} t(\tau)=\mathbf{x}_{0}+\mathbf{v} \tau\left(1-v^{2}\right)^{-1 / 2} \tag{1.82}
\end{equation*}
$$

where $\mathrm{x}_{0}$ and $\mathbf{v}$ are constants and $|\mathbf{v}|<1$. For such a situation the transition amplitude (1.78) is proportional to a Dirac delta function, i.e. we obtain that

$$
\begin{equation*}
\mathcal{A}_{\text {ine }}(\Omega)=\frac{\mathcal{M}}{\sqrt{4 \pi \omega}} e^{-i \mathbf{k} \cdot \mathbf{x}_{0}} \delta_{D}\left(\Omega+(\omega-\mathbf{k} \cdot \mathbf{v})\left(1-\mathbf{v}^{2}\right)^{-1 / 2}\right)=0 \tag{1.83}
\end{equation*}
$$

The last equality in the above equation follows from noting that since k.v $\leq$ $|\mathbf{k}||\mathbf{v}|<\omega$ and $\Omega>0$; the argument of the delta function is always greater than zero. The transition in the detector is essentially forbidden on the grounds of energy conservation which is a direct consequence of Poincaré invariance. The Unruh-DeWitt detector does not respond in the Minkowski vacuum state when in inertial motion in flat spacetime and therefore satisfies the demand we had made of realistic detectors.

If, on the other hand, instead of an inertial trajectory and the Minkowski vacuum state, we had chosen a more complicated trajectory and an arbitrary initial state $\left|\Psi_{i}\right\rangle$, the integral (1.78) would not have yielded a delta function and the result would, in general, be nonzero. In such a case, it is of interest to calculate
the transition probability to all possible final states $\left|\Psi_{f}\right\rangle$ of the quantum field. This can be obtained by squaring the modulus of the transition amplitude and then summing over the complete set of final states $\left|\Psi_{f}\right\rangle$. The transition probability can then be expressed in a more formal and concise manner as follows:

$$
\begin{equation*}
\mathcal{P}(\Omega)=\sum_{\left|\Psi_{f}\right\rangle}|\mathcal{A}(\Omega)|^{2}=|\mathcal{M}|^{2} \mathcal{F}(\Omega), \tag{1.84}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{F}(\Omega)=\int_{-\infty}^{\infty} d \tau \int_{-\infty}^{\infty} d \tau^{\prime} e^{-i \Omega\left(\tau-\tau^{\prime}\right)} G^{+}\left[x(\tau), x\left(\tau^{\prime}\right)\right] \tag{1.85}
\end{equation*}
$$

(Since the quantity $|\mathcal{M}|^{2}$ depends only on the internal structure of the detector and not on its motion, we will hereafter drop this term and concentrate on the detector response function $\mathcal{F}(\Omega)$.) The detector response function $\mathcal{F}(\Omega)$ is independent of the details of the detector and is determined completely by the Wightman function $G^{+}\left[x(\tau), x\left(\tau^{\prime}\right)\right]$ which is defined to be

$$
\begin{equation*}
G^{+}\left[x(\tau), x\left(\tau^{\prime}\right)\right]=\left\langle\Psi_{i}\right| \Phi(x) \Phi\left(x^{\prime}\right)\left|\Psi_{i}\right\rangle . \tag{1.86}
\end{equation*}
$$

For trajectories in flat spacetime which are integral curves of timelike Killing vector fields, for e.g. the inertial and the accelerated trajectories, the Wightman function corresponding to the Minkowski vacuum state is invariant under time translations in the reference frame of the detector [10]. Hence

$$
\begin{align*}
G^{+}\left[x(\tau), x\left(\tau^{\prime}\right)\right] & =\left\langle 0_{M}\right| \Phi(x) \Phi\left(x^{\prime}\right)\left|0_{M}\right\rangle \\
& =G^{+}\left(\tau-\tau^{\prime}\right) \\
& =G^{+}(\Delta \tau) \tag{1.87}
\end{align*}
$$

and the double integration in (1.85) for such a Wightman function reduces to a Fourier transform of the Wightman function multiplied by an infinite time interval. The transition probability is divergent, simply because the detector is kept
switched on for an infinite time interval. Such a divergent integral is frequently encountered in quantum theory, like, for instance, when transition probabilities are evaluated in time dependent perturbation theory using Fermi's golden rule [61]. This divergence is usually handled by concentrating on the transition probability rate rather than on the transition probability itself. We can, therefore, interpret the Fourier transform of the Wightman function as the probability of transition per unit time of the detector. That is, the transition probability rate of the detector is described by the following integral:

$$
\begin{equation*}
\mathcal{R}(\Omega)=\int_{-\infty}^{\infty} d \Delta \tau e^{-i \Omega \Delta \tau} G^{+}(\Delta \tau) . \tag{1.88}
\end{equation*}
$$

### 1.3.2 Inertial and uniformly accelerated Unruh-DeWitt detectors in flat spacetime

Let us now evaluate the transition probability rate of inertial and uniformly accelerated Unruh-DeWitt detectors in flat spacetime. In this subsection, we shall assume the initial state of the quantum field to be the Minkowski vacuum state, i.e. $\left|\Psi_{i}\right\rangle=\left|0_{M}\right\rangle$ and we shall work in $(3+1)$ dimensions.

The Wightman function for a massless scalar field in $(3+1)$ dimensions in the Minkowski vacuum state is given by (see, for instance, ref. [1], pp. 52-53)

$$
\begin{align*}
G^{+}\left(x, x^{\prime}\right) & =\left\langle 0_{M}\right| \Phi(x) \Phi\left(x^{\prime}\right)\left|0_{M}\right\rangle \\
& =\frac{-1}{4 \pi^{2}\left(\left(t-t^{\prime}-i \epsilon\right)^{2}-\left|\mathrm{x}-\mathrm{x}^{\prime}\right|^{2}\right)} \tag{1.89}
\end{align*}
$$

where $\epsilon \rightarrow 0^{+}$.

## Transition probability rate of an inertial detector

For the case of an inertial trajectory in flat spacetime (cf. equation (1.82)) the Wightman function in the Minkowski vacuum state (1.89) reduces to

$$
\begin{equation*}
G_{i n e}^{+}(\Delta \tau)=\frac{-1}{4 \pi^{2}(\Delta \tau-i \epsilon)^{2}}, \tag{1.90}
\end{equation*}
$$

where, we have absorbed a positive factor $\left(1-v^{2}\right)^{-1 / 2}$ into $\epsilon$. Substituting this Wightman function in equation (1.88), we find that the rate of transition probability is described by the integral

$$
\begin{equation*}
\mathcal{R}_{i n e}(\Omega)=-\frac{1}{4 \pi^{2}} \int_{-\infty}^{\infty} d \Delta \tau\left(\frac{e^{-i \Omega \Delta \tau}}{(\Delta \tau-i \epsilon)^{2}}\right) . \tag{1.91}
\end{equation*}
$$

Since $\Omega>0$, this integral can be performed with the aid of an infinite semicircular contour in the lower half of the complex $\Delta \tau$-plane. Since the pole of the two point function (1.90) is at $\Delta \tau=i \epsilon$, it does not contribute to the integral and the detector response is zero. In other words an inertial detector does not respond in the Minkowski vacuum, the conclusion we had reached earlier by analyzing the transition amplitude of the inertial detector.

## Transition probability rate of a uniformly accelerated detector

Now, consider the following transformations of the Minkowski coordinates [39, 40]

$$
\begin{equation*}
t=\xi \sinh (g \tau) \quad ; \quad x=\xi \cosh (g \tau) \quad ; \quad y=y \quad \text { and } \quad z=z, \tag{1.92}
\end{equation*}
$$

where $g$ is a constant. (Note that these transformations correspond to choosing $\lambda=\xi^{-1}$ and $s=(g \xi \tau)$ in equation (1.27).) In terms of the Rindler coordinates ( $\tau, \xi, y, z$ ), the line element in flat spacetime reduces to

$$
\begin{equation*}
d s^{2}=g^{2} \xi^{2} d \tau^{2}-d \xi^{2}-d y^{2}-d z^{2} . \tag{1.93}
\end{equation*}
$$

The Rindler coordinates cover only the right quarter of flat spacetime which corresponds to the region $x>|t|$ in the $t-x$ plane. From equation (1.92) it can easily seen that

$$
\begin{equation*}
x^{2}-t^{2}=\xi^{2} \quad \text { and } \quad \tanh (g \tau)=(t / x) . \tag{1.94}
\end{equation*}
$$

These relations then imply that curves of constant $\tau$ are straight lines passing through the origin, while curves of constant $\xi$ are hyperbolae in the $t$-x plane. As we had noted in subsection 1.1.2, each of these hyperbolae then represent the spacetime trajectory of an observer who is accelerating uniformly along the $x$-axis with a proper acceleration $\xi^{-1}$.

The Wightman function corresponding to such a uniformly accelerated observer is obtained by substituting the Rindler transformations (1.92) in equation (1.89). The result is

$$
\begin{equation*}
G_{a c c}^{+}(\Delta \tau)=-\left\{16 \pi^{2} g^{-2} \sinh ^{2}(g \Delta \tau / 2-i \epsilon)\right\}^{-1} \tag{1.95}
\end{equation*}
$$

where, without any loss of generality, we have set $\xi=g^{-1}$. Using the expansion

$$
\begin{equation*}
\operatorname{cosec}^{2} \pi x=\pi^{-2} \sum_{n=-\infty}^{\infty}(x-n)^{-2} \tag{1.96}
\end{equation*}
$$

we can express (1.95) as

$$
\begin{equation*}
G_{a c c}^{+}(\Delta \tau)=-\frac{1}{4 \pi^{2}} \sum_{n=-\infty}^{\infty}\left(\Delta \tau-i \epsilon+2 \pi i n g^{-1}\right)^{-2} \tag{1.97}
\end{equation*}
$$

Substituting (1.97) into (1.88) we obtain that

$$
\begin{equation*}
\mathcal{R}_{a c c}(\Omega)=-\frac{1}{4 \pi^{2}} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d \tau\left(\frac{e^{-i \Omega \Delta \tau}}{\left(\Delta \tau-i \epsilon+2 \pi i n g^{-1}\right)^{2}}\right) . \tag{1.98}
\end{equation*}
$$

This integral can be performed on an infinite semicircular contour in the lower half of the complex $\Delta \tau$-plane. The poles in the lower half of the complex $\Delta \tau$-plane
contribute to the integral and we obtain the transition probability rate of the uniformly accelerated detector to be

$$
\begin{equation*}
\mathcal{R}_{a c c}(\Omega)=\frac{1}{2 \pi} \frac{\Omega}{\left(e^{2 \pi g}-1\right)}, \tag{1.99}
\end{equation*}
$$

which is a thermal spectrum with a temperature $T=(g / 2 \pi)$ (we have set the Boltzmann's constant to unity).

Earlier, in subsection 1.1.3, we had found that the expectation value of the Rindler number operator in the Minkowski vacuum state was a thermal distribution. We had also noted that the Rindler transformations correspond to trajectories of uniformly accelerated observers. We now find that the response of a uniformly accelerated Unruh-DeWitt detector in the Minkowski vacuum state is a thermal spectrum. From the concurrence of these two results, we may be tempted to conclude that the uniformly accelerating Unruh-DeWitt detector is detecting the Rindler particles in the Minkowski vacuum state and hence is a particle detector. But this reasoning would be incorrect, simply because there exists a clear counter example to such a reasoning. If the canonical quantization is carried out in a uniformly rotating coordinate system in flat spacetime, the expectation value of the rotational number operator in the Minkowski vacuum state turns out to be zero; whereas the response of a uniformly rotating UnruhDeWitt detector proves to be nonzero [9, 10]. Therefore, we must not think of the Unruh-DeWitt detector as detecting particles. In fact, the transition probability rate of the Unruh-DeWitt detector is proportional the power spectrum of the two point correlation function of the quantum field. Hence, we should think of the Unruh-DeWitt detector as a 'fluctuometer' rather than as a particle detector.

Here, we have evaluated the response of inertial and uniformly accelerated

Unruh-DeWitt detectors assuming that the quantum field is in the Minkowski vacuum state. The response of these detectors in $n$-particle and coherent states of the quantum field have also been analyzed in literature [62].

### 1.3.3 Unruh-DeWitt detectors in Schwarzschild and deSitter spacetimes

We shall now evaluate the transition probability rates of Unruh-Dewitt detectors that are stationed at a constant radius in Schwarzschild and de-Sitter spacetimes. Since the normal modes for the Schwarzschild and de-Sitter spacetimes are not known in a closed form in $(3+1)$ dimensions, we shall carry out our analysis of the detector response in $(1+1)$ dimensions. The quantum field we shall consider here is a massless, real scalar field.

The Schwarzschild spacetime in $(1+1)$ dimensions is described by the line element (see, for instance, ref. [42], section 100)

$$
\begin{equation*}
d s^{2}=\left(1-\frac{2 M}{r}\right) d t^{2}-\left(1-\frac{2 M}{r}\right)^{-1} d r^{2} \tag{1.100}
\end{equation*}
$$

Under the transformation (cf. ref. [63], equation (25.31))

$$
\begin{equation*}
r^{*}=r+2 M \ln \left(\frac{r}{2 M}-1\right), \tag{1.101}
\end{equation*}
$$

the Schwarzschild line element reduces to the Regge-Wheeler metric given by

$$
\begin{equation*}
d s^{2}=\left(1-\frac{2 M}{r}\right)\left(d t^{2}-d r^{* 2}\right) \tag{1.102}
\end{equation*}
$$

The Kruskal-Szekeres (KS, hereafter) coordinate system is related to the ReggeWheeler (RW, hereafter) coordinate system by the following transformations (see, for e.g., ref. [63], section 31.4)

$$
\begin{equation*}
v=e^{r^{*} / 4 M} \sinh (t / 4 M) \quad ; \quad u=e^{r^{*} / 4 M} \cosh (t / 4 M) \tag{1.103}
\end{equation*}
$$

and the line element in the KS coordinate system is given by

$$
\begin{equation*}
d s^{2}=\left(\frac{32 M^{3}}{r}\right) e^{-r / 2 M}\left(d v^{2}-d u^{2}\right) . \tag{1.104}
\end{equation*}
$$

The metrics in the RW and the KS coordinate systems, given by equations (1.102) and (1.104), respectively, are conformally related to the flat space metric. Since the action for a massless scalar field in $(1+1)$ dimensions is conformally invariant, the normal modes of the scalar field in these coordinates are just plane waves.

We can define a vacuum state with respect to KS time coordinate $v$ and study the response of a detector stationed at a constant $r^{*}$ in the RW coordinate system [41]. It is easy to see from equation (1.103) that the curves of constant $r^{*}$ are hyperbolae in the $v-u$ plane of the KS coordinates. Hence they are similar in form to the trajectories of a uniformly accelerated observer in the Minkowski $t-x$ plane. It turns out that the response of an Unruh-Dewitt detector stationed at constant $r^{*}$ in the vacuum state defined with respect to the KS time coordinate $v$ is exactly similar to the response of an accelerated detector in the Minkowski vacuum (see, for instance, ref. [1], section 8.3; also see ref. [14]). This well known result can be obtained as follows.

The Wightman function for a massless scalar field in $(1+1)$ dimensions in the KS coordinate system is given by (cf. ref. [1], section 8.3)

$$
\begin{equation*}
G^{+}\left(x, x^{\prime}\right)=-\frac{1}{4 \pi} \ln \left\{\left|\left(v-v^{\prime}-i \epsilon\right)^{2}-\left(u-u^{\prime}\right)^{2}\right|\right\} . \tag{1.105}
\end{equation*}
$$

For an observer stationed at a constant $r^{*}$, when the transformations (1.103) are substituted in the Wightman function (1.105), it reduces to

$$
\begin{equation*}
G^{+}\left(x, x^{\prime}\right)=-\frac{1}{2 \pi} \ln \left\{\left|2 e^{r^{*} / M} \sinh \left((1-2 M / r)^{-1 / 2} \Delta \tau / 8 M-i \epsilon\right)\right|\right\} \tag{1.106}
\end{equation*}
$$

where $\tau$ is the proper time in the detector's frame and $r$ is related to $r^{*}$ by (1.101). The proper time $\tau$ in the frame of the detector is related to the Schwarzschild
coordinate $t$ as follows: $\tau=(1-2 M / r)^{1 / 2} t$. Since the above Wightman function is invariant with respect to translations in the detector's proper time $\tau$ we can define the Fourier transform of Wightman function to be the transition probability rate of a detector stationed at a constant $r^{*}$. Let us now assume that the detector is stationed at $r^{*}=\infty$ (i.e. $r=\infty$ ). Then, the detector's proper time is the same as the Schwarzschild time $t$. Substituting the KS Wightman function for an observer at $r^{*}=\infty$ in the integral for the transition probability rate (1.88) we obtain that

$$
\begin{equation*}
\mathcal{R}(\Omega)=-\frac{1}{2 \pi} \int_{-\infty}^{\infty} d \Delta t e^{-i \Omega \Delta t} \ln \left\{\left|2 e^{r^{*} / M} \sinh \left(\frac{\Delta t}{8 M}-i \epsilon\right)\right|\right\} \tag{1.107}
\end{equation*}
$$

Integrating this expression twice by parts, we find that it reduces to the following integral (see, for instance, ref. [64], section 4.4):

$$
\begin{equation*}
\mathcal{R}(\Omega)=-\frac{1}{2 \pi} \int_{-\infty}^{\infty} d \Delta t e^{-i \Omega \Delta t}\left\{8 M \Omega \sinh \left(\frac{\Delta t}{8 M}-i \epsilon\right)\right\}^{-2} \tag{1.108}
\end{equation*}
$$

which is the integral we had dealt with in the last subsection. The result is a thermal spectrum with a temperature $T=(1 / 8 \pi M)$, i.e. [48]

$$
\begin{equation*}
\mathcal{R}(\Omega) \propto \frac{1}{\Omega\left(e^{8 \pi M \Omega}-1\right)} \tag{1.109}
\end{equation*}
$$

For a detector stationed at a finite $r(>2 M)$, its transition probability rate again proves to be a thermal spectrum with a temperature that is related to the temperature measured at $r=\infty$ by the corresponding red-shift factor (see, for instance, ref. [63], section 25.4)

$$
\begin{equation*}
T(r)=(8 \pi M \sqrt{1-2 M / r})^{-1} \tag{1.110}
\end{equation*}
$$

(We have set the Boltzmann's constant to be unity.)

A similar analysis can be carried out for the case of the de-Sitter spacetime. In $(1+1)$ dimensions, de-Sitter spacetime is described by line element $[65,66]$

$$
\begin{equation*}
d s^{2}=\left(1-H^{2} r^{2}\right) d t^{2}-\left(1-H^{2} r^{2}\right)^{-1} d r^{2} \tag{1.111}
\end{equation*}
$$

where $H$ is a constant. Defining a new coordinate $r^{*}$ which is related to $r$ as follows:

$$
\begin{equation*}
r^{*}=\frac{1}{2 H} \ln \left\{\left|\frac{1+H r}{1-H r}\right|\right\}, \tag{1.112}
\end{equation*}
$$

we find that the de-Sitter spacetime in terms of the new coordinate $r^{*}$ is described by the line element

$$
\begin{equation*}
d s^{2}=\left(1-H^{2} r^{2}\right)\left(d t^{2}-d r^{* 2}\right) . \tag{1.113}
\end{equation*}
$$

This metric is conformal to the flat space metric. Performing the following transformations

$$
\begin{equation*}
v=e^{H r^{*}} \sinh (H t) \quad \text { and } \quad u=e^{H r^{*}} \cosh (H t), \tag{1.114}
\end{equation*}
$$

we find that the line element (1.111) reduces to

$$
\begin{equation*}
d s^{2}=H^{-2}(1-H r)^{2}\left(d v^{2}-d u^{2}\right) . \tag{1.115}
\end{equation*}
$$

Just as constant $r^{*}$ trajectories in KS coordinate system and the uniformly accelerated trajectories in the Minkowski $t-x$ plane are hyperbolae, constant $r^{*}$ trajectories in de-Sitter spacetime are also hyperbolae in the $v$ - $u$ plane. The study of the response of a detector stationed at constant $r^{*}$ in a vacuum defined with respect to the time coordinate $v$ in the de-Sitter spacetime is hence similar to the study of the detector response in the Schwarzschild spacetime discussed above. For a detector that is stationed at $r=0$ and is kept on for an infinite time interval we obtain a thermal response with a temperature $T=(H / 2 \pi)$ (see refs. [67, 68]; for a different derivation, see ref. [69], section 9.4). The temperature as measured
by detectors stationed at a nonzero $r\left(r<H^{-1}\right)$ is related to the temperature as measured at $r=0$ by the corresponding red-shift factor (see, for e.g., ref. [63], section 25.4), i.e.

$$
\begin{equation*}
T(r)=\left(2 \pi H^{-1} \sqrt{1-H^{2} r^{2}}\right)^{-1}, \tag{1.116}
\end{equation*}
$$

where, as before, we have set the Boltzmann's constant to unity.

We have restricted our discussion in this section to the response of the Unruh-DeWitt detectors. As we have mentioned earlier, the Unruh-DeWitt detector is coupled to the quantum field through a monopole coupling. Detectors can be coupled to the quantum field in different ways. The response of detectors that are coupled to the quantum field through a derivative coupling as well the response of detectors that are coupled to the energy-momentum tensor of quantum field have been studied in literature [70, 71, 72]. In general, the response of these detectors turns out to be different from the response of the Unruh-DeWitt detector.

### 1.4 Pair production in a constant electric field background

In the last three sections, we have been discussing the behavior of quantum fields in classical gravitational backgrounds. In particular, we have been interested in the following aspects of quantum fields in curved spacetimes: (i) the concept of a particle and (ii) the phenomenon of particle production. We find that in a curved spacetime, the particle concept, in general, proves to be coordinate dependent. This feature is a hurdle that will have to be overcome if we are to provide a covariant description of the phenomenon of particle production.

As we have mentioned at the beginning of this chapter, phenomena such vacuum polarization and particle production take place in classical electromagnetic backgrounds too. Just as the evolution of a quantum field in a particular spacetime can be studied in different coordinate systems, its evolution in a given electromagnetic background can be analyzed in different gauges related by gauge transformations. The evolution of quantum fields in classical electromagnetic backgrounds has been studied in literature with the hope that such a study will offer some insight to understand the gravitational case better [33, 38]. In this section and the next we shall study the evolution of a quantum field in a constant electromagnetic background by the method of normal mode analysis and the effective Lagrangian approach, respectively.

The system we shall consider consists of a complex scalar field $\Phi$ interacting with the electromagnetic field represented by the vector potential $A^{\mu}$. It is described by the action (see, for e.g., ref. [73], p. 98)

$$
\begin{align*}
\mathcal{S}\left[\Phi, A^{\mu}\right]= & \int d^{4} x \mathcal{L}\left(\Phi, A^{\mu}\right) \\
= & \int d^{4} x\left\{\left(\partial_{\mu} \Phi+i q A_{\mu} \Phi\right)\left(\partial^{\mu} \Phi^{*}-i q A^{\mu} \Phi^{*}\right)\right. \\
& \left.-m^{2} \Phi \Phi^{*}-\frac{1}{4} F^{\mu \nu} F_{\mu \nu}\right\}, \tag{1.117}
\end{align*}
$$

where $q$ and $m$ are the charge and the mass associated with a single quantum of the complex scalar field, the asterisk, as usual, denotes complex conjugation and

$$
\begin{equation*}
F_{\mu \nu}=\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu} . \tag{1.118}
\end{equation*}
$$

We shall assume that the electromagnetic field behaves classically, hence $A^{\mu}$ is just a $c$-number, while we shall assume the complex scalar field to be a quantum field so that $\Phi$ is an operator valued distribution. Varying the action (1.117)
with respect to the complex scalar field $\Phi$, we obtain the following Klein-Gordon equation:

$$
\begin{equation*}
\left(\left(\partial_{\mu}+i q A_{\mu}\right)\left(\partial^{\mu}+i q A^{\mu}\right)+m^{2}\right) \Phi=0 \tag{1.119}
\end{equation*}
$$

The electromagnetic background we shall consider in this section is a constant electric field described by field vector $\mathbf{E}=E \hat{\mathbf{x}}$, where $E$ is a constant and $\hat{\mathrm{x}}$ is the unit vector along the positive $x$-axis. This electromagnetic background can be described by either the time dependent vector potential

$$
\begin{equation*}
A_{1}^{\mu}=(0,-E t, 0,0) \tag{1.120}
\end{equation*}
$$

or the space dependent one

$$
\begin{equation*}
A_{2}^{\mu}=(-E x, 0,0,0) . \tag{1.121}
\end{equation*}
$$

In the following two subsections we shall illustrate how the creation of particles corresponding to the quantum field $\Phi$ is described in the two gauges $A_{1}^{\mu}$ and $A_{2}^{\mu}$.

### 1.4.1 Quantization in the time dependent gauge: Bogolubov coefficients

Let us begin by quantizing the complex scalar field $\Phi$ in the time dependent gauge $A_{1}^{\mu}[38,74,75,76]$. In the case of the time dependent gravitational example we had discussed earlier in section 1.2, the metric we had considered, viz. (1.56), was Minkowskian in the asymptotic past as well as in the asymptotic future. Because of this feature we were able to define a particle in the asymptotic domains unambiguously. But the vector potential $A_{1}^{\mu}$ representing the constant electric field has a time dependence without these asymptotic features and hence it is not easy to provide a particle interpretation. The usual strategy adopted in such
cases is the following: We obtain a complete set of orthonormal solutions which can be identified as positive and negative frequency solutions in the asymptotic past, i.e. as $t \rightarrow-\infty$. We can then identify as positive frequency modes those solutions which have a decreasing phase, say, in the WKB limit. We can obtain, in a similar manner, the positive and negative frequency modes in the asymptotic future, i.e. as $t \rightarrow \infty$. Because of the time dependence of the vector potential $A_{1}^{\mu}$, a mode which is purely positive frequency in the infinite past will evolve into a combination of positive and negative frequency modes in the infinite future; a phenomenon we had interpreted earlier as particle production.

Substituting the vector potential (1.120) in the Klein-Gordon equation (1.119), we obtain that

$$
\begin{equation*}
\left(\partial_{t}^{2}-\nabla^{2}-2 i q E t \partial_{x}+q^{2} E^{2} t^{2}+m^{2}\right) \Phi(t, \mathbf{x})=0 \tag{1.122}
\end{equation*}
$$

The mode functions for scalar field $\Phi$ can be decomposed as

$$
\begin{equation*}
u_{\mathbf{k}}(t, \mathbf{x}) \propto f_{\mathbf{k}}(t) \exp i \mathbf{k} \cdot \mathbf{x} \tag{1.123}
\end{equation*}
$$

where $\mathbf{k} \equiv\left(k_{x}, k_{y}, k_{z}\right)=\left(k_{x}, \mathbf{k}_{\perp}\right)$, the function $f_{\mathbf{k}}(t)$ satisfies the following differential equation:

$$
\begin{equation*}
\frac{d^{2} f_{\mathrm{k}}}{d t^{2}}+\left(m^{2}+k_{\perp}^{2}+\left(k_{x}+q E t\right)^{2}\right) f_{\mathrm{k}}=0 \tag{1.124}
\end{equation*}
$$

and $k_{\perp}=\left|\mathbf{k}_{\perp}\right|$. Introducing the new variables

$$
\left.\begin{array}{l}
\tau=\sqrt{q E} t+\left(k_{x} / \sqrt{q E}\right)  \tag{1.125}\\
\lambda=\left(k_{\perp}^{2}+m^{2}\right) / q E \\
\nu=-(1-i \lambda) / 2,
\end{array}\right\}
$$

we find that, in terms of these variables, the differential equation satisfied by the function $f_{\mathrm{k}}$ reduces to

$$
\begin{equation*}
\frac{d^{2} f_{\mathrm{k}}}{d \tau^{2}}+\left(\tau^{2}+\lambda\right) f_{\mathrm{k}}=0 \tag{1.126}
\end{equation*}
$$

If $f_{\mathbf{k}}(\lambda ; \tau)$ is a solution, then so are the functions $f_{\mathbf{k}}^{*}(\lambda ; \tau), f_{\mathbf{k}}(\lambda ;-\tau)$ and $f_{\mathbf{k}}^{*}(\lambda ;-\tau)$. This solution set can be taken to be

$$
\begin{equation*}
\left\{D_{\nu^{*}}((1+i) \tau), D_{\nu}((1-i) \tau), D_{\nu^{*}}(-(1+i) \tau), D_{\nu}(-(1-i) \tau)\right\} \tag{1.127}
\end{equation*}
$$

where $D_{\nu}(z)$ is the parabolic cylinder function (see, for e.g, ref. [77], p. 1067). Only two of these four functions are linearly independent.

From the asymptotic properties of the parabolic cylinder functions (see, for instance, ref. [77], pp. 1065-1066), we find that as $\tau \rightarrow-\infty$

$$
\begin{equation*}
D_{\nu}(-(1-i) \tau) \longrightarrow(\sqrt{2}|\tau|)^{\nu} e^{-i \pi \nu / 4} \exp i\left(\tau^{2} / 2\right) \tag{1.128}
\end{equation*}
$$

and

$$
\begin{equation*}
D_{\nu^{*}}(-(1+i) \tau) \longrightarrow(\sqrt{2}|\tau|)^{\nu^{*}} e^{i \pi \nu^{*} / 4} \exp -i\left(\tau^{2} / 2\right) \tag{1.129}
\end{equation*}
$$

Whereas, as $\tau \rightarrow \infty$

$$
\begin{equation*}
D_{\nu}((1-i) \tau) \longrightarrow(\sqrt{2} \tau)^{\nu} e^{-i \pi \nu / 4} \exp i\left(\tau^{2} / 2\right) \tag{1.130}
\end{equation*}
$$

and

$$
\begin{equation*}
D_{\nu^{*}}((1+i) \tau) \longrightarrow(\sqrt{2} \tau)^{\nu^{*}} e^{i \pi \nu^{*} / 4} \exp -i\left(\tau^{2} / 2\right) \tag{1.131}
\end{equation*}
$$

It is clear from the asymptotic forms of the parabolic cylinder functions that $D_{\nu}(-(1-i) \tau)$ is the positive frequency mode as $\tau \rightarrow-\infty$ (since the positive frequency mode should have a decreasing phase in this limit), whereas $D_{\nu^{*}}((1+i) \tau)$ is the positive frequency mode as $\tau \rightarrow \infty$. Evolving $D_{\nu}(-(1-i) \tau)$ to $\tau \rightarrow \infty$, we
find that (cf. ref. [77], p. 1066)

$$
\begin{align*}
D_{\nu}(-(1-i) \tau)=-\left(\frac{\sqrt{2 \pi}}{\Gamma(-\nu)}\right) e^{i \pi(\nu-1) / 2} & D_{\nu^{*}}((1+i) \tau) \\
& +e^{i \pi \nu} D_{\nu}((1-i) \tau) \tag{1.132}
\end{align*}
$$

where $\Gamma(-\nu)$ is the Gamma function. The Bogolubov coefficients can be read off from the above expression; we find that

$$
\begin{equation*}
\alpha(\mathbf{k})=\left(\frac{\sqrt{2 \pi} e^{-(\lambda-i) \pi / 4}}{\Gamma[(1-i \lambda) / 2]}\right) \quad \text { and } \quad \beta(\mathbf{k})=e^{-(\lambda+i) \pi / 2} \tag{1.133}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
|\alpha(\mathbf{k})|^{2}=1+\exp -(\pi \lambda) \quad \text { and } \quad|\beta(\mathbf{k})|^{2}=\exp -(\pi \lambda) ; \tag{1.134}
\end{equation*}
$$

clearly

$$
\begin{equation*}
|\alpha(\mathbf{k})|^{2}-|\beta(\mathbf{k})|^{2}=1 . \tag{1.135}
\end{equation*}
$$

These results imply that the constant electric field background produces $|\beta(\mathbf{k})|^{2}=$ $\exp -\left(\pi\left(m^{2}+k_{\perp}^{2}\right) / q E\right)$ number of particles corresponding to the quantum scalar field. (Note that $|\beta(\mathbf{k})|^{2}$ is independent of $k_{x}$.)

### 1.4.2 Quantization in the space dependent gauge: tunneling probability

Let us now carry out the normal mode analysis in the space dependent gauge $A_{2}^{\mu}$ given by equation (1.121) [38, 78, 79]. Substituting the vector potential $A_{2}^{\mu}$ in the Klein-Gordon equation (1.119), we obtain that

$$
\begin{equation*}
\left(\partial_{t}^{2}-\nabla^{2}-2 i q E x \partial_{t}-q^{2} E^{2} x^{2}+m^{2}\right) \Phi(t, \mathrm{x})=0 \tag{1.136}
\end{equation*}
$$

Since the vector potential $A_{2}^{\mu}$ is independent of time coordinate $t$ as well as the $y$ and $z$ coordinates, the normal modes of the scalar field can be decomposed in
this gauge as

$$
\begin{equation*}
u_{\omega \mathbf{k}_{\perp}}(t, \mathbf{x}) \propto e^{-i \omega t} e^{i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}} g_{\omega \mathbf{k}_{\perp}}(x), \tag{1.137}
\end{equation*}
$$

where, as before, $\mathbf{k}_{\perp}=\left(k_{y}, k_{z}\right)$ and $g_{\omega \mathbf{k}_{\perp}}(x)$ satisfies the following differential equation:

$$
\begin{equation*}
\frac{d^{2} g_{\omega \mathbf{k}_{\perp}}}{d x^{2}}+\left((\omega+q E x)^{2}-k_{\perp}^{2}-m^{2}\right) g_{\omega k_{\perp}}=0 . \tag{1.138}
\end{equation*}
$$

(Note that $k_{\perp}=\left|\mathbf{k}_{\perp}\right|$.) A difficulty arises if we attempt here the same analysis we had carried out in the time dependent gauge $A_{1}^{\mu}$. Since the time dependence of the normal modes above are of the form $\exp -i \omega t$ for all times, the Bogolubov coefficient $\beta$ is trivially zero. The vacuum state defined with respect to the positive frequency modes $\exp -i \omega t$ remains a vacuum for all times and we will not obtain any particle production in the manner we had obtained in the time dependent gauge. Therefore, if only a nonzero $\beta$ is to be interpreted as particle production we will then be led to results that are gauge dependent. It is in such a situation that the tunneling interpretation comes to our rescue.

Let us look at the situation more closely. Substituting the following variables

$$
\left.\begin{array}{l}
\rho=\sqrt{q E} x+(\omega / \sqrt{q E})  \tag{1.139}\\
\lambda=\left(k_{\perp}^{2}+m^{2}\right) / q E \\
\nu=-(1-i \lambda) / 2
\end{array}\right\}
$$

in the differential equation for $g_{\omega \mathbf{k}_{\perp}}$, we find that it reduces to

$$
\begin{equation*}
\frac{d^{2} g_{\omega \mathbf{k}_{\perp}}}{d \rho^{2}}+\left(\rho^{2}-\lambda\right) g_{\omega \mathbf{k}_{\perp}}=0 . \tag{1.140}
\end{equation*}
$$

This differential equation is similar to the one we had encountered in the time dependent gauge (for the function $f_{\mathbf{k}}(t)$ ) with the sign of $\lambda$ changed; this change
is equivalent to $\nu \leftrightarrow \nu^{*}$. So the solution set for $g_{\omega \mathbf{k}_{\perp}}$ is still the same with some change of signs:

$$
\begin{equation*}
\left\{D_{\nu}((1+i) \rho), D_{\nu^{*}}((1-i) \rho), D_{\nu}(-(1+i) \rho), D_{\nu^{*}}(-(1-i) \rho)\right\} . \tag{1.141}
\end{equation*}
$$

The first pair are the left and right moving modes in the far right (i.e. as $\rho \rightarrow \infty$ ), while the second pair corresponds to right and left moving modes in the far left (i.e. as $\rho \rightarrow-\infty$ ). (We define a right moving mode as the one which has an increasing phase in space.) A meaningful theory can be constructed out of any independent pair of these solutions.

Notice that the differential equation (1.140) can be rewritten as

$$
\begin{equation*}
-\frac{d^{2} g_{\omega \mathbf{k}_{\perp}}}{d \rho^{2}}-\rho^{2} g_{\omega \mathbf{k}_{\perp}}=-\lambda g_{\omega \mathbf{k}_{\perp}} \tag{1.142}
\end{equation*}
$$

which then resembles a Schrödinger equation for an inverted oscillator corresponding to an energy eigenvalue $-\lambda<0$ (cf. equation (1.139)). What is usually done in literature at this stage is the following: Since the natural definition of particles in the far left does not match with the natural definition of particles in the far right, one can attempt an interpretation for particle creation in terms of 'tunneling' across the inverted oscillator potential in the 'effective Schrödinger equation' (1.142) above (see, for e.g., ref. [80], pp. 284-285). This approach leads to the same result we had obtained in the time dependent gauge. To see this, consider a mode which is right moving in the $\rho>0$ region (i.e. as $\rho \rightarrow \infty$ ). This is given by $D_{\nu^{*}}((1-i) \rho)$. We look at its behavior in the far left region, i.e. as $\rho \rightarrow-\infty$; we can express $D_{\nu^{*}}((1-i) \rho)$ as a superposition of $D_{\nu}(-(1+i) \rho)$ and $D_{\nu^{*}}(-(1-i) \rho)$ as follows (see, for instance, ref. [77], p. 1066)

$$
\begin{align*}
D_{\nu^{*}}((1-i) \rho)=e^{i \pi \nu^{*}} & D_{\nu^{*}}(-(1-i) \rho) \\
& +\left(\frac{\sqrt{2 \pi}}{\Gamma\left(-\nu^{*}\right)}\right) e^{i \pi\left(\nu^{*}+1\right) / 2} D_{\nu}(-(1+i) \rho) . \tag{1.143}
\end{align*}
$$

Asymptotically, as $\rho \rightarrow \infty$ (see, for e.g., ref. [77], pp. 1065-1066)

$$
\begin{align*}
D_{\nu^{*}}((1-i) \rho) & \longrightarrow(\sqrt{2} \rho)^{\nu^{*}} e^{-i \pi \nu^{*} / 4} \exp i\left(\rho^{2} / 2\right) \\
& \equiv B \exp i\left(\rho^{2} / 2\right) \tag{1.144}
\end{align*}
$$

while, as $\rho \rightarrow-\infty$

$$
\begin{align*}
& D_{\nu^{*}}((1-i) \rho) \longrightarrow\left(\frac{\sqrt{2 \pi} e^{i \pi\left(\nu^{*}+1\right) / 2}}{\Gamma\left(-\nu^{*}\right)}\right)(\sqrt{2}|\rho|)^{\nu} e^{i \pi \nu / 4} \exp -i\left(\rho^{2} / 2\right) \\
&+(\sqrt{2}|\rho|)^{\nu^{*}} e^{3 i \pi \nu^{*} / 4} \exp i\left(\rho^{2} / 2\right) \\
& \equiv A|\rho|^{\nu} \exp -i\left(\rho^{2} / 2\right)+C|\rho|^{\nu^{*}} \exp i\left(\rho^{2} / 2\right) \tag{1.145}
\end{align*}
$$

In this expression the first term represents the incident wave and the second the reflected wave (since the direction of propagation of the wave is that in which its phase increases in the relevant limit). Let us now assume that a wave of amplitude $R$ is incident on the potential, $T$ of which is transmitted through the potential and a wave of unit amplitude is scattered back. We can then identify the coefficients $R$ and $T$ by comparing equations (1.143), (1.144) and (1.145). We find that

$$
\begin{equation*}
R(\mathbf{k})=\left(\frac{A}{C}\right)=\left(\frac{\sqrt{2 \pi} e^{-(\lambda+i) \pi / 4}}{\Gamma\left(-\nu^{*}\right)}\right) \quad ; \quad T(\mathbf{k})=\left(\frac{B}{C}\right)=\exp -i \pi \nu^{*} \tag{1.146}
\end{equation*}
$$

therefore

$$
\begin{equation*}
|R(\mathbf{k})|^{2}=1+\exp -(\pi \lambda) \quad ; \quad|T(\mathbf{k})|^{2}=\exp -(\pi \lambda) \tag{1.147}
\end{equation*}
$$

and

$$
\begin{equation*}
|R(\mathbf{k})|^{2}-|T(\mathbf{k})|^{2}=1 . \tag{1.148}
\end{equation*}
$$

It is this 'tunneling probability' $|T(\mathbf{k})|^{2}=\exp -\left(\pi\left(m^{2}+k_{\perp}^{2}\right) / q E\right)$ that is interpreted in literature as rate at which particles are being produced by the background electric field. Also, this result exactly matches the quantity $|\beta(\mathbf{k})|^{2}$ we had
obtained in the time dependent gauge. The tunneling interpretation thus rescues us out of a gauge dependent result we would have obtained had we considered only a nonzero Bogolubov coefficient $\beta$ to imply particle production in both the gauges $A_{1}^{\mu}$ and $A_{2}^{\mu}$.

Thus, we find that, the phenomenon of particle production has to be described by two different approaches, viz. Bogolubov transformations in time dependent gauges and the tunneling interpretation in the case of time independent ones, if we are to obtain gauge independent results. In such a situation, it is desirable to look for a single approach that can lead us directly to results that are gauge invariant. We shall find that the effective Lagrangian approach is able to provide us with such a feature. In the following section, we introduce the effective Lagrangian approach for a simple toy model with two interacting degrees of freedom and then go on to illustrate as to how this approach can help us obtain gauge invariant results for the case of a constant electromagnetic background.

### 1.5 The effective Lagrangian approach

Consider a theory which describes the interaction of two mechanical systems having the dynamical variables $C$ and $q$. (This discussion closely follows the discussion in section 3 of ref. [38].) The quantum theory of the complete system can be constructed from the exact path integral [81, 82]

$$
\begin{equation*}
K\left(C_{2}, q_{2}, t_{2} \mid C_{1}, q_{1}, t_{1}\right)=\int \mathcal{D} C \int \mathcal{D} q \exp i \mathcal{S}(C, q) \tag{1.149}
\end{equation*}
$$

where

$$
\begin{equation*}
S[C, q]=\int d t \mathcal{L}[C, q] \tag{1.150}
\end{equation*}
$$

is the action describing the total system. The above path integral is often impossible to evaluate. It would therefore be useful to have some approximate ways of studying the system. The effective Lagrangian method is a reliable approximation scheme that has been developed for handling (1.149). This method is of value when one of the variables, say, $C$, behaves nearly classically while the other variable is fully quantum mechanical. In such a case, the problem can be attacked in the following manner.

Let us suppose that the path integral over the variable $q$ can be performed exactly for an arbitrary $C(t)$. That is, we can evaluate the quantity

$$
\begin{align*}
F\left(q_{2}, t_{2} \mid q_{1}, t_{1}\right)_{C(t)} & =\int \mathcal{D} q \exp i \mathcal{S}[C, q] \\
& \equiv \exp i \mathcal{S}_{\epsilon f f}[C(t)], \tag{1.151}
\end{align*}
$$

treating $C(t)$ as an arbitrary function of time. If we can then perform the path integral

$$
\begin{equation*}
K\left(C_{2}, q_{2}, t_{2} \mid C_{1}, q_{1}, t_{1}\right)=\int \mathcal{D} C \exp i \mathcal{S}_{e f f}[C] \tag{1.152}
\end{equation*}
$$

exactly, we would have completely solved the problem. Since this is not possible, we can evaluate (1.152) by invoking the fact that $C$ is almost classical. This means that most of the contribution to (1.152) comes from nearly classical paths satisfying the condition

$$
\begin{equation*}
\frac{\delta \mathcal{S}_{e f f}[C]}{\delta C}=0 \tag{1.153}
\end{equation*}
$$

It is easy to evaluate (1.152) in this approximation and thereby obtain an approximate solution to our problem. In fact, quite often, we will be content with obtaining the solutions to (1.153), and will not bother to calculate (1.152). Equation (1.153) will contain some of the effects of the quantum fluctuations in $q$ on $C$ and is often called the semiclassical equation. The quantity $\mathcal{S}_{e f f}$ is called the
effective action for the $C$ system. It will not always be possible to express the functional $\mathcal{S}_{e f f}[C(t)]$ as an integral over time of a local density. Whenever it is possible, we can define an effective Lagrangian through the following relation:

$$
\begin{equation*}
\mathcal{S}_{e f f}=\int d t \mathcal{L}_{e f f} . \tag{1.154}
\end{equation*}
$$

The way we have defined our expressions, the quantities $K$ and $\mathcal{S}_{\text {eff }}$ depend on the boundary conditions ( $q_{2}, t_{2}, q_{1}, t_{1}$ ). It is preferable to have an effective action which is completely independent of the $q$-degree of freedom. The most natural way of achieving this is to integrate out the effect of $q$ for all times by considering the limit of $t_{2} \rightarrow \infty$ and $t_{1} \rightarrow-\infty$ in our definition of the effective action. We will also assume, as is usual, that $C(t)$ becomes a constant asymptotically. In such a limit the kernel essentially represents the amplitude for the $q$ system to make a transition from the ground state in the infinite past to the ground state in the infinite future. Hence

$$
\begin{align*}
F\left(q_{2}, t_{2} \rightarrow \infty \mid q_{1}, t_{1} \rightarrow-\infty\right)_{C(t)} & =\exp i \mathcal{S}_{e f f}[C(t)] \\
& =N\left(q_{2}, q_{1}\right)\left\langle 0_{o u t} \mid 0_{i n}\right\rangle_{C(t)}, \tag{1.155}
\end{align*}
$$

where $\left\langle 0_{\text {out }} \mid 0_{i n}\right\rangle_{C(t)}$ stands for the vacuum-to-vacuum transition amplitude for the $q$-system in the presence of the external source $C(t)$ and $N\left(q_{2}, q_{1}\right)$ is a normalization factor independent of $C(t)$. Taking logarithms on both sides of the above equation, we obtain

$$
\begin{equation*}
\mathcal{S}_{e f f}[C(t)] \equiv-i \ln \left(\left|\left\langle 0_{o u t} \mid 0_{i n}\right\rangle_{C(t)}\right|\right)+\text { constant } . \tag{1.156}
\end{equation*}
$$

Since the constant term is independent of $C$ it will not contribute in (1.153). Therefore, for the purpose of our calculation we may take the effective action to be defined by the relation

$$
\begin{equation*}
\mathcal{S}_{e f f}[C(t)] \equiv-i \ln \left(\left|\left\langle 0_{o u t} \mid 0_{\text {in }}\right\rangle_{C(t)}\right|\right) \tag{1.157}
\end{equation*}
$$

in which all references to the quantum mode $q$ have been eliminated. Notice that the way we have defined $F$, the effective action $\mathcal{S}_{e f f}$ contains the kinetic energy of $C$ and any potential energy of $C$ which depends only on $C$. That is, if the original Lagrangian was of the form

$$
\begin{equation*}
\mathcal{L}(C, q)=\mathcal{L}_{C}(C)+\mathcal{L}_{q}(q)+\mathcal{L}_{\text {int }}(q, C), \tag{1.158}
\end{equation*}
$$

then the effective Lagrangian will have the form

$$
\begin{equation*}
\mathcal{L}_{e f f}(C)=\mathcal{L}_{C}(C)+\mathcal{L}_{\text {corr }}(C) ; \tag{1.159}
\end{equation*}
$$

the first term $\mathcal{L}_{C}$ goes for a ride and second term $\mathcal{L}_{\text {corr }}$ is the result of integrating out the degree of freedom $q$.

An external perturbation can cause transitions from the initial ground state to an excited state. In other words the probability for the system to be in the ground state in the infinite future (even though it started in the ground state in the infinite past) could be less than unity. This implies that the effective action $\mathcal{S}_{\text {eff }}$ need not be real. The imaginary part of $\mathcal{S}_{\text {eff }}$ contains information about the rate of transitions induced in the $q$-system by the presence of $C(t)$. Also, if we use $\mathcal{S}_{e f f}$ directly in (1.153) we have no assurance that the solution to $C$ will be real.

Let us suppose that the action $\mathcal{S}[C, q]$ is of the form

$$
\begin{align*}
\mathcal{S}[C, q] & =\mathcal{S}_{C}[C]+\mathcal{S}_{q}[q]+\mathcal{S}_{\text {int }}[c, q] \\
& =\int d t\left\{\mathcal{L}_{C}(C)+\mathcal{L}_{q}(q)+\mathcal{L}_{\text {int }}(C, q)\right\}, \tag{1.160}
\end{align*}
$$

where $\mathcal{L}_{C}$ and $\mathcal{L}_{q}$ are the free parts of the Lagrangian corresponding to the $C$ and the $q$ degrees of freedom and $\mathcal{L}_{\text {int }}$ represents the interaction between $C$ and $q$. Let
us also assume that the variation of $C(t)$ is adiabatic. In such a situation, it can be shown that the vacuum-to-vacuum transition amplitude is the given by (see either, ref. [73], p. 180 or, ref. [83], section 4.2)

$$
\begin{align*}
& \lim _{t_{2} \rightarrow \infty} \lim _{1} \rightarrow-\infty \\
& F\left(q_{2}, t_{2} \mid q_{1}, t_{1}\right)_{C(t)} \equiv \exp i \mathcal{S}_{e f f}[C] \\
& \equiv \int \mathcal{D} q \exp i \mathcal{S}[C, q] \\
&= \exp i \mathcal{S}_{C}[C] \times \int \mathcal{D} q \exp i\left(\mathcal{S}_{q}[q]+\mathcal{S}_{\text {int }}[C, q]\right) \\
&= \exp i\left(\mathcal{S}_{C}[C]+\mathcal{S}_{\text {corr }}[C]\right) \\
&= \exp i\left\{\int d t\left(\mathcal{L}_{C}(C)+\mathcal{L}_{\text {corr }}(C)\right)\right\} \\
&= \exp i \mathcal{S}_{C}[C]  \tag{1.161}\\
& \quad \times \text { constant } \exp -i\left(\int d t E_{0}(C)\right),
\end{align*}
$$

where $E_{0}(C)$ is the ground state energy of the $q$ system in the presence of $C$. From the above equation it is easy to identify that the quantity $\mathcal{L}_{\text {corr }}$ is related to the ground state energy of the $q$ mode as follows:

$$
\begin{equation*}
\mathcal{L}_{\text {corr }}=-E_{0}(C) . \tag{1.162}
\end{equation*}
$$

This result, which is valid when $C(t)$ varies adiabatically with time, provides a means of computation of the effective Lagrangian if the $C$ dependence of the ground state can be ascertained.

The transitions to higher states, indicated by the existence of an imaginary part to $\mathcal{S}_{\text {corr }}$, can also be discussed in terms of the above relation. $\mathcal{S}_{\text {corr }}$ can become complex only if $E_{0}$ (and therefore $\mathcal{L}_{\text {corr }}$ ) turns out to be complex. The appearance of an imaginary part to the ground state energy indicates an exponential decay probability for this state which is precisely what we expect if transitions to higher states are possible.

Though, we have presented our discussion here assuming that $C$ and $q$ are systems with a single of freedom, our discussion is applicable to field theoretic situations as well. For instance, the $C$ could describe a set of variables like the components of a vector field and $q$ those of a complex scalar field. In the context of field theory, the vacuum-to-vacuum transition amplitude would correspond to vacuum polarization and the transitions to excited states would correspond to particle production. Notice that the effective Lagrangian approach is applicable only when $C$ behaves almost classically (see our discussion following (1.152)). Therefore, the effective Lagrangian approach can be used to study phenomena such as production of particles corresponding to a quantum field by a classical background.

### 1.5.1 Effective Lagrangian for a constant electromagnetic background

We shall now apply the formalism we have developed above to the evaluate the effective Lagrangian for a constant electromagnetic background.

The system we shall consider here consists of a complex scalar field $\Phi$ interacting with an electromagnetic field represented by the vector potential $A^{\mu}$ and is described by the action (1.117). (Notice that the vector potential $A^{\mu}$ would correspond to the degree of freedom $C$ and the complex scalar field to $q$ in our discussion above.) The effective Lagrangian for the electromagnetic field can be obtained by integrating the degrees of freedom corresponding to the quantum field $\Phi$. It can be expressed as

$$
\begin{equation*}
\mathcal{L}_{e f f}=\mathcal{L}_{e m}+\mathcal{L}_{c o r r}, \tag{1.163}
\end{equation*}
$$

where $\mathcal{L}_{e m}$ is the Lagrangian density for the free electromagnetic field, viz. the third term under the integral in the action (1.117) and $\mathcal{L}_{\text {corr }}$ is implicitly given by

$$
\begin{array}{r}
\exp i \int d^{4} x \mathcal{L}_{\text {corr }}\left(A^{\mu}\right) \\
=\int \mathcal{D} \Phi \int \mathcal{D} \Phi^{*} \exp i \int d^{4} x\left\{\left(\partial_{\mu} \Phi+i q A_{\mu} \Phi\right)\left(\partial^{\mu} \Phi^{*}-i q A^{\mu} \Phi^{*}\right)\right. \\
\left.-m^{2} \Phi \Phi^{*}\right\} \tag{1.164}
\end{array}
$$

Thus, we need to evaluate the functional integral over $\Phi$ for a given background electromagnetic field.

As we have mentioned earlier, the evaluation of the above functional integral is an impossible task if $A^{\mu}(x)$ is an arbitrary background field. Therefore, let us assume that $A_{\mu}(x)$ varies slowly with the spacetime coordinates $x^{\mu}$ so that we can write

$$
\begin{equation*}
A_{\mu}(x) \simeq-\frac{1}{2} F_{\mu \nu} x^{\nu}+O\left((\partial F) x^{2}\right) \tag{1.165}
\end{equation*}
$$

where $F_{\mu \nu}$ 's are treated as constants. This corresponds to assuming that the classical background is a constant electromagnetic field $F_{\mu \nu}$. We have seen earlier that, in the adiabatic limit $\mathcal{L}_{\text {corr }}$ is proportional to the ground state energy of the system. The ground state energy $E_{0}\left(F_{\mu \nu}\right)$ of the complex scalar field $\Phi$ in a constant $F_{\mu \nu}$ will then determine the effective Lagrangian for the constant electromagnetic background.

The task of evaluating the ground state energy is particularly easy if the background field satisfies the conditions $(\mathbf{E} . \mathbf{B})=0$ and $\left(\mathbf{B}^{2}-\mathbf{E}^{2}\right)>0$, where $\mathbf{E}$ and $\mathbf{B}$ are the constant electric and magnetic fields respectively. (The following derivation is adapted from ref. [84], section 129.) In such a case, the electromagnetic background can be expressed as a purely magnetic field in some Lorentz frame. Let $\mathbf{B}=B \hat{\mathbf{y}}$, where $B$ is a constant and $\hat{\mathbf{y}}$ is the unit vector along the
positive $y$-direction. We can choose the gauge $A^{\mu}=(0,0,0,-B x)$ to describe such a background. The Klein-Gordon equation (1.119) in such a gauge is then given by

$$
\begin{equation*}
\left(\partial_{t}^{2}-\nabla^{2}-2 i q B x \partial_{z}+q^{2} B^{2} x^{2}+m^{2}\right) \Phi(t, \mathbf{x})=0 \tag{1.166}
\end{equation*}
$$

Because the vector potential is independent of time $t$ and also the $x$ and $y$ coordinates, the normal modes of the scalar field $\Phi$ can be decomposed as follows:

$$
\begin{equation*}
u_{\omega \mathbf{k}_{\perp}} \propto e^{-i \omega t} e^{i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}} f_{\omega \mathbf{k}_{\perp}}(x), \tag{1.167}
\end{equation*}
$$

where, as usual, $\mathbf{k}_{\perp}=\left(k_{y}, k_{z}\right), \mathbf{x}_{\perp}=(y, z)$ and $f_{\omega \mathbf{k}_{\perp}}(x)$ satisfies the differential equation

$$
\begin{equation*}
\frac{d^{2} f_{\omega \mathbf{k}_{\perp}}}{d x^{2}}+\left(\omega^{2}-\left(q B x-k_{z}\right)^{2}\right) f_{\omega \mathbf{k}_{\perp}}=\left(m^{2}+k_{y}^{2}\right) f_{\omega \mathbf{k}_{\perp}} . \tag{1.168}
\end{equation*}
$$

This differential equation can be rewritten as

$$
\begin{equation*}
-\frac{d^{2} f_{\omega \mathbf{k}_{\perp}}}{d \xi^{2}}+q^{2} B^{2} \xi^{2} f_{\omega \mathbf{k}_{\perp}}=\varepsilon f_{\omega \mathbf{k}_{\perp}}, \tag{1.169}
\end{equation*}
$$

where

$$
\begin{equation*}
\xi=x-\frac{k_{z}}{q B} \quad \text { and } \quad \varepsilon=\omega^{2}-m^{2}-k_{y}^{2} . \tag{1.170}
\end{equation*}
$$

Equation (1.169) resembles the Schrödinger equation for a harmonic oscillator with mass $(1 / 2)$ and frequency $2 q B$. So, if $f_{\omega \mathbf{k}_{\perp}}$ has to be bounded for large $x$, the energy $\varepsilon$ of the oscillator must be quantized, i.e.

$$
\begin{equation*}
\varepsilon_{n}=2 q B\left(n+\frac{1}{2}\right)=\left(\omega^{2}-m^{2}-k_{y}^{2}\right) . \tag{1.171}
\end{equation*}
$$

Therefore, the allowed set of frequencies for the normal modes are

$$
\begin{equation*}
\omega_{n}=\left(m^{2}+k_{y}^{2}+q B(2 n+1)\right)^{1 / 2} . \tag{1.172}
\end{equation*}
$$

The ground state energy per mode is $2\left(\omega_{n} / 2\right)=\omega_{n}$ because the complex scalar field has twice as many degrees of freedom as a real scalar field. The total ground
state energy is given by the sum over all modes $k_{y}$ and $n$. The weightage factor for the discrete sum over $n$, in a magnetic field is obtained by the correspondence

$$
\begin{equation*}
\frac{d k_{x}}{2 \pi} \frac{d k_{y}}{2 \pi} \longrightarrow \sum_{n}\left(\frac{q B}{2 \pi}\right) \frac{d k_{y}}{2 \pi} . \tag{1.173}
\end{equation*}
$$

Hence the ground state energy is

$$
\begin{equation*}
E_{0}=-\mathcal{L}_{\text {corr }}=\left(\frac{q B}{2 \pi}\right) \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{d k_{y}}{2 \pi}\left(k_{y}^{2}+m^{2}+q B(2 n+1)\right)^{1 / 2} . \tag{1.174}
\end{equation*}
$$

Now, consider the quantity

$$
\begin{equation*}
I \equiv-\left(\frac{2 \pi}{q B}\right) \frac{\partial^{2} E_{0}}{\partial\left(m^{2}\right)^{2}}=\left(\frac{2 \pi}{q B}\right) \frac{\partial^{2} \mathcal{L}_{\text {corr }}}{\partial\left(m^{2}\right)^{2}}, \tag{1.175}
\end{equation*}
$$

which can be evaluated in the following manner:

$$
\begin{align*}
I & =\frac{1}{4} \sum_{n=0}^{\infty} \int \frac{d k_{y}}{2 \pi}\left(k_{y}^{2}+m^{2}+q B(2 n+1)\right)^{-3 / 2} \\
& =\frac{1}{4 \pi} \sum_{n=0}^{\infty}\left(m^{2}+q B(2 n+1)\right)^{-1} \\
& =\frac{1}{4 \pi} \sum_{n=0}^{\infty} \int_{0}^{\infty} d s e^{-m^{2} s} \exp -q B(2 n+1) s \\
& =\frac{1}{4 \pi} \int_{0}^{\infty} d s e^{-m^{2} s}\left(\frac{\exp -(q B s)}{1-\exp -(2 q B s)}\right) \\
& =\frac{1}{8 \pi} \int_{0}^{\infty} d s\left(\frac{e^{-m^{2} s}}{\sinh (q B s)}\right) \tag{1.176}
\end{align*}
$$

Then, $\mathcal{L}_{\text {corr }}$ can be obtained by integrating the above expression twice with respect to $m^{2}$. We obtain that

$$
\begin{equation*}
\mathcal{L}_{\text {corr }}=\frac{1}{16 \pi^{2}} \int_{0}^{\infty} \frac{d s}{s^{3}} e^{-m^{2} s}\left(\frac{q B s}{\sinh (q B s)}\right) \tag{1.177}
\end{equation*}
$$

This expression has a divergence in the lower limit of the integration. This divergence can be regularized by subtracting the contribution due to $\mathcal{L}_{\text {corr }}$ with the constant $B$ set to zero. Also, the integration with respect to $m^{2}$ produces a term like ( $c_{1} m^{2}+c_{2}$ ) with two (divergent) integration constants $c_{1}$ and $c_{2}$. These two
divergences can be handled by redefining the field strengths (renormalization) and hence we shall ignore these divergences and carry on with our discussion here.

The quantity $\mathcal{L}_{\text {corr }}$ is the quantum correction to the Lagrangian density describing the classical electromagnetic background. Being a Lagrangian density, we would expect $\mathcal{L}_{\text {corr }}$ to be a Lorentz as well as a gauge invariant quantity. In fact, Schwinger, using his proper time formalism (we will discuss this formalism later in section 3.1), has been able to explicitly show that this is indeed true, at least for the case of a constant electromagnetic background [33]. The only nonzero gauge invariant quantities that can be constructed out of a constant electromagnetic field are $\left(\mathbf{E}^{2}-\mathbf{B}^{2}\right)$ and (E.B). Hence, the effective Lagrangian for a constant electromagnetic background can depened only on these two quantities. Let us now define two constants $a$ and $b$ by the relations

$$
\begin{equation*}
\mathbf{E}^{2}-\mathbf{B}^{2}=a^{2}-b^{2} \quad \text { and } \quad \mathbf{E} . \mathbf{B}=a b . \tag{1.178}
\end{equation*}
$$

Then, for a constant electromagnetic background $\mathcal{L}_{\text {corr }}=\mathcal{L}_{\text {corr }}(a, b)$. For the case of the constant magnetic field we are considering here $a=0$ and $b=B$. Therefore, $\mathcal{L}_{\text {corr }}$ can be written in a manifestly invariant way as follows:

$$
\begin{equation*}
\mathcal{L}_{\text {corr }}=\frac{1}{16 \pi^{2}} \int_{0}^{\infty} \frac{d s}{s^{3}} e^{-m^{2} s}\left(\frac{q b s}{\sinh (q b s)}\right) . \tag{1.179}
\end{equation*}
$$

Since $\mathcal{L}_{\text {corr }}$ has to be Lorentz invariant, it must be valid in any frame in which $\left(\mathbf{B}^{2}-\mathbf{E}^{2}\right)>0$ and $(\mathbf{E} . \mathbf{B})=0$. In all such cases,

$$
\begin{equation*}
\mathcal{L}_{\text {corr }}=\frac{1}{16 \pi^{2}} \int_{0}^{\infty} \frac{d s}{s^{3}} e^{-m^{2} s}\left(\frac{q s \sqrt{\mathbf{B}^{2}-\mathbf{E}^{2}}}{\sinh \left(q s \sqrt{\mathbf{B}^{2}-\mathbf{E}^{2}}\right)}\right) . \tag{1.180}
\end{equation*}
$$

Notice that this expression for $\mathcal{L}_{\text {corr }}$ is invariant under the following transformation: $|\mathbf{E}| \rightarrow i|\mathbf{B}|,|\mathbf{B}| \rightarrow-i|\mathbf{E}|$. We shall make use of this property later in our calculation.

We shall now consider the case with arbitrary $\mathbf{E}$ and $\mathbf{B}$ for which $a$ and $b$ are not simaltaneously zero. It is well-known that by choosing our Lorentz frame suitably, we can make $\mathbf{E}$ and $\mathbf{B}$ parallel, say along the $y$-axis. We will describe this field $(\mathbf{E}=E \hat{\mathbf{y}} ; \mathbf{B}=B \hat{\mathbf{y}})$ by the vector potential $A^{\mu}=(-E y, 0,0, B x)$. ( $\hat{\mathrm{y}}$ is the unit vector along the direction of positive $y$-axis.) The Klein-Gordon equation (1.119) corresponding to this vector potential is given by

$$
\begin{equation*}
\left(\partial_{t}^{2}-\nabla^{2}-2 i q E y \partial_{t}+2 i q B x \partial_{z}-q^{2} E^{2} y^{2}+q^{2} B^{2} x^{2}+m^{2}\right) \Phi(t, \mathbf{x})=0 \tag{1.181}
\end{equation*}
$$

Since the vector potential is independent of $t$ and $z$, the normal modes for the scalar field $\Phi$ can now be decomposed as follows:

$$
\begin{equation*}
u_{\omega k_{z}}(t, \mathrm{x}) \propto \exp -i\left(\omega t-k_{z} z\right) f_{\omega k_{z}}(x, y), \tag{1.182}
\end{equation*}
$$

where $f_{w k_{z}}(x, y)$ satisfies the differential equation

$$
\begin{equation*}
\left(\partial_{x}^{2}+\partial_{y}^{2}+(\omega+q E y)^{2}-\left(k_{z}-q B x\right)^{2}\right) f_{\omega k_{z}}=m^{2} f_{\omega k_{z}} \tag{1.183}
\end{equation*}
$$

which clearly separates into $x$ and $y$ modes. Writing

$$
\begin{equation*}
f_{\omega k_{z}}(x, y)=g_{k_{z}}(x) Q_{\omega}(y) \tag{1.184}
\end{equation*}
$$

we find that $g_{k_{z}}(x)$ satisfies the Schrödinger equation for a harmonic oscillator

$$
\begin{equation*}
-\frac{d^{2} g_{k_{z}}}{d x^{2}}+\left(k_{z}-q B x\right)^{2} g_{k_{z}}=2 q B\left(n+\frac{1}{2}\right) g_{k_{z}} \tag{1.185}
\end{equation*}
$$

and $Q_{\omega}(y)$ satisfies the following differential equation:

$$
\begin{equation*}
\frac{d Q_{\omega}}{d y^{2}}+(\omega+q E y)^{2} Q_{\omega}=\left\{m^{2}+2 q B\left(n+\frac{1}{2}\right)\right\} Q_{\omega} . \tag{1.186}
\end{equation*}
$$

Changing to the dimensionless variable

$$
\begin{equation*}
\eta=y \sqrt{q E}+\frac{\omega}{\sqrt{q E}}, \tag{1.187}
\end{equation*}
$$

we obtain

$$
\begin{equation*}
\frac{d^{2} Q_{\omega}}{d \eta^{2}}+\eta^{2} Q_{\omega}=\frac{1}{q E}\left(m^{2}+q B(2 n+1)\right) Q_{\omega} . \tag{1.188}
\end{equation*}
$$

This expression shows that the only dimensionless combination which appears in the presence of an electric field is

$$
\begin{equation*}
\tau=\frac{1}{q E}\left(m^{2}+q B(2 n+1)\right) . \tag{1.189}
\end{equation*}
$$

Thus, purely from dimensional considerations, we expect the ground state energy to have the form

$$
\begin{equation*}
E_{0}=\sum_{n=0}^{\infty} 2 q B G(\tau), \tag{1.190}
\end{equation*}
$$

where $G(\tau)$ is a function to be determined. Introducing the Laplace transform $F$ of $G$, by the relation

$$
\begin{equation*}
G(\tau)=\int_{0}^{\infty} d k F(k) e^{-k \tau} \tag{1.191}
\end{equation*}
$$

we can write

$$
\begin{equation*}
\mathcal{L}_{\text {corr }}=2 q B \sum_{n=0}^{\infty} \int_{0}^{\infty} d k F(k) \exp -\left\{\left(m^{2}+q B(2 n+1)\right) k / q E\right\} . \tag{1.192}
\end{equation*}
$$

Summing the geometric series and redefining the variable $k=q E s$, we obtain that

$$
\begin{align*}
\mathcal{L}_{\text {corr }} & =2(q B)(q E) \int_{0}^{\infty} d s e^{-m^{2} s} F(q E s)\left(\frac{\exp -(q B s)}{1-\exp -(2 q B s)}\right) \\
& =(q B)(q E) \int_{0}^{\infty} d s e^{-m^{2} s}\left(\frac{F(q E s)}{\sinh (q B s)}\right) . \tag{1.193}
\end{align*}
$$

We can now determine the form of $F$ by using the fact that $\mathcal{L}_{\text {corr }}$ must be invariant under the following transformation: $|\mathbf{E}| \rightarrow i|\mathbf{B}|,|\mathbf{B}| \rightarrow-i|\mathbf{E}|$; a property of $\mathcal{L}_{\text {corr }}$ we had pointed out earlier. Under such a transfomation, we find

$$
\begin{equation*}
\mathcal{L}_{c o r r}=-(q B)(q E) \int_{0}^{\infty} d s e^{-m^{2} s}\left(\frac{F(i q E s)}{\sinh (i q E s)}\right) \tag{1.194}
\end{equation*}
$$

Comparing the two expressions (1.193) and (1.194) and using the uniqueness of Laplace transforms with respect to $m^{2}$, we obtain that

$$
\begin{equation*}
\left(\frac{F(q E s)}{\sinh (q B s)}\right)=-\left(\frac{F(i q B s)}{\sinh (i q E s)}\right) \tag{1.195}
\end{equation*}
$$

or equivalently

$$
\begin{equation*}
F(q E s) \sin (q E s)=F(i q B s) \sin (i q B s) . \tag{1.196}
\end{equation*}
$$

Since each side depends only on either $E$ or $B$ alone, each side must be independent of $E$ and $B$. Therefore,

$$
\begin{equation*}
F(q E s) \sin (q E s)=F(i q B s) \sin (i q B s)=A(s) \tag{1.197}
\end{equation*}
$$

with the result

$$
\begin{equation*}
\mathcal{L}_{\text {corr }}=(q B)(q E) \int_{0}^{\infty} d s e^{-m^{2} s}\left(\frac{A(s)}{\sin (q E s) \sinh (q B s)}\right) \tag{1.198}
\end{equation*}
$$

In the limit of $E \rightarrow 0$ this $\mathcal{L}_{\text {corr }}$ then reduces to

$$
\begin{equation*}
\mathcal{L}_{\text {corr }}=(q B) \int_{0}^{\infty} \frac{d s}{s} e^{-m^{2} s}\left(\frac{A(s)}{\sinh (q B s)}\right) \tag{1.199}
\end{equation*}
$$

Comparing this expression with equation (1.177) we obtain that

$$
\begin{equation*}
A(s)=\left(\frac{1}{16 \pi^{2} s}\right) \tag{1.200}
\end{equation*}
$$

Thus, we arrive at the final answer for $\mathcal{L}_{\text {corr }}$ for a constant electromagnetic background

$$
\begin{equation*}
\mathcal{L}_{\text {corr }}=\frac{1}{16 \pi^{2}} \int_{0}^{\infty} \frac{d s}{s^{3}} e^{-m^{2} s}\left(\frac{q E s}{\sin (q E s)}\right)\left(\frac{q B s}{\sinh (q B s)}\right) . \tag{1.201}
\end{equation*}
$$

In the situation we are considering here $\mathbf{E}$ and $\mathbf{B}$ are parallel making $\left(\mathbf{E}^{2}-\mathbf{B}^{2}\right)=$ $\left(a^{2}-b^{2}\right)$ and $\mathbf{E} . \mathbf{B}=a b$. Therefore, the results above can be written in a manifestly invariant form as

$$
\begin{equation*}
\mathcal{L}_{\text {corr }}=\frac{1}{16 \pi^{2}} \int_{0}^{\infty} \frac{d s}{s^{3}} e^{-m^{2} s}\left(\frac{q a s}{\sin (q a s)}\right)\left(\frac{q b s}{\sinh (q b s)}\right) \tag{1.202}
\end{equation*}
$$

This result is now valid for any $\left(\mathbf{E}^{2}-\mathbf{B}^{2}\right)$ and (E.B).

The effective Lagrangian for a constant electric field background (the configuration we had considered in the last section) can be obtained by setting $b=0$ in the above expression for $\mathcal{L}_{\text {corr }}$. Setting $b=0$, we obtain that

$$
\begin{equation*}
\mathcal{L}_{\text {corr }}=\frac{1}{16 \pi^{2}} \int_{0}^{\infty} \frac{d s}{s^{3}} e^{-m^{2} s}\left(\frac{q a s}{\sin (q a s)}\right) . \tag{1.203}
\end{equation*}
$$

The sine function in this integral has poles along the path of integration at $s=$ $(n \pi / q a)$ where $n=1,2, \ldots$. This integral can be evaluated by going around each one of these poles on small semicircles in the upper half of the complex $s$ plane. (This choice of the upper half plane is suggested by the general principle in field theory that $m^{2}$ should be treated as the limit of ( $m^{2}-i \epsilon$ ), where $\epsilon \rightarrow 0^{+}$. In the above integral for $\mathcal{L}_{\text {corr }}$, this is equivalent to treating ( $\left.q a\right)$ as the limit of $(q a+i \epsilon)$.) The residues of all these poles contribute to the imaginary part of $\mathcal{L}_{\text {corr }}$ with the result that

$$
\begin{equation*}
\operatorname{Im} \mathcal{L}_{\text {corr }}=\frac{(q E)^{2}}{16 \pi^{3}} \sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n^{2}}\right) \exp -\left(n \pi m^{2} / q E\right) \tag{1.204}
\end{equation*}
$$

where we have set $a=E$. The $n$-th term in this expression then corresponds to the probability of $n$-pairs of particles being produced (per unit volume per unit time) by the background electric field. Note that the above expression for $\mathcal{L}_{\text {corr }}$ is non-analytic in $q$; a perturbative series expansion even to all orders in $q$ would not have produced this result.

In the last section, when we had carried out a normal mode analysis of the quantum scalar field in the time dependent gauge $A_{1}^{\mu}$, we had obtained the number of particles produced in a single mode to be $|\beta(\mathbf{k})|^{2}$ (where $|\beta(\mathbf{k})|^{2}$ is given by equation (1.134)). The relative probability for pair creation in a single mode
is then given by

$$
\begin{equation*}
\mathcal{R}(\mathbf{k})=\left(\frac{|\beta(\mathbf{k})|^{2}}{|\alpha(\mathbf{k})|^{2}}\right)=\left(\frac{\exp -(\pi \lambda)}{1+\exp -(\pi \lambda)}\right), \tag{1.205}
\end{equation*}
$$

where $\lambda$ is given by equation (1.125). Therefore, the probability that no pair creation occurs in a single mode is then given by

$$
\begin{equation*}
\mathcal{P}(\mathbf{k})=(1-\mathcal{R}(\mathbf{k}))=\left(\frac{1}{1+\exp -(\pi \lambda)}\right) . \tag{1.206}
\end{equation*}
$$

The vacuum persistence probability will then be given by

$$
\begin{align*}
\left|\left\langle 0_{\text {out }} \mid 0_{\text {in }}\right\rangle\right|^{2}=\prod_{\mathbf{k}} \mathcal{P}(\mathbf{k}) & =\prod_{\mathbf{k}}\left(\frac{1}{1+\exp -(\pi \lambda)}\right) \\
& =\exp -\left\{\sum_{\mathbf{k}} \ln (1+\exp -(\pi \lambda))\right\} \\
& =\exp -2 \int d^{4} x \operatorname{Im} \mathcal{L}_{\text {corr }}, \tag{1.207}
\end{align*}
$$

where in the last equality we have introduced the imaginary part of the effective Lagrangian in the standard manner. This allows us to identify

$$
\begin{align*}
2 \int d^{4} x \operatorname{Im} \mathcal{L}_{\text {corr }} & =\sum_{\mathbf{k}} \ln (1+\exp -(\pi \lambda)) \\
& =\sum_{\mathbf{k}, n}\left(\frac{(-1)^{n+1}}{n}\right) \exp -(n \pi \lambda) . \tag{1.208}
\end{align*}
$$

Changing the summation to an integration by the rule

$$
\begin{equation*}
\sum_{\mathrm{k}} \longrightarrow V \int_{-\infty}^{\infty} \frac{d k_{x}}{2 \pi} \int_{-\infty}^{\infty} \frac{d k_{y}}{2 \pi} \int_{-\infty}^{\infty} \frac{d k_{z}}{2 \pi}=\frac{V}{(2 \pi)^{3}} \int_{-\infty}^{\infty} d k_{x} \int_{0}^{\infty} 2 \pi k_{\perp} d k_{\perp}, \tag{1.209}
\end{equation*}
$$

where, again, we have used the notation $k_{\perp}=\left|\mathbf{k}_{\perp}\right|$. We can now rewrite the $n$-th term in the above summation as

$$
\begin{align*}
\left(\frac{(-1)^{n+1}}{n}\right) & \left(\frac{V}{(2 \pi)^{3}}\right) \int_{-\infty}^{\infty} d k_{x} \int_{0}^{\infty} \pi d\left(k_{\perp}^{2}\right) \exp -\left\{n \pi\left(k_{\perp}^{2}+m^{2}\right) / q E\right\} \\
= & \left(\frac{(-1)^{n+1}}{n}\right)\left(\frac{V}{(2 \pi)^{3}}\right) \int_{-\infty}^{\infty} d k_{x}\left(\frac{q E}{n}\right) \exp -\left(n \pi m^{2} / q E\right) \\
= & \left(\frac{q E V}{(2 \pi)^{3}}\right) \int_{-\infty}^{\infty} d k_{x}\left(\frac{(-1)^{n+1}}{n^{2}}\right) \exp -\left(n \pi m^{2} / q E\right) \\
= & \left(\frac{(q E)^{2} V T}{(2 \pi)^{3}}\right)\left(\frac{(-1)^{n+1}}{n^{2}}\right) \exp -\left(n \pi m^{2} / q E\right) \tag{1.210}
\end{align*}
$$

In arriving at the last expression, we have interpreted a $\delta_{D}(0)$ as giving the rate per unit volume per unit time of physical process; since $k_{x}$ and ( $q E t$ ) have the same dimensions we had performed the integral over $k_{x}$ as an integral over ( $q E t$ ) for an interval $T$. We thus obtain the final result

$$
\begin{equation*}
\operatorname{Im} \mathcal{L}_{\text {corr }}=\frac{(q E)^{2}}{16 \pi^{3}} \sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n^{2}}\right) \exp -\left(n \pi m^{2} / q E\right) \tag{1.211}
\end{equation*}
$$

which exactly matches the result (1.204) we had obtained by evaluating the effective Lagrangian from the ground state energy of the quantum field.

On the other hand, if we set $a=0$ (this condition corresponds to the case of a pure magnetic field) in the expression (1.202), we obtain that

$$
\begin{equation*}
\mathcal{L}_{c o r r}=\frac{1}{16 \pi^{2}} \int_{0}^{\infty} \frac{d s}{s^{3}} e^{-m^{2} s}\left(\frac{q b s}{\sinh (q b s)}\right) \tag{1.212}
\end{equation*}
$$

(Note that with $b=B$, this expression is the same as equation (1.177).) This integral has no poles along the path of integration and hence does not have an imaginary part to it, which implies that a constant magnetic field does not produce particles. From these results, we can clearly conclude that a constant electromagnetic background can produce particles if and only if $\left(a^{2}-b^{2}\right)>0$, which is the same as the gauge invariant condition $\left(\mathbf{E}^{2}-\mathbf{B}^{2}\right)>0$.

### 1.6 Backreaction on the classical background

Until now, we have been studying the evolution of a quantum field in a given electromagnetic or gravitational background completely neglecting the backreaction of the quantum field on the classical background. If a particular background is capable of producing particles then the particles that have been produced will
certainly react back on the classical background. For instance, consider the electric field between a pair of capacitor plates. We would expect such a background to produce particles. The particles that have been produced will be attracted towards the capacitor plates thereby reducing the strength of the electric field between the plates.

Even in the absence of particle production, the polarization of the vacuum will effect the classical background non-trivially. For e.g., it is the vacuum polarization that leads to a nonzero attraction between Casimir plates (the Casimir force). This attraction will reduce the distance between the Casimir plates unless they are are held behind by an external agency. Such effects have to be accounted for if we are to study the evolution of the quantum fields in classical backgrounds more completely. In this section, we shall discuss a particular proposal that attempts to take into account the backreaction effects due to vacuum polarization as well as particle production.

Let us now consider a system which consists of a massless, real scalar field $\Phi$ coupled minimally to gravity. Such a system is described by the action (see, for instance, ref. [1], p. 43)

$$
\begin{align*}
\mathcal{S}\left[g_{\mu \nu}, \Phi\right] & =\int d^{4} x \sqrt{-g} \mathcal{L}\left(g_{\mu \nu}, \Phi\right) \\
& =\int d^{4} x \sqrt{-g}\left\{\frac{R}{16 \pi}+\frac{1}{2} g_{\mu \nu} \partial^{\mu} \Phi \partial^{\nu} \Phi\right\} \tag{1.213}
\end{align*}
$$

where $g_{\mu \nu}$ is the metric tensor describing the gravitational background and we have set $G=1$ for convenience. Just as we had defined an effective Lagrangian for the electromagnetic background in the last section, we can define an effective Lagrangian for the gravitational background by integrating the degrees of freedom corresponding to the quantum scalar field as follows (see, for e.g., ref. [4],
section 6.11):

$$
\begin{align*}
\exp i \mathcal{S}_{e f f}\left[g_{\mu \nu}\right] & =\exp i \int d^{4} x \sqrt{-g} \mathcal{L}_{e f f}\left(g_{\mu \nu}\right) \\
& \equiv \int \mathcal{D} \Phi \exp i \mathcal{S}\left[\Phi, g_{\mu \nu}\right] \tag{1.214}
\end{align*}
$$

The variation of the effective action $\mathcal{S}_{\text {eff }}$ with respect to the metric tensor then leads to the following equation (see, for e.g., ref. [1], section 6.1):

$$
\begin{equation*}
G_{\mu \nu}=R_{\mu \nu}-\frac{1}{2} R g_{\mu \nu}=8 \pi\left\langle 0_{o u t}\right| \hat{T}_{\mu \nu}\left|0_{i n}\right\rangle, \tag{1.215}
\end{equation*}
$$

where $\left|0_{\text {in }}\right\rangle$ and $\left|0_{\text {out }}\right\rangle$ are the in and the out-vacuum states respectively and $\hat{T}_{\mu \nu}$ is the energy-momentum operator corresponding to the quantum scalar field.

As we have mentioned in our discussion in the last section, the effective action is, in general, a complex quantity. Hence, the solutions to the semiclassical equation for the classical background (that is obtained by varying the effective action) will not always be real. For the gravitational case we are considering here, the metric $g_{\mu \nu}$ induced by the transition element $\left\langle 0_{o u t}\right| \hat{T}_{\mu \nu}\left|0_{\text {in }}\right\rangle$ in the semiclassical equation (1.215), will, in general, be a complex quantity. This is an undesirable feature. A simple prescription to avoid such a feature would be to throw away imaginary part of the effective action, thereby clearly ensuring that the solutions to the semiclassical equation are always real. But such a prescription would be completely ad hoc. Also, since it is the imaginary part of the effective action that reflects particle production, by throwing away the imaginary part we would in effect neglect the backreaction of the particles that have been produced on the classical background [32]. For these reasons, it is generally assumed that the backreaction of a quantum field on the classical metric is given by the expectation value of the energy-momentum tensor of the quantum field [27, 28, 29, 30, 31]. Since an expectation value is a real quantity, such a proposal ensures that the metric
induced by the quantum field is always real. Also, the expectation value of the energy-momentum tensor of the quantum field reflects both vacuum polarization as well as particle production. Thus, this proposal takes both these effects into account in the backreaction.

But, the semiclassical theory we are considering here is incapable of specifying a particular state for the quantum field. Hence, the expectation value of the energy-momentum tensor of the quantum field has to be evaluated in a state that is specified by hand. So, the complete analysis of the backreaction problem amounts to solving the semiclassical Einstein's equations

$$
\begin{equation*}
G_{\mu \nu}=8 \pi\left\langle\hat{T}_{\mu \nu}\right\rangle, \tag{1.216}
\end{equation*}
$$

where $\left\langle\hat{T}_{\mu \nu}\right\rangle$ is the expectation value (evaluated in a specified state) of the energymomentum operator corresponding to the quantum field $\Phi$ and the following Klein-Gordon equation satisfied by $\Phi$ :

$$
\begin{equation*}
\frac{1}{\sqrt{-g}} \partial_{\mu}\left(\sqrt{-g} g^{\mu \nu} \partial_{\nu}\right) \Phi=0 \tag{1.217}
\end{equation*}
$$

self-consistently.

## Chapter 2

## Finite time detectors

The original motivation behind the idea of detectors was to provide an operational definition for the concept of a particle in a general curved spacetime (see our discussion at the beginning of section 1.3). With this motivation in mind, the response of different types of detectors (the Unruh-DeWitt detector, derivative coupled detectors, a detector that is coupled to the energy-momentum tensor of the quantum field etc.) have been studied in literature [11, 12, 70, 71, 72]. In subsections 1.3.2 and 1.3.3, we had reviewed the analysis of the response of inertial and uniformly accelerated Unruh-DeWitt detectors in flat spacetime as well as the response of these detectors when they are stationed at a constant radius in Schwarzschild and de-Sitter spacetimes. The response of detectors have always been evaluated for their entire history, viz. from the infinite past to the infinite future in the detector's proper time. But, in any realistic situation, detectors can be kept switched on only for a finite period of time. Due to this reason the study of the response of a detector for a finite proper time interval becomes important.

There also exist other motivations to study the response of finite time detectors. Consider a detector that is coupled to the field in such a way that it responds
to the energy-momentum content of the quantum field. We can possibly utilize this detector to analyze the backreaction of the quantum field on the gravitational background as follows. This detector can be set in motion on a certain trajectory, in the spacetime of our interest and switched on for a finite proper time interval during its motion. Since we have always assumed a detector to be a point like object which can be described by a single classical worldline (see our discussion at the beginning of section 1.3), the response of this finite time detector will then reflect the particle content of the quantum field in that localized region of spacetime. We can then attempt to relate the response of this detector to the term that is responsible for the backreaction of the quantum field on the background metric in that localized region of spacetime.

Another motivation to study the finite time response of detectors is as follows. In a time dependent background without any asymptotically flat regions, like for instance, a matter dominated Friedmann universe, a timelike Killing vector will not be available at all. In the absence of a timelike Killing vector field, positive frequency modes and hence particles can not be defined unambiguously. In such a situation, a finite time detector can be used to provide an operational definition of the particle concept. Consider a comoving particle detector in the Friedmann universe that is switched on for a finite proper time interval. The response of such a detector will then reflect the particle content of the quantum field during the period when the detector was kept switched on.

The original idea of a finite time detector is due to Grove [85]. There has been a few attempts in literature in the recent past, when the response of a detector has been actually evaluated for a finite proper time interval [86, 87, 88]. The authors in ref. [86] study the response of a Unruh-Dewitt detector that is
turned on and off abruptly with the aid of a rectangular window function. They encounter an ultraviolet divergence and resort to a regularization procedure to remove this divergence. But, no realistic detector can be switched on and off abruptly. With this motivation, the authors in ref. [87] analyze the response of a Unruh-DeWitt detector that is switched on and off with a smooth window function. They point out that no divergences arise in the response function of the detector when it is switched on and off smoothly. They also show that in the limit when their window function matches a rectangular window function the ultraviolet divergence reported in ref. [86] does appear in the detector response function.

We reanalyze this problem in this chapter [89]. We begin by noting that a detector which is kept on only for a finite time interval $T$ will be affected by the transients related to the process of switching. This has the consequence that, even an inertial detector in flat spacetime will respond in the Minkowski vacuum if it is switched on for a finite $T$. This effect, as we shall see, needs to be clearly identified before one studies the response of a detector on an arbitrary trajectory for a finite $T$. Further, we expect the response to vanish in the limit of $T \rightarrow 0$ for any realistic detector on any trajectory. This is simply a physical requirement arising from the demand that 'a detector which was never switched on should not detect anything'. While this demand sounds reasonable, its mathematical implementation turns out to be fairly subtle. We will see that spurious results can arise if one does not implement the limiting procedure with care.

The response of a detector, as we had mentioned in section 1.3, depends on the following three elements: (i) the state of the quantum field, (ii) the trajectory of the detector and (iii) the nature of coupling that exists between the field and
the detector. Here, we shall assume the coupling between the detector and the field to be of the linear monopole type, i.e. the detector is the Unruh-DeWitt one. We consider inertial and uniformly accelerated detectors in flat spacetime. The quantum field we consider here is a massless scalar field and we shall assume that the quantum field is in the Minkowski vacuum state. We shall study the response of these detectors when they are switched on for a finite time interval smoothly as well as abruptly. Studying the response of detectors for these different window functions can help us identify the origin of the divergences that may arise in the detector response functions.

This chapter is organized as follows. In section 2.1, we comment on certain limiting procedures in the response function of the Unruh-DeWitt detector. In section 2.2 , we study the response of the detector which it is operational only for a finite interval of time; the cases of smooth window functions as well as that of abrupt switching are considered. In section 2.3, we discuss the conclusions that can be drawn from the analysis we have carried out in subsections 2.2.1, 2.2.2 and 2.2.3. In the same section, we also discuss the wider implications of our analysis. Finally, in section 2.4, we present the limitations of the detector concept.

### 2.1 Aspects of finite time detection

In this section, we point out certain aspects of finite time detection. We also illustrate here how spurious results can arise if the limiting procedures are not implemented with care.

We had seen earlier, in subsection 1.3.1, that up to the first order in per-
turbation theory, the amplitude for transition of a Unruh-DeWitt detector from its ground state with energy $E_{0}$ to an excited state with energy $E$, is given by

$$
\begin{equation*}
\mathcal{A}(\Omega)=\mathcal{M} \int_{-\infty}^{\infty} d \tau e^{i \Omega \tau}\left\langle\Psi_{f}\right| \Phi[x(\tau)]\left|\Psi_{i}\right\rangle \tag{2.1}
\end{equation*}
$$

where $\Omega=\left(E-E_{0}\right)$ and

$$
\begin{equation*}
\mathcal{M}=i c\langle E| m(0)\left|E_{0}\right\rangle, \tag{2.2}
\end{equation*}
$$

$m$ being the detector's monopole operator. (We will hereafter drop the term $\mathcal{M}$ in the transition amplitude for the same reasons we had given in subsection 1.3.1.) In the expression for the transition amplitude above, $\left|\Psi_{i}\right\rangle$ is the initial state of the quantum field, $\left|\Psi_{f}\right\rangle$ is the state of the quantum field after its interaction with the detector and $x^{\mu}(\tau)$ is the spacetime trajectory of the detector at proper time $\tau$. We will hereafter assume that the initial state of quantum field is the Minkowski vacuum state, i.e. $\left|\Psi_{i}\right\rangle=\left|0_{M}\right\rangle$.

If we now expand the scalar field $\Phi$ in terms of the standard Minkowski plane wave modes, it is clear from equation (2.1) that the nonzero contribution to the transition amplitude arises only from the state $\left|\Psi_{f}\right\rangle=\left|1_{\mathbf{k}}\right\rangle$ (since $\left|\Psi_{i}\right\rangle=\left|0_{M}\right\rangle$ ). For the case of an inertial trajectory in (1+1) dimensions, i.e.

$$
\begin{equation*}
x(\tau)=x_{0}+v t(\tau)=x_{0}+v \gamma \tau \tag{2.3}
\end{equation*}
$$

where $x_{0}$ and $v$ are constants, $\gamma=\left(1-v^{2}\right)^{-1 / 2}$ and $|v|<1$, the transition amplitude (2.1) turns out to be

$$
\begin{equation*}
\mathcal{A}_{\text {ine }, \omega}(\Omega)=\frac{e^{-i k x_{0}}}{\sqrt{4 \pi \omega}} \int_{-\infty}^{\infty} d \tau e^{i \Omega \tau} e^{i \gamma \tau(\omega-k v)} \tag{2.4}
\end{equation*}
$$

where $\omega=|k|$. The result of this integral is a Dirac delta function, i.e. we obtain that

$$
\begin{equation*}
\mathcal{A}_{i n e, \omega}(\Omega)=\sqrt{\frac{\pi}{\omega}} e^{-i k x_{0}} \delta_{D}(a)=0 \tag{2.5}
\end{equation*}
$$

where $a=(\Omega+\gamma(\omega-k v))$. The last equality in the above equation follows from noting that since, $k v \leq|k||v|<\omega$ and $\Omega>0$, the argument of the delta function is always greater than zero. As we had noted in the last chapter, the transition in the detector is essentially forbidden on the grounds of energy conservation.

The following points should be stressed regarding the above-apparently simple-calculation: The amplitude is being calculated for the system to make a transition from the state $\left|E_{0}\right\rangle$ in the infinite past, to the state $|E\rangle$ in the infinite future. To do so we need to know the trajectory $x^{\mu}(\tau)$ for all $\tau$, i.e. for $-\infty<\tau<$ $\infty$. No realistic detector can be kept switched on forever. Suppose the inertial detector was kept switched on only during the time interval $-T \leq \tau \leq T$; then the amplitude will be nonzero:

$$
\begin{align*}
\mathcal{A}_{\text {ine }, \omega}(\Omega, T) & =\frac{e^{-i k x_{0}}}{\sqrt{4 \pi \omega}} \int_{-T}^{T} d \tau e^{i \Omega \tau} e^{i \gamma \tau(\omega-k v)} \\
& =\frac{e^{-i k x_{0}}}{\sqrt{4 \pi \omega}}\left(\frac{2 \sin (a T)}{a}\right) \tag{2.6}
\end{align*}
$$

And, the probability for transition for a fixed $\omega$ will be

$$
\begin{align*}
\mathcal{P}_{\text {ine }, \omega}(\Omega, T) & =\left|\mathcal{A}_{\text {ine }, \omega}(\Omega, T)\right|^{2} \\
& =\frac{1}{\pi \omega}\left(\frac{\sin (a T)}{a}\right)^{2} \tag{2.7}
\end{align*}
$$

which is finite for all finite $T$. For small $T, \mathcal{P}_{\text {ine }, \omega} \propto T^{2}$ and hence vanishes as $T \rightarrow 0$; for large $T$, we use the relations

$$
\begin{align*}
\lim _{T \rightarrow \infty}\left\{\frac{\sin (a T)}{\pi a}\right\}^{2} & =\lim _{T \rightarrow \infty}\left\{\left(\lim _{T \rightarrow \infty} \frac{\sin (a T)}{\pi a}\right)\left(\frac{\sin (a T)}{\pi a}\right)\right\} \\
& =\lim _{T \rightarrow \infty}\left\{\delta_{D}(a) \frac{\sin (a T)}{\pi a}\right\} \\
& =\lim _{T \rightarrow \infty}\left\{\frac{T}{\pi} \delta_{D}(a)\right\} \tag{2.8}
\end{align*}
$$

i.e.

$$
\begin{equation*}
\lim _{T \rightarrow \infty}\left\{\frac{\mathcal{P}_{\text {ine }, \omega}(\Omega, T)}{T}\right\}=\frac{1}{\omega} \delta_{D}(a) . \tag{2.9}
\end{equation*}
$$

Clearly, the rate of transitions $\mathcal{R}_{\text {ine }, \omega}(\Omega, T)=\left(\mathcal{P}_{\text {ine, } \omega}(\Omega, T) / T\right)$ has the following behavior: $\mathcal{R}_{\text {ine, } \omega} \propto T$ for small $T$ and $\mathcal{R}_{\text {ine, } \omega} \propto \delta_{D}(a)$ for large $T$. Hence $\mathcal{R}_{\text {ine, } \omega}$ vanishes in both the limits.

The above analysis should teach us the following lessons. Firstly, even an inertial detector will respond if it is switched on and off. This is merely a manifestation of the energy-time uncertainty principle; a detection process lasting for a time $2 T$ can not measure energy differences with an accuracy greater than $(1 / 2 T)$. So for $(a 2 T) \lesssim 1$, the rate $\mathcal{R}_{\text {ine, } \omega}$ will be significantly nonzero. Secondly, the rate $\mathcal{R}_{\text {ine, } \omega}$ is a more reliable quantity to compute than $\mathcal{P}_{\text {ine }, \omega}$, especially if one is considering the $T \rightarrow \infty$ limit. In particular, $\mathcal{P}_{\text {ine }, \omega}$ is infinite if we take $T \rightarrow \infty$ limit naively in (2.7). Thirdly, if we want to study the response of accelerated detectors which are switched on only for a finite time, we must subtract out the result which is already present in the inertial case. This subtraction is mandatory since we want the response of the detector to reflect the effects that are uniquely due to its acceleration. Finally, the limits also need to be handled with care to obtain sensible results. We shall say more about the limiting procedures later on.

For the case of a uniformly accelerated trajectory in $(1+1)$ dimensions, the transformations from the Minkowski to the accelerated frame are

$$
\begin{equation*}
x=\xi \cosh (g \tau) \quad \text { and } \quad t=\xi \sinh (g \tau) \tag{2.10}
\end{equation*}
$$

where $\tau$ is the proper time of an observer with a proper acceleration $\xi^{-1}$ (cf. subsection 1.3.2). In what follows we shall set $\xi=g^{-1}$ without any loss of generality. The transition amplitude for a detector on such an accelerated trajectory turns
out to be

$$
\begin{equation*}
\mathcal{A}_{a c c, \omega}(\Omega)=\frac{1}{\sqrt{4 \pi \omega}} \int_{-\infty}^{\infty} d \tau e^{i \Omega \tau} \exp i g^{-1}(\omega \sinh g \tau-k \cosh g \tau) . \tag{2.11}
\end{equation*}
$$

For a wave traveling to the right, i.e. when $k=\omega$, the above integral can be expressed in a closed form and the result is

$$
\begin{equation*}
\mathcal{A}_{a c c, \omega}(\Omega)=\frac{1}{\sqrt{4 \pi \omega}} g^{-1}\left(\omega g^{-1}\right)^{i \Omega g^{-1}} \Gamma\left(-i \Omega g^{-1}\right) \exp -(\pi \Omega / 2 g) \tag{2.12}
\end{equation*}
$$

where $\Gamma(z)$ is the gamma function [77]. This is clearly nonzero. The probability for transition $\mathcal{P}_{\text {acc, } \omega}$ for a particular $\omega$ will then be given by

$$
\begin{align*}
\mathcal{P}_{a c c, \omega}(\Omega) & =\left|\mathcal{A}_{a c c, \omega}(\Omega)\right|^{2} \\
& =\frac{g^{-2}}{4 \pi \omega}\left|\Gamma\left(-i \Omega g^{-1}\right)\right|^{2} e^{-\pi \Omega g^{-1}} \\
& =\frac{1}{2 \omega g}\left(\frac{1}{\Omega\left(e^{2 \pi \Omega g^{-1}}-1\right)}\right) \tag{2.13}
\end{align*}
$$

which is a thermal spectrum in $\Omega$ with a temperature $T=(g / 2 \pi)$. We had encountered this thermal spectrum earlier, in subsection 1.3.2 when we had evaluated the transition probability rate of a uniformly accelerated Unruh-DeWitt detector.

The finite proper time integral for the transition amplitude of the accelerated detector, obtained after substituting for $x$ and $t$ from (2.10) in (2.1) is given by

$$
\begin{equation*}
\mathcal{A}_{a c c, \omega}(\Omega, T)=\frac{1}{\sqrt{4 \pi \omega}} J(\Omega, T) \tag{2.14}
\end{equation*}
$$

where

$$
\begin{equation*}
J(\Omega, T)=\int_{-T}^{T} d \tau e^{i \Omega \tau} \exp -\left(i \omega g^{-1} e^{-g \tau}\right) \tag{2.15}
\end{equation*}
$$

and we have assumed that $k=\omega$, i.e. the Minkowski normal mode is traveling to the right. This integral for $J(\Omega, T)$ can be rewritten as
$J(\Omega, T)=\int_{-\infty}^{\infty} d \tau e^{i \Omega \tau} e^{-i \omega g^{-1} e^{-g \tau}}-\int_{-\infty}^{T} d \tau e^{i \Omega \tau} e^{-i \omega g^{-1} e^{-g \tau}}-\int_{T}^{\infty} d \tau e^{i \Omega \tau} e^{-i \omega g^{-1} e^{-g \tau}}$.

After some simple algebraic manipulations, we obtain that

$$
\begin{align*}
J(\Omega, T)= & g^{-1}\left(\omega g^{-1}\right)^{i \Omega g^{-1}} e^{-\pi \Omega / 2 g}\left\{\Gamma\left(-i \Omega g^{-1}\right)\right. \\
& \left.-\int_{0}^{i \omega g^{-1} s^{-1}} d \tau e^{-\tau} \tau^{-i \Omega g^{-1}-1}-\int_{i \omega g^{-1} s}^{i \infty} d \tau e^{-\tau} \tau^{-i \Omega g^{-1}-1}\right\}, \tag{2.17}
\end{align*}
$$

where $s=e^{g T}$. Consider now the limit $T \rightarrow 0$, i.e. when the detector is not switched on at all. In this limit, $s \rightarrow 1$ and two integrals in the above expression add up to be the gamma function thereby reducing $J(\Omega, T)$ to be zero. In the other limit, i.e. when $T \rightarrow \infty, s \rightarrow \infty$ and $s^{-1} \rightarrow 0$, so that

$$
\begin{align*}
J(\Omega) & =J(\Omega, T \rightarrow \infty) \\
& =g^{-1}\left(\omega g^{-1}\right)^{i \Omega g^{-1}} \Gamma\left(-i \Omega g^{-1}\right) \exp -(\pi \Omega / 2 g) \tag{2.18}
\end{align*}
$$

and $\left(|J(\Omega)|^{2} / 4 \pi \omega\right)$ yields the thermal spectrum we had obtained earlier in (2.13). Thus we obtain reasonable results for both the limits $T \rightarrow 0$ as well as $T \rightarrow \infty$.

There is another feature that needs emphasis as regards both (2.13) and (2.7). These are probabilities for transition to fixed final states $\left|1_{k}\right\rangle$ characterized by a given momentum $k$. Normally one would like to integrate over all $k$ so as to find the net probability for the detector to have made a transition from $\left|E_{0}\right\rangle$ to $|E\rangle$. This will lead to an integral

$$
\begin{equation*}
I_{\text {ine }}=\int_{0}^{\infty} \frac{d \omega}{\omega}\left(\frac{\sin ^{2}((\Omega+\omega) T)}{(\Omega+\omega)^{2}}\right) \tag{2.19}
\end{equation*}
$$

in the case of (2.7) and to an integral

$$
\begin{equation*}
I_{a c c}=\int_{0}^{\infty} \frac{d \omega}{\omega} \tag{2.20}
\end{equation*}
$$

in the case of (2.13). Both these integrals are formally divergent. However, consider the limit

$$
\lim _{T \rightarrow \infty}\left(\frac{I_{\text {ine }}}{T}\right)=\int_{0}^{\infty} \frac{d \omega}{\omega}\left\{\lim _{T \rightarrow \infty}\left(\frac{1}{T} \frac{\sin ^{2}((\Omega+\omega) T)}{(\Omega+\omega)^{2}}\right)\right\}
$$

$$
\begin{equation*}
=\frac{1}{\pi} \int_{0}^{\infty} \frac{d \omega}{\omega} \delta_{D}(\Omega+\omega) \tag{2.21}
\end{equation*}
$$

If $\Omega>0, \omega>0$ the integrand identically vanishes and we may take this integral to be zero, thereby recovering the earlier result. (Also see ref. [87] for a similar discussion.) This result shows that ( $I_{\text {ine }} / T$ ) is formally divergent for all finite $T$ but can be interpreted to be zero as $T \rightarrow \infty$ ! Such a contradiction arises because of an illegitimate interchange of limits. We will elaborate on the limiting procedures later on.

The integral (2.19) is divergent in both the lower and the upper limits of $\omega$. The divergence for small $\omega$ (infra-red divergence) is a feature of massless scalar fields in $(1+1)$ dimensions. For the $(3+1)$ dimensional case, we will find, later in this chapter, that no infra-red divergences arise and only logarithmic divergences for large $\omega$ (ultra-violet divergences) are encountered. These ultraviolet divergences are the divergences that have been reported earlier in refs. [86] and [87]. We shall see later that the divergences in (2.19) for a finite $T$ can be attributed to the abrupt switching of the detector.

We shall gather here some of the results from subsection 1.3.1, we will need for our further discussion. The probability of transition of the Unruh-DeWitt detector is determined by the detector response function $\mathcal{F}(\Omega)$ which is described by the following integral (cf. equation (1.85)):

$$
\begin{equation*}
\mathcal{F}(\Omega)=\int_{-\infty}^{\infty} d \tau \int_{-\infty}^{\infty} d \tau^{\prime} e^{-i \Omega\left(\tau-\tau^{\prime}\right)} G^{+}\left[x(\tau), x\left(\tau^{\prime}\right)\right] \tag{2.22}
\end{equation*}
$$

Since we have assumed the initial state of the quantum field to be the Minkowski vacuum state, the Wightman function is defined as

$$
\begin{equation*}
G^{+}\left[x(\tau), x\left(\tau^{\prime}\right)\right]=\left\langle 0_{M}\right| \Phi(x) \Phi\left(x^{\prime}\right)\left|0_{M}\right\rangle . \tag{2.23}
\end{equation*}
$$

For inertial and uniformly accelerated trajectories in flat spacetime, the Wightman function corresponding to the Minkowski vacuum state is invariant under time translations in the reference frame of the detector. For such trajectories, the transition probability rate of the Unruh-DeWitt detector is given by the integral (cf. equation (1.88))

$$
\begin{equation*}
\mathcal{R}(\Omega)=\int_{-\infty}^{\infty} d \Delta \tau e^{-i \Omega \Delta \tau} G^{+}(\Delta \tau) \tag{2.24}
\end{equation*}
$$

The Wightman function for a massless scalar field corresponding to the Minkowski vacuum state in $(3+1)$ dimensions is given by equation (1.89). The trajectories of inertial and accelerated uniformly accelerated detectors are given by equations (1.82) and (1.92) respectively. The Wightman function (1.89) then reduces to

$$
\begin{equation*}
G_{i n e}^{+}(\Delta \tau)=\frac{-1}{4 \pi^{2}(\Delta \tau-i \epsilon)^{2}} \tag{2.25}
\end{equation*}
$$

for the case of the inertial trajectory and

$$
\begin{equation*}
G_{a c c}^{+}(\Delta \tau)=-\frac{1}{4 \pi^{2}} \sum_{n=-\infty}^{\infty}\left(\Delta \tau-i \epsilon+2 \pi i n g^{-1}\right)^{-2} \tag{2.26}
\end{equation*}
$$

for the case of the accelerated trajectory (cf. equations (1.90) and (1.97)). Note that the Wightman function (2.26) corresponds to that of a uniformly accelerated detector with a proper acceleration $g$.

To understand some of the subtlities mentioned earlier regarding the limiting procedures, we shall now present the following discussion.

Consider a Unruh-DeWitt detector which is moving on a trajectory $x^{\mu}(\tau)$ and is switched on during the interval $\tau=-T$ to $\tau=T$. The response of such a detector is governed by the integral

$$
\begin{equation*}
\mathcal{F}(\Omega, T)=\int_{-T}^{T} d \tau \int_{-T}^{T} d \tau^{\prime} e^{-i \Omega\left(\tau-\tau^{\prime}\right)} G^{+}\left(\tau, \tau^{\prime}\right) . \tag{2.27}
\end{equation*}
$$

We shall further assume that the trajectory of the detector is along the integral curve of a timelike Killing vector field so that $G^{+}\left(\tau, \tau^{\prime}\right)=G^{+}\left(\tau-\tau^{\prime}\right)$. It is clear from the above equation that $\mathcal{F} \rightarrow 0$ as $T \rightarrow 0$ irrespective of any other details. Also, we should recover the standard results when $T \rightarrow \infty$.

We shall now rewrite the integral (2.27) in different variables and then take the limits $T \rightarrow 0$ and $T \rightarrow \infty$. Changing the variables to

$$
\begin{equation*}
x=\left(\tau-\tau^{\prime}\right) \quad \text { and } \quad y=\left(\tau+\tau^{\prime}\right) \tag{2.28}
\end{equation*}
$$

we obtain that

$$
\begin{equation*}
\int_{-T}^{T} d \tau \int_{-T}^{T} d \tau^{\prime} e^{-i \Omega\left(\tau-\tau^{\prime}\right)} G^{+}\left(\tau-\tau^{\prime}\right)=\frac{1}{2} \int_{-2 T}^{2 T} d x \int_{-2 T+|x|}^{2 T-|x|} d y e^{-i \Omega x} G^{+}(x) \tag{2.29}
\end{equation*}
$$

where the factor $(1 / 2)$ is the Jacobian of the transformation from the $\left(\tau, \tau^{\prime}\right)$ coordinates to the $(x, y)$ coordinates. After integrating with respect to $y$, we find that

$$
\begin{equation*}
\mathcal{F}(\Omega, T)=\int_{-2 T}^{2 T} d x e^{-i \Omega x} G^{+}(x)(2 T-|x|) \tag{2.30}
\end{equation*}
$$

Let us now consider the limits $T \rightarrow \infty$ and $T \rightarrow 0$ of this integral. When $T \rightarrow \infty$, we get

$$
\begin{align*}
\mathcal{F}(\Omega) & =\mathcal{F}(\Omega, T \rightarrow \infty) \\
& =\lim _{T \rightarrow \infty}\left\{(2 T) \tilde{G}^{+}(\Omega)-\int_{-2 T}^{2 T} d x e^{-i \Omega x} G^{+}(x)|x|\right\}, \tag{2.31}
\end{align*}
$$

where $\tilde{G}^{+}(\Omega)$ is the Fourier transform of $G^{+}(x)$. Clearly,

$$
\begin{align*}
\mathcal{R}(\Omega) & =\lim _{T \rightarrow \infty}\left\{\frac{\mathcal{F}(\Omega, T)}{2 T}\right\} \\
& =\lim _{T \rightarrow \infty}\left\{\tilde{G}^{+}(\Omega)-\frac{1}{2 T} \int_{-\infty}^{\infty} d x e^{-i \Omega x} G^{+}(x)|x|\right\} \\
& =\tilde{G}^{+}(\Omega) \tag{2.32}
\end{align*}
$$

provided the second integral is well defined. This expression is finite and represents a constant rate of transition; we have thus recovered the standard result in the $T \rightarrow \infty$ limit.

Let us next consider the $T \rightarrow 0$ limit which is rather tricky. We need to evaluate

$$
\begin{equation*}
\mathcal{F}(\Omega, T \rightarrow 0)=\lim _{T \rightarrow 0}\left\{\int_{-2 T}^{2 T} d x e^{-i \Omega x} G^{+}(x)(2 T-|x|)\right\} . \tag{2.33}
\end{equation*}
$$

The integral over $x$ is confined to a small range $(-2 T, 2 T)$ around the origin. This implies that we can expand the integrand in a Taylor series around the origin to obtain the required limit. We write

$$
\begin{align*}
& e^{-i \Omega x} G^{+}(x) \simeq\left(1-i \Omega x-\frac{\Omega^{2} x^{2}}{2}+\cdots\right) \\
& \times\left(G^{+}(0)+G^{+^{\prime}}(0) x+G^{+^{\prime \prime}}(0) \frac{x^{2}}{2}+\cdots\right) . \tag{2.34}
\end{align*}
$$

Substituting this expression into (2.33) and performing the integration we obtain that

$$
\begin{align*}
\mathcal{F}(\Omega, T) & \simeq 4 T^{2} G^{+}(0)+\frac{4 T^{4}}{3}\left(G^{+^{\prime \prime}}(0)-\Omega^{2} G^{+}(0)-2 i \Omega G^{+^{\prime}}(0)\right)+O\left(\Omega^{4} T^{4}\right) \\
& \simeq 4 T^{2} G^{+}(0) \tag{2.35}
\end{align*}
$$

to the lowest order. All derivatives of $G^{+}(x)$ in $(3+1)$ dimensions behave as $\epsilon^{-n}$ at origin and in particular, $G^{+}(0)=\left(1 / 4 \pi^{2} \epsilon^{2}\right)$ giving

$$
\begin{equation*}
\mathcal{F}(\Omega, T) \simeq\left(\frac{T^{2}}{\pi^{2} \epsilon^{2}}\right) \tag{2.36}
\end{equation*}
$$

The above expression shows that care should be exercised when the limits $T \rightarrow 0$ and $\epsilon \rightarrow 0$ are taken. It is clear from the fundamental definition of the integral in (2.27) that we must have $\mathcal{F}(\Omega, T=0)=0$ for all regular integrands. If the integrand has a pole in the real axis (requiring an $i \epsilon$ prescription to give
meaning to the integral) then we should arrange the limiting procedure in such a way that $\mathcal{F}(\Omega, T=0)=0$. This can be achieved by using the rule that $\epsilon \rightarrow 0$ limit should be taken right at the end, after the limit $T \rightarrow 0$ has been taken. Since

$$
\begin{equation*}
\lim _{\epsilon \rightarrow 0}\left\{\lim _{T \rightarrow 0} \frac{T^{2}}{\epsilon^{2}}\right\}=0 \tag{2.37}
\end{equation*}
$$

while

$$
\begin{equation*}
\lim _{T \rightarrow 0}\left\{\lim _{\epsilon \rightarrow 0} \frac{T^{2}}{\epsilon^{2}}\right\}=\infty \tag{2.38}
\end{equation*}
$$

only the former ordering will provide physically reasonable results. This prescription is also necessary to ensure that $G^{+}(0), G^{+^{\prime}}(0), \ldots$ etc. exist in the Taylor expansion for $G^{+}(x)$. For $\epsilon=0$, this expansion ceases to exist.

In $(1+1)$ dimensions $G^{+}(x)$ has a logarithmic dependence in $x$; hence in the limit of small $T$ the detector response function will be modified to the form

$$
\begin{equation*}
\mathcal{F}(\Omega, T) \propto T^{2} \ln \left(\epsilon^{2}\right) \tag{2.39}
\end{equation*}
$$

Taking $T \rightarrow 0$ limit first will give the sensible result $\mathcal{F}(\Omega, T=0)=0$ while if $\epsilon \rightarrow 0$ limit is taken first we will obtain a logarithmic divergence. We had mentioned this logarithmic divergence earlier in the discussion following equations (2.19) and (2.20). We shall see explicit examples of such ambiguities (and their resolution) in the following section.

Having thus pointed out some generic features of finite time detection, we shall now analyze the response of detectors that are switched on for a finite proper time interval with different window functions.

### 2.2 Detector response with window functions

We shall now calculate the response of inertial and uniformly accelerated UnruhDeWitt detectors that are switched on and off with the aid of three different window functions. Hence, instead of working with (2.27), we will consider the integral of the form

$$
\begin{equation*}
\mathcal{F}(\Omega, T)=\int_{-\infty}^{\infty} d \tau \int_{-\infty}^{\infty} d \tau^{\prime} e^{-i \Omega\left(\tau-\tau^{\prime}\right)} W(\tau, T) W\left(\tau^{\prime}, T\right) G^{+}\left[x(\tau), x\left(\tau^{\prime}\right)\right] \tag{2.40}
\end{equation*}
$$

where $W(\tau, T)$ is a window function with the following properties:

$$
W(\tau, T) \approx \begin{cases}1 & \text { for }|\tau| \ll T  \tag{2.41}\\ 0 & \text { for }|\tau| \gg T\end{cases}
$$

The abrupt switching corresponds to

$$
\begin{equation*}
W_{3}(\tau, T)=\Theta(T-\tau)+\Theta(T+\tau) \tag{2.42}
\end{equation*}
$$

More gradual switching on and off can be achieved with the window functions like

$$
\begin{equation*}
W_{1}(\tau, T)=\exp -\left(\frac{\tau^{2}}{2 T^{2}}\right) \tag{2.43}
\end{equation*}
$$

or

$$
\begin{equation*}
W_{2}(\tau, T)=\exp -\left(\frac{|\tau|}{T}\right) \tag{2.44}
\end{equation*}
$$

The motivation to study the detector response with smooth window functions $W_{1}$ and $W_{2}$ are twofold. One is to carefully identify any divergence that may arise when a finite time detection is performed. And, the other is to check whether a certain lack of the smoothness in the window function is responsible for the appearance of divergences in the detector response, as it has been reported in refs. [86] and [87].

In the following three subsections we shall evaluate the response of inertial and uniformly accelerated finite time detectors that are switched on and off with the window functions $W_{1}, W_{2}$ and $W_{3}$, in that order.

### 2.2.1 Gaussian window function

The detector response integral with the window function $W_{1}$ is given by the integral

$$
\begin{align*}
& \mathcal{F}(\Omega, T)=\int_{-\infty}^{\infty} d \tau \int_{-\infty}^{\infty} d \tau^{\prime} e^{-i \Omega\left(\tau-\tau^{\prime}\right)} G^{+}\left[x(\tau), x\left(\tau^{\prime}\right)\right] \\
& \times \exp -(\tau / T)^{2} \exp -\left(\tau^{\prime} / T\right)^{2} \tag{2.45}
\end{align*}
$$

which can be rewritten as

$$
\begin{align*}
\mathcal{F}(\Omega, T)=\int_{-\infty}^{\infty} d \tau & \int_{-\infty}^{\infty} d \tau^{\prime} e^{-i \Omega\left(\tau-\tau^{\prime}\right)} G^{+}\left[x(\tau), x\left(\tau^{\prime}\right)\right] \\
& \times \exp -\left\{\frac{1}{2 T^{2}}\left[\left(\tau+\tau^{\prime}\right)^{2}+\left(\tau-\tau^{\prime}\right)^{2}\right]\right\} \tag{2.46}
\end{align*}
$$

Let us first consider the response of a detector on an inertial trajectory. Substituting the Wightman function (2.25) for the inertial trajectory in the above integral and performing the transformations (2.28) the integral for the detector response function simplifies to

$$
\begin{align*}
& \mathcal{F}_{\text {ine }}(\Omega, T)=-\left(\frac{1}{8 \pi^{2}}\right) \int_{-\infty}^{\infty} d y \exp -\left(y^{2} / 2 T^{2}\right) \\
& \times \int_{-\infty}^{\infty} d x e^{-i \Omega x}\left(\frac{\exp -\left(x^{2} / 2 T^{2}\right)}{(x-i \epsilon)^{2}}\right) \\
&=-\left(\frac{T}{\sqrt{32 \pi^{3}}}\right) I(\Omega, T), \tag{2.47}
\end{align*}
$$

where

$$
\begin{equation*}
I(\Omega, T)=\int_{-\infty}^{\infty} d x e^{-i \Omega x}\left(\frac{\exp -\left(x^{2} / 2 T^{2}\right)}{(x-i \epsilon)^{2}}\right) \tag{2.48}
\end{equation*}
$$

Writing the gaussian function in the above integral as a Fourier transform

$$
\begin{equation*}
\exp -\left(x^{2} / 2 T^{2}\right)=\left(\frac{T}{\sqrt{2 \pi}}\right) \int_{-\infty}^{\infty} d k e^{i k x} \exp -\left(k^{2} T^{2} / 2\right) \tag{2.49}
\end{equation*}
$$

and interchanging the order of integration, we obtain

$$
\begin{equation*}
I(\Omega, T)=\left(\frac{T}{\sqrt{2 \pi}}\right) \int_{-\infty}^{\infty} d k \exp -\left(k^{2} T^{2} / 2\right) \int_{-\infty}^{\infty} d x\left(\frac{e^{i(k-\Omega) x}}{(x-i \epsilon)^{2}}\right) \tag{2.50}
\end{equation*}
$$

When $k>\Omega$, the integral over $x$ can be performed as a contour integral by closing the contour in the upper half of the complex $x$-plane and the second order pole at $x=i \epsilon$ gives a non-trivial contribution to the integral. When $k<\Omega$ the contour has to be closed in the lower-half of the complex $x$-plane and since the integrand is analytic in this half the integral vanishes. Hence the lower limit of the $k$-integral is $\Omega$. After some manipulations and substituting this result in (2.47), we obtain

$$
\begin{equation*}
\mathcal{F}_{\text {ine }}(\Omega, T)=\left(\frac{e^{\Omega \epsilon}}{2 \pi}\right) e^{\epsilon^{2} / 2 T^{2}} \int_{r}^{\infty} d p e^{-p^{2}}(p-r) \tag{2.51}
\end{equation*}
$$

where

$$
\begin{equation*}
p=\frac{1}{\sqrt{2}}\left(k T+\frac{\epsilon}{T}\right) \quad \text { and } \quad r=\frac{1}{\sqrt{2}}\left(\Omega T+\frac{\epsilon}{T}\right) . \tag{2.52}
\end{equation*}
$$

Before proceeding further let us check whether the expression (2.51) gives sensible results for the limits $T \rightarrow 0$ and $T \rightarrow \infty$. Since this is an inertial detector we must have $\mathcal{F}(\Omega, T \rightarrow \infty)=0$; also for a detector on any trajectory we demand that $\mathcal{F}(\Omega, T=0)=0$. These two limits can be obtained from the above result. When $T \rightarrow \infty$, the lower and the upper limits of the $p$-integral in (2.51) coincide thereby giving a null result as expected for the inertial detector. (Note that for large $r$, the expression

$$
\begin{equation*}
r \int_{r}^{\infty} d p e^{-p^{2}} \simeq\left(\frac{e^{-r^{2}}}{2}\right)\left\{1+O\left(\frac{1}{r^{2}}\right)\right\} \tag{2.53}
\end{equation*}
$$

vanishes exponentially.) Hence, there is no ambiguity in this result.

Studying the limit $T \rightarrow 0$ of (2.51), when the window function is sharply peaked at the origin, has to be done more carefully. In this case, it matters crucially whether the limit $T \rightarrow 0$ is taken first and the condition $\epsilon \rightarrow 0$ is incorporated later or vice-versa. The earlier alternative is to be adopted (as mentioned earlier) for the reason that $\epsilon$ helps us to identify the poles in the contour integrals; hence unless and until all the other limits in the problem have already been taken care of, the limit on $\epsilon$ should not be incorporated. Then, as $T \rightarrow 0, r \rightarrow(\epsilon / \sqrt{2} T)$ and $\mathcal{F}_{\text {ine }}(\Omega, T)$ can be rewritten as

$$
\begin{align*}
\mathcal{F}_{\text {ine }}(\Omega, T)=\left(\frac{e^{\Omega \epsilon}}{2 \pi}\right) & e^{\epsilon^{2} / 2 T^{2}}\left\{\int_{(\epsilon / \sqrt{2} T)}^{\infty} d p e^{-p^{2}} p\right. \\
& \left.-\frac{\epsilon}{\sqrt{2} T}\left(\int_{0}^{\infty} d p e^{-p^{2}}-\int_{0}^{(\epsilon / \sqrt{2} T)} d p e^{-p^{2}}\right)\right\} . \tag{2.54}
\end{align*}
$$

The last term in the above expression is the error function and its asymptotic form for large arguments is as follows

$$
\begin{equation*}
\frac{2}{\sqrt{\pi}} \int_{0}^{x} d v e^{-v^{2}}=1-\frac{e^{-x^{2}}}{\sqrt{\pi}}\left(\frac{1}{x}-\frac{1}{2 x^{3}}+\frac{3}{4 x^{5}} \cdots\right) . \tag{2.55}
\end{equation*}
$$

Substituting the above expression in (2.54), we obtain the detector response as $T \rightarrow 0$ to be

$$
\begin{equation*}
\mathcal{F}_{\text {ine }}(\Omega, T \rightarrow 0)=\left(\frac{e^{\Omega \epsilon} T^{2}}{4 \pi \epsilon^{2}}\right) \rightarrow 0 \tag{2.56}
\end{equation*}
$$

for finite $\epsilon$. This expression has the same form as (2.36) and clearly illustrates the need to keep $\epsilon \neq 0$ till the end. Note that the detector response function as well the rate of transition $\mathcal{R}_{\text {ine }}(\Omega, T)=\left(\mathcal{F}_{\text {ine }}(\Omega, T) / T\right)$ vanish as $T \rightarrow 0$. The non-commutativity of the limiting procedure as regards $T \rightarrow 0, \epsilon \rightarrow 0$ in the detector response functions is evident due to the presence of factors like $(\epsilon / T)$.

If the condition $\epsilon \rightarrow 0$ is incorporated first in (2.51), the expression factorizes to

$$
\begin{equation*}
\mathcal{F}_{\text {ine }}^{\prime}(\Omega, T)=\frac{1}{2 \pi} \int_{(\Omega T / \sqrt{2})}^{\infty} d p e^{-p^{2}}\left(p-\frac{\Omega T}{\sqrt{2}}\right) . \tag{2.57}
\end{equation*}
$$

If we now take the limit $T \rightarrow 0$ we obtain that

$$
\begin{equation*}
\mathcal{F}^{\prime}{ }_{\text {ine }}(\Omega, T=0)=\frac{1}{2 \pi} \int_{0}^{\infty} d p e^{-p^{2}} p=\frac{1}{4 \pi} . \tag{2.58}
\end{equation*}
$$

As we have mentioned earlier, we expect the detector response function to go to zero in the limit of $T \rightarrow 0$ irrespective of any other details. We find that $\mathcal{F}_{\text {ine }}(\Omega, T)$ does not go to zero if we set $T=0$ after we have set $\epsilon=0$. On the other hand, $\mathcal{F}_{\text {ine }}(\Omega, T)$ vanishes if we take the limit $T \rightarrow 0$ before we set $\epsilon=0$. Therefore, it is quite clear that the procedure of setting $\epsilon$ to zero only after the $T \rightarrow 0$ limit has been taken is the proper one.

If we are only interested in finite, nonzero values of $T$ then we can set $\epsilon=0$ in the integral (2.51). The response of the inertial detector for a finite $T$ can then be written in a closed form as

$$
\begin{equation*}
\mathcal{F}_{\text {ine }}(\Omega, T)=\frac{1}{4 \pi}\left\{\exp -\left(\Omega^{2} T^{2} / 2\right)-(\Omega T / \sqrt{2}) \Gamma\left(\frac{1}{2}, \frac{\Omega^{2} T^{2}}{2}\right)\right\}, \tag{2.59}
\end{equation*}
$$

where $\Gamma(a, b)$ is the incomplete gamma function [77]. For $\Omega T \gg 1$, this expression has the asymptotic form

$$
\begin{equation*}
\mathcal{F}_{\text {ine }}(\Omega, T) \simeq\left(\frac{\exp -\left(\Omega^{2} T^{2} / 2\right)}{4 \pi \Omega^{2} T^{2}}\right) \tag{2.60}
\end{equation*}
$$

This shows that an inertial detector, switched on for a finite period of time, gives a nonzero response which goes to zero as $T \rightarrow \infty$.

Let us now carry out the same analysis for the accelerated detector. For this case, when the Wightman function (2.26) is substituted into (2.46) and the transformations (2.28) are performed, we find that

$$
\begin{align*}
\mathcal{F}_{a c c}(\Omega, T)=-\left(\frac{1}{8 \pi^{2}}\right) & \int_{-\infty}^{\infty} d y \exp -\left(y^{2} / 2 T^{2}\right) \\
& \times \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d x e^{-i \Omega x}\left(\frac{\exp -\left(x^{2} / 2 T^{2}\right)}{\left(x-i b_{n}\right)^{2}}\right) \tag{2.61}
\end{align*}
$$

where $b_{n}=\left(\epsilon-2 \pi g^{-1} n\right)$. With the aid of (2.49), the above integral can then be simplified to the form

$$
\begin{equation*}
\mathcal{F}_{a c c}(\Omega, T)=-\left(\frac{T}{\sqrt{32 \pi^{3}}}\right) \sum_{n=-\infty}^{\infty} I_{n}(\Omega, T) \tag{2.62}
\end{equation*}
$$

where

$$
\begin{equation*}
I_{n}(\Omega, T)=\left(\frac{T}{\sqrt{2 \pi}}\right) \int_{-\infty}^{\infty} d k \exp -\left(k^{2} T^{2} / 2\right) \int_{-\infty}^{\infty} d x\left(\frac{e^{i(k-\Omega) x}}{\left(x-i b_{n}\right)^{2}}\right) \tag{2.63}
\end{equation*}
$$

When $k>\Omega$, the $x$ integration can be performed by closing the contour in the upper half of the complex $x$-plane and the poles corresponding to the values of $n$ between $-\infty$ and zero contribute non-trivially to $\mathcal{F}_{\text {acc }}(\Omega, T)$ giving

$$
\begin{equation*}
\mathcal{F}_{a c c 1}(\Omega, T)=\frac{1}{2 \pi} \sum_{n=-\infty}^{0} e^{\Omega b_{n}} e^{b_{n}^{2} / 2 T^{2}} \int_{r^{\prime}}^{\infty} d p^{\prime} e^{-p^{\prime 2}}\left(p^{\prime}-r^{\prime}\right) \tag{2.64}
\end{equation*}
$$

where

$$
\begin{equation*}
p^{\prime}=\frac{1}{\sqrt{2}}\left(k T+\frac{b_{n}}{T}\right) \quad \text { and } \quad r^{\prime}=\frac{1}{\sqrt{2}}\left(\Omega T+\frac{b_{n}}{T}\right) . \tag{2.65}
\end{equation*}
$$

When $k<\Omega$, the contour can be closed in the lower half of the complex $x$-plane and the poles corresponding to the values of $n$ between one and infinity contribute non-trivially, with the result

$$
\begin{equation*}
\mathcal{F}_{a c c 2}(\Omega, T)=\frac{1}{2 \pi} \sum_{n=1}^{\infty} e^{\Omega b_{n}} e^{b_{n}^{2} / 2 T^{2}} \int_{-r^{\prime}}^{\infty} d p^{\prime} e^{-p^{\prime 2}}\left(p^{\prime}+r^{\prime}\right) \tag{2.66}
\end{equation*}
$$

The complete result is

$$
\begin{equation*}
\mathcal{F}_{a c c}(\Omega, T)=\mathcal{F}_{a c c 1}(\Omega, T)+\mathcal{F}_{a c c 2}(\Omega, T) \tag{2.67}
\end{equation*}
$$

i.e.

$$
\begin{align*}
\mathcal{F}_{a c c}(\Omega, T)= & \frac{1}{2 \pi} \sum_{n=-\infty}^{0} e^{\Omega b_{n}} e^{b_{n}^{2} / 2 T^{2}} \int_{r^{\prime}}^{\infty} d p^{\prime} e^{-p^{\prime 2}}\left(p^{\prime}-r^{\prime}\right) \\
& +\frac{1}{2 \pi} \sum_{n=1}^{\infty} e^{\Omega b_{n}} e^{b_{n}^{2} / 2 T^{2}} \int_{-r^{\prime}}^{\infty} d p^{\prime} e^{-p^{\prime 2}}\left(p^{\prime}+r^{\prime}\right) \tag{2.68}
\end{align*}
$$

Let us again check the two relevant limits. In the limit $T \rightarrow \infty$ the lower limits of the first and the second integrals in the above expression reduce to $\infty$ and $-\infty$ respectively, so that only $\mathcal{F}_{\text {acc } 2}(\Omega, T)$ contributes to the detector response. Evaluating the integral and then setting $\epsilon=0$, we obtain the standard result:

$$
\begin{align*}
\mathcal{R}_{a c c}(\Omega) & =\lim _{T \rightarrow \infty}\left\{\frac{\mathcal{F}_{a c c}(\Omega, T)}{T}\right\} \\
& =\frac{1}{\sqrt{8 \pi}}\left(\frac{\Omega}{e^{2 \pi g^{-1} \Omega}-1}\right) . \tag{2.69}
\end{align*}
$$

When $T \rightarrow 0$, we can perform the same analysis we had carried out earlier for the inertial detector. Since only the $n=0$ term in the series (2.68) contributes non-trivially; we obtain that

$$
\begin{equation*}
\mathcal{F}_{a c c}(\Omega, T \rightarrow 0)=\left(\frac{e^{\Omega \epsilon} T^{2}}{2 \pi \epsilon^{2}}\right) \rightarrow 0 . \tag{2.70}
\end{equation*}
$$

This is identical to the inertial detector result and shows that the transition probability (as well the rate) will go to zero as $T \rightarrow 0$.

The fact that both accelerated and inertial detectors give identical results for the $T \rightarrow 0$ limit is to be expected on physical grounds. The curvature of the trajectory can not make its presence felt for infinitesimal intervals and the response of the detector can not depend on parameters like $g$ which characterize the detector trajectory.

Note that, for any $T$, the detection is now due to two effects: (i) The trajectory being noninertial and (ii) the detector being kept switched on only for a finite time. The second effect is present even for a detector on an inertial trajectory. As we have mentioned earlier, it will be physically more useful to subtract the inertial response from the accelerated detector response to obtain
the effects that are uniquely due to (i). In this case, $\mathcal{F}_{\text {net }}(\Omega, T)=\left(\mathcal{F}_{\text {acc }}(\Omega, T)-\right.$ $\left.\mathcal{F}_{\text {ine }}(\Omega, T)\right)$ vanishes trivially for $T \rightarrow 0$.

It is possible to state some of these results in a greater generality for the gaussian window function. Note that for a detector moving along any trajectory for which $G^{+}\left[x(\tau), x\left(\tau^{\prime}\right)\right]=G^{+}\left(\tau-\tau^{\prime}\right)$ the response function is

$$
\begin{align*}
\mathcal{F}(\Omega, T) & =\int_{-\infty}^{\infty} d \tau \int_{-\infty}^{\infty} d \tau^{\prime} e^{-i \Omega\left(\tau-\tau^{\prime}\right)} G^{+}\left(\tau-\tau^{\prime}\right) \exp -\left\{\frac{1}{2 T^{2}}\left[\tau^{2}+\tau^{\prime 2}\right]\right\} \\
& =\frac{1}{2} \int_{-\infty}^{\infty} d y \exp -\left(y^{2} / 2 T^{2}\right) \int_{-\infty}^{\infty} d x e^{-i \Omega x} G^{+}(x) \exp -\left(x^{2} / 2 T^{2}\right) \\
& =\sqrt{\frac{\pi}{2}} T \int_{-\infty}^{\infty} d x e^{-i \Omega x} G^{+}(x) \exp -\left(x^{2} / 2 T^{2}\right) \tag{2.71}
\end{align*}
$$

We can write

$$
\begin{equation*}
f(x)\left[e^{-i \Omega x} G^{+}(x)\right]=f\left(i \frac{\partial}{\partial \Omega}\right)\left[e^{-i \Omega x} G^{+}(x)\right] \tag{2.72}
\end{equation*}
$$

for any function $f(x)$ which has a power series expansion around $x=0$. Hence we have

$$
\begin{align*}
\mathcal{F}(\Omega, T) & =\sqrt{\frac{\pi}{2}} T \int_{-\infty}^{\infty} d x \exp \left(\frac{1}{T} \frac{\partial^{2}}{\partial \Omega^{2}}\right)\left[e^{-i \Omega x} G^{+}(x)\right] \\
& =\exp \left(\frac{1}{2 T^{2}} \frac{\partial^{2}}{\partial \Omega^{2}}\right)[\mathcal{F}(\Omega)] \tag{2.73}
\end{align*}
$$

The expression in the square brackets is the result for the infinite time detector. (Note that $\mathcal{F}(\Omega)=\mathcal{F}(\Omega, T \rightarrow \infty)$.) The corresponding rates are

$$
\begin{equation*}
\mathcal{R}(\Omega, T)=\exp \left(\frac{1}{2 T^{2}} \frac{\partial^{2}}{\partial \Omega^{2}}\right)[\mathcal{R}(\Omega)] \tag{2.74}
\end{equation*}
$$

(Also note that $\mathcal{R}(\Omega)=\mathcal{R}(\Omega, T \rightarrow \infty)$.) This formula allows us to systematically calculate finite time corrections as a series in $(1 / T)$. To the lowest order, the correction is

$$
\begin{equation*}
\mathcal{R}(\Omega, T)=\mathcal{R}(\Omega)+\frac{1}{2 T^{2}} \frac{\partial^{2} \mathcal{R}(\Omega)}{\partial \Omega^{2}}+O\left(\frac{1}{T^{4}}\right) \tag{2.75}
\end{equation*}
$$

In the case of uniformly accelerated detector, up to the lowest order, we obtain that

$$
\begin{align*}
\mathcal{R}_{a c c}(\Omega, T) \simeq \mathcal{R}_{a c c}(\Omega) & \left\{1-\frac{2 \pi}{g \Omega T^{2}}\left(\frac{e^{2 \pi \Omega_{g}-1}}{\left(e^{2 \pi \Omega g^{-1}}-1\right)^{2}}\right)\right. \\
& \left.\times\left[e^{2 \pi \Omega g^{-1}}\left(1-\pi \Omega g^{-1}\right)-1-\pi \Omega g^{-1}\right]\right\} \tag{2.76}
\end{align*}
$$

The above expression thus gives corrections to the standard thermal response of an accelerated detector up to order $\left(1 / T^{2}\right)$ for a large $T$.

### 2.2.2 Window function with an exponential cut-off

Having studied the detector response with a gaussian window function, we shall now study the same with the window function $W_{2}$. In this case the response function turns out to be

$$
\begin{equation*}
\mathcal{F}(\Omega, T)=\int_{-\infty}^{\infty} d \tau \int_{-\infty}^{\infty} d \tau^{\prime} e^{-i \Omega\left(\tau-\tau^{\prime}\right)} G^{+}\left[x(\tau), x\left(\tau^{\prime}\right)\right] \exp -\left\{\frac{1}{T}\left(|\tau|+\left|\tau^{\prime}\right|\right)\right\} \tag{2.77}
\end{equation*}
$$

Introducing the window functions as Fourier transforms, i.e.

$$
\begin{equation*}
\exp -(|\tau| / T)=\int_{-\infty}^{\infty} d k f(k) e^{i k \tau} \quad \text { where } \quad f(k)=\frac{T}{\pi}\left(\frac{1}{1+k^{2} T^{2}}\right) \tag{2.78}
\end{equation*}
$$

and substituting the transformations (2.28), we obtain the response function of an inertial detector to be

$$
\begin{align*}
\mathcal{F}_{\text {ine }}(\Omega, T)=-\left(\frac{1}{8 \pi^{2}}\right) \int_{-\infty}^{\infty} d k f(k) & \int_{-\infty}^{\infty} d q f(q) \int_{-\infty}^{\infty} d y e^{i[y(k+q) / 2]} \\
& \times \int_{-\infty}^{\infty} d x\left(\frac{e^{i x[(k-q) / 2-\Omega]}}{(x-i \epsilon)^{2}}\right) \tag{2.79}
\end{align*}
$$

When the $y$ and the $q$-integrals in the above expression are performed, in that order, the result is

$$
\begin{equation*}
\mathcal{F}_{\text {ine }}(\Omega, T)=-\left(\frac{1}{2 \pi}\right) \int_{-\infty}^{\infty} d k f(k) f(-k) \int_{-\infty}^{\infty} d x\left(\frac{e^{i(k-\Omega) x}}{(x-i \epsilon)^{2}}\right) \tag{2.80}
\end{equation*}
$$

Performing the contour integral after substituting for $f(k)$, we find that the detector response function reduces to

$$
\begin{equation*}
\mathcal{F}_{\text {ine }}(\Omega, T)=\left(\frac{e^{\Omega \epsilon}}{6 \pi^{2}}\right) T^{2} \int_{\Omega}^{\infty} d k\left(\frac{(k-\Omega) e^{-\epsilon k}}{\left(1+k^{2} T^{2}\right)^{2}}\right) \tag{2.81}
\end{equation*}
$$

If $\epsilon$ is kept nonzero, the above expression, up to the lowest order in $T$, clearly dies down as $T^{2}$ as $T \rightarrow 0$. We can rewrite the above integral as

$$
\begin{equation*}
\mathcal{F}_{\text {ine }}(\Omega, T)=\left(\frac{e^{\Omega \epsilon}}{\pi^{2}}\right) \int_{\Omega T}^{\infty} d p e^{-p \epsilon / T}\left(\frac{p-\Omega T}{\left(1+p^{2}\right)^{2}}\right) \tag{2.82}
\end{equation*}
$$

where $p=k T$. When $T \rightarrow \infty$ the limits of the above integral coincide giving a null result as expected.

We again note the crucial role played by the $\epsilon$ factor. The limits $\epsilon \rightarrow 0$, $T \rightarrow 0$ do not (again!) commute in the function $\exp -(p \epsilon / T)$ :

$$
\begin{equation*}
\lim _{T \rightarrow 0}\left\{\lim _{\epsilon \rightarrow 0} \exp -(p \epsilon / T)\right\}=1 \tag{2.83}
\end{equation*}
$$

whereas

$$
\begin{equation*}
\lim _{\epsilon \rightarrow 0}\left\{\lim _{T \rightarrow 0} \exp -(p \epsilon / T)\right\}=0 \tag{2.84}
\end{equation*}
$$

If $\epsilon$ is set to zero in the integral (2.82), we obtain that

$$
\begin{equation*}
\mathcal{F}_{\text {ine }}^{\prime}(\Omega, T)=\left(\frac{1}{\pi^{2}}\right) \int_{\Omega T}^{\infty} d p\left(\frac{p-\Omega T}{\left(1+p^{2}\right)^{2}}\right) . \tag{2.85}
\end{equation*}
$$

When the limit $T \rightarrow 0$ is taken in the above integral, it reduces to

$$
\begin{equation*}
\mathcal{F}_{\text {ine }}^{\prime}(\Omega, T)=\left(\frac{1}{\pi^{2}}\right) \int_{0}^{\infty} \frac{d p p}{\left(1+p^{2}\right)^{2}}=\left(\frac{1}{2 \pi^{2}}\right) \tag{2.86}
\end{equation*}
$$

i.e. the detector response is nonzero even as $T \rightarrow 0$. As we have emphasised several times by now, a physically sensible result (that the response of the detector is zero when it is not switched on at all) can be obtained only if $\epsilon$ is kept nonzero until all the other limits have been taken.

If we are interested only in the $T \neq 0$ case, then we can set $\epsilon=0$ in (2.82). When $\epsilon$ is set to zero, we find that $\mathcal{F}_{\text {ine }}(\Omega, T)$ can be expressed in a closed form as follows:

$$
\begin{align*}
& \mathcal{F}_{\text {ine }}(\Omega, T)=\frac{1}{2 \pi^{2}}\left\{\frac{1}{\left(1+\Omega^{2} T^{2}\right)}-\frac{\Omega T}{2}(\pi-2 \arctan (\Omega T)\right. \\
& -\sin 2[\arctan (\Omega T)])\} . \tag{2.87}
\end{align*}
$$

For $\Omega T \gg 1$, this function behaves as

$$
\begin{equation*}
\mathcal{F}_{\text {ine }}(\Omega, T) \simeq\left(\frac{1}{6 \pi^{2} \Omega^{2} T^{2}}\right) \tag{2.88}
\end{equation*}
$$

We once again see that the inertial detector responds in the Minkowski vacuum if it is switched on only for a finite $T$. As $T \rightarrow \infty$, this response dies as $\left(1 / T^{2}\right)$.

Let us next consider the case of the accelerated detector. The response function of the accelerated detector is given by the integral

$$
\begin{align*}
& \mathcal{F}_{\text {acc }}(\Omega, T)=-\left(\frac{1}{8 \pi^{2}}\right) \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d k f(k) \int_{-\infty}^{\infty} d q f(q) \int_{-\infty}^{\infty} d y e^{i y(k+q) / 2} \\
& \times \int_{-\infty}^{\infty} d x\left(\frac{e^{i x[(k-q) / 2-\Omega]}}{\left(x-i b_{n}\right)^{2}}\right) \tag{2.89}
\end{align*}
$$

where $b_{n}=\left(\epsilon-2 \pi g^{-1} n\right)$. When the $y$ and the $q$-integrals are carried out, in that order, the detector response function reduces to

$$
\begin{equation*}
\mathcal{F}_{a c c}(\Omega, T)=-\left(\frac{1}{2 \pi}\right) \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d k f(k) f(-k) \int_{-\infty}^{\infty} d x\left(\frac{e^{i(k-\Omega)}}{\left(x-i b_{n}\right)^{2}}\right) \tag{2.90}
\end{equation*}
$$

The above contour integral can be performed in the same fashion as it was carried out in the previous subsection to give the following result:

$$
\begin{align*}
\mathcal{F}_{a c c}(\Omega, T)=\frac{1}{\pi^{2}} & \sum_{n=-\infty}^{0} e^{\Omega b_{n}} \int_{\Omega T}^{\infty} d p e^{-p b_{n} / T}\left(\frac{p-\Omega T}{\left(1+p^{2}\right)^{2}}\right) \\
& +\frac{1}{\pi^{2}} \sum_{n=1}^{\infty} e^{\Omega b_{n}} \int_{-\Omega T}^{\infty} d p e^{p b_{n} / T}\left(\frac{p+\Omega T}{\left(1+p^{2}\right)^{2}}\right), \tag{2.91}
\end{align*}
$$

where $p=k T$. When $T \rightarrow \infty$, the $\exp -\left(p b_{n} / T\right)$ factors in the integrand reduce to unity and the lower limit of the integrals are $\infty$ and $-\infty$ respectively. Since the limits coincide, the first integral vanishes. In the second integral, only the second term contributes, the first term being an odd function it reduces to zero on integration under symmetric limits. Thus, in the $T \rightarrow \infty$ limit, we recover the thermal spectrum after $\epsilon$ is set to zero:

$$
\begin{align*}
\mathcal{R}_{a c c}(\Omega) & =\lim _{T \rightarrow \infty}\left\{\frac{\mathcal{F}_{a c c}(\Omega, T)}{T}\right\} \\
& =\frac{\Omega}{2 \pi} \sum_{n=1}^{\infty} e^{-2 \pi g^{-1} \Omega n} \\
& =\frac{1}{2 \pi}\left(\frac{\Omega}{e^{2 \pi \Omega g^{-1}}-1}\right) . \tag{2.92}
\end{align*}
$$

As we have mentioned several times by now, $\left(\mathcal{F}_{\text {acc }}(\Omega, T) / T\right)$ is to be interpreted as the transition probability rate of the detector. When the $T \rightarrow 0$ limit is considered keeping $\epsilon \neq 0$, all the integrands in (2.91) decay exponentially thereby giving a null result.

Before concluding this section we shall provide an asymptotic formula for the detector response with any smooth window function of the form $W(\tau / T)$. This is a direct generalization of the results in (2.71) to (2.75). For such a window function we can write

$$
\begin{align*}
\mathcal{F}(\Omega, T) & =\int_{-\infty}^{\infty} d \tau \int_{-\infty}^{\infty} d \tau^{\prime} e^{-i \Omega\left(\tau-\tau^{\prime}\right)} W(\tau, T) W\left(\tau^{\prime}, T\right) G^{+}\left(\tau-\tau^{\prime}\right) \\
& =W\left(i \frac{\partial}{\partial \Omega}, T\right) W\left(-i \frac{\partial}{\partial \Omega}, T\right) \mathcal{F}(\Omega) \tag{2.93}
\end{align*}
$$

Expanding $W(\tau, T)=W(\tau / T)$ as a Taylor series around $\tau=0$ and assuming that $W(0)=1, W^{\prime}(0)=0$, i.e.

$$
\begin{align*}
W\left(\frac{\tau}{T}\right) & \simeq W(0)+W^{\prime}(0)\left(\frac{\tau}{T}\right)+\frac{1}{2} W^{\prime \prime}(0)\left(\frac{\tau}{T}\right)^{2} \\
& \simeq 1+\frac{1}{2} W^{\prime \prime}(0)\left(\frac{\tau}{T}\right)^{2}, \tag{2.94}
\end{align*}
$$

we obtain that

$$
\begin{align*}
\mathcal{F}(\Omega, T) & \simeq\left(1-\frac{W^{\prime \prime}(0)}{2 T^{2}} \frac{\partial^{2}}{\partial \Omega^{2}}\right)^{2} \mathcal{F}(\Omega) \\
& \simeq \mathcal{F}(\Omega)-\left(\frac{W^{\prime \prime}(0)}{T^{2}}\right) \frac{\partial^{2}[\mathcal{F}(\Omega)]}{\partial \Omega^{2}} . \tag{2.95}
\end{align*}
$$

This gives the rate to be

$$
\begin{equation*}
\mathcal{R}(\Omega, T)=\mathcal{R}(\Omega)-\left(\frac{W^{\prime \prime}(0)}{T^{2}}\right) \frac{\partial^{2}[\mathcal{R}(\Omega)]}{\partial \Omega^{2}}+O\left(\frac{1}{T^{4}}\right) \tag{2.96}
\end{equation*}
$$

for any window function and trajectory. Note that the response of a detector for a finite $T$ depends on the derivatives of the window function-for e.g. $W^{\prime \prime}(0)$. Hence, if the detector is switched on abruptly, these derivatives will diverge thereby leading to divergent responses. We shall discuss such a case explicitly in the following subsection.

### 2.2.3 A rectangular window function (sum of two step functions)

In this section we study the response of a detector that has been switched on and off abruptly. The detector response integral for this case is given by (2.27) and when the transformations (2.28) are carried out it reduces to (2.30), i.e.

$$
\begin{equation*}
\mathcal{F}(\Omega, T)=\int_{-2 T}^{2 T} d x e^{-i \Omega x} G^{+}(x)(2 T-|x|) \tag{2.97}
\end{equation*}
$$

For the response of an inertial detector that is turned on and off abruptly, the integrals to be evaluated are

$$
\begin{equation*}
\mathcal{F}_{\text {ine1 }}(\Omega, T)=-\left(\frac{T}{2 \pi^{2}}\right) \int_{-2 T}^{2 T} d x \frac{e^{-i \Omega x}}{(x-i \epsilon)^{2}} \tag{2.98}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathcal{F}_{\text {ine } 2}(\Omega, T)=\frac{1}{4 \pi^{2}} \int_{-2 T}^{2 T} d x \frac{e^{-i \Omega x}|x|}{(x-i \epsilon)^{2}}, \tag{2.99}
\end{equation*}
$$

so that

$$
\begin{equation*}
\mathcal{F}_{\text {ine }}(\Omega, T)=\mathcal{F}_{\text {ine } 1}(\Omega, T)+\mathcal{F}_{\text {ine } 2}(\Omega, T) \tag{2.100}
\end{equation*}
$$

The evaluation of the above integrals is discussed in detail in appendix A.1. The result is

$$
\begin{array}{r}
\mathcal{F}_{\text {ine }}(\Omega, T)=\frac{1}{4 \pi^{2}}\left\{2 \int_{0}^{\infty} d v \frac{e^{-\Omega v} v}{(v+\epsilon)^{2}}-e^{2 i \Omega T} \int_{0}^{\infty} d v \frac{e^{-\Omega v} v}{(v+\epsilon-2 i T)^{2}}\right. \\
\left.-e^{-2 i \Omega T} \int_{0}^{\infty} d v \frac{e^{-\Omega v} v}{(v+\epsilon+2 i T)^{2}}\right\} . \tag{2.101}
\end{array}
$$

For a finite $T$, if we take the limit $\epsilon \rightarrow 0$, the second and the third integrals in the above result remain finite; but the first integral diverges logarithmically. Hence $\mathcal{F}_{\text {ine }}(\Omega, T)$ is divergent for all finite $T$. This is the ultra-violet divergence that has been reported in ref. [86].

It was shown towards the end of the previous subsection that the response of a finite time detector and its rate involve the derivatives of the window function. The rectangular window function we consider here is continuous but has derivatives which diverge at $\tau=-T$ and $\tau=T$. The origin of the logarithmic divergences in $\mathcal{F}_{\text {ine }}$ and $\mathcal{R}_{\text {ine }}$ when $\epsilon$ is set to zero for a finite $T$ can be attributed to these divergent derivatives.

The two relevant limits, viz. $T \rightarrow 0$ and $T \rightarrow \infty$, however, give sensible results. When $T \rightarrow 0$, the second and the third integrals exactly cancel the first and hence $\mathcal{F}_{\text {ine }}(\Omega, T=0)=0$, provided we keep $\epsilon \neq 0$. For large $T$, i.e. when $T \rightarrow \infty$ the rate $\mathcal{R}_{\text {ine }}(\Omega, T)=\left(\mathcal{F}_{\text {ine }}(\Omega, T) / T\right)$ vanishes because $\mathcal{F}_{\text {ine }}$ is bounded (when $\epsilon \neq 0$ ) and well defined.

For a small $T$ and a finite $\epsilon$, such that $T<\epsilon$ the integrands in (2.101) can
be Taylor expanded in $T$ and the result up to $O\left(T^{2}\right)$ is

$$
\begin{equation*}
\mathcal{F}_{\text {ine }}(\Omega, T) \simeq\left(\frac{1}{\pi^{2}}\right)\left\{2 \Omega^{2} T^{2} I_{1}(\Omega)+\frac{2 \Omega T^{2}}{\epsilon} I_{2}(\Omega)+\frac{6 T^{2}}{\epsilon^{2}} I_{3}(\Omega)\right\} \tag{2.102}
\end{equation*}
$$

where

$$
\begin{equation*}
I_{1}(\Omega)=\int_{0}^{\infty} d y \frac{e^{-\Omega \epsilon y} y}{(y+1)^{2}}, \quad ; \quad I_{2}(\Omega)=\int_{0}^{\infty} d y \frac{e^{-\Omega \epsilon y} y}{(y+1)^{3}} \tag{2.103}
\end{equation*}
$$

and

$$
\begin{equation*}
I_{3}(\Omega)=\int_{0}^{\infty} d y \frac{e^{-\Omega \epsilon y} y}{(y+1)^{4}} \tag{2.104}
\end{equation*}
$$

Since the quantities $I_{1}, I_{2}$ and $I_{3}$ are independent of $T$, from the above expression it is easy to see that $\mathcal{F}_{\text {ine }}(\Omega, T)$ dies down as $T^{2}$ in the limit of $T \rightarrow 0$.

For the finite time response of an accelerated detector, the integrals to be evaluated are almost similar to those of the inertial case. The response function is given by

$$
\begin{equation*}
\mathcal{F}_{\text {acc }}(\Omega, T)=-\frac{1}{4 \pi^{2}} \sum_{n=-\infty}^{\infty} \int_{-T}^{T} d \tau^{\prime} \int_{-T}^{T} d \tau \frac{e^{-i \Omega\left(\tau-\tau^{\prime}\right)}}{\left(\tau-\tau^{\prime}-i b_{n}\right)^{2}}, \tag{2.105}
\end{equation*}
$$

where $b_{n}=\left(\epsilon-2 \pi g^{-1} n\right)$. After carrying out the transformations (2.28) we obtain the response function to be

$$
\begin{equation*}
\mathcal{F}_{a c c}(\Omega, T)=\sum_{n=-\infty}^{\infty} \mathcal{F}_{a c c 1 n}(\Omega, T)+\mathcal{F}_{a c c 2 n}(\Omega, T), \tag{2.106}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{F}_{a c c 1 n}(\Omega, T)=-\left(\frac{T}{2 \pi^{2}}\right) \int_{-2 T}^{2 T} d x \frac{e^{-i \Omega x}}{\left(x-i b_{n}\right)^{2}} \tag{2.107}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathcal{F}_{a c c 2 n}(\Omega, T)=\frac{1}{4 \pi^{2}} \int_{-2 T}^{2 T} d x \frac{e^{-i \Omega x}|x|}{\left(x-i b_{n}\right)^{2}} . \tag{2.108}
\end{equation*}
$$

The evaluation of the above integrals is discussed in detail in appendix A.2. The result is

$$
\mathcal{F}_{a c c}(\Omega, T)=\frac{1}{4 \pi^{2}} \sum_{n=-\infty}^{\infty}\left\{4 \pi \Omega T \Theta(n) e^{\Omega b_{n}}+2 \int_{0}^{\infty} d v \frac{e^{-\Omega v} v}{\left(v+b_{n}\right)^{2}}\right.
$$

$$
\begin{align*}
& -e^{2 i \Omega T} \int_{0}^{\infty} d v \frac{e^{-\Omega v} v}{\left(v+b_{n}-2 i T\right)^{2}} \\
& \left.-e^{-2 i \Omega T} \int_{0}^{\infty} d v \frac{e^{-\Omega v} v}{\left(v+b_{n}+2 i T\right)^{2}}\right\} \tag{2.109}
\end{align*}
$$

where $\Theta(n)=1$ for $n>0$ and zero otherwise.

The nature of divergence in this expression is the same as that of (2.101). This is because, for a finite $T$ when $\epsilon$ is set to zero it is the second integral in the above expression that diverges logarithmically (when $n=0$ in the sum), which is exactly the term that exhibits a divergence for case of the inertial detector. As we have mentioned before, the response of the inertial detector has to be subtracted from the response of the accelerated detector to give sensible results.

In the limit $T \rightarrow 0$, for a nonzero $\epsilon$, the expression (2.109) reduces to zero, the first term vanishing identically being proportional to $T$; the second term being cancelled by the third and the fourth. Whereas in the infinite time limit, concentrating on the transition probability rate we obtain that

$$
\begin{align*}
\mathcal{R}_{a c c}(\Omega) & =\lim _{T \rightarrow \infty}\left\{\frac{\mathcal{F}_{a c c}(\Omega, T)}{2 T}\right\} \\
& =\frac{1}{4 \pi^{2}} \sum_{n=-\infty}^{\infty} 2 \pi \Omega \Theta(n) e^{\Omega b_{n}}=\frac{1}{2 \pi}\left(\frac{\Omega}{e^{2 \pi \Omega g^{-1}}-1}\right), \tag{2.110}
\end{align*}
$$

the thermal spectrum seen by the accelerated detector, the other terms in (2.109) vanishing when divided by the infinite time interval.

### 2.3 Discussion

To clearly illustrate the conclusions we wish to draw from the analysis we have carried until now in this chapter, we tabulate here the response of an inertial detector for $T \rightarrow 0$, finite $T$ and $T \rightarrow \infty$, when the limits on $\epsilon$ and $T$ are taken

Table 2.1: $\mathcal{F}_{\text {ine }}(\Omega, \epsilon, T)$ and $\mathcal{R}_{\text {ine }}(\Omega, \epsilon, T)$ in different limits

|  |  | Gaussian | Exponential | Rectangular |
| :---: | :---: | :---: | :---: | :---: |
| $\lim _{T \rightarrow 0} \lim _{\epsilon \rightarrow 0}$ | $\mathcal{F}_{\text {ine }}(\Omega, \epsilon, T)$ | $(1 / 4 \pi)$ | $\left(1 / 2 \pi^{2}\right)$ | $\ln (\epsilon)-\ln (T)$ <br> (Divergence) |
| $\lim _{\epsilon \rightarrow 0} \lim _{T \rightarrow 0}$ | $\mathcal{F}_{\text {ine }}(\Omega, \epsilon, T)$ | 0 | 0 | 0 |
| $\lim _{T \rightarrow \infty} \lim _{\epsilon \rightarrow 0}$ | $\mathcal{R}_{\text {ine }}(\Omega, \epsilon, T)$ | 0 | 0 | $\begin{gathered} \ln (\epsilon) \\ \text { (Divergence) } \end{gathered}$ |
| $\lim _{\epsilon \rightarrow 0} \lim _{T \rightarrow \infty}$ | $\mathcal{R}_{\text {ine }}(\Omega, \epsilon, T)$ | 0 | 0 | 0 |
| $\lim _{\epsilon \rightarrow 0} T \neq 0$ | $\mathcal{F}_{\text {ine }}(\Omega, \epsilon, T)$ | Finite | Finite | $\ln (\epsilon)$ <br> (Divergence) |
| $\lim _{T \rightarrow 0} \epsilon \neq 0$ | $\mathcal{F}_{\text {ine }}(\Omega, \epsilon, T)$ | 0 | 0 | 0 |
| $\lim _{T \rightarrow \infty} \epsilon \neq 0$ | $\mathcal{R}_{\text {ine }}(\Omega, \epsilon, T)$ | 0 | 0 | 0 |

in different orders. Note that $\mathcal{F}_{\text {ine }}$ and $\mathcal{R}_{\text {ine }}$ are functions of $\epsilon$ before it is set to zero. In the last column of table 2.1, whenever divergences arise we have just quoted the divergent terms dropping the finite expressions. The second and the fourth rows of the above table imply that when $\epsilon$ is kept nonzero the response of an inertial detector and its rate go to zero as $T \rightarrow 0$ and $T \rightarrow \infty$, respectively, for all window functions. This is just reiterated in the last two rows. When the $\epsilon \rightarrow 0$ limit is taken first, and the $T$ is set to zero after, as the first row of the
above table shows, the detector response does not go to zero and in fact, for the case of the rectangular window function, logarithmic divergences are encountered. When the $T \rightarrow \infty$ limit is considered after having set $\epsilon=0$ (third row) and when the detector has been switched on with the rectangular window function, logarithmic divergences appear in the detector response rate. Finally, for a finite T when $\epsilon$ has been set to zero (fifth row) logarithmic divergences arise again in the detector response for the case of the rectangular window function. The divergences that are listed in the first and the fifth rows of the above table for the case of the rectangular window function have been reported earlier in literature [86].

The role played by $\epsilon$ in producing the finite result for the different limits is by now obvious. In fact, by keeping $\epsilon$ finite till the end we are effectively introducing an ultra-violet cut-off. This can be seen by expressing the Wightman function (1.89) as

$$
\begin{align*}
G^{+}\left(x, x^{\prime}\right) & =\int \frac{d^{3} \mathbf{k}}{(2 \pi)^{3} 2 \omega} e^{-i \omega\left(t-t^{\prime}-i \epsilon\right)+i \mathbf{k} \cdot\left(\mathbf{x}-\mathrm{x}^{\prime}\right)} \\
& =\int \frac{d^{3} \mathbf{k}}{(2 \pi)^{3} 2 \omega} e^{-i \omega\left(t-t^{\prime}\right)+i \mathbf{k} \cdot\left(\mathrm{x}-\mathrm{x}^{\prime}\right)} e^{-\epsilon \omega} \tag{2.111}
\end{align*}
$$

where $\omega=|\mathbf{k}|$. The results for the limits $T \rightarrow 0$ and $T \rightarrow \infty$ remain sensible even after the cut-off is removed, provided it is done right at the end.

The logarithmic divergences that appear in the response of a detector (for a finite $T$ ) when it is switched on abruptly can be attributed to the discontinuities that arise in the derivatives of the window function. These divergences are certainly not the infinities that are inherent to quantum field theory, for had they been so, the response functions of the detectors would have diverged irrespective of the manner in which the detectors are switched on and off.

We shall now touch upon the relevance of the analysis we have carried in
this chapter in a somewhat broader context.

In bringing together the principles of quantum theory and general relativity one notices a major issue of conflict: General relativity is inherently local in its description while the conventional formulation of field theory uses global structures to define even the most primitive concepts like the vacuum state. This point has been repeatedly made in the literature related to quantum gravity. However, it should also be noted that there is another, operational angle to the quantum theory as well. Quantum mechanics emphasizes the role of operational definition of physical quantities including that of the quantum state. As a matter of principle the same philosophy should be applicable to the field theory as well. In other words, one would like to define concepts like vacuum state etc. in field theory using purely operational procedures similar to the ones used, for example in defining the spin of an electron by using a magnetic field selector.

It is, however, well-known that such procedures are exceedingly difficult to formulate in the case of a relativistic field. The role of detectors assumes special importance in this context. The work by Unruh and DeWitt comes closest to the operational definition of quantum states in field theory. In a simplified sense this detector model captures the essence of the actual particle detection which takes place in the laboratory. There is, however, one difficulty in the original UnruhDeWitt model. This model uses the definition for detection which is based on asymptotic states. The calculations are done to estimate the transition probability from past infinity to future infinity. In any laboratory context, any detection is local in both space and time.

The analysis we have carried out in this chapter makes a first attempt at
investigating the possibility of a localized detection, in space as well as time. We have resolved the difficulties which arise in such a detection and we have provided general formulas to calculate the response of detectors which have been coupled to the field only for a finite interval of time. It will be worthwhile to investigate how these finite time detectors respond in curved spacetimes while on geodesic and non-geodesic trajectories. (Earlier, in subsection 1.3.3, we had found that the response of a Unruh-DeWitt detector that is stationed at a constant radius in Schwarzschild and de-Sitter spacetimes is similar to the response of a uniformly accelerated detector in Minkowski vacuum state. Hence, the analysis we have carried out in this chapter can be trivially extended to detectors stationed at a constant radius in these two spacetimes.) Since these toy-models mimic the physical situation as regards locality in space and time, we can expect the results to shed some light on the operational definition of quantum processes in curved spacetimes.

### 2.4 Limitations of the detector concept

It has been repeatedly pointed out in literature that, though the transition probability rate of a uniformly accelerated Unruh-DeWitt detector (for an infinite time interval) yields the same result as the expectation value of the Rindler number operator in the Minkowski vacuum state, this concurrence is purely coincidental. There exist other noninertial frames in which the expectation value of the number operator (corresponding to the noninertial coordinate system) in the Minkowski vacuum does not match the response of the Unruh-DeWitt detector [10]. For example, in a rotating coordinate system the expectation value of its number operator in the Minkowski vacuum state proves to be zero, whereas the transition
probability rate of a rotating Unruh-DeWitt detector turns out to be nonzero (see, for instance, refs. [10, 13]; however, see ref. [90]). This should not come as a surprise since the Unruh-DeWitt detector does not respond to the particle content of the quantum field but acts as a fluctuometer that measures the power spectrum of fluctuations in the quantum field. (We had, in fact, discussed this aspect earlier towards the end of subsection 1.3.2.)

In fact, all the detectors that have been constructed along the lines of the Unruh-DeWitt detector (the derivative coupled detector, the detector coupled to the energy-momentum tensor of the quantum field) respond to fluctuations in the term that couples the detector to the quantum field. For instance, a detector that is coupled to the energy-momentum tensor of the quantum field responds to the power spectrum of the fluctuations in the energy-momentum tensor of the quantum field, whereas, ideally, we would have liked our detector to measure the expectation value of the energy-momentum tensor [72].

Another drawback of the detector idea is that the response of different detectors have always been evaluated only up to the first order in perturbation theory. Evaluating higher order corrections to the detector response is an involved task and it is not clear whether any generic statements can be made about these corrections. These corrections can prove to be important when we attempt to compare the results obtained from the canonical quantization procedure with those of the detector response. Also, a detector that we have considered here has a classically well-defined trajectory and hence occupies a single worldline. Whereas a coordinate system covers an entire patch. The fact that a particular coordinatization of a spacetime actually matches the worldines of certain observers in that spacetime is a very special feature. One could equally well choose a different coordinatiza-
tion of that spacetime which nevertheless happens to coincide in the vicinity of one particular detector's worldline. This feature has in fact forced Padmanabhan and Singh to conclude that while it may be possible to maintain formal covariance with elaborate regularization procedures, operational covariance is completely lost at a very fundamental level in quantum theory (see ref. [72]; also see ref. [91]).

Recently, Ford and Roman have put forward a proposal for measuring the energy-momentum content of a quantum scalar field with the help of detectors [92]. They set a bunch of finite time Unruh-DeWitt and derivative coupled detectors in motion on certain trajectories and then attempt to relate the combined response of all these detectors to the expectation value of the energy-momentum tensor of the quantum field. Other than the fact that these detectors have to be switched very rapidly to actually measure the expectation value of the energy-momentum tensor at any particular point in spacetime, the prescription due to Ford and Roman still possesses all the drawbacks we have discussed in the last paragraph.

There are quantum states; there are detector measurements. What we mean by a particle can not be sensibly expressed without any reference to a detector. All we can predict and discuss are the experiences of detectors. A finite time particle detector is an operational idea that offers some scope for a localized view of a quantum particle. However, in quantum field theory, the concept of a particle, as defined through Fock spaces is a global one. Also, in a curved spacetime, in general, the definition of a particle is not unique. Until we understand these different aspects better, the connection between particles and the response of detectors is bound to be obscure. The construction of a detector that actually responds to the particle content of the quantum field would possibly help us bridge this gap in our understanding.

## Chapter 3

## Quantum field theory in classical electromagnetic backgrounds

Just as there exists a semiclassical regime for the gravitational field, wherein we can analyze the behavior of quantum fields in classical gravitational backgrounds, a similar domain exists for the electromagnetic field too [93, 94]. The existence of such a domain allows us to study the evolution of quantum fields in classical electromagnetic backgrounds. Phenomena such as vacuum polarization and particle production take place in electromagnetic backgrounds too (see our discussion in section 1.4 and subsection 1.5.1). In this chapter, we shall study the evolution of quantum fields in classical electromagnetic backgrounds with the same motivation we had mentioned earlier, viz. that such a study will provide us with some insights to understand the gravitational case. In fact, we will see later in this chapter, that there do exist some common features in the behavior of quantum fields in electromagnetic and gravitational backgrounds which can be exploited to help us improve our understanding of the semiclassical regime.

This chapter is organized as follows. In section 3.1, we outline Schwinger's proper time formalism to evaluate the effective Lagrangian for a given classical
electromagnetic background. In section 3.2 , we examine the validity of the tunneling interpretation by comparing this approach with the effective Lagrangian approach for a time independent magnetic field background. We also discuss in detail the implications of this comparison to the study of particle production in time independent electromagnetic and gravitational backgrounds. In section 3.3, we present the limitations of the Klein approach that is invoked to explain the phenomenon of particle production in time independent electromagnetic backgrounds. In section 3.4, we propose the conjecture that the effective Lagrangian will prove to be zero if all the invariant scalars (gauge invariant scalars in the case of electromagnetism and covariant scalars in the case of gravity) describing the classical background vanish identically. We also present examples of electromagnetic and gravitational backgrounds to support our conjecture. Evaluating the effective Lagrangian explicitly, using Schwinger's proper time formalism, we show that it vanishes in these backgrounds. In the same section, we also discuss the wider implications of our conjecture. We conclude this chapter with section 3.5, wherein we make a few remarks regarding the boundary conditions that are implicitly assumed in the evaluation of the effective Lagrangian using Schwinger's formalism.

### 3.1 Schwinger's proper time formalism for evaluating effective Lagrangians

The system we shall mostly deal with in this chapter is the same system we had considered earlier in section 1.4 and also in subsection 1.5.1. It consists of a complex scalar field $\Phi$ interacting with an electromagnetic field represented by
the vector potential $A^{\mu}$. It is described by the action

$$
\begin{align*}
\mathcal{S}\left[\Phi, A^{\mu}\right]= & \int d^{4} x \mathcal{L}\left(\Phi, A_{\mu}\right) \\
= & \int d^{4} x\left\{\left(\partial_{\mu} \Phi+i q A_{\mu} \Phi\right)\left(\partial^{\mu} \Phi^{*}-i q A^{\mu} \Phi^{*}\right)\right. \\
& \left.-m^{2} \Phi \Phi^{*}-\frac{1}{4} F^{\mu \nu} F_{\mu \nu}\right\} \tag{3.1}
\end{align*}
$$

where, as before, $q$ and $m$ correspond to the charge and the mass associated with a single quantum of the complex scalar field, the asterisk denotes complex conjugation and

$$
\begin{equation*}
F_{\mu \nu}=\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu} \tag{3.2}
\end{equation*}
$$

We shall assume that the electromagnetic field behaves classically and we shall consider the complex scalar field to be a quantum field. Also, we shall assume that the electromagnetic field is given to us a priori, i.e. we will not take into account the backreaction of the quantum field on the classical background. In such a situation, we can obtain an effective Lagrangian for the classical electromagnetic background by integrating out the degrees of freedom corresponding to the quantum field as follows (see our discussion at the beginning of subsection 1.5.1):

$$
\begin{equation*}
\exp i \int d^{4} x \mathcal{L}_{e f f}\left(A_{\mu}\right) \equiv \int \mathcal{D} \Phi \int \mathcal{D} \Phi^{*} \exp i \int d^{4} x \mathcal{L}\left(\Phi, A_{\mu}\right) \tag{3.3}
\end{equation*}
$$

(Note that we have set $\hbar=c=1$.) The effective Lagrangian can be expressed as

$$
\begin{equation*}
\mathcal{L}_{e f f}=\mathcal{L}_{e m}+\mathcal{L}_{\text {corr }}, \tag{3.4}
\end{equation*}
$$

where $\mathcal{L}_{e m}$ is the Lagrangian density for the free electromagnetic field, the third term under the integral in action (3.1) and $\mathcal{L}_{\text {corr }}$ is implicitly given by

$$
\begin{array}{r}
\exp i \int d^{4} x \mathcal{L}_{\text {corr }}\left(A_{\mu}\right) \\
=\int \mathcal{D} \Phi \int \mathcal{D} \Phi^{*} \exp i \int d^{4} x\left\{\left(\partial_{\mu} \Phi+i q A_{\mu} \Phi\right)\left(\partial^{\mu} \Phi^{*}-i q A^{\mu} \Phi^{*}\right)\right. \\
\left.-m^{2} \Phi \Phi^{*}\right\} \tag{3.5}
\end{array}
$$

Integrating the action for the scalar field in the above equation by parts and dropping the resulting surface terms, we obtain that (see, for instance, ref. [73], p. 193)

$$
\begin{align*}
\exp i \int d^{4} x \mathcal{L}_{\text {corr }}\left(A_{\mu}\right) & =\int \mathcal{D} \Phi \int \mathcal{D} \Phi^{*} \exp -i \int d^{4} x \Phi^{*} \hat{D} \Phi \\
& =(\operatorname{det} \hat{D})^{-1} \tag{3.6}
\end{align*}
$$

where the operator $\hat{D}$ is given by

$$
\begin{equation*}
\hat{D} \equiv D_{\mu} D^{\mu}+m^{2} \quad \text { and } \quad D_{\mu} \equiv\left(\partial_{\mu}+i q A_{\mu}\right) \tag{3.7}
\end{equation*}
$$

The determinant in equation (3.6) can be expressed as follows

$$
\begin{align*}
\exp i \int d^{4} x \mathcal{L}_{\text {corr }} & =(\operatorname{det} \hat{D})^{-1} \\
& =\exp -\operatorname{Tr}(\ln \hat{D}) \\
& =\exp -\int d^{4} x\langle t, \mathbf{x}| \ln \hat{D}|t, \mathbf{x}\rangle \tag{3.8}
\end{align*}
$$

and in arriving at the last expression, following Schwinger [33, 34], we have chosen a complete and orthonormal set of basis vectors $|t, \mathbf{x}\rangle$ to evaluate the trace of the operator $\ln \hat{D}$. From the above equation it is easy to identify that

$$
\begin{equation*}
\mathcal{L}_{\text {corr }}=i\langle t, \mathrm{x}| \ln \hat{D}|t, \mathrm{x}\rangle . \tag{3.9}
\end{equation*}
$$

Using the following integral representation for the operator $\ln \hat{D}$

$$
\begin{equation*}
\ln \hat{D} \equiv-\int_{0}^{\infty} \frac{d s}{s} \exp -i(\hat{D}-i \epsilon) s \tag{3.10}
\end{equation*}
$$

where $\epsilon \rightarrow 0^{+}$; the expression for $\mathcal{L}_{\text {corr }}$ can be written as

$$
\begin{equation*}
\mathcal{L}_{\text {corr }}=-i \int_{0}^{\infty} \frac{d s}{s} e^{-i\left(m^{2}-i \epsilon\right) s} K(t, \mathbf{x}, s \mid t, \mathbf{x}, 0), \tag{3.11}
\end{equation*}
$$

where

$$
\begin{equation*}
K(t, \mathrm{x}, s \mid t, \mathrm{x}, 0)=\langle t, \mathrm{x}| e^{-i \hat{H} s}|t, \mathrm{x}\rangle \tag{3.12}
\end{equation*}
$$

and

$$
\begin{equation*}
\hat{H} \equiv D_{\mu} D^{\mu}=\left(\partial_{\mu}+i q A_{\mu}\right)\left(\partial^{\mu}+i q A^{\mu}\right) . \tag{3.13}
\end{equation*}
$$

That is, $K(t, \mathbf{x}, s \mid t, \mathbf{x}, 0)$ is the kernel for a quantum mechanical particle (in 4 dimensions) described by the Hamiltonian operator $\hat{H}$. The variable $s$, that was introduced in (3.10) when the operator $\ln \hat{D}$ was expressed in an integral form, acts as the time parameter for the quantum mechanical system.

The integral representation for the operator $\ln \hat{D}$ we have used above is divergent in the lower limit of the integral, i.e. near $s=0$. This divergence is usually regularized in field theory by subtracting from it another divergent integral, viz. the integral representation of an operator $\ln \hat{D}_{0}$, where $\hat{D}_{0}=\left(\partial^{\mu} \partial_{\mu}+\right.$ $\mathrm{m}^{2}$ ), the operator corresponding to that of a free quantum field. That is, to avoid the divergence, the integral representation for $\ln \hat{D}$ is actually considered to be

$$
\begin{equation*}
\ln \hat{D}-\ln \hat{D}_{0} \equiv-\int_{0}^{\infty} \frac{d s}{s}\left(\exp -i(\hat{D}-i \epsilon) s-\exp -i\left(\hat{D}_{0}-i \epsilon\right) s\right) \tag{3.14}
\end{equation*}
$$

Or equivalently, the quantity $\mathcal{L}_{\text {corr }}^{0}$ which corresponds to the case of a free quantum field, can be subtracted from $\mathcal{L}_{\text {corr }}$ to obtain finite results. The quantum mechanical kernel $K^{0}(t, \mathrm{x}, s \mid t, \mathrm{x}, 0)$ corresponding to the operator $\hat{D}_{0}$ is the kernel for a free particle in four dimensions in the coincidence limit. It is given by $K^{0}(t, \mathrm{x}, s \mid t, \mathrm{x}, 0)=\left(1 / 16 \pi^{2} i s^{2}\right)$ (see, for instance, ref. [81], p. 42). Substituting this quantity in the expression for $\mathcal{L}_{\text {corr }}$ above, we obtain that

$$
\begin{equation*}
\mathcal{L}_{\text {corr }}^{0}=-\left(\frac{1}{16 \pi^{2}}\right) \int_{0}^{\infty} \frac{d s}{s^{3}} e^{-i\left(m^{2}-i \epsilon\right) s} . \tag{3.15}
\end{equation*}
$$

This is the expression which has to be subtracted from $\mathcal{L}_{\text {corr }}$ to yield a finite result.

### 3.2 Examining the validity of the tunneling interpretation

In subsection 1.5.1, we had obtained an effective Lagrangian for a constant electromagnetic background by integrating out the degrees of freedom corresponding to the quantum scalar field. We had found that the effective Lagrangian thus obtained can be expressed in terms of the two gauge invariant quantities $\mathcal{G}=F^{\mu \nu} F_{\mu \nu}=2\left(\mathbf{B}^{2}-\mathbf{E}^{2}\right)$ and $\mathcal{F}=\epsilon^{\mu \nu \lambda \rho} F_{\mu \nu} F_{\lambda \rho}=-8(\mathbf{E} . \mathbf{B})$, where $\mathbf{E}$ and $\mathbf{B}$ are the constant electric and the magnetic fields respectively. Also, we had found that the effective Lagrangian had an imaginary part only when $\mathcal{G}<0$. The appearance of an imaginary part in the effective Lagrangian implies an instability of the vacuum and we had attributed the cause of this instability to the production of particle, anti-particle pairs corresponding to the quantum field by the electromagnetic background. We had then interpreted the imaginary part of the effective Lagrangian as the number of pairs that have been produced, per unit four-volume, by the external electromagnetic field.

The derivation of the effective Lagrangian for a constant electromagnetic background we had presented in subsection 1.5.1 is adapted from ref. [84] and is originally due to Heisenberg and Euler [95]. This derivation wherein the effective Lagrangian is related to the ground state energy of the quantum field is applicable only when the background is varying adiabatically. In a more generic situation, wherein the background is dependent on space and/or time coordinates, one can utilize Schwinger's proper time formalism, we have introduced in the last section, to evaluate the effective Lagrangian. Throughout this chapter we shall adopt Schwinger's formalism to evaluate effective Lagrangians.

As we had mentioned in subsection 1.5.1, the evaluation of the effective Lagrangian for an arbitrary electromagnetic background proves to be an impossible task. Due to this reason, there has been numerous attempts in literature to obtain the effective Lagrangian using Schwinger's technique for a given nontrivial electromagnetic background [96, 97, 98, 99, 100, 101, 102]. In spite of all these efforts, there exist very few examples for which the effective Lagrangian is known in a closed form. Quite often, the phenomenon of particle production in classical electromagnetic backgrounds is studied in literature by the method of normal mode analysis. In this approach, as we saw in section 1.4, the normal modes of the quantum field are obtained by solving the wave equation it satisfies in a given electromagnetic background (in a particular gauge). The coefficients of the positive frequency normal modes of the quantum field are then identified to be the annihilation operators. The evolution of these operators therefore follow the evolution of the normal modes. Then, by relating these operators defined in the asymptotic regions (either in space or in time) the number of particles that have been produced by the electromagnetic background can be computed.

Consider an electromagnetic background that can be represented by a time dependent gauge. If we choose to study the evolution of the quantum field in such a gauge, then a positive frequency normal mode of the quantum field at late times will, in general, prove to be a linear superposition of the positive and negative frequency modes defined at early times. The coefficients in such a superposition are the Bogolubov coefficients $\alpha$ and $\beta$ (see subsection 1.1.3). A nonzero Bogolubov coefficient $\beta$ would then imply that the $i n$-vacuum state is not the same as the out-vacuum state. This in turn implies that the in-out transition amplitude is less than unity which can be attributed to the excitation of the modes of the quantum
field by the electromagnetic background $[74,75,76,103,104,105]$. These excitations manifest themselves as real particles corresponding to the quantum field (see our discussion in section 1.2 and subsection 1.4.1).

On the other hand, consider an electromagnetic background that can be described by a space dependent gauge (by which we mean a gauge that is completely independent of time). If the evolution of the quantum field is studied in such a gauge, then due to the lack of dependence on time, the Bogolubov coefficient $\beta$ proves to be trivially zero. This could then imply that the electromagnetic background which is being considered does not produce particles.

An interesting situation arises when the same electromagnetic field can be described by a (purely) space dependent gauge as well as a (purely) time dependent gauge. If we choose to study the evolution of the quantum field in the time dependent gauge, in general, $\beta$ will prove to be nonzero thereby implying (as discussed above) that particles are being produced by the electromagnetic background. But, in the space dependent gauge, $\beta$ is trivially zero thereby disagreeing with the result obtained in the time dependent gauge. Therefore, to obtain results that are gauge invariant, the phenomenon of particle production has to be somehow explained in the space dependent gauge. In literature, a tunneling interpretation is usually invoked to explain the phenomenon of particle production in such a situation $[106,107,108,109]$. In this approach, an effective Schrödinger equation is obtained after the quantum field is decomposed into normal modes in the space dependent gauge. The nonzero tunneling probability for this Schrödinger equation is then attributed to the production of particles by the electromagnetic background.

We had encountered exactly such a situation in section 1.4 when we had reviewed the quantization of a complex scalar field in a constant electric field background. We had found that, in the time dependent gauge $A_{1}^{\mu}$, the positive frequency normal modes of the quantum field at $t=+\infty$ are related by a nonzero Bogolubov coefficient $\beta$ to the positive frequency modes at $t=-\infty$. We had then interpreted the quantity $|\beta|^{2}$ as the number of particles that have been produced in a single mode of the quantum field at late times in the $i n$-vacuum. Whereas, when we had analyzed the evolution of the quantum field in the space dependent gauge $A_{2}^{\mu}$, because of time independence, $\beta$ proved to be trivially zero thereby disagreeing with the result we had obtained in the gauge $A_{1}^{\mu}$. We had invoked the tunneling interpretation in such a situation to explain particle production in the space dependent gauge $A_{2}^{\mu}$. We had obtained, after the normal mode decomposition of the quantum field, an effective Schrödinger equation along the $x$-direction (cf. equation 1.142)). We had then interpreted the nonzero tunneling probability, $|T|^{2}$, for this Schrödinger equation as the number of particles that have been produced in a single mode of the quantum field. The tunneling probability $|T|^{2}$ evaluated in the gauge $A_{2}^{\mu}$, in fact, exactly matched the quantity $|\beta|^{2}$ obtained in the gauge $A_{1}^{\mu}$ (cf. equations (1.134) and (1.147)). Also, these two quantities agreed with the pair creation rate we had later obtained from the imaginary part of the effective Lagrangian (see our discussion following equation (1.204)).

The fact that the quantities $|\beta|^{2}$ and $|T|^{2}$ agree, not only with each other, but also with the pair creation rate obtained from the effective Lagrangian, for the case of a constant electric field has given certain credibility to the tunneling interpretation. In fact, we have not seen in literature another example of a time independent electric field background for which the tunneling probability $|T|^{2}$ has
been shown to match the imaginary part of the effective Lagrangian. Our aim, in this section, is to probe the validity of the tunneling interpretation.

Now, consider an arbitrary electromagnetic background that can be described by a space dependent gauge. Also assume that, when the evolution of the quantum field is analyzed in such a gauge, there exists a nonzero tunneling probability for the effective Schrödinger equation. Can such a nonzero tunneling probability be always interpreted as particle production? We attempt to answer this question in this section by comparing the results obtained from the effective Lagrangian with those obtained from the tunneling approach. We carry out our analysis for a spatially varying, time independent magnetic field when it is described by a space dependent gauge. We find that there exists-in general-a lack of consistency between the results obtained from the tunneling approach and those obtained from the effective Lagrangian [110]. This inconsistency clearly calls into question the validity of the tunneling interpretation as it is presently understood in literature.

In the subsection that follows immediately, we show that the imaginary part of the effective Lagrangian for a time independent, but otherwise arbitrary, magnetic field is zero. In subsection 3.2.2, we calculate the tunneling probability, which happens to be nonzero, for a particular spatially confined and time independent magnetic field when it is represented by a space dependent gauge. And, in subsection 3.2.3, we discuss the wider implications of our analysis to the study of particle production in time independent electromagnetic and gravitational backgrounds.

### 3.2.1 Effective Lagrangian for a time independent magnetic field background

Consider a background electromagnetic field described by the vector potential

$$
\begin{equation*}
A^{\mu}=(0,0, A(x), 0), \tag{3.16}
\end{equation*}
$$

where $A(x)$ is an arbitrary function of $x$. This vector potential does not produce an electric field but gives rise to a magnetic field $\mathbf{B}=(d A / d x) \hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ is the unit vector along the positive $z$-axis. According to the Maxwell's equations, in the absence of an electric field, the magnetic field is related to the current $\mathbf{j}(x)$ as follows

$$
\begin{equation*}
\nabla \times \mathbf{B}=\mathbf{j} . \tag{3.17}
\end{equation*}
$$

Then, the current that can give rise to the time independent magnetic field we consider here is given by

$$
\begin{equation*}
\mathbf{j}=-\left(\frac{d^{2} A}{d x^{2}}\right) \hat{\mathbf{y}}, \tag{3.18}
\end{equation*}
$$

where $\hat{\mathbf{y}}$ is the unit vector along the positive $y$-axis. If we assume that $\mathbf{j}$ is finite and continuous everywhere and also vanishes as $|x| \rightarrow \infty$, then the magnetic field we consider here will be confined to a finite extent along the $x$-axis.

The operator $\hat{H}$ (cf. equation (3.13)) corresponding to the vector potential (3.16) is given by

$$
\begin{equation*}
\hat{H} \equiv \partial_{t}^{2}-\nabla^{2}+2 i q A \partial_{y}+q^{2} A^{2} . \tag{3.19}
\end{equation*}
$$

Then, the kernel for the quantum mechanical particle described by this Hamiltonian can be formally written as

$$
\begin{equation*}
K(t, \mathrm{x}, s \mid t, \mathrm{x}, 0)=\langle t, \mathrm{x}| \exp -i\left(\partial_{t}^{2}-\nabla^{2}+2 i q A \partial_{y}+q^{2} A^{2}\right) s|t, \mathrm{x}\rangle . \tag{3.20}
\end{equation*}
$$

Using the translational invariance of the Hamiltonian operator $\hat{H}$ along the time coordinate $t$ and the spatial coordinates $y$ and $z$, we can express the above kernel as follows

$$
\begin{align*}
& K(t, \mathbf{x}, s \mid t, \mathbf{x}, 0)=\int_{-\infty}^{\infty} \frac{d \omega}{2 \pi} \int_{-\infty}^{\infty} \frac{d p_{y}}{2 \pi} \int_{-\infty}^{\infty} \frac{d p_{z}}{2 \pi} e^{i\left(\omega^{2}-p_{z}^{2}\right) s} \\
& \times\langle x| \exp -i\left(-d_{x}^{2}+\left(p_{y}-q A\right)^{2}\right) s|x\rangle \tag{3.21}
\end{align*}
$$

where we have used the notation $d_{x}^{2}$ to represent the differential operator $\left(d^{2} / d x^{2}\right)$. Performing the $\omega$ and $p_{z}$ integrations, we obtain that

$$
\begin{equation*}
K(t, \mathbf{x}, s \mid t, \mathbf{x}, 0)=\frac{1}{4 \pi s} \int_{-\infty}^{\infty} \frac{d p_{y}}{2 \pi}\langle x| e^{-i \hat{G} s}|x\rangle \tag{3.22}
\end{equation*}
$$

where

$$
\begin{equation*}
\hat{G} \equiv-\frac{d^{2}}{d x^{2}}+\left(p_{y}-q A\right)^{2} . \tag{3.23}
\end{equation*}
$$

The quantity $\langle x| e^{-i \hat{G} s}|x\rangle$ is then the kernel for a one dimensional quantum mechanical system described by the effective Hamiltonian operator $\hat{G}$. It can expressed, using the Feynman-Kac formula, as (see, for instance, ref. [82], chapter 7)

$$
\begin{equation*}
\langle x| \exp -i \hat{G} s|x\rangle=\sum_{E}\left|\Psi_{E}(x)\right|^{2} e^{-i E s}, \tag{3.24}
\end{equation*}
$$

where $\Psi_{E}$ is the eigenfunction of the operator $\hat{G}$ corresponding to an eigenvalue E, i.e.

$$
\begin{equation*}
\hat{G} \Psi_{E} \equiv-\frac{d^{2} \Psi_{E}}{d x^{2}}+\left(p_{y}-q A\right)^{2} \Psi_{E}=E \Psi_{E} \tag{3.25}
\end{equation*}
$$

so that $K(t, \mathbf{x}, s \mid t, \mathbf{x}, 0)$ reduces to

$$
\begin{equation*}
K(t, \mathbf{x}, s \mid t, \mathbf{x}, 0)=\frac{1}{4 \pi s} \int_{-\infty}^{\infty} \frac{d p_{y}}{2 \pi} \sum_{E}\left|\Psi_{E}(x)\right|^{2} e^{-i E s} . \tag{3.26}
\end{equation*}
$$

(It is assumed that the summation over $E$ stands for integration over the relevant range when $E$ varies continuously.) Since the potential term, $\left(p_{y}-q A(x)\right)^{2}$, in the Hamiltonian operator $\hat{G}$ is a positive definite quantity, the eigenvalue $E$ can only
lie in the range $(0, \infty)$. Substituting the expression for $K(t, \mathbf{x}, s \mid t, \mathbf{x}, 0)$ in (3.11), we find that $\mathcal{L}_{\text {corr }}$ is given by

$$
\begin{equation*}
\mathcal{L}_{\text {corr }}=-\frac{i}{4 \pi} \int_{-\infty}^{\infty} \frac{d p_{y}}{2 \pi} \sum_{E}\left|\Psi_{E}(x)\right|^{2} \int_{0}^{\infty} \frac{d s}{s^{2}} e^{-i\left(m^{2}+E-i \epsilon\right) s} . \tag{3.27}
\end{equation*}
$$

(We will not bother here to subtract the quantity $\mathcal{L}_{\text {corr }}^{0}$ from $\mathcal{L}_{\text {corr }}$ since this regularization is not necessary for the conclusions we wish to draw from our analysis.) Differentiating the above expression for $\mathcal{L}_{\text {corr }}$ twice with respect to $m^{2}$ and then carrying out the integration over the variable $s$, we obtain that

$$
\begin{equation*}
\mathcal{L}_{\text {corr }}^{\prime \prime}=\frac{\partial^{2} \mathcal{L}_{\text {corr }}}{\partial\left(m^{2}\right)^{2}}=\frac{1}{4 \pi} \int_{-\infty}^{\infty} \frac{d p_{y}}{2 \pi} \sum_{E}\left(\frac{\left|\Psi_{E}(x)\right|^{2}}{m^{2}+E-i \epsilon}\right) \tag{3.28}
\end{equation*}
$$

The quantity $\left(m^{2}+E-i \epsilon\right)^{-1}$ in the above expression, can be written as

$$
\begin{equation*}
\left(\frac{1}{m^{2}+E-i \epsilon}\right)=\mathcal{P}\left(\frac{1}{m^{2}+E}\right)+i \pi \delta_{D}\left(m^{2}+E\right) \tag{3.29}
\end{equation*}
$$

where $\mathcal{P}$ is the principal value of the corresponding argument. Since $E$ is a positive semi-definite quantity, the argument of the delta function above never reduces to zero. Therefore the second term in the above expression vanishes with the result that $\mathcal{L}_{\text {corr }}^{\prime \prime}$ is a real quantity thereby implying that $\mathcal{L}_{\text {corr }}$ is also a real quantity. In fact, integrating $\mathcal{L}_{\text {corr }}^{\prime \prime}$ twice with respect to $m^{2}$, we find that $\mathcal{L}_{\text {corr }}$ can be expressed as

$$
\begin{equation*}
\mathcal{L}_{\text {corr }}=\frac{1}{4 \pi} \int_{-\infty}^{\infty} \frac{d p_{y}}{2 \pi} \sum_{E}\left|\Psi_{E}(x)\right|^{2} \alpha(\ln \alpha-1) \tag{3.30}
\end{equation*}
$$

where $\alpha=\left(m^{2}+E\right)>0$ and $\epsilon$ has been set to zero. Then, clearly $\mathcal{L}_{\text {corr }}$ is a real quantity. (To be rigorous, one has to take into account the two constants of integration that will appear on integrating $\mathcal{L}_{\text {corr }}^{\prime \prime}$ with respect to $m^{2}$ (see our discussion following equation (1.175)), but these constants are irrelevant for our arguments here.)

Though we are unable to evaluate the effective Lagrangian for an arbitrary time independent magnetic field in a closed form, we have been able to show that it certainly does not have an imaginary part. Therefore we can unambiguously conclude that time independent background magnetic fields do not produce particles. This, of course, agrees with Schwinger's result for the constant magnetic field background [33].

### 3.2.2 Tunneling probability in a time independent magnetic field background

We shall now calculate the tunneling probability for the a specific time independent background magnetic field in a space dependent gauge. Consider the vector potential

$$
\begin{equation*}
A^{\mu}=\left(0,0, B_{0} L \tanh (x / L), 0\right) \tag{3.31}
\end{equation*}
$$

where $B_{0}$ and $L$ are arbitrary constants. This vector potential does not produce an electric field but gives rise to the following magnetic field

$$
\begin{equation*}
\mathbf{B}=B_{0} \operatorname{sech}^{2}(x / L) \hat{\mathbf{z}} \tag{3.32}
\end{equation*}
$$

where $\hat{\mathbf{z}}$ is the unit vector along the positive $z$-axis. The magnetic field $\mathbf{B}$ goes to zero as $|x| \rightarrow \infty$, i.e its strength is confined to an effective width $L$ along the $x$ axis. In the absence of an electric field, according to the Maxwell's equation (3.17), the magnetic field given by (3.32) can be produced by the current

$$
\begin{equation*}
\mathbf{j}=\left(\frac{2 B_{0}}{L}\right) \operatorname{sech}(x / L) \tanh (x / L) \hat{\mathbf{y}}, \tag{3.33}
\end{equation*}
$$

where, as before, $\hat{\mathbf{y}}$ denotes the unit vector along the positive $y$-axis. The current j is finite and continuous everywhere and also goes to zero as $|x| \rightarrow \infty$.

In an electromagnetic background, described by the vector potential $A^{\mu}$, the complex scalar field satisfies the following Klein-Gordon equation

$$
\begin{equation*}
\left(D_{\mu} D^{\mu}+m^{2}\right) \Phi=\left\{\left(\partial_{\mu}+i q A_{\mu}\right)\left(\partial_{\mu}+i q A_{\mu}\right)+m^{2}\right\} \Phi=0 \tag{3.34}
\end{equation*}
$$

Substituting the vector potential (3.31) in the above equation, we obtain that

$$
\begin{equation*}
\left(\partial_{t}^{2}-\nabla^{2}+2 i q B_{0} L \tanh (x / L) \partial_{y}+q^{2} B_{0}^{2} L^{2} \tanh ^{2}(x / L)+m^{2}\right) \Phi(t, \mathbf{x})=0 \tag{3.35}
\end{equation*}
$$

Since the vector potential (3.31) is dependent only on the spatial coordinate $x$, the normal modes of the scalar field $\Phi$ can be decomposed as follows

$$
\begin{equation*}
g_{\omega \mathbf{k}_{\perp}}(t, \mathbf{x}) \propto e^{-i \omega t} e^{i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}} \psi_{\omega \mathbf{k}_{\perp}}(x), \tag{3.36}
\end{equation*}
$$

where $\mathbf{k}_{\perp} \equiv\left(k_{y}, k_{z}\right)$ and $\mathbf{x}_{\perp} \equiv(y, z)$. Substituting the normal mode $g_{\omega \mathbf{k}_{\perp}}$ in (3.35), we find that $\psi_{\omega \mathbf{k}_{\perp}}$ satisfies the following differential equation

$$
\begin{equation*}
\frac{d^{2} \psi}{d \rho^{2}}+\left(\omega^{2}-\left(k_{y}-q B_{0} L \tanh \rho\right)^{2}-k_{z}^{2}-m^{2}\right) L^{2} \psi=0 \tag{3.37}
\end{equation*}
$$

where $\rho=(x / L)$ and we have dropped the subscripts on $\psi$. This differential equation can be rewritten as

$$
\begin{equation*}
-\frac{d^{2} \psi}{d \rho^{2}}+\left(k_{y} L-q B_{0} L^{2} \tanh \rho\right)^{2} \psi=\left(\omega^{2}-k_{z}^{2}-m^{2}\right) L^{2} \psi \tag{3.38}
\end{equation*}
$$

which then resembles a time independent Schrödinger equation corresponding to a potential $\left(k_{y} L-q B_{0} L^{2} \tanh \rho\right)^{2} / 2$ and energy eigenvalue $\left(\omega^{2}-k_{z}^{2}-m^{2}\right) L^{2} / 2$. The potential term in the effective Schrödinger equation above reduces to a finite constant as $|x| \rightarrow \infty$. Therefore, there exist solutions for $\psi$ which reduce to $e^{ \pm i k_{L} x}$ as $x \rightarrow-\infty$ and $e^{ \pm i k_{R} x}$ as $x \rightarrow+\infty$, where $k_{L}$ and $k_{R}$ are given by

$$
\left.\begin{array}{l}
k_{L}=\left(\omega^{2}-\left(k_{y}+q B_{0} L\right)^{2}-k_{z}^{2}-m^{2}\right)^{1 / 2}  \tag{3.39}\\
k_{R}=\left(\omega^{2}-\left(k_{y}-q B_{0} L\right)^{2}-k_{z}^{2}-m^{2}\right)^{1 / 2}
\end{array}\right\}
$$

We shall confine our attention to values of $\omega$ and $k_{\perp}$ such that $k_{L}$ and $k_{R}$ are real.

The differential equation (3.37) can be solved by the following ansatz [111]

$$
\begin{equation*}
\psi=e^{-a \rho} \operatorname{sech}^{b} \rho f(\rho) \tag{3.40}
\end{equation*}
$$

where

$$
\begin{equation*}
a=i k_{-} L \quad ; \quad b=i k_{+} L \quad \text { and } \quad k_{ \pm}=\left(k_{R} \pm k_{L}\right) / 2 . \tag{3.41}
\end{equation*}
$$

Substituting the above ansatz in (3.37), we find that $f$ satisfies the following differential equation

$$
\begin{equation*}
u(u-1) \frac{d^{2} f}{d u^{2}}+(1+a+b-2(b+1) u) \frac{d f}{d u}+\left(q^{2} B_{0}^{2} L^{4}-b(b+1)\right) f=0 \tag{3.42}
\end{equation*}
$$

where the variable $u$ is related to $\rho$ by the equation: $u=(1-\tanh \rho) / 2$. The above equation is a hypergeometric differential equation and its general solution is a linear combination of two hypergeometric functions (cf. [59], pp. 562 and 563), i.e.

$$
\begin{align*}
f(u)=A F & \left(b+\frac{1}{2}+c, b+\frac{1}{2}-c, 1+a+b, u\right) \\
& +B u^{-a-b} F\left(\frac{1}{2}-a+c, \frac{1}{2}-a-c, 1-a-b, u\right), \tag{3.43}
\end{align*}
$$

where $A$ and $B$ are arbitrary constants and

$$
\begin{equation*}
c=\left(\frac{1}{4}+q^{2} B_{0}{ }^{2} L^{4}\right)^{1 / 2} \tag{3.44}
\end{equation*}
$$

To calculate the tunneling probability for the effective Schrödinger equation (3.38), we have to choose the constants $A$ and $B$ such that $\psi \sim e^{i k_{R} x}$ as $x \rightarrow+\infty$ i.e. when $u \rightarrow 0$. This can be achieved by setting $A=0$ and $B=2^{-b}$, so that

$$
\begin{equation*}
f(u)=2^{-b} u^{-a-b} F\left(\frac{1}{2}-a+c, \frac{1}{2}-a-c, 1-a-b, u\right) . \tag{3.45}
\end{equation*}
$$

Substituting the above solution in (3.40) and using the relation (cf. [59], p. 559)

$$
\begin{align*}
F\left(\frac{1}{2}\right. & \left.-a+c, \frac{1}{2}-a-c, 1-a-b, u\right) \\
= & P F\left(\frac{1}{2}-a+c, \frac{1}{2}-a-c, 1-a+b, 1-u\right) \\
& +Q(1-u)^{a-b} F\left(\frac{1}{2}-b-c, \frac{1}{2}-b+c, 1+a-b, 1-u\right), \tag{3.46}
\end{align*}
$$

where

$$
\begin{equation*}
P=\left(\frac{\Gamma(1-a-b) \Gamma(a-b)}{\Gamma\left(\frac{1}{2}-b-c\right) \Gamma\left(\frac{1}{2}-b+c\right)}\right) \tag{3.47}
\end{equation*}
$$

and

$$
\begin{equation*}
Q=\left(\frac{\Gamma(1-a-b) \Gamma(b-a)}{\Gamma\left(\frac{1}{2}-a+c\right) \Gamma\left(\frac{1}{2}-a-c\right)}\right), \tag{3.48}
\end{equation*}
$$

we find that, as $x \rightarrow-\infty$, i.e when $(1-u) \rightarrow 0$,

$$
\begin{equation*}
\psi \longrightarrow P e^{i k_{L} x}+Q e^{-i k_{L} x} \tag{3.49}
\end{equation*}
$$

Consider a solution of the effective Schrödinger equation (3.38) which goes as $\left(R e^{i k_{L} x}+e^{-i k_{L} x}\right)$ as $x \rightarrow-\infty$ and goes over to $\left(T e^{i k_{R} x}\right)$ as $x \rightarrow+\infty$ (see our discussion in subsection 1.4.2). Then it is easy to identify the expressions for $R$ and $T$ from equation (3.49). They are given by

$$
\begin{align*}
R & =\left(\frac{P}{Q}\right)=\left(\frac{\Gamma\left(\frac{1}{2}-a+c\right) \Gamma\left(\frac{1}{2}-a-c\right) \Gamma(a-b)}{\Gamma\left(\frac{1}{2}-b-c\right) \Gamma\left(\frac{1}{2}-b+c\right) \Gamma(b-a)}\right) \\
T & =\left(\frac{1}{Q}\right)=\left(\frac{\Gamma\left(\frac{1}{2}-a+c\right) \Gamma\left(\frac{1}{2}-a-c\right)}{\Gamma(1-a-b) \Gamma(b-a)}\right) \tag{3.50}
\end{align*}
$$

so that

$$
\begin{equation*}
|R|^{2}=\left(\frac{\cosh 2 \pi k_{+} L+\cos 2 \pi c}{\cosh 2 \pi k_{-} L+\cos 2 \pi c}\right) \tag{3.51}
\end{equation*}
$$

and

$$
\begin{equation*}
|T|^{2}=\left(\frac{k_{L}}{k_{R}}\right)\left(\frac{\cosh 2 \pi k_{+} L-\cosh 2 \pi k_{-} L}{\cosh 2 \pi k_{-} L+\cos 2 \pi c}\right) . \tag{3.52}
\end{equation*}
$$

The Wronskian condition for the effective Schrödinger equation (3.38) then leads us to the following relation

$$
\begin{equation*}
|R|^{2}-\left(\frac{k_{R}}{k_{L}}\right)|T|^{2}=1 \tag{3.53}
\end{equation*}
$$

So, the tunneling probability is nonzero for the time independent magnetic field we have considered here. It is, in fact, given by $|T|^{2}$ in equation (3.52).

The implications of our analysis are discussed in the following subsection.

### 3.2.3 Implications

A time independent magnetic field does not give rise to an electric field (in a particular Lorentz frame) and a pure magnetic field cannot do any work on charged particles. Therefore it seems plausible that such a magnetic field does not produce particles. This expectation is, in fact, corroborated by the result we have obtained in section 3.2.1, viz. that the imaginary part of the effective Lagrangian for a time independent, but otherwise arbitrary, magnetic field is zero. Our analysis in sections 3.2.1 and 3.2.2 has been carried out assuming that the time independent magnetic field is described by a space dependent gauge. In such a gauge, $\beta$ is trivially zero and if we had considered only a nonzero $\beta$ to imply particle production, then this result would have proved to be consistent with the result we have obtained in section 3.2.1.

But this is not the whole story. According to the tunneling interpretation, in a time independent gauge it is the tunneling probability for the effective Schrödinger equation that has to be interpreted as particle production. In section 3.2.2, we find that there exists a nonzero tunneling probability for a spatially confined and time independent magnetic field. If the tunneling interpretation is
true, this result would then imply that such a background can produce particles thereby contradicting the result we have obtained in section 3.2.1.

The tunneling probability can, in fact, prove to be nonzero in a more general case and is certainly not an artifact of our specific example. This can be seen as follows: Consider an arbitrary electromagnetic field described by the vector potential

$$
\begin{equation*}
A^{\mu}=(\phi(x), 0, A(x), 0) \tag{3.54}
\end{equation*}
$$

where $\phi(x)$ and $A(x)$ are arbitrary functions of $x$. If the decomposition of the normal modes is carried out as was done in (3.36), then the effective Schrödinger equation for the $x$-coordinate corresponding to the above vector potential turns out to be

$$
\begin{equation*}
-\frac{d^{2} \psi}{d x^{2}}+\left(\left(k_{y}-q A\right)^{2}-(\omega-q \phi)^{2}\right) \psi=\left(-k_{z}^{2}-m^{2}\right) \psi \tag{3.55}
\end{equation*}
$$

If we also assume that $\phi(x)$ and $A(x)$ vanish at the spatial infinities, then it is clear that the solutions for $\psi$ will reduce to plane waves as $|x| \rightarrow \infty$. When such solutions are possible, in general, there is bound to exist a nonzero tunneling probability for the effective Schrödinger equation. Thus, quite generally, the tunneling interpretation will force us to conclude that the electromagnetic field described by the above potential produces particles. In particular, the tunneling probability will prove to be be nonzero even when $\phi=0$ - the case which corresponds to a pure time independent magnetic field. But for such a case, we have shown in section 3.2.1 that the effective Lagrangian is real and hence there can be no particle production. Thus we again reach a contradiction between the results obtained from the tunneling interpretation and those obtained from the effective Lagrangian.

On the other hand, consider the following situation. If we choose $A(x)$ to be zero and $\phi(x)$ to be nonzero in the above vector potential, then such a vector potential will give rise to a time independent electric field. Such an electric field is always expected to produce particles. But in the space dependent gauge we have chosen here $\beta$ is trivially zero and if we consider only a nonzero $\beta$ to imply particle production, then we will be forced to conclude that time independent electric fields will not produce particles! It is to salvage such a situation, that the tunneling interpretation has been repeatedly invoked in literature. But then, our analysis in the last two sections show that tunneling probability can be nonzero even if the effective Lagrangian has no imaginary part!

There appears to be three possible ways of reacting to this contradiction. We shall examine each of them below:
(i) We may begin by noticing that in quantum field theory, there is always a tacit assumption that not only the fields but also the potentials should vanish at spatial infinities. If we take this requirement seriously, we may disregard the results for constant electromagnetic fields (the only case for which explicit results are known by more than one method!) as unphysical. Then we only need to provide a gauge invariant criterion for particle production in electromagnetic fields described by potentials which vanish at infinity.

This turns out to be a difficult task, even conceptually. To begin with, we do not know how to generalize Schwinger's analysis and compute the effective Lagrangian for a spatially varying electromagnetic field. The only other procedure available for us to study the evolution of the quantum field in such backgrounds are based on the method of normal mode analysis where we go on to
obtain the tunneling probability $|T|^{2}$. But then, the potential term in the effective Schrödinger equation is not gauge invariant, as can be easily seen from its form in equation (3.55). So the tunneling interpretation, even if it is adhered to, has the problem that it may not yield results that are gauge invariant. In fact, the situation is more serious; the entire tunneling approach can be used only after a particular gauge has been chosen. In some sense, the battle has been lost already.

Operationally also, it is doubtful whether the tunneling approach will yield results that are always consistent with the effective Lagrangian. As the analysis in the last two subsections shows, there is at least one case-that of a spatially confined magnetic field-for which one can obtain a formal expression for effective Lagrangian and compare it with the results obtained from the normal mode analysis. These results are clearly in contradiction with each other.
(ii) One may take the point of view that particle production in an electromagnetic field is a gauge dependent phenomenon. It appears to be a remedy worse than disease and is possibly not acceptable. In addition to philosophical objections one can also rule out this possibility by the following argument. We note that we can produce electromagnetic fields in the laboratory by choosing charges and current distributions but we have no operational way of implementing a gauge. So, given a particular electromagnetic field, in some region of the laboratory, we will either see particles being produced or not. It is hard to see how the gauge can enter this result.

This point has some interesting similarities (and differences) with the question of particle definition in a gravitational field. If we assume that the choice of gauge in electromagnetic backgrounds is similar to the choice of a coordinate
system in gravity, then one would like to ask whether the concept of particle is dependent on the coordinate choice. People seem to have no difficulty in accepting a coordinate dependence of particles (and particle production) in the case of gravity though the same people might not like the particle concept to be gauge dependent in the case of electromagnetism! To some extent, this arises from the belief that a coordinate choice is implementable by choosing a special class of observers, say, while a gauge choice in electromagnetism is not implementable in practice.
(iii) Finally, one may take the point of view that tunneling interpretation is completely invalid and one should rely entirely on the effective Lagrangian for interpreting the particle production. In this approach one would calculate the effective Lagrangian for a given electromagnetic field (possibly by numerical techniques, say) and will claim that particle production takes place only if the effective Lagrangian has an imaginary part. Further one would confine oneself to those potentials which vanish at infinity, thereby ensuring proper asymptotic behavior

This procedure is clearly gauge invariant in the sense that the effective Lagrangian is (at least formally) gauge invariant. Of course, one needs to provide a procedure for calculating the effective Lagrangian without having to choose a particular gauge. Given such a procedure, we have an unambiguous, gauge invariant criterion for particle production for all potentials which vanish asymptotically. In fact, the effective Lagrangian for a spatially varying electromagnetic background can be formally expressed in terms of gauge invariant quantities that involve the derivatives of the potentials and the fields.

This point could also have an interesting implication for gravitational back-
grounds. The analogue of a constant electromagnetic background in gravity corresponds to spacetimes whose $R_{\mu \nu \rho \sigma}$ 's are constants. The effective Lagrangian in gravity can then possibly be expressed in terms of coordinate invariant quantities constructed from $R_{\mu \nu \rho \sigma}$ 's, just as it was possible to express the effective Lagrangian for a constant electromagnetic background in terms of gauge invariant quantities involving $F_{\mu \nu}$ 's.

We would like to stress here the following points. Equations (3.38) and (3.55) resemble a Schrödinger equation only in a formal sense. The actual time dependent differential equation that we ought to deal with is the functional Schrödinger equation defined on the configuration space of all fields [94, 112]. It is possible that such an approach would lead to an unambiguous way of dealing with particle creation. The results obtained from an analysis of the functional Schrödinger equation might not, in general, agree with the tunneling probability calculated for equations such as (3.38) or (3.55). It would be interesting to know the conditions under which the particle creation rate obtained from an analysis of the functional Schrödinger equation coincides with the tunneling probability evaluated, say, for equation (3.55). However, given the mathematical difficulties associated with solving functional differential equations, it is difficult to arrive at clear conclusions regarding the results for arbitrary electromagnetic backgrounds.

Comparing the three choices listed above, it seems that the third one is the most reasonable. Therefore, we conclude that the results obtained from the effective Lagrangian can be relied upon whereas the tunneling approach has to be treated with caution. It is likely, however, that the tunneling interpretation will prove to be consistent with the effective Lagrangian approach if we demand that an auxiliary gauge invariant criterion has to be satisfied by the electromagnetic
background before we can attribute a nonzero tunneling probability to particle production. But it is not obvious as to how such a condition can be obtained from the normal mode analysis.

### 3.3 Limitations of the Klein approach

A criticism of our analysis in the last section would be that the nonzero tunneling probability we have calculated corresponds to just a scattering by the time independent magnetic field and does not correspond to particle production. It can be claimed that there ought to arise a Klein paradox for a nonzero tunneling probability to be interpreted as particle production [113, 114, 115, 116, 117]. But the Klein paradox and its eventual resolution in terms of pair creation can not adequately explain particle production in time independent electromagnetic backgrounds. In this section, we point out the inadequacies of the Klein approach through a couple of examples.

Consider the following vector potential

$$
\begin{equation*}
A^{\mu}=(\phi(x), 0,0,0) \tag{3.56}
\end{equation*}
$$

Let us assume that the function $\phi(x) \rightarrow \phi_{ \pm}$as $x \rightarrow \pm \infty$, where $\phi_{ \pm}$are finite constants. Since the vector potential is independent of time as well as the $y$ and $z$ coordinates the normal modes of scalar field $\Phi$ can be decomposed as follows:

$$
\begin{equation*}
u_{\omega \mathbf{k}_{\perp}}(t, \mathbf{x}) \propto e^{-i \omega t} e^{i k_{\perp} \cdot x_{\perp}} \psi_{\omega \mathbf{k}_{\perp}}(x), \tag{3.57}
\end{equation*}
$$

where $\psi_{\omega \mathbf{k}_{\perp}}$ satisfies the following differential equation (set $A(x)=0$ in equation (3.55))

$$
\begin{equation*}
-\frac{d^{2} \psi}{d x^{2}}-(\omega-q \phi)^{2} \psi=\left(-\left|\mathbf{k}_{\perp}\right|^{2}-m^{2}\right) \psi \tag{3.58}
\end{equation*}
$$

and we have dropped the subscripts on $\psi$. The conserved four-current, in the presence of a vector potential $A^{\mu}$, is given by

$$
\begin{equation*}
j^{\mu}=-i\left\{\Phi\left(D^{\mu} \Phi\right)^{*}-\Phi^{*}\left(D^{\mu} \Phi\right)\right\} \tag{3.59}
\end{equation*}
$$

where $D_{\mu}$ is given by (3.7). It is easy to verify with the help of the Klein-Gordon equation (3.34) that

$$
\begin{equation*}
\partial_{\mu} j^{\mu}=0 . \tag{3.60}
\end{equation*}
$$

Let us now assume that

$$
\begin{equation*}
\psi(x) \longrightarrow e^{i k_{L} x}+R e^{-i k_{L} x} \tag{3.61}
\end{equation*}
$$

as $x \rightarrow-\infty$ and

$$
\begin{equation*}
\psi(x) \longrightarrow T e^{i k_{R} x} \tag{3.62}
\end{equation*}
$$

as $x \rightarrow \infty$, where

$$
\left.\begin{array}{l}
k_{R}=\left(\left(\omega-q \phi_{+}\right)^{2}-\left|\mathbf{k}_{\perp}\right|^{2}-m^{2}\right)^{1 / 2}  \tag{3.63}\\
k_{L}=\left(\left(\omega-q \phi_{-}\right)^{2}-\left|\mathbf{k}_{\perp}\right|^{2}-m^{2}\right)^{1 / 2}
\end{array}\right\}
$$

and we shall concentrate on values for $\omega$ and $\mathbf{k}_{\perp}$ such that both $k_{R}$ an $k_{L}$ are real. Also, we shall assume that $k_{R}$ and $k_{L}$ are positive definite quantities. For such a case, the conservation of the $x$-component of the four current $j^{\mu}$ leads us to the following Wronskian condition:

$$
\begin{equation*}
|R|^{2}+\left(k_{R} / k_{L}\right)|T|^{2}=1 . \tag{3.64}
\end{equation*}
$$

The incident, reflected and the transmitted current densities (viz. the zeroth
component of the four current $j^{\mu}$ ) have the following forms:

$$
\left.\begin{array}{l}
j_{i}^{0}=\left(\omega-q \phi_{-}\right)  \tag{3.65}\\
j_{r}^{0}=|R|^{2}\left(\omega-q \phi_{-}\right) \\
j_{t}^{0}=|T|^{2}\left(\omega-q \phi_{+}\right)
\end{array}\right\}
$$

respectively. Let us now set $\mathbf{k}_{\perp}=0$. Since we have assumed that $k_{R}$ and $k_{L}$ are real and positive definite quantities, if we now also assume that $\left(\omega-q \phi_{-}\right)>$ $m$, then it is clear from the above equation that the incident and the reflected currents correspond to a flux of particles. If we choose values for $\phi_{ \pm}$such that $\left|q\left(\phi_{+}-\phi_{-}\right)\right|>2 m$, then the current density $j_{t}^{0}$ will be negative for a certain range of values of $\omega$. But a negative current density corresponds to anti-particles. Also, a current of anti-particles to the right is equivalent to a current of particles to the left. Hence, for those values of $\omega$ for which $j_{t}^{0}$ is negative, we will have to flip the sign of $k_{R}$ in the Wronskian condition (3.64). This leads to

$$
\begin{equation*}
|R|^{2}=1+\left(k_{R} / k_{L}\right)|T|^{2} \tag{3.66}
\end{equation*}
$$

which implies that $|R|^{2}>1$. (Note that $k_{R}$ and $k_{L}$ are positive definite quantities.) That is, the reflected flux of particles is greater than the flux that was incident on the electromagnetic potential. This is usually attributed in literature to pair creation by the electromagnetic background. But, is this interpretation correct in a more general background? We have constructed below at least two examples where this interpretation will prove to be inadequate to study pair production.

Consider the case when $\phi_{ \pm}=0$ but $\phi(x)$ is otherwise arbitrary for any finite $x$. Then the sign flip that was expected in the current densities at the left and right extremes will not occur and $|R|^{2}$ will not be greater than unity. On the other
hand, for any finite $x$ if the potential $\phi(x)$ varies sufficiently one would expect the background to produce particles. But the standard interpretation would not be able to help us obtain the number of particles that have been produced by the background since it depends only on the currents at the asymptotics. Since the electric field is the derivative of the potential, if the potential is not a monotonically increasing function, then the resulting electric field will certainly produce pairs but will accelerate them in opposite directions thereby possibly even nulling the currents at the asymptotics. Hence, the standard interpretation which depends so strongly on asymptotic currents will prove to be inadequate to give us the number of particles that have been produced by the background.

Now, consider the following vector potential: $A^{\mu}=(\phi(x), 0, A(x), 0)$. The function $\phi(x)$ will give rise to an electric field and $A(x)$ to a magnetic field. Choose any $\phi(x)$ such that the difference between the maximum and the minimum of $\phi(x)$ is certainly greater than $2 m$. In such a situation, the background is expected to produce particles. But if we choose $A(x)$ such that it produces a strong magnetic field for large $x$ then even if the electric field is able to produce particles the magnetic field will confine the resultant currents so that the currents actually die down as $|x| \rightarrow \infty$. And, in the absence of transmitted currents we would be forced to conclude that no particles are being produced by the background. The worst case is when $A(x)=B x$ where $B$ is a finite constant. This gives rise to a constant magnetic field for large $|x|$ and there simply will not be any currents at the right and left extremes for any finite but otherwise arbitrary $\phi(x)$. And, this example is just as physical or unphysical as the Schwinger's example of a constant electromagnetic background!

Also, some of the discussion we had presented as a criticism of the tunneling
approach applies to the Klein approach too. For instance, even the Klein approach does not give a gauge invariant criterion for an electromagnetic background to produce particles. In fact, the condition $\left|q\left(\phi_{+}-\phi_{-}\right)\right|>2 m$ that has to be satisfied for $|R|^{2}>1$ is not even Lorentz invariant!

These two examples clearly point out the inadequacy of the Klein approach to explain the phenomenon of particle production by time independent electromagnetic backgrounds. The effective Lagrangian approach on the other hand holds more promise. But, as we have mentioned repeatedly, evaluating the effective Lagrangian even for a given classical background proves to be a difficult task. In such a situation, it will interesting to ask whether we can we say anything about the effective Lagrangian by knowing certain features of a classical background. In the following section, we show that this is indeed possible and can be exploited successfully.

### 3.4 Effective Lagrangian: a conjecture

As we have discussed in the last two sections, the effective Lagrangian approach is probably the most unambiguous approach available at present to study the evolution of quantum fields in classical electromagnetic backgrounds. This is not only true of electromagnetic backgrounds but applies to gravitational backgrounds too $[118,119,120,121,122,123,124,125]$. Several non-perturbative features of the theory can be understood if the effective Lagrangian can be evaluated exactly for an arbitrary background field configuration.

Symmetry considerations suggest that it should be possible to express the effective Lagrangian, at least formally, in terms of invariant scalars describing the
classical background (gauge invariant quantities involving the field tensor $F_{\mu \nu}$ and its derivatives in the case of electromagnetism and coordinate invariant scalars involving the Riemann curvature tensor $R_{\mu \nu \lambda \rho}$ and its derivatives in the case of gravity). The existence of an imaginary part to the effective Lagrangian-and other features-should be related to the actual values of some of these scalars. We had seen in subsection 1.5.1 that the effective Lagrangian for a constant electromagnetic background depends only on the two gauge invariant quantities $\mathcal{G}=F^{\mu \nu} F_{\mu \nu}=2\left(\mathbf{B}^{2}-\mathbf{E}^{2}\right)$ and $\mathcal{F}=\epsilon^{\mu \nu \lambda \rho} F_{\mu \nu} F_{\lambda \rho}=-8(\mathbf{E} . \mathbf{B})$. Further, we saw that the effective Lagrangian had an imaginary part only if $\mathcal{G}<0$, thereby implying that constant magnetic fields cannot produce particles while constant electric fields can. This result, of course, had been obtained only for constant $F_{\mu \nu}$ 's and it is not easy to evaluate the effective Lagrangian for a more general case. Also, for an arbitrary electromagnetic background, there is no a priori reason as to why the effective Lagrangian cannot depend on invariant quantities involving the derivatives of $F_{\mu \nu}$ 's, for instance, say, $\partial_{\lambda} F^{\mu \nu} \partial^{\lambda} F_{\mu \nu}$.

The situation is still worse in the case of gravitational backgrounds. The gravitational analogue of Schwinger's electromagnetic example would be the case of a constant gravitational field, i.e. a spacetime whose $R_{\mu \nu \lambda \rho}$ 's are constants. It would certainly be a worthwhile effort to evaluate the effective Lagrangian for such a background. Though, considerable amount work has been done in this direction in literature (see, for instance, refs. [126, 127, 128]), we are yet to have a covariant criterion for particle production by constant gravitational fields (analogous to the criterion $\mathcal{G}<0$ Schwinger had obtained for the constant electromagnetic background). Also, since the gravitational interaction is not renormalizable, it is not easy at all to regularize the effective Lagrangian (see, for e.g., ref. [4],
sections 6.11 and 6.12).

In this section, we investigate a related but more restricted question. We ask: Can one find non-trivial background field configurations for which the (regularized) effective Lagrangian vanishes identically? That is, we are interested in finding classical field configurations in which neither vacuum polarization nor particle production takes place. Such configurations certainly enjoy some special status because these are the ones for which lowest order semiclassical corrections vanish. The vanishing of the semiclassical corrections imply that the presence of the quantum field does not affect the classical background at all. Or, in other words, classical field configurations for which the effective Lagrangian is zero are stable in the sense that such backgrounds are immune to backreaction effects of the quantum field. What kind of classical field configurations will have such a feature?

The effective Lagrangian for the constant electromagnetic background reduces to zero when the gauge invariant quantities $\mathcal{F}$ and $\mathcal{G}$ are set to zero. Apart form this case, at least one more non-trivial electromagnetic field configuration is already known in literature for which the effective Lagrangian proves to be zero. Schwinger, in his pioneering paper [33], also calculates the effective Lagrangian for a plane electromagnetic wave background (for which gauge invariant quantities $\mathcal{F}$ and $\mathcal{G}$ are zero) and shows that it vanishes identically. These results suggest the following conjecture: The effective Lagrangian will be zero if all the scalar invariants describing the background vanish identically. In this section, we present examples of non-trivial electromagnetic and gravitational backgrounds with vanishing scalar invariants to support our conjecture. We evaluate the effective Lagrangian explicitly using Schwinger's proper time formalism for the case
of a quantized scalar field and show that it identically vanishes in these backgrounds [129].

In subsection 3.4.1, we present a time independent example from electromagnetism and in subsection 3.4.2, we evaluate the effective Lagrangian for the electromagnetic wave background using our technique. In subsection 3.4.3, we present an example from gravity. We explicitly evaluate the effective Lagrangian and show that it vanishes identically in these backgrounds. Finally, in subsection 3.4.4, we discuss the wider implications of our analysis.

### 3.4.1 A time independent electromagnetic example

Consider a time independent electromagnetic background described by the vector potential

$$
\begin{equation*}
A^{\mu}=(\phi(x, y), 0,0, \phi(x, y)) \tag{3.67}
\end{equation*}
$$

where $\phi(x, y)$ is an arbitrary function of the coordinates $x$ and $y$. The resulting electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ are then given by

$$
\begin{equation*}
\mathbf{E}=-\left(\frac{\partial \phi}{\partial x} \hat{\mathbf{x}}+\frac{\partial \phi}{\partial y} \hat{\mathbf{y}}\right) \quad \text { and } \quad \mathbf{B}=\left(\frac{\partial \phi}{\partial y} \hat{\mathbf{x}}-\frac{\partial \phi}{\partial x} \hat{\mathbf{y}}\right) \tag{3.68}
\end{equation*}
$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are the unit vectors along the positive $x$ and $y$ axes respectively. According to Maxwell's equations, in the absence of time dependence, the charge and the current densities, viz. $\rho$ and $\mathbf{j}$ that give rise to the above field configuration are

$$
\begin{equation*}
\rho=\nabla \cdot \mathbf{E}=-\left(\frac{\partial^{2} \phi}{\partial x^{2}}+\frac{\partial^{2} \phi}{\partial y^{2}}\right) \quad \text { and } \quad \mathbf{j}=\nabla \times \mathbf{B}=-\left(\frac{\partial^{2} \phi}{\partial x^{2}}+\frac{\partial^{2} \phi}{\partial y^{2}}\right) \hat{\mathbf{z}}, \tag{3.69}
\end{equation*}
$$

where $\hat{\mathbf{z}}$ is the unit vector along the positive $z$-axis. Therefore, if the functions $\rho$ and $\mathbf{j}$ are chosen such that they are finite and continuous everywhere and also
vanish as $\left(x^{2}+y^{2}\right) \rightarrow \infty$, then the corresponding electric and magnetic fields given by equation (3.68) will be confined to a finite extent in the $x-y$ plane.

It is obvious from equation (3.68) that $\mathcal{G}=2\left(\mathbf{B}^{2}-\mathbf{E}^{2}\right)=0$ and $\mathcal{F}=$ $-8(\mathbf{E} . \mathbf{B})=0$ for this background field configuration. (As an aside, note that this is an example of a field configuration other than that of a wave, for which $\left(\mathbf{E}^{2}-\mathbf{B}^{2}\right)$ as well as (E.B) are zero.) It is, therefore, a good candidate to test our conjecture. The operator $\hat{H}$ (cf. equation (3.13)) that corresponds to the vector potential (3.67) is given by

$$
\begin{equation*}
\hat{H} \equiv \partial_{t}^{2}-\nabla^{2}+2 i q \phi\left(\partial_{t}+\partial_{z}\right) . \tag{3.70}
\end{equation*}
$$

The kernel for the quantum mechanical particle described by the Hamiltonian operator above can then be formally written as

$$
\begin{equation*}
K(t, \mathbf{x}, s \mid t, \mathbf{x}, 0)=\langle t, \mathbf{x}| \exp -i\left[\left(\partial_{t}{ }^{2}-\nabla^{2}+2 i q \phi\left(\partial_{t}+\partial_{z}\right)\right) s\right]|t, \mathbf{x}\rangle . \tag{3.71}
\end{equation*}
$$

Using the translational invariance of the Hamiltonian operator $\hat{H}$ along the time coordinate $t$ and the spatial coordinate $z$, we can express the above kernel as follows

$$
\begin{align*}
K(t, \mathrm{x}, s \mid & t, \mathrm{x}, 0) \\
= & \int_{-\infty}^{\infty} \frac{d \omega}{2 \pi} \int_{-\infty}^{\infty} \frac{d p_{z}}{2 \pi} e^{i\left(\omega^{2}-p_{z}^{2}\right) s} \\
& \quad \times\langle x, y| \exp -i\left[\left(-\partial_{x}^{2}-\partial_{y}^{2}+2 q\left(\omega-p_{z}\right) \phi\right) s\right]|x, y\rangle \tag{3.72}
\end{align*}
$$

Changing variables of integration in the expression above to $p_{u}=\left(p_{z}-\omega\right) / 2$ and $p_{v}=\left(p_{z}+\omega\right) / 2$, we find that

$$
\begin{align*}
& K(t, \mathrm{x}, s \mid t, \mathrm{x}, 0) \\
& \begin{aligned}
=\left(\frac{1}{2 \pi^{2}}\right) & \int_{-\infty}^{\infty} d p_{u} \int_{-\infty}^{\infty} d p_{v} e^{-4 i p_{u} p_{v} s} \\
& \times\langle x, y| \exp -i\left[\left(-\partial_{x}^{2}-\partial_{y}^{2}-4 q p_{u} \phi\right) s\right]|x, y\rangle .
\end{aligned}
\end{align*}
$$

Performing the integrations over $p_{v}$ and the $p_{u}$ in that order, we obtain that

$$
\begin{align*}
K(t, \mathbf{x}, s \mid t, \mathbf{x}, 0)= & \left(\frac{1}{\pi}\right) \int_{-\infty}^{\infty} d p_{u} \delta_{D}\left(4 p_{u} s\right) \\
& \times\langle x, y| \exp -i\left[\left(-\partial_{x}^{2}-\partial_{y}^{2}-4 q p_{u} \phi\right) s\right]|x, y\rangle \\
= & \left(\frac{1}{4 \pi s}\right)\langle x, y| \exp -i\left[\left(-\partial_{x}^{2}-\partial_{y}^{2}\right) s\right]|x, y\rangle \\
= & \left(\frac{1}{16 \pi^{2} i s^{2}}\right) \tag{3.74}
\end{align*}
$$

Substituting this expression for $K(t, \mathbf{x}, s \mid t, \mathbf{x}, 0)$ in (3.11) we find that the resulting $\mathcal{L}_{\text {corr }}$ is the same as that of a free field (cf. equation (3.15)). So, on regularization $\mathcal{L}_{\text {corr }}$ identically reduces to zero. This result then implies that neither any particle production nor any vacuum polarization takes place in the time independent electromagnetic background we have considered here.

In arriving at the above result we have carried out the $p_{v}$ and the $p_{u}$ integrals first and then evaluated the matrix element. We shall now illustrate that such an interchange of operations is valid by testing it for the case of a simple example. Consider the case when $\phi(x, y)=x$. This corresponds to a constant electromagnetic background with the electric and magnetic fields given by $\mathbf{E}=-\hat{\mathbf{x}}$ and $\mathbf{B}=-\hat{\mathbf{y}}$. For this case, the operator $\hat{H}$ is given by

$$
\begin{equation*}
\hat{H}=\partial_{t}^{2}-\nabla^{2}+2 i q x\left(\partial_{t}+\partial_{z}\right) \tag{3.75}
\end{equation*}
$$

The translational invariance of the above operator along the $t, y$ and $z$ directions can then be exploited to express the quantum mechanical kernel for the above operator as follows

$$
\begin{align*}
& K(t, \mathbf{x}, s \mid t, \mathbf{x}, 0)= \int_{-\infty}^{\infty} \\
& \frac{d \omega}{2 \pi} \int_{-\infty}^{\infty} \frac{d p_{z}}{2 \pi} \int_{-\infty}^{\infty} \frac{d p_{y}}{2 \pi} e^{i\left(\omega^{2}-p_{y}^{2}-p_{z}^{2}\right) \cdot s}  \tag{3.76}\\
& \times\langle x| \exp -i\left[\left(-d_{x}^{2}+2 q\left(\omega-p_{z}\right) x\right) s\right]|x\rangle
\end{align*}
$$

Carrying out the $p_{y}$-intergration and changing variables to $p_{u}=\left(p_{z}-\omega\right) / 2$ and $p_{v}=\left(p_{z}+\omega\right) / 2$, we obtain that

$$
\begin{array}{r}
K(t, \mathbf{x}, s \mid t, \mathbf{x}, 0)=\left(\frac{1}{2 \pi^{2}(4 \pi i s)^{1 / 2}}\right) \int_{-\infty}^{\infty} d p_{v} \int_{-\infty}^{\infty} d p_{u} e^{-4 i p_{u} p_{v} s} \\
\times\langle x| \exp -i\left[\left(-d_{x}^{2}-4 q p_{u} x\right) s\right]|x\rangle . \tag{3.77}
\end{array}
$$

The matrix element in the above equation corresponds to that of a quantum mechanical particle subjected to a constant force along the $x$-axis. The matrix element above is then given by (see ref. [130], p. 194)

$$
\begin{equation*}
\langle x| \exp -i\left[\left(-d_{x}^{2}-4 q p_{u} x\right) s\right]|x\rangle=\left(\frac{1}{(4 \pi i s)^{1 / 2}}\right) \exp -4 i\left(q p_{u} x s+\frac{1}{3} q^{2} p_{u}^{2} s^{3}\right) . \tag{3.78}
\end{equation*}
$$

Substituting this expression in the kernel (3.77), we obtain that

$$
\begin{align*}
K(t, \mathbf{x}, s \mid t, \mathbf{x}, 0)= & \left(\frac{1}{8 \pi^{3} i s}\right) \int_{-\infty}^{\infty} d p_{v} \int_{-\infty}^{\infty} d p_{u} \exp -4 i\left\{\frac{1}{3} q^{2} p_{u}^{2} s^{3}\right. \\
& \left.+p_{u} s\left(q x+p_{v}\right)\right\} \\
= & \left(\frac{1}{8 \pi^{3} i s}\right)\left(\frac{3 \pi}{4 i q^{2} s^{3}}\right)^{1 / 2} \int_{-\infty}^{\infty} d p_{v} \exp \left(\frac{3 i}{q^{2} s}\left(p_{v}+q x\right)^{2}\right) \\
= & \left(\frac{1}{16 \pi^{2} i s^{2}}\right) \tag{3.79}
\end{align*}
$$

which is the result we have obtained in equation (3.74). This discussion confirms the fact that, in equation (3.74), the evaluation of the matrix element after the $p_{v}$ and $p_{u}$ integrals are carried out is a valid exchange of operations.

As we have mentioned earlier, the effective Lagrangian Schwinger had obtained for the constant electromagnetic background identically vanishes when the gauge invariant quantities $\mathcal{G}$ and $\mathcal{F}$ are set to zero [33]. Our result above agrees with Schwinger's result since a constant electromagnetic background would just correspond to choosing the function $\phi(x, y)$ above to be linear in the coordinates
$x$ and/or $y$. Having said that, we would like to stress here the following fact. In evaluating the effective Lagrangian above we have not made any assumptions at all on the form of the function $\phi(x, y)$. Hence, our result above holds good for any time independent electromagnetic background with vanishing $\mathcal{G}$ and $\mathcal{F}$. Thus, in a way, our result here is more generic than Schwinger's result.

### 3.4.2 Effective Lagrangian for a plane electromagnetic wave background

In this subsection, we rederive Schwinger's result for the electromagnetic wave background using our technique. The plane electromagnetic wave can be described by the vector potential

$$
\begin{equation*}
A_{\mu}=(0,1,0,0) f(t-z) \tag{3.80}
\end{equation*}
$$

where $f(t-z)$ is an arbitrary function of $(t-z)$. The operator $\hat{H}$ corresponding to this vector potential is then given by

$$
\begin{equation*}
\hat{H}=\partial_{t}^{2}-\partial_{x}^{2}-\partial_{y}^{2}-\partial_{z}^{2}+2 i q f \partial_{x}+q^{2} f^{2} \tag{3.81}
\end{equation*}
$$

and in terms of the null coordinates $u=(t-z)$ and $v=(t+z)$ the above operator reduces to

$$
\begin{equation*}
\hat{H}=4 \partial_{u} \partial_{v}-\partial_{x}^{2}-\partial_{y}^{2}+2 i q f(u) \partial_{x}+q^{2} f^{2}(u) \tag{3.82}
\end{equation*}
$$

The corresponding quantum mechanical kernel can then be formally expressed as

$$
\begin{align*}
& K(u, x, y, v, s \mid u, x, y, v, 0) \\
& \quad=\langle u, x, y, v| \exp -i\left[\left(4 \partial_{u} \partial_{v}-\partial_{x}^{2}-\partial_{y}^{2}+2 i q f \partial_{x}+q^{2} f^{2}\right) s\right]|u, x, y, v\rangle . \tag{3.83}
\end{align*}
$$

Exploiting the translational invariance of the operator $\hat{H}$ along the $x, y$ and the $v$ coordinates we can write the above kernel as

$$
K(u, x, y, v, s \mid u, x, y, v, 0)
$$

$$
\begin{align*}
= & \int_{-\infty}^{\infty} \frac{d p_{x}}{2 \pi} \int_{-\infty}^{\infty} \frac{d p_{y}}{2 \pi} \int_{-\infty}^{\infty} \frac{d p_{v}}{2 \pi} e^{-i\left(p_{x}^{2}+p_{y}^{2}\right) s} \\
& \quad \times 2\langle u| \exp -i\left[\left(-4 i p_{v} \partial_{u}-2 q p_{x} f+q^{2} f^{2}\right) s\right]|u\rangle \tag{3.84}
\end{align*}
$$

where the factor 2 is the Jacobian of the transformation between the conjugate momenta ( $\omega, p_{x}$ ) and ( $p_{u}, p_{v}$ ) corresponding to the coordinates $(t, z)$ and $(u, v)$ respectively.

The matrix element in the above equation corresponds to the quantum mechanical kernel for a time evolution operator given by

$$
\begin{equation*}
\hat{H}_{1}=-4 i p_{v} d_{u}-2 q p_{x} f(u)+q^{2} f^{2}(u) . \tag{3.85}
\end{equation*}
$$

The normalized solution $\psi_{E}(u)$ to the time independent Schrödinger equation for the operator $\hat{H}_{1}$ corresponding to an energy eigenvalue $E$ is then given by

$$
\begin{equation*}
\psi_{E}(u)=\left(\frac{1}{8 \pi p_{v}}\right)^{1 / 2} e^{i q E / 4 p_{v}} \exp -i\left(h(u) / 4 p_{v}\right) \tag{3.86}
\end{equation*}
$$

where

$$
\begin{equation*}
h(u)=-\int d u\left(2 q p_{x} f(u)-q^{2} f^{2}(u)\right) . \tag{3.87}
\end{equation*}
$$

The matrix element can now be evaluated with the help of the Feynman-Kac formula as follows (see, for instance, ref. [82], chapter 7):

$$
\begin{align*}
\langle u| e^{-i \hat{H}_{1} s}\left|u^{\prime}\right\rangle= & \int_{-\infty}^{\infty} d E \psi_{E}(u) \psi_{E}^{*}\left(u^{\prime}\right) e^{-i E s} \\
= & \left(\frac{1}{8 \pi p_{v}}\right) \exp -i\left\{\left(h(u)-h\left(u^{\prime}\right)\right) / 4 p_{v}\right\} \\
& \times \int_{-\infty}^{\infty} d E \exp i\left(E\left(u-u^{\prime}\right) / 4 p_{v}\right) e^{-i E s} \\
= & \exp -i\left\{\left(h(u)-h\left(u^{\prime}\right)\right) / 4 p_{v}\right\} \delta_{D}\left(u-u^{\prime}-s / 4 p_{v}\right) \tag{3.88}
\end{align*}
$$

and in the coincidence limit $u=u^{\prime}$, the matrix element reduces to a Dirac delta function i.e.

$$
\begin{equation*}
\langle u| \exp -i \hat{H}_{1} s|u\rangle=\delta_{D}\left(4 p_{v} s\right) . \tag{3.89}
\end{equation*}
$$

Substituting this result in equation (3.84), we obtain that

$$
\begin{align*}
K(u, x, y, v, s \mid u, x, y, v, 0) & =\left(\frac{2}{(4 \pi i s)^{1 / 2}}\right) \int_{-\infty}^{\infty} \frac{d p_{x}}{2 \pi} e^{-i p_{x}^{2} s} \int_{-\infty}^{\infty} \frac{d p_{v}}{2 \pi} \delta_{D}\left(4 p_{v} s\right) \\
& =\left(\frac{2}{4 \pi i s}\right) \int_{-\infty}^{\infty} \frac{d p_{v}}{2 \pi}\left(\frac{1}{4 s}\right) \delta_{D}\left(p_{v}\right) \\
& =\left(\frac{1}{16 \pi^{2} i s^{2}}\right) \tag{3.90}
\end{align*}
$$

When this kernel is substituted in equation (3.11) we find that the resulting $\mathcal{L}_{\text {corr }}$ is the same as that of $\mathcal{L}_{\text {corr }}^{0}$ (cf. equation (3.15)), which on regularization reduces identically to zero. Therefore, neither vacuum polarization nor particle production takes place in an electromagnetic wave background (also, see ref. [131]).

### 3.4.3 An example from gravity

In this subsection, we shall present a gravitational background for which the effective Lagrangian proves to be identically zero. The system we shall consider in this subsection consists of a massive, real scalar field $\Phi$ coupled minimally to gravity. It is described by the action

$$
\begin{align*}
\mathcal{S}\left[g_{\mu \nu}, \Phi\right] & =\int d^{4} x \sqrt{-g} \mathcal{L}\left(g_{\mu \nu}, \Phi\right) \\
& =\int d^{4} x \sqrt{-g}\left\{\frac{R}{16 \pi}+\frac{1}{2} g_{\mu \nu} \partial^{\mu} \Phi \partial^{\nu} \Phi-\frac{1}{2} m^{2} \Phi^{2}\right\} \tag{3.91}
\end{align*}
$$

where $m$ is the mass of a single quantum of the scalar field and $g_{\mu \nu}$ is the metric tensor describing the gravitational background and we have set $G=1$ for convenience. An effective Lagrangian can be defined for the gravitational background as follows (see our discussion in section 1.6):

$$
\begin{equation*}
\exp i \int d^{4} x \sqrt{-g} \mathcal{L}_{e f f}\left(g_{\mu \nu}\right) \equiv \int \mathcal{D} \Phi \exp i \mathcal{S}\left[\Phi, g_{\mu \nu}\right] \tag{3.92}
\end{equation*}
$$

The effective Lagrangian can then be expressed as $\mathcal{L}_{\text {eff }}=\mathcal{L}_{\text {grav }}+\mathcal{L}_{\text {corr }}$, where $\mathcal{L}_{\text {grav }}=(R / 16 \pi)$, the Lagrangian density for the gravitational background. Inte-
grating the action for the scalar field in the above equation by parts and dropping the resulting surface terms, we find that $\mathcal{L}_{\text {corr }}$ can then be expressed as (see, for instance, ref. [1], p. 193)

$$
\begin{align*}
\exp i \int d^{4} x \sqrt{-g} \mathcal{L}_{\text {corr }}\left(g_{\mu \nu}\right) & =\int \mathcal{D} \Phi \exp -i \int d^{4} x \sqrt{-g}(\Phi \hat{D} \Phi) \\
& =(\operatorname{det} \hat{D})^{-1 / 2} \\
& =\exp -\frac{1}{2} \operatorname{Tr}(\ln \hat{D}) \\
& =\exp -\frac{1}{2} \int d^{4} x \sqrt{-g}\langle t, \mathbf{x}| \ln \hat{D}|t, \mathbf{x}\rangle \tag{3.93}
\end{align*}
$$

where the operator $\hat{D}$ is given by

$$
\begin{equation*}
\hat{D} \equiv \frac{1}{\sqrt{-g}} \partial_{\mu}\left(g^{\mu \nu} \sqrt{-g} \partial_{\nu}\right)+m^{2} \tag{3.94}
\end{equation*}
$$

and, as we had done earlier done in the case of electromagnetism, we have introduced a complete set of orthonormal vectors $|t, \mathrm{x}\rangle$, to evaluate the trace. From equation (3.93) it is easy to identify that $\mathcal{L}_{\text {corr }}=(i / 2)\langle t, \mathrm{x}| \ln \hat{D}|t, \mathrm{x}\rangle$. Using equation (3.10) $\mathcal{L}_{\text {corr }}$ can then be written as

$$
\begin{equation*}
\mathcal{L}_{c o r r}=-\frac{i}{2} \int_{0}^{\infty} \frac{d s}{s} e^{-i\left(m^{2}-i \epsilon\right) s} K(t, \mathrm{x}, s \mid t, \mathrm{x}, 0), \tag{3.95}
\end{equation*}
$$

where

$$
\begin{equation*}
K(t, \mathbf{x}, s \mid t, \mathbf{x}, 0)=\langle t, \mathbf{x}| e^{-i \hat{H} s}|t, \mathbf{x}\rangle \tag{3.96}
\end{equation*}
$$

and the operator $\hat{H}$ is now given by

$$
\begin{equation*}
\hat{H} \equiv \frac{1}{\sqrt{-g}} \partial_{\mu}\left(g^{\mu \nu} \sqrt{-g} \partial_{\nu}\right) . \tag{3.97}
\end{equation*}
$$

To obtain finite results, the quantity that has to be subtracted from $\mathcal{L}_{\text {corr }}$ is then given by

$$
\begin{equation*}
\mathcal{L}_{\text {corr }}^{0}=-\left(\frac{1}{32 \pi^{2}}\right) \int_{0}^{\infty} \frac{d s}{s^{3}} e^{-i\left(m^{2}-i \epsilon\right) s} \tag{3.98}
\end{equation*}
$$

which corresponds to setting $g_{\mu \nu}=\eta_{\mu \nu}$ in the operator $\hat{H}$ above. $\left(\mathcal{L}_{\text {corr }}^{0}\right.$ given by equation (3.15) is twice the $\mathcal{L}_{\text {corr }}^{0}$ above because the complex scalar field we had considered in the last two subsections has twice the number of degrees of freedom as a real scalar field we are considering here.)

A gravitational background can be described by fourteen independent scalar invariants constructed out of the Riemann curvature tensor [132, 133]. To verify our conjecture, we should evaluate $\mathcal{L}_{\text {corr }}$ defined in equation (3.95) for a background for which all these invariants vanish. And, of course, we need a background which is sufficiently simple for allowing the evaluation of $\mathcal{L}_{\text {corr }}$ in a closed form.

One such example is given by the spacetime described by the line element

$$
\begin{equation*}
d s^{2}=(1+f(x, y)) d t^{2}-2 f(x, y) d t d z-(1-f(x, y)) d z^{2}-d x^{2}-d y^{2}, \tag{3.99}
\end{equation*}
$$

where $f(x, y)$ is an arbitrary function of the coordinates $x$ and $y$. (This metric is a special case of the metric that appears in [134]. It can be shown that all the fourteen algebraic invariants for this metric vanish identically [135].) The nonzero components of the Ricci tensor for the above metric are

$$
\begin{equation*}
R^{00}=R^{33}=R^{30}=\left(\frac{1}{2}\right)\left(\frac{\partial^{2} f}{\partial x^{2}}+\frac{\partial^{2} f}{\partial y^{2}}\right) \tag{3.100}
\end{equation*}
$$

and the Ricci scalar $R$ is zero. Since the Ricci scalar $R$ is zero, the Einstein tensor is given by $G^{\mu \nu}=R^{\mu \nu}$ and the Einstein's equations reduce to $R^{\mu \nu}=8 \pi T^{\mu \nu}$. A pressureless steady flow of null dust with energy density $\rho=R^{00}$ traveling along the $z$-direction satisfies the above Einstein's equations and therefore gives rise to the metric (3.99). Since $\operatorname{det}\left(g_{\mu \nu}\right)=-1$, the operator $\hat{H}$ (cf. equation (3.97)) corresponding to this metric is given by

$$
\begin{equation*}
\hat{H}=\partial_{t}^{2}-\partial_{z}^{2}-\partial_{x}^{2}-\partial_{y}^{2}-f\left(\partial_{t}^{2}+\partial_{z}^{2}+2 \partial_{t} \partial_{z}\right) . \tag{3.101}
\end{equation*}
$$

Using the translational invariance along the $t$ and $z$ directions the kernel for the time evolution operator above can be written as

$$
\begin{align*}
K(t, \mathbf{x}, s \mid t, \mathbf{x}, 0)= & \int_{-\infty}^{\infty} \frac{d \omega}{2 \pi} \int_{-\infty}^{\infty} \frac{d p_{z}}{2 \pi} e^{i\left(\omega^{2}-p_{z}^{2}\right) s} \\
& \times\langle x, y| \exp -i\left[\left(-\partial_{x}^{2}-\partial_{y}^{2}+\left(\omega-p_{z}\right)^{2} f\right) s\right]|x, y\rangle \tag{3.102}
\end{align*}
$$

Changing the variables of integration to $p_{u}=\left(p_{z}-\omega\right) / 2$ and $p_{v}=\left(p_{z}+\omega\right) / 2$, we obtain that

$$
\begin{align*}
& K(t, \mathbf{x}, s \mid t, \mathbf{x}, 0)=\left(\frac{1}{2 \pi^{2}}\right) \int_{-\infty}^{\infty} d p_{u} \\
& \quad \int_{-\infty}^{\infty} d p_{v} e^{-4 i p_{u} p_{v} s} \\
& \quad \times\langle x, y| \exp -i\left[\left(-\partial_{x}^{2}-\partial_{y}^{2}+4 p_{u}^{2} f\right) s\right]|x, y\rangle \\
&=\left(\frac{1}{\pi}\right) \int_{-\infty}^{\infty} d p_{u} \\
& \delta_{D}\left(4 p_{u} s\right) \\
& \times\langle x, y| \exp -i\left[\left(-\partial_{x}^{2}-\partial_{y}^{2}+4 p_{u}^{2} f\right) s\right]|x, y\rangle  \tag{3.103}\\
&=\left(\frac{1}{4 \pi s}\right)\langle x, y| \exp -i\left[\left(-\partial_{x}^{2}-\partial_{y}^{2}\right) s\right]|x, y\rangle \\
&=\left(\frac{1}{16 \pi^{2} i s^{2}}\right) .
\end{align*}
$$

Substituting the above result in equation (3.95) we find that

$$
\begin{equation*}
\mathcal{L}_{\text {corr }}=-\left(\frac{1}{32 \pi^{2}}\right) \int_{0}^{\infty} \frac{d s}{s^{3}} e^{-i\left(m^{2}-i \epsilon\right) s} \tag{3.104}
\end{equation*}
$$

which on subtracting the quantity $\mathcal{L}_{\text {corr }}^{0}$ given by equation (3.98) reduces to zero. This result again implies that in the gravitational background we have considered here neither any particle production nor any vacuum polarization takes place.

### 3.4.4 Discussion

The effective Lagrangian provides a simple way of estimating the amount of vacuum polarization and particle production in a classical background. For example,
the background field is expected to induce vacuum instability and produce particles if and only if the effective Lagrangian has an imaginary part. If the effective Lagrangian vanishes for a particular background field, then no vacuum polarization or particle production takes place in such a field configuration.

In principle, this is an observable phenomenon since physical effects occur if the effective Lagrangian happens to be nonzero. For example, consider a constant electric field confined in space, say, the electric field between a pair of capacitor plates. In such a case, the imaginary part of effective Lagrangian will be nonzero and the particle production will take place. These particles that have been produced will get attracted towards the capacitor plates thereby reducing the strength of the electric field between the plates. To maintain the original configuration intact, an external agency has to correct for this effect. We can therefore conclude that the above configuration-viz., that of a constant electric field in a confined region-is not immune to quantum backreaction effects. Such, physically observable, effects do occur even if the effective Lagrangian does not have an imaginary part. A typical example would be the Casimir effect in flat spacetime. It can be shown that for such a case the effective Lagrangian is nonzero and real. The effective Lagrangian which depends on the separation between the plates, can be related to the Casimir energy. The resulting observable physical effect is the attraction between the Casimir plates. Left to themselves, the Casimir plates will move towards each other because of a force which is a quantum backreaction effect arising from the nonzero real part of the effective Lagrangian. Once again, to maintain the original configuration-viz., the original separation between the plates-an external agency has to correct for the quantum backreaction effect.

In contrast to the above examples, backgrounds with vanishing effective

Lagrangian are 'self-consistent' in the sense that no backreaction of the quantum field on the classical background occurs in these configurations. This is a feature of certain backgrounds which does not seem to have been noted in literature before. This aspect seems to be worthy of further study.

It should be possible to express the determinant of the operator $\hat{D}$ (and hence the quantity $\mathcal{L}_{\text {corr }}$ ) appearing in equations (3.6) and (3.93), at least formally, in terms of the invariant quantities describing the background. In particular, one would expect the effective Lagrangian to contain only those terms that are simple algebraic functions of the scalar invariants (otherwise renormalization would not be possible). If so, the effective Lagrangian would prove to be zero if all the invariants describing the background vanish identically. Motivated by this fact, we put forward the conjecture that the regularized $\mathcal{L}_{\text {corr }}$ will prove to be zero for background field configurations for which all scalar invariants are zero. In other words, our conjecture implies that integrating out the degrees of freedom corresponding to the quantum field does not introduce any quantum corrections to the Lagrangian describing classical backgrounds with vanishing scalar invariants.

We had also tested our conjecture with some specific examples. For the electromagnetic background we have considered in section 3.4.1 we had pointed out that the gauge invariant quantities $\mathcal{G}$ and $\mathcal{F}$ are zero and it can be easily shown that quantities such as $\partial_{\lambda} F^{\mu \nu} \partial^{\lambda} F_{\mu \nu}$ and $\epsilon^{\lambda \rho \mu \nu} \partial_{\eta} F_{\lambda_{\rho}} \partial^{\eta} F_{\mu \nu}$ also vanish identically. It is likely that all the gauge invariant quantities that can be constructed out of the vector potential (3.67) vanish identically. For the gravitational example considered in section 3.4.3, as mentioned before, it can be shown that all the fourteen algebraic invariants that can be constructed out of the Riemann tensor for the metric (3.99) vanish identically [135]. Therefore, the vanishing of $\mathcal{L}_{\text {corr }}$ for
these backgrounds is consistent with-and supports-our conjecture.

We would like to point out here the following fact. The classical backgrounds we have presented in sections 3.4.1 and 3.4.3 are non-trivial though all the scalar invariants may vanish. They are not just flat space presented in an arbitrary gauge or a coordinate system. The fact that a particle in these backgrounds will experience non-trivial forces acting on it ascertains this fact.

The examples that we had presented in sections 3.4.1 and 3.4.3 are time independent examples. As we saw in 3.4.2, an example of a time dependent background for which the effective Lagrangian proves to be zero is that of a plane electromagnetic wave. For the electromagnetic wave background, all the gauge invariant quantities vanish identically. This can be argued as follows (see, ref. [131]). The characteristic of a plane electromagnetic wave is that its field strength is of the form

$$
\begin{equation*}
F_{\mu \nu}=f_{\mu \nu} F\left(n_{\mu} x^{\mu}\right), \tag{3.105}
\end{equation*}
$$

where $n_{\mu}$ is a null vector and the amplitudes $f_{\mu \nu}$ 's are constants. Also, $F_{\mu \nu}$ and ${ }^{*} F_{\mu \nu}\left({ }^{*} F_{\mu \nu}\right.$ is the dual of $\left.F_{\mu \nu}\right)$ are orthogonal to $n_{\mu}$. It is then clear that all gauge invariant quantities with explicit derivatives vanish, since any $n_{\mu}$ must contract either with $F_{\mu \nu}$ or itself. Thus, only polynomials in $F_{\mu \nu}$ and ${ }^{*} F_{\mu \nu}$ remain. But, any scalar function involving $F_{\mu \nu}$ and ${ }^{*} F_{\mu \nu}$ can be written only in terms of the invariants $\mathcal{G}$ and $\mathcal{F}$, both of which vanish here. Therefore, all gauge invariant quantities involving the field tensor and its derivatives are zero for an electromagnetic wave background. Hence, the vanishing of the effective Lagrangian in such a background clearly supports our conjecture.

Ideally, one would have liked to evaluate the effective Lagrangian for an
arbitrary classical background field configuration. But, as we have pointed out repeatedly, evaluating the effective Lagrangian for an arbitrary classical background proves to be an impossible task. Due to this reason, our approach to this entire problem has been a more practical one. The conjecture we have put forward in this section is only the first step in this approach. There exist deeper reasons in proposing this conjecture (with the danger of sounding obvious) and attempting to establish its validity with some specific examples. These motivations are as follows. The effective Lagrangian may indeed prove to be zero for classical backgrounds for which all the scalar invariants are zero, but the converse need not be true. That is, the effective Lagrangian may prove to be zero even though some of the scalar invariants describing the background are nonzero. Backgrounds with vanishing effective Lagrangians but non-vanishing scalar invariants can help us identify the terms that will appear in the effective Lagrangian for the most general case. Classifying such backgrounds will certainly prove to be a worthwhile exercise when evaluating the effective Lagrangian for an arbitrary background is proving to be an impossible task.

### 3.5 Some remarks on the Schwinger's formalism

Our discussion in the last three sections clearly points to the following fact: The effective Lagrangian approach proves to be more reliable than the other approaches available at present to study phenomena such as vacuum polarization and particle production in classical backgrounds. Also, we have been able to utilize the formalism due to Schwinger to evaluate the effective Lagrangian for non-trivial electromagnetic and gravitational backgrounds.

But, Schwinger's formalism implicitly chooses a particular boundary (or initial) condition in the evaluation of the effective Lagrangian. It is not at all obvious as to what would such a boundary condition correspond to in a general situation. Let us say that we are evaluating the effective Lagrangian for the Schwarzschild spacetime using Schwinger's formalism. What would the boundary condition that is implicitly chosen in Schwinger's formalism correspond to in such a case? Will it correspond to choosing the initial state of the quantum field to be the Boulware vacuum state or will it prove to be the Unruh vacuum? If it is the former, then we would expect the effective Lagrangian to have no imaginary part, whereas we would expect its imaginary part to be nonzero for the latter condition. Also, if it is the latter condition, we would expect the nonzero imaginary part to correspond to the total energy emitted by the black hole due to Hawking radiation.

Some insight into this aspect of Schwinger's formalism can be gained by evaluating the Feynman Green's function using the same formalism for the constant electric field background in the two gauges $A_{1}^{\mu}$ and $A_{2}^{\mu}$ given by equations (1.120) and (1.121), respectively. The Feynman Green's function $G_{F}\left(x, x^{\prime}\right)$ satisfies the following differential equation

$$
\begin{equation*}
\hat{D}_{x} G_{F}\left(x, x^{\prime}\right)=-\delta_{D}\left(x-x^{\prime}\right), \tag{3.106}
\end{equation*}
$$

where the operator $\hat{D}$ is given by equation (3.7) and the subscript $x$ on the operator $\hat{D}$ denotes that the differentials are with respect to coordinates $(t, x)$. Using a complete orthonormal set of vectors $|t, \mathbf{x}\rangle$ the Feynman Green's function can be expressed as follows:

$$
\begin{equation*}
G_{F}\left(x, x^{\prime}\right)=\langle t, \mathbf{x}| \hat{G}_{F}\left|t^{\prime}, \mathbf{x}^{\prime}\right\rangle . \tag{3.107}
\end{equation*}
$$

Then, the differential equation satisfied by $G_{F}\left(x, x^{\prime}\right)$ can be formally written as

$$
\begin{equation*}
\hat{D}_{x}\langle t, \mathbf{x}| \hat{G}_{F}\left|t^{\prime}, \mathbf{x}^{\prime}\right\rangle=-\left\langle t, \mathbf{x} \mid t^{\prime}, \mathbf{x}^{\prime}\right\rangle \tag{3.108}
\end{equation*}
$$

or simply as $\hat{D} \hat{G}_{F}=-1$, which then implies that $\hat{G}_{F} \equiv-\hat{D}^{-1}$ (see, for e.g., ref. [136]). Therefore

$$
\begin{equation*}
G_{F}\left(x, x^{\prime}\right)=\langle t, \mathrm{x}| \hat{D}^{-1}\left|t^{\prime}, \mathrm{x}^{\prime}\right\rangle \tag{3.109}
\end{equation*}
$$

Using the following integral representation for the operator $\hat{D}^{-1}$ :

$$
\begin{equation*}
\hat{D}^{-1}=i \int_{0}^{\infty} d s \exp -i(\hat{D}-i \epsilon) s \tag{3.110}
\end{equation*}
$$

(where $\epsilon \rightarrow 0^{+}$), the Feynman Green's function can be written as

$$
\begin{align*}
G_{F}\left(x, x^{\prime}\right) & =-i \int_{0}^{\infty} d s\langle t, \mathbf{x}| e^{-i(\hat{D}-i \epsilon) s}\left|t^{\prime}, \mathbf{x}^{\prime}\right\rangle \\
& =-i \int_{0}^{\infty} d s e^{-i\left(m^{2}-i \epsilon\right) s} K\left(t, \mathbf{x}, s \mid t^{\prime}, \mathbf{x}^{\prime}, 0\right) \tag{3.111}
\end{align*}
$$

where

$$
\begin{equation*}
K\left(t, \mathrm{x}, s \mid t^{\prime}, \mathrm{x}^{\prime}, 0\right)=\langle t, \mathrm{x}| e^{-i H s}\left|t^{\prime}, \mathrm{x}^{\prime}\right\rangle \tag{3.112}
\end{equation*}
$$

is the quantum mechanical kernel corresponding to the Hamiltonian operator $\hat{H}$ (given by equation (3.13)) we have encountered earlier. The Feynman's Green's function for the case of a constant electric field background can be evaluated in the two gauges $A_{1}^{\mu}$ and $A_{2}^{m u}$ using the above technique. It is given by the expression

$$
\begin{align*}
G_{F}^{1}\left(x, x^{\prime}\right)=-\left(\frac{q E}{16 \pi^{2}}\right) & \exp -i q E\left(t+t^{\prime}\right)\left(x-x^{\prime}\right) / 2 \\
& \times \int_{0}^{\infty} \frac{d s}{s \sinh (q E s)} e^{-i\left(m^{2}-i \epsilon\right) s} \exp \frac{i}{4 s}\left\{\left(y-y^{\prime}\right)^{2}+\left(z-z^{\prime}\right)^{2}\right\} \\
& \times \exp -\frac{i q E}{4 \tanh (q E s)}\left\{\left(t-t^{\prime}\right)^{2}+\left(x-x^{\prime}\right)^{2}\right\} \tag{3.113}
\end{align*}
$$

in the time dependent gauge $A_{1}^{\mu}$ and by the expression

$$
G_{F}^{2}\left(x, x^{\prime}\right)=-\left(\frac{q E}{16 \pi^{2}}\right) \exp i q E\left(t-t^{\prime}\right)\left(x+x^{\prime}\right) / 2
$$

$$
\begin{align*}
& \times \int_{0}^{\infty} \frac{d s}{s \sinh (q E s)} e^{-i\left(m^{2}-i \epsilon\right) s} \exp \frac{i}{4 s}\left\{\left(y-y^{\prime}\right)^{2}+\left(z-z^{\prime}\right)^{2}\right\} \\
& \quad \times \exp -\frac{i q E}{4 \tanh (q E s)}\left\{\left(t-t^{\prime}\right)^{2}+\left(x-x^{\prime}\right)^{2}\right\} \tag{3.114}
\end{align*}
$$

in the space dependent gauge $A_{2}^{\mu}$. From these two expressions, it can be easily seen that

$$
\begin{equation*}
G_{F}^{2}\left(x, x^{\prime}\right)=G_{F}^{1}\left(x, x^{\prime}\right) \exp i q E\left(t x-t^{\prime} x^{\prime}\right) . \tag{3.115}
\end{equation*}
$$

The gauge transformed mode in the gauge $A_{2}^{\mu}$ is a product of the normal mode in the gauge $A_{1}^{\mu}$ and the gauge factor $\exp i(q E x t)$. Hence, in the above relation the phase that relates $G_{F}^{2}$ to $G_{F}^{1}$ is the corresponding gauge factor. Therefore, the above relation implies that the Feynman Green's functions evaluated in the two gauges using Schwinger's formalism are gauge transforms of each other. On the other hand, compare the normal modes we had obtained earlier in these two gauges $A_{1} \mu$ and $A_{2}^{\mu}$. They are given by equations (1.123) and (1.137). It can be easily seen that the normal mode in the gauge $A_{2}^{\mu}$ is not equal to the product of the normal mode in the gauge $A_{1}^{\mu}$ and the gauge factor $\exp i(q E x t)$. Therefore, the normal modes in these two gauges are not gauge transforms of each other.

Our analysis in this section shows that Schwinger's formalism is able to choose a particular boundary condition such that the resulting effective Lagrangian yields a gauge invariant result. It will be a worthwhile effort to carry out a similar analysis in different coordinates corresponding to a particular curved spacetime, say, for instance, the black hole spacetime. Such an analysis might provide us with some clues to understand the reason behind the coordinate dependence of the particle concept in a curved spacetime.

## Chapter 4

## Limited validity of the semiclassical theory

Our aim in the last two chapters has been two fold: (i) to improve our understanding of the particle concept and (ii) to look for an invariant description of the phenomenon of particle production. To a certain extent, we have been successful in our efforts. In chapter 2, we found that a finite time detector can possibly be utilized to provide a localized definition of the particle concept. And, in chapter 3, we illustrated as to how the effective Lagrangian approach proves to be more reliable than the other approaches that are available at present to study the evolution of quantum fields in classical backgrounds. Not only that, the effective Lagrangian is an invariant quantity and hence it can directly lead to an invariant description of the phenomenon of particle production. In the last two chapters, we had studied the evolution of quantum fields in a given classical background and we had not taken into account the backreaction of the quantum field on the classical background. In this chapter we shall analyze some issues of the backreaction problem.

As we had discussed in section 1.6, the effective Lagrangian approach itself
can be used to study the backreaction problem. Say, we are able to evaluate the effective Lagrangian for an arbitrary background field configuration. Then, we can vary the resulting effective Lagrangian and obtain the equations of motion for the classical background field, thereby possibly even taking into the account the backreaction of the quantum field on the classical background. Of course, as we have repeatedly mentioned, the evaluation of the effective Lagrangian for an arbitrary background proves to be an impossible task. Even if we had been able to do so, such an approach has another drawback which we had pointed out, earlier, in section 1.6. The effective Lagrangian is, in general, a complex quantity and hence the resulting equations of motion for the classical background can prove to be complex. The backreaction of the quantum field on the classical background can then possibly be studied by dropping the imaginary part of the effective Lagrangian and retaining only the real part. But, such prescription would be ad hoc. Also, since the imaginary part of the effective Lagrangian reflects the amount of particle produced by the background, dropping the imaginary part of the effective Lagrangian would correspond to neglecting the effect of particle production on the classical background.

In the case of gravitational backgrounds, as we have discussed in section 1.6, a more natural and plausible proposal would be to consider the expectation value of the energy-momentum operator of the quantum field as the term that induces the non-trivial geometry $[27,28,29,30,31,32]$. (See refs. [137, 138, 139], for a discussion of the backreaction of the quantum field on classical electromagnetic backgrounds.) Since the theory we are considering here, by itself, is incapable of providing us with a preferred state for the quantum matter field, the expectation value $\left\langle\hat{T}_{\mu \nu}\right\rangle$ has to be evaluated in a state specified by hand that is also consistent
with the dynamics. Therefore, the complete analysis of the backreaction of a quantum field, say a massless scalar field, on the classical background metric reduces to that of solving the Einstein's equations

$$
\begin{equation*}
G_{\mu \nu}=R_{\mu \nu}-\frac{1}{2} g_{\mu \nu} R=8 \pi\left\langle\hat{T}_{\mu \nu}\right\rangle, \tag{4.1}
\end{equation*}
$$

where $\left\langle\hat{T}_{\mu \nu}\right\rangle$ is the expectation value of the energy-momentum operator (in the specified state) of the scalar field and the following Klein-Gordon equation

$$
\begin{equation*}
\frac{1}{\sqrt{-g}} \partial_{\mu}\left(\sqrt{-g} g^{\mu \nu} \partial_{\nu}\right) \Phi=0, \tag{4.2}
\end{equation*}
$$

self-consistently.

Apart from the fact that the energy scales involved should be far below the Planck scale ( $\sim 10^{19} \mathrm{GeV}$ ) for the semiclassical theory as proposed above to be valid, the fluctuations in the energy-momentum densities of the quantum field should not be too large either [140], i.e. we must demand

$$
\begin{equation*}
\left\langle\hat{T}_{\alpha \beta}(x) \hat{T}_{\mu \nu}(y)\right\rangle \approx\left\langle\hat{T}_{\alpha \beta}(x)\right\rangle\left\langle\hat{T}_{\mu \nu}(y)\right\rangle . \tag{4.3}
\end{equation*}
$$

So, equation (4.1) will prove to be inadequate to describe a situation when the fluctuations in the energy-momentum densities are large. The goal of this present chapter is to check the validity of the semiclassical theory that is based on the equations (4.1) and (4.2) in time dependent background metrics like, for instance, a Friedmann universe, for different states prescribed for the quantum field [141].

The calculations necessary for drawing the limits on the validity of the semiclassical theory, with aid of the condition (4.3), will involve evaluating the expectation values of the operators $\hat{T}_{\mu \nu}$ and $\hat{T}_{\mu \nu} \hat{T}_{\alpha \beta}$. These calculations will involve divergences of quantum field theory, which arise because of the infinite
degrees of freedom associated with the fields, and these infinities will have to be regularized in a systematic manner. Since these issues will eventually sidetrack our main concern, in this chapter, we shall study the backreaction problem for a minisuperspace model of a Friedmann universe with a quantized massless scalar field when all but one mode of the scalar field are 'frozen'. (For a discussion on minisuperspace, see, for instance, ref. [142].) In such a case, the divergences that may arise because of the infinite degrees of freedom are avoided.

This chapter is organized as follows. In section 4.1, we discuss the minisuperspace model we intend to study and in section 4.2 , we extend the criterion that has been suggested earlier by Kuo and Ford to draw the limits on the validity of the semiclassical theory to our model. In section 4.3, we utilize this criterion to study the reliability of the semiclassical theory for our model when the quantized scalar field mode is assumed to be in a (i) vacuum, (ii) $n$-particle or (iii) coherent state. In section 4.4, we discuss the implications of our analysis on field theory and we close this chapter with section 4.5 , wherein we present the conclusions that can drawn from our analysis.

### 4.1 Friedmann universe with a massless scalar field: minisuperspace model

A massless scalar field $\Phi$ that is coupled minimally to gravity is described by the following action:

$$
\begin{equation*}
\mathcal{S}\left[g_{\mu \nu}, \Phi\right]=\int d^{4} x \sqrt{-g}\left(\frac{R}{16 \pi}+\frac{1}{2} g_{\mu \nu} \partial^{\mu} \Phi \partial^{\nu} \Phi\right) . \tag{4.4}
\end{equation*}
$$

Consider a homogeneous and and isotropic spacetime described by the line element

$$
\begin{equation*}
d s^{2}=N^{2}(t) d t^{2}-a^{2}(t)\left(d x^{2}+d y^{2}+d z^{2}\right), \tag{4.5}
\end{equation*}
$$

where $N(t)$ is an arbitrary function of the time coordinate $t$. We can exploit the homogeneity of such a spacetime and decompose the scalar field into its Fourier modes. For the case of the metric (4.5), the above action, after the $\ddot{a}$ terms have been integrated away by parts and the scalar field has been decomposed into its Fourier modes, reduces to

$$
\begin{equation*}
\mathcal{S}\left[a, q_{k}\right]=\int d t a^{3}\left(-\frac{3 V}{8 \pi N}\left\{\frac{\dot{a}^{2}}{a^{2}}\right\}+\sum_{\mathbf{k}} \frac{1}{2}\left\{\frac{\left|\dot{q}_{k}\right|^{2}}{N}-N \omega^{2}\left|q_{k}\right|^{2}\right\}\right), \tag{4.6}
\end{equation*}
$$

where $q_{k}(t)$ are the spatial Fourier transforms of the scalar field, $\omega(t)=(k / a(t))$, $k=|\mathbf{k}|$ and $V$ is the volume of the universe.

As is well-known, a quantum field has infinite degrees of freedom associated with it. Due to this reason, divergences arise when we evaluate expectation values, say, that of the energy-momentum tensor of the quantum field. Therefore, to avoid these divergences, as we had mentioned at the beginning of this chapter, we shall consider the evolution of just a single mode $\mathbf{k}$ of the scalar field. That is, we shall carry out our analysis for a system with only a finite number of degrees of freedom (in fact, just two) which is described by the following action:

$$
\begin{equation*}
\mathcal{S}[a, q]=\int d t a^{3}\left(-\frac{3 V}{8 \pi N}\left\{\frac{\dot{a}^{2}}{a^{2}}\right\}+\frac{1}{2}\left\{\frac{\dot{q}^{2}}{N}-N \omega^{2} q^{2}\right\}\right) . \tag{4.7}
\end{equation*}
$$

Varying the above action with respect to $N$ and setting $N=1$ after the variation, we obtain the following equation of motion for the degree of freedom $a$ :

$$
\begin{equation*}
\frac{\dot{a}^{2}}{a^{2}}=\frac{8 \pi}{3 V}\left\{\frac{1}{2}\left(\dot{q}^{2}+\omega^{2} q^{2}\right)\right\}, \tag{4.8}
\end{equation*}
$$

which is the Friedmann equation (or rather, its minisuperspace version) we will be interested in.

In the semiclassical domain, when the single mode $q$ of the scalar field is
quantized it satisfies the following Heisenberg equation of motion

$$
\begin{equation*}
\frac{d^{2} \hat{q}}{d t^{2}}+3 \frac{\dot{a}}{a} \frac{d \hat{q}}{d t}+\omega^{2} \hat{q}=0 \tag{4.9}
\end{equation*}
$$

Let us now express the operator $\hat{q}$ as follows:

$$
\begin{equation*}
\hat{q}(t)=\hat{A} Q(t)+\hat{A}^{\dagger} Q^{*}(t) \tag{4.10}
\end{equation*}
$$

where $\hat{A}$ is an operator independent of time and $Q$ satisfies the same differential equation as the operator $\hat{q}$. As we have discussed towards the end of section 1.1, in a gravitational background a timelike Killing vector field is essential to define the positive frequency modes of the quantum field unambiguously. But the Friedmann universe we are considering here is time dependent, and, in general, it will not possess a timelike Killing vector field. We had encountered such a time dependent situation, earlier, in subsection 1.4.1, when we had analyzed the evolution of a quantum field in a constant electric field background in the time dependent gauge $A_{1}^{\mu}$. We had then defined the positive frequency modes of the quantum field in the WKB limit. In the case of the Friedmann universe we are considering here, we can carry out a similar decomposition of $Q$ in the WKB limit. Note that in the action (4.6) the mode $q$ resembles a time dependent oscillator with a mass $a^{3}$ and frequency $\omega$ (when $N$ is assumed to be unity). Therefore, $Q$ can be decomposed in the WKB limit as follows [21, 23, 24]:

$$
\begin{equation*}
Q=\alpha(t) f(t)+\beta(t) f^{*}(t), \tag{4.11}
\end{equation*}
$$

where $f(t)$ is to be identified as the positive frequency component of the scalar field mode $\hat{q}$. It is given by

$$
\begin{equation*}
f(t)=\frac{1}{\sqrt{2 \omega a^{3}}} \exp -i\left\{\int_{t_{0}}^{t} d t^{\prime} \omega\left(t^{\prime}\right)\right\} \tag{4.12}
\end{equation*}
$$

where $t_{0}$ is an early time when the initial conditions for the differential equation (4.9) are specified. If we now define $\dot{Q}$ to be

$$
\begin{equation*}
\dot{Q}=-i \omega\left(\alpha(t) f(t)-\beta(t) f^{*}(t)\right), \tag{4.13}
\end{equation*}
$$

then, we find that $\alpha$ and $\beta$ satisfy the following set of coupled differential equations

$$
\left.\begin{array}{l}
\dot{\alpha}=(\dot{a} / a) \beta \exp 2 i\left\{\int_{t_{0}}^{t} d t^{\prime} \omega\left(t^{\prime}\right)\right\}  \tag{4.14}\\
\dot{\beta}=(\dot{a} / a) \alpha \exp -2 i\left\{\int_{t_{0}}^{t} d t^{\prime} \omega\left(t^{\prime}\right)\right\}
\end{array}\right\}
$$

If the initial conditions for $Q$ are chosen such that $\alpha\left(t_{0}\right)=1$ and $\beta\left(t_{0}\right)=0$, then the Wronskian condition corresponding to the differential equation (4.9) is

$$
\begin{equation*}
|\alpha|^{2}-|\beta|^{2}=1 . \tag{4.15}
\end{equation*}
$$

If we now substitute equation (4.11) in (4.10), we find that the operator $\hat{q}$ is given by

$$
\begin{equation*}
\hat{q}(t)=\hat{a}(t) f(t)+\hat{a}^{\dagger}(t) f^{*}(t) \tag{4.16}
\end{equation*}
$$

where

$$
\begin{equation*}
\hat{a}(t)=\alpha(t) \hat{A}+\beta^{*}(t) \hat{A}^{\dagger} \tag{4.17}
\end{equation*}
$$

and $\hat{a}\left(t_{0}\right)=\hat{A}$. From the above relation it is easy to see that the quantities $\alpha$ and $\beta$ are the Bogolubov coefficients that relate the annihilation and the creation operators at the initial time $t_{0}\left(\hat{A}\right.$ and $\left.\hat{A}^{\dagger}\right)$ to those at any later time $t(\hat{a}$ and $\left.\hat{a}^{d a g}\right)$. The Hamiltonian corresponding to the scalar field mode at any time $t \geq t_{0}$ is given by

$$
\begin{align*}
\hat{H} & =\left(\frac{a^{3}}{2}\right)\left(\hat{\dot{q}}^{2}+\omega^{2} \hat{q}^{2}\right) \\
& =\left(\hat{a}^{\dagger} \hat{a}+(1 / 2)\right) \omega . \tag{4.18}
\end{align*}
$$

The decomposition of operator $\hat{q}$, as we have carried out in equation (4.16 corresponds to an instantaneous diagonalization of the Hamiltonian $\hat{H}$.

In the semiclassical domain, when the single mode $q$ of the scalar field has been quantized as discussed above, the semiclassical equation corresponding to (4.1) for our minisuperspace model is then given by

$$
\begin{equation*}
\frac{\dot{a}^{2}}{a^{2}}=\frac{8 \pi}{3 V a^{3}}\langle\psi| \hat{H}|\psi\rangle, \tag{4.19}
\end{equation*}
$$

where $|\psi\rangle$ is the state of the scalar field mode and $\hat{H}$ is given by (4.18). In the Heisenberg picture we are considering here, the quantum state $|\psi\rangle$ is independent of time and it can be defined at the same time $t_{0}$ when the initial conditions for the differential equation (4.9) are specified. In the following sections of this chapter, we shall examine the validity of the semiclassical equation (4.19) for different states of the quantized scalar field mode $\hat{q}$. The three quantum states of the scalar field mode we will be interested in are the (i) vacuum (|0才), (ii) $n$-particle $(|n\rangle)$ and (iii) coherent $(|\lambda\rangle)$ states. They are defined as follows:

$$
\left.\begin{array}{rl}
\hat{A}|0\rangle & =0  \tag{4.20}\\
\hat{A}^{\dagger} \hat{A}|n\rangle & =n|n\rangle \\
\hat{A}|\lambda\rangle & =\lambda|\lambda\rangle .
\end{array}\right\}
$$

### 4.2 Criterion for drawing the limits on the validity of the semiclassical theory

The semiclassical theory as described by the equations (4.1) and (4.2) does not take into account the fluctuations in the energy-momentum densities of the quantum field. Hence, as we had discussed at the beginning of this chapter, this
semiclassical theory can be relied upon only when the fluctuations in the energymomentum densities of the quantum field are small when compared to their expectation values.

Motivated by this fact, Kuo and Ford have suggested that the dimensionless quantity [143]

$$
\begin{equation*}
\Delta_{\alpha \beta \mu \nu}(x, y) \equiv\left|\frac{\left\langle: \hat{T}_{\alpha \beta}(x) \hat{T}_{\mu \nu}(y):\right\rangle-\left\langle: \hat{T}_{\alpha \beta}(x):\right\rangle\left\langle: \hat{T}_{\mu \nu}(x):\right\rangle}{\left\langle: \hat{T}_{\alpha \beta}(x) \hat{T}_{\mu \nu}(y):\right\rangle}\right| \tag{4.21}
\end{equation*}
$$

(where the colons represent normal ordering) be considered as a measure of the fluctuations in the energy-momentum densities of the quantum field. When the fluctuations in the energy-momentum densities are negligible, this quantity will be far less than unity and the semiclassical theory as described by equations (4.1) and (4.2) will prove to be quite sound. And, when the fluctuations are large the above quantity is expected to be of order unity reflecting a breakdown of the theory.

The numerous components and the dependence on the two spacetime points make the quantity $\Delta_{\alpha \beta \mu \nu}(x, y)$ an extremely cumbersome object to handle. For the sake of simplicity, as Kuo and Ford themselves suggest, we can confine our attention to either the evaluation of the purely temporal component of this quantity in the coincidence limit (i.e when $x \rightarrow y$ )

$$
\begin{equation*}
\Delta_{K F 2}(x)=\left|\frac{\left\langle: \hat{T}_{00}(x) \hat{T}_{00}(x):\right\rangle-\left\langle: \hat{T}_{00}(x):\right\rangle^{2}}{\left\langle: \hat{T}_{00}(x) \hat{T}_{00}(x):\right\rangle}\right| \tag{4.22}
\end{equation*}
$$

(subscript $K F$ stands for Kuo and Ford) or the quantity

$$
\begin{equation*}
\Delta_{K F 1}(x)=\left|\frac{\left\langle: \hat{T}_{00}(x) \hat{T}_{00}(x):\right\rangle-\left\langle: \hat{T}_{00}(x):\right\rangle^{2}}{\left\langle: \hat{T}_{00}(x):\right\rangle^{2}}\right| . \tag{4.23}
\end{equation*}
$$

The quantities $\Delta_{K F 1}$ and $\Delta_{K F 2}$ are related to each other by the equation

$$
\begin{equation*}
\Delta_{K F 2}=\left(\frac{\Delta_{K F 1}}{\Delta_{K F 1}+1}\right) . \tag{4.24}
\end{equation*}
$$

In (4.19), the semiclassical equation for our minisuperspace model, the backreaction term is the expectation value of the Hamiltonian operator of the scalar field mode. The validity of equation (4.19) will then depend on the magnitude of fluctuations in the expectation value of the operator $\hat{H}$. Since the minisuperspace model we are considering here, has only a finite number of degrees of freedom, no divergences occur in the expectation values. Hence no regularization needs to be carried out. Then, the quantities that correspond $\Delta_{K F 1}$ and $\Delta_{K F 2}$ for the case of our minisuperspace model are

$$
\begin{equation*}
\Delta_{S C 1}(t) \equiv\left|\frac{\left\langle\hat{H}^{2}\right\rangle-\langle\hat{H}\rangle^{2}}{\langle\hat{H}\rangle^{2}}\right| \tag{4.25}
\end{equation*}
$$

(subscript SC stands for semiclassical) and

$$
\begin{equation*}
\Delta_{S C 2}(t) \equiv\left|\frac{\left\langle\hat{H}^{2}\right\rangle-\langle\hat{H}\rangle^{2}}{\left\langle\hat{H}^{2}\right\rangle}\right| \tag{4.26}
\end{equation*}
$$

The magnitude of the two quantities $\Delta_{S C 1}$ and $\Delta_{S C 2}$ will then reflect the amount of fluctuations in $\langle\hat{H}\rangle$ and therefore on the validity of the semiclassical equation (4.19). The two quantities $\Delta_{S C 1}$ and $\Delta_{S C 2}$ are related to each other by the equation

$$
\begin{equation*}
\Delta_{S C 2}=\left(\frac{\Delta_{S C 1}}{\Delta_{S C 1}+1}\right) \tag{4.27}
\end{equation*}
$$

( $\Delta_{S C 1}$ and $\Delta_{S C 2}$ are expected to yield equivalent results.)

In the adiabatic limit, i.e. when the background metric is evolving very slowly, the ground state energy of each mode of the quantum field just gets shifted and no excitation of these modes takes place. Or, in other words, no particle creation takes place. In this limit the semiclassical equation (4.1) proves to be quite reliable [144]. On the other hand, when the metric is evolving very rapidly, a large number of particles get created, with the result that the expectation value of the
energy-momentum density of the quantum field ceases to account for the backreaction adequately. The adiabatic limit for our minisuperspace model corresponds to the case when the scale factor $a$ of the Friedmann universe is a slowly varying function of time, i.e. when $(\dot{a} / a) \rightarrow 0$. In this limit, for the initial conditions we have chosen viz. $\alpha\left(t_{0}\right)=1$ and $\beta\left(t_{0}\right)=0$, equation (4.14) implies that $\beta \rightarrow 0$. So, when $\beta \rightarrow 0$, we expect $\Delta_{S C 1}$ and $\Delta_{S C 2}$ to vanish thus suggesting a perfect validity of equation (4.19). And, when $\beta \rightarrow \infty$, i.e. when $(\dot{a} / a)$ is large, we expect $\Delta_{S C 1}$ and $\Delta_{S C 2}$ to be of order unity implying that (4.19) does not describe the backreaction problem adequately.

## 4.3 $\Delta_{S C}$ for different quantum states of the scalar field mode

In the following three subsections we shall evaluate the quantities $\Delta_{S C 1}$ and $\Delta_{S C 2}$ for the (i) vacuum, (ii) $n$-particle and (iii) coherent states of the quantized scalar field mode $\hat{q}$. The evaluation of these quantities are quite straight forward.

### 4.3.1 For a vacuum state

Let us assume that the state of the scalar field mode $\hat{q}$ is a vacuum state at the time $t=t_{0}$. Then, the expectation values of the operators $\hat{H}$ and $\hat{H}^{2}$ in such a state are

$$
\begin{equation*}
\langle\hat{H}\rangle=\langle 0|\left(\hat{a}^{\dagger} \hat{a}+(1 / 2)\right)|0\rangle \omega=\left(|\beta|^{2}+(1 / 2)\right) \omega \tag{4.28}
\end{equation*}
$$

and

$$
\begin{align*}
\left\langle\hat{H}^{2}\right\rangle & =\langle 0|\left(\hat{a}^{\dagger} \hat{a}+(1 / 2)\right)\left(\hat{a}^{\dagger} \hat{a}+(1 / 2)\right)|0\rangle \omega^{2} \\
& =\left(3|\beta|^{2}+3|\beta|^{4}+(1 / 4)\right) \omega^{2} . \tag{4.29}
\end{align*}
$$

Substituting these quantities in the expressions for $\Delta_{S C 1}$ and $\Delta_{S C 2}$ above, we find that they are given by

$$
\begin{align*}
\Delta_{S C 1} & =\left(\frac{2|\beta|^{2}+2|\beta|^{4}}{|\beta|^{2}+|\beta|^{4}+(1 / 4)}\right)  \tag{4.30}\\
\Delta_{S C 2} & =\left(\frac{2|\beta|^{2}+2|\beta|^{4}}{3|\beta|^{2}+3|\beta|^{4}+(1 / 4)}\right) . \tag{4.31}
\end{align*}
$$

### 4.3.2 For a $n$-particle state

If the quantum state of the scalar field mode $\hat{q}$ is assumed to be a $n$-particle state, then the expectation values of the operators $\hat{H}$ and $\hat{H}^{2}$ are given by

$$
\begin{align*}
\langle\hat{H}\rangle & =\langle n|\left(\hat{a}^{\dagger} \hat{a}+(1 / 2)\right)|n\rangle \omega \\
& =\left((2 n+1)|\beta|^{2}+n+(1 / 2)\right) \omega \tag{4.32}
\end{align*}
$$

and

$$
\begin{align*}
\left\langle\hat{H}^{2}\right\rangle= & \langle n|\left(\hat{a}^{\dagger} \hat{a}+(1 / 2)\right)\left(\hat{a}^{\dagger} \hat{a}+(1 / 2)\right)|n\rangle \omega^{2} \\
= & \left\{\left(n^{2}+n\right)\left(1+6|\beta|^{2}+6|\beta|^{4}\right)\right. \\
& \left.+\left(3|\beta|^{2}+3|\beta|^{4}+(1 / 4)\right)\right\} \omega^{2} . \tag{4.33}
\end{align*}
$$

When these quantities are substituted in equations (4.25) and (4.26), we find that $\Delta_{S C 1}$ and $\Delta_{S C 2}$ are given by the following expressions:

$$
\begin{equation*}
\Delta_{S C 1}=\left\{\left(\frac{2|\beta|^{2}+2|\beta|^{4}}{1+4|\beta|^{2}+4|\beta|^{4}}\right)\left(\frac{n^{2}+n+1}{n^{2}+n+(1 / 4)}\right)\right\} \tag{4.34}
\end{equation*}
$$

and

$$
\begin{equation*}
\Delta_{S C 2}=\left\{\left(\frac{2|\beta|^{2}+2|\beta|^{4}}{1+6|\beta|^{2}+6|\beta|^{4}}\right)\left(\frac{n^{2}+n+1}{n^{2}+n+(1 / 2)}\right)\right\} . \tag{4.35}
\end{equation*}
$$

### 4.3.3 For a coherent state

When the quantum state for $q$ is specified to be a coherent state, the expectation values of $\hat{H}$ and $\hat{H}^{2}$ are

$$
\begin{align*}
\langle\hat{H}\rangle & =\langle\lambda|\left(\hat{a}^{\dagger} \hat{a}+(1 / 2)\right)|\lambda\rangle \omega \\
& =\left\{|\lambda|^{2}\left(1+2|\beta|^{2}\right)+\lambda^{2} \alpha \beta+\lambda^{* 2} \alpha^{*} \beta^{*}+|\beta|^{2}+(1 / 2)\right\} \omega \tag{4.36}
\end{align*}
$$

and

$$
\begin{align*}
\left\langle\hat{H}^{2}\right\rangle= & \langle\lambda|\left(\hat{a}^{\dagger} \hat{a}+(1 / 2)\right)\left(\hat{a}^{\dagger} \hat{a}+(1 / 2)\right)|\lambda\rangle \omega^{2} \\
= & \left\{\left(|\lambda|^{4}+2|\lambda|^{2}\right)\left(1+6|\beta|^{2}+6|\beta|^{4}\right)\right. \\
& +\left(2|\lambda|^{2}+3\right)\left(\lambda^{2} \alpha \beta+\lambda^{* 2} \alpha^{*} \beta^{*}\right)\left(1+2|\beta|^{2}\right) \\
& \left.+\left(\lambda^{4} \alpha^{2} \beta^{2}+\lambda^{* 4} \alpha^{* 2} \beta^{* 2}\right)+\left(3|\beta|^{2}+3|\beta|^{4}\right)+(1 / 4)\right\} \omega^{2} . \tag{4.37}
\end{align*}
$$

The expressions for the $\Delta_{S C 1}$ and $\Delta_{S C 2}$ corresponding to the coherent state $|\lambda\rangle$ are then given by

$$
\begin{align*}
\Delta_{S C 1}= & \left\{|\lambda|^{2}\left(1+8|\beta|^{2}+8|\beta|^{4}\right)\right. \\
& \left.+2\left(1+2|\beta|^{2}\right)\left(\lambda^{2} \alpha \beta+\lambda^{* 2} \alpha^{*} \beta^{*}\right)+2\left(|\beta|^{2}+|\beta|^{4}\right)\right\} \\
& \times\left\{|\lambda|^{2}\left(1+2|\beta|^{2}\right)+\lambda^{2} \alpha \beta+\lambda^{* 2} \alpha^{*} \beta^{*}+|\beta|^{2}+(1 / 2)\right\}^{-2} \tag{4.38}
\end{align*}
$$

and

$$
\begin{align*}
\Delta_{S C 2}= & \left\{|\lambda|^{2}\left(1+8|\beta|^{2}+8|\beta|^{4}\right)\right. \\
& \left.+2\left(1+2|\beta|^{2}\right)\left(\lambda^{2} \alpha \beta+\lambda^{* 2} \alpha^{*} \beta^{*}\right)+2\left(|\beta|^{2}+|\beta|^{4}\right)\right\} \\
& \times\left\{\left(|\lambda|^{4}+2|\lambda|^{2}\right)\left(1+6|\beta|^{2}+6|\beta|^{4}\right)\right. \\
& +\left(2|\lambda|^{2}+3\right)\left(\lambda^{2} \alpha \beta+\lambda^{* 2} \alpha^{*} \beta^{*}\right)\left(1+2|\beta|^{2}\right) \\
& \left.+\left(\lambda^{4} \alpha^{2} \beta^{2}+\lambda^{* 4} \alpha^{* 2} \beta^{* 2}\right)+\left(3|\beta|^{2}+3|\beta|^{4}\right)+(1 / 4)\right\}^{-1} . \tag{4.39}
\end{align*}
$$

Table 4.1: $\Delta_{S C}$ in the limit of $\beta \rightarrow 0$

|  | Vacuum | $n$-particle | Coherent |
| :---: | :---: | :---: | :---: |
| $\Delta_{S C 1}$ | 0 | 0 | $\left(\frac{\|\lambda\|^{2}}{\|\lambda\|^{4}+\|\lambda\|^{2}+(1 / 4)}\right)$ |
| $\Delta_{S C 2}$ | 0 | 0 | $\left(\frac{\|\lambda\|^{2}}{\|\lambda\|^{4}+2\|\lambda\|^{2}+(1 / 4)}\right)$ |

Table 4.2: $\Delta_{S C}$ in the limit of $\beta \rightarrow \infty$

|  | Vacuum | $n$-particle | Coherent |
| :---: | :---: | :---: | :---: |
| $\Delta_{S C 1}$ | 2 | $\left(\frac{n^{2}+n+1}{2 n^{2}+2 n+(1 / 2)}\right)$ | $\left(\frac{\|\lambda\|^{2}\left(8+4 c_{1}\right)+2}{\left(\|\lambda\|^{2}\left(2+c_{1}\right)+1\right)^{2}}\right)$ |
| $\Delta_{S C 2}$ | $(2 / 3)$ | $\left(\frac{n^{2}+n+1}{3 n^{2}+3 n+(3 / 2)}\right)$ | $\left(\frac{\|\lambda\|^{2}\left(8+4 c_{1}\right)+2}{\|\lambda\|^{4}\left(6+4 c_{1}+c_{2}\right)+\|\lambda\|^{2}\left(12+6 c_{1}\right)+3}\right)$ |

$\Delta_{S C 1}$ and $\Delta_{S C 2}$ for the three quantum states in the two limits $\beta \rightarrow 0$ and $\beta \rightarrow \infty$ are tabulated in tables 4.1 and 4.2 , respectively. The quantities $c_{1}$ and $c_{2}$ in table 4.2 are given by

$$
\begin{equation*}
c_{1}=2 \cos (a+b+2 l) \quad ; \quad c_{1}=2 \cos (2 a+2 b+4 l), \tag{4.40}
\end{equation*}
$$

where $a, b$ and $l$ are the arguments of the complex quantities $\alpha, \beta$ and $\lambda$, respectively.

The results tabulated in tables 4.1 and 4.2 show that in the adiabatic limit, i.e when $\beta \rightarrow 0, \Delta_{S C 1}$ and $\Delta_{S C 2}$ identically vanish for the vacuum and $n$-particle states whereas, for coherent states with a large value of $\lambda$ they die down as $|\lambda|^{-2}$. And, in the limit when the Friedmann metric is evolving rapidly, i.e when $\beta \rightarrow \infty$, we find that $\Delta_{S C 1}$ and $\Delta_{S C 2}$ are of order unity for the vacuum as well as the $n$ particle states (even when $n$ is large). This result then implies that the fluctuations in the backreaction term in the semiclassical equation (4.19) are large in vacuum and $n$-particle states when a large amount of particles are being produced by the gravitational background. Whereas, for the coherent state with a large $\lambda$ the quantities $\Delta_{S C 1}$ and $\Delta_{S C 2}$ die down as $|\lambda|^{-2}$ even when $\beta \rightarrow \infty$. These results then imply that the semiclassical theory for our minisuperspace model as described by equation (4.19) is valid, during all stages of evolution, only if the scalar field mode is assumed to be in states like coherent states.

## 4.4 $\Delta_{K F}$ for different quantum states of the scalar field mode

Had we been dealing with the complete field theory instead of a minisuperspace model we would have encountered divergences when evaluating the expectation
values of the operators involving quantum fields. These infinities would have had to be systematically removed. In particular, it would have been necessary to normal order the operators.

In this section, we shall evaluate the quantities that correspond to $\Delta_{K F 1}$ and $\Delta_{K F 2}$ for our model. These quantities would be

$$
\begin{equation*}
\Delta_{K F 1}(t)=\left|\frac{\left\langle: \hat{H}^{2}:\right\rangle-\langle: \hat{H}:\rangle^{2}}{\langle: \hat{H}:\rangle^{2}}\right| \tag{4.41}
\end{equation*}
$$

and

$$
\begin{equation*}
\Delta_{K F 2}(t)=\left|\frac{\left\langle: \hat{H}^{2}:\right\rangle-\langle: \hat{H}:\rangle^{2}}{\left\langle: \hat{H}^{2}:\right\rangle}\right| \tag{4.42}
\end{equation*}
$$

where the colons imply normal ordering. For our model the operators have to be normal ordered with respect to $\hat{a}$. This has to be so, because, if the expression for $\langle 0| \hat{H}|0\rangle$ is normal ordered with respect to $\hat{A}$, instead of $\hat{a}$, it will kill the $|\beta|^{2}$ term in (4.28) which otherwise will contribute to the backreaction. Alternatively, one can try to regularize the expectation values by subtracting the vacuum contribution, i.e the $\langle 0|\left(\hat{A}^{\dagger} \hat{A}+(1 / 2)\right)|0\rangle \omega=(\omega / 2)$ and $\langle 0|\left(\hat{A}^{\dagger} \hat{A}+(1 / 2)\right)\left(\hat{A}^{\dagger} \hat{A}+(1 / 2)\right)|0\rangle \omega^{2}=\left(\omega^{2} / 4\right)$ terms can be removed from $\langle\hat{H}\rangle$ and $\left\langle\hat{H}^{2}\right\rangle$, respectively. The goal of this section is to point out a drawback when the magnitude of either $\Delta_{K F 1}$ or $\Delta_{K F 2}$ is used to decide the validity of the semiclassical theory in the adiabatic limit.

### 4.4.1 For a vacuum state

Let us now assume that the state of the scalar field mode $\hat{q}$ is the vacuum state $|0\rangle$. When operators $\hat{H}$ and $\hat{H}^{2}$ are normal ordered with respect to $\hat{a}$, we obtain that

$$
\begin{align*}
\langle: \hat{H}:\rangle_{N O} & =\langle 0| \hat{a}^{\dagger} \hat{a}|0\rangle \omega \\
& =|\beta|^{2} \omega \tag{4.43}
\end{align*}
$$

and

$$
\begin{align*}
\left\langle: \hat{H}^{2}:\right\rangle_{N O} & =\langle 0| \hat{a}^{\dagger} \hat{a}^{\dagger} \hat{a} \hat{a}|0\rangle \omega^{2} \\
& =\left(|\beta|^{2}+3|\beta|^{4}\right) \omega^{2} . \tag{4.44}
\end{align*}
$$

When the vacuum terms are subtracted from the expectation values of $\hat{H}$ and $\hat{H}^{2}$ as follows:

$$
\begin{equation*}
\langle: \hat{H}:\rangle_{V S}=\langle 0|\left(\hat{a}^{\dagger} \hat{a}+(1 / 2)\right)|0\rangle \omega-(\omega / 2) \tag{4.45}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\langle: \hat{H}^{2}:\right\rangle_{V S}=\langle 0|\left(\hat{a}^{\dagger} \hat{a}+(1 / 2)\right)\left(\hat{a}^{\dagger} \hat{a}+(1 / 2)\right)|0\rangle \omega^{2}-\left(\omega^{2} / 4\right), \tag{4.46}
\end{equation*}
$$

then the expressions for $\langle: \hat{H}:\rangle_{V S}$ and $\left\langle: \hat{H}^{2}:\right\rangle_{V S}$ are the same as the quantities $\langle\hat{H}\rangle$ and $\left\langle\hat{H}^{2}\right\rangle$ given by equations (4.28) and (4.29), but without the ( $\omega / 2$ ) and ( $\omega^{2} / 4$ ) terms, respectively. Substituting these expressions in the equations (4.41) and (4.42), we obtain that

$$
\begin{align*}
\Delta_{K F 1(N O)} & =\left(\frac{1+2|\beta|^{2}}{|\beta|^{2}}\right), \\
\Delta_{K F 2(N O)} & =\left(\frac{1+2|\beta|^{2}}{1+3|\beta|^{2}}\right) \tag{4.47}
\end{align*}
$$

and

$$
\begin{align*}
\Delta_{K F 1(V S)} & =\left(\frac{3+2|\beta|^{2}}{|\beta|^{2}}\right), \\
\Delta_{K F 2(V S)} & =\left(\frac{3+2|\beta|^{2}}{3+3|\beta|^{2}}\right), \tag{4.48}
\end{align*}
$$

where the subscripts $N O$ and $V S$ represent regularization by normal ordering and vacuum subtraction, respectively.

### 4.4.2 For a $n$-particle state

Let us now evaluate the expectation values of $\hat{H}$ and $\hat{H}^{2}$ in the $n$-particle state when these operators are normal ordered with respect to $\hat{a}$. They are given by the following expressions:

$$
\begin{align*}
\langle: \hat{H}:\rangle_{N O} & =\langle n| \hat{a}^{\dagger} \hat{a}|n\rangle \omega \\
& =\left(|\beta|^{2}(2 n+1)+n\right) \omega \tag{4.49}
\end{align*}
$$

and

$$
\begin{align*}
\left\langle: \hat{H}^{2}:\right\rangle_{N O}= & \langle n| \hat{a}^{\dagger} \hat{a}^{\dagger} \hat{a} \hat{a}|n\rangle \omega^{2} \\
= & \left\{n^{2}\left(1+6|\beta|^{2}+6|\beta|^{4}\right)\right. \\
& \left.+n\left(-1+2|\beta|^{2}+6|\beta|^{4}\right)+\left(|\beta|^{2}+3|\beta|^{4}\right)\right\} \omega^{2} \tag{4.50}
\end{align*}
$$

When the vacuum terms have been subtracted from the expectation values, i.e

$$
\begin{equation*}
\langle: \hat{H}:\rangle_{V S}=\langle n|\left(\hat{a}^{\dagger} \hat{a}+(1 / 2)\right)|n\rangle \omega-(\omega / 2) \tag{4.51}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\langle: \hat{H}^{2}:\right\rangle_{V S}=\langle n|\left(\hat{a}^{\dagger} \hat{a}+(1 / 2)\right)\left(\hat{a}^{\dagger} \hat{a}+(1 / 2)\right)|n\rangle \omega^{2}-\left(\omega^{2} / 4\right) \tag{4.52}
\end{equation*}
$$

the expressions for $\langle: \hat{H}:\rangle_{V S}$ and $\left\langle: \hat{H}^{2}:\right\rangle_{V S}$ are the same as the quantities $\langle\hat{H}\rangle$ and $\left\langle\hat{H}^{2}\right\rangle$ in equations (4.32) and (4.33), but without the $(\omega / 2)$ and $\left(\omega^{2} / 4\right)$ terms, respectively. Substituting these quantities we have evaluated in equations (4.41) and (4.42), we obtain that

$$
\begin{gather*}
\Delta_{K F 1(N O)}=\left(\left|\frac{|\beta|^{4}\left(2 n^{2}+2 n+2\right)+|\beta|^{2}(2 n+1)-n}{|\beta|^{4}\left(4 n^{2}+4 n+1\right)+|\beta|^{2}\left(4 n^{2}+2 n\right)+n^{2}}\right|\right),  \tag{4.53}\\
\Delta_{K F 2(N O)}=\left(\left|\frac{|\beta|^{4}\left(2 n^{2}+2 n+2\right)+|\beta|^{2}(2 n+1)-n}{|\beta|^{4}\left(6 n^{2}+6 n+3\right)+|\beta|^{2}\left(6 n^{2}+2 n+1\right)+\left(n^{2}-n\right)}\right|\right), \tag{4.54}
\end{gather*}
$$

$$
\begin{equation*}
\Delta_{K F 1(V S)}=\left(\frac{|\beta|^{4}\left(2 n^{2}+2 n+2\right)+|\beta|^{2}\left(2 n^{2}+4 n+3\right)+n}{|\beta|^{4}\left(4 n^{2}+4 n+1\right)+|\beta|^{2}\left(4 n^{2}+2 n\right)+n^{2}}\right), \tag{4.55}
\end{equation*}
$$

and

$$
\begin{equation*}
\Delta_{K F 2(V S)}=\left(\frac{|\beta|^{4}\left(2 n^{2}+2 n+2\right)+|\beta|^{2}\left(2 n^{2}+4 n+3\right)+n}{\left(|\beta|^{4}+|\beta|^{2}\right)\left(6 n^{2}+6 n+3\right)+\left(n^{2}+n\right)}\right) . \tag{4.56}
\end{equation*}
$$

### 4.4.3 For a coherent state

Let us now assume that the scalar field mode $\hat{q}$ is in a coherent state. When the operators $\hat{H}$ and $\hat{H}^{2}$ are normal ordered with respect to $\hat{a}$, we find that

$$
\begin{align*}
\langle: \hat{H}:\rangle_{N O} & =\langle\lambda| \hat{a}^{\dagger} \hat{a}|\lambda\rangle \omega \\
& =\left(|\lambda|^{2}\left(1+2|\beta|^{2}\right)+\lambda^{2} \alpha \beta+\lambda^{* 2} \alpha^{*} \beta^{*}+|\beta|^{2}\right) \omega \tag{4.57}
\end{align*}
$$

and

$$
\begin{align*}
\left\langle: \hat{H}^{2}:\right\rangle_{N O}= & \langle\lambda| \hat{a}^{\dagger} \hat{a}^{\dagger} \hat{a} \hat{a}|\lambda\rangle \omega^{2} \\
= & \left\{|\lambda|^{4}\left(1+6|\beta|^{2}+6|\beta|^{4}\right)+|\lambda|^{2}\left(8|\beta|^{2}+12|\beta|^{4}\right)\right. \\
& +\left(\lambda^{2} \alpha \beta+\lambda^{* 2} \alpha^{*} \beta^{*}\right)\left\{1+6|\beta|^{2}+|\lambda|^{2}\left(2+4|\beta|^{2}\right)\right\} \\
& \left.+\left(\lambda^{4} \alpha^{2} \beta^{2}+\lambda^{* 4} \alpha^{* 2} \beta^{* 2}\right)+|\beta|^{2}+3|\beta|^{4}\right\} \omega^{2} . \tag{4.58}
\end{align*}
$$

For the case, when the vacuum terms are subtracted from the expectation values of $\hat{H}$ and $\hat{H}^{2}$ as follows:

$$
\begin{equation*}
\langle: \hat{H}:\rangle_{V S}=\langle\lambda|\left(\hat{a}^{\dagger} \hat{a}+(1 / 2)\right)|\lambda\rangle \omega-(\omega / 2) \tag{4.59}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\langle: \hat{H}^{2}:\right\rangle_{V S}=\langle\lambda|\left(\hat{a}^{\dagger} \hat{a}+(1 / 2)\right)\left(\hat{a}^{\dagger} \hat{a}+(1 / 2)\right)|\lambda\rangle \omega^{2}-\left(\omega^{2} / 4\right), \tag{4.60}
\end{equation*}
$$

the expectation values, $\langle: \hat{H}:\rangle_{V S}$ and $\left\langle: \hat{H}^{2}:\right\rangle_{V S}$ are the same as the quantities $\langle\hat{H}\rangle$ and $\left\langle\hat{H}^{2}\right\rangle$ in equations (4.36) and (4.37) but without the $(\omega / 2)$ and $\left(\omega^{2} / 4\right)$
terms, respectively. For the coherent state, $\Delta_{K F 1}$ and $\Delta_{K F 2}$ are given by the following expressions:

$$
\begin{align*}
\Delta_{K F 1(N O)}= & \left\{|\lambda|^{2}\left(6|\beta|^{2}+8|\beta|^{4}\right)\right. \\
& \left.+\left(1+4|\beta|^{2}\right)\left(\lambda^{2} \alpha \beta+\lambda^{* 2} \alpha^{*} \beta^{*}\right)+\left(|\beta|^{2}+2|\beta|^{4}\right)\right\} \\
& \times\left\{|\lambda|^{2}\left(1+2|\beta|^{2}\right)+\lambda^{2} \alpha \beta+\lambda^{* 2} \alpha^{*} \beta^{*}+|\beta|^{2}\right\}^{-2} \tag{4.61}
\end{align*}
$$

$$
\begin{aligned}
\Delta_{K F 2(N O)}= & \left\{|\lambda|^{2}\left(6|\beta|^{2}+8|\beta|^{4}\right)\right. \\
& \left.+\left(1+4|\beta|^{2}\right)\left(\lambda^{2} \alpha \beta+\lambda^{* 2} \alpha^{*} \beta^{*}\right)+\left(|\beta|^{2}+2|\beta|^{4}\right)\right\} \\
& \times\left\{|\lambda|^{4}\left(1+6|\beta|^{2}+6|\beta|^{4}\right)+|\lambda|^{2}\left(8|\beta|^{2}+12|\beta|^{4}\right)\right. \\
& +\left(\lambda^{2} \alpha \beta+\lambda^{* 2} \alpha^{*} \beta^{*}\right)\left\{1+6|\beta|^{2}+|\lambda|^{2}\left(2+4|\beta|^{2}\right)\right\} \\
& \left.+\left(\lambda^{4} \alpha^{2} \beta^{2}+\lambda^{* 4} \alpha^{* 2} \beta^{* 2}\right)+|\beta|^{2}+3|\beta|^{4}\right\}^{-1}
\end{aligned}
$$

$$
\Delta_{K F 1(V S)}=\left\{|\lambda|^{2}\left(2+10|\beta|^{2}+8|\beta|^{4}\right)\right.
$$

$$
\left.+\left(3+4|\beta|^{2}\right)\left(\lambda^{2} \alpha \beta+\lambda^{* 2} \alpha^{*} \beta^{*}\right)+\left(3|\beta|^{2}+2|\beta|^{4}\right)\right\}
$$

$$
\begin{equation*}
\times\left\{|\lambda|^{2}\left(1+2|\beta|^{2}\right)+\lambda^{2} \alpha \beta+\lambda^{* 2} \alpha^{*} \beta^{*}+|\beta|^{2}\right\}^{-2} \tag{4.63}
\end{equation*}
$$

and

$$
\begin{align*}
\Delta_{K F 2(V S)}= & \left\{|\lambda|^{2}\left(2+10|\beta|^{2}+8|\beta|^{4}\right)\right. \\
& \left.+\left(3+4|\beta|^{2}\right)\left(\lambda^{2} \alpha \beta+\lambda^{* 2} \alpha^{*} \beta^{*}\right)+\left(3|\beta|^{2}+2|\beta|^{4}\right)\right\} \\
& \times\left\{\left(|\lambda|^{4}+2|\lambda|^{2}\right)\left(1+6|\beta|^{2}+6|\beta|^{4}\right)\right. \\
& +\left(2|\lambda|^{2}+3\right)\left(\lambda^{2} \alpha \beta+\lambda^{* 2} \alpha^{*} \beta^{*}\right)\left(1+2|\beta|^{2}\right) \\
& \left.+\left(\lambda^{4} \alpha^{2} \beta^{2}+\lambda^{* 4} \alpha^{* 2} \beta^{* 2}\right)+\left(3|\beta|^{2}+3|\beta|^{4}\right)\right\}^{-1} \tag{4.64}
\end{align*}
$$

Table 4.3: $\Delta_{K F}$ in the limit of $\beta \rightarrow 0$

|  | Vacuum | $n$-particle | Coherent |
| :---: | :---: | :---: | :---: |
| $\Delta_{K F 1(N O)}$ | $\infty$ | $(1 / n)$ | 0 |
| $\Delta_{K F 2(N O)}$ | 1 | $\left(\left\|\frac{1}{n-1}\right\|\right)$ | 0 |
| $\Delta_{K F 1(V S)}$ | $\infty$ | $(1 / n)$ | $\left(\frac{2}{\|\lambda\|^{2}}\right)$ |
| $\Delta_{K F 2(V S)}$ | 1 | $\left(\left\|\frac{1}{n-1}\right\|\right)$ | $\left(\frac{2}{2+\|\lambda\|^{2}}\right)$ |

Table 4.4: $\Delta_{K F}$ in the limit of $\beta \rightarrow \infty$

|  | Vacuum | $n$-particle | Coherent |
| :--- | :---: | :---: | :---: |
| $\Delta_{K F 1(N O)}$ | 2 | $\left(\frac{n^{2}+n+1}{2 n^{2}+2 n+(1 / 2)}\right)$ | $\left(\frac{\|\lambda\|^{2}\left(8+4 c_{1}\right)+2}{\left(\|\lambda\|^{2}\left(2+c_{1}\right)+1\right)^{2}}\right)$ |
| $\Delta_{K F 2(N O)}$ | $(2 / 3)$ | $\left(\frac{n^{2}+n+1}{3 n^{2}+3 n+(3 / 2)}\right)$ | $\left(\frac{\|\lambda\|^{2}\left(8+4 c_{1}\right)+2}{\|\lambda\|^{4}\left(6+4 c_{1}+c_{2}\right)+\|\lambda\|^{2}\left(12+6 c_{1}\right)+3}\right)$ |
| $\Delta_{K F 1(V S)}$ | 2 | $\left(\frac{n^{2}+n+1}{2 n^{2}+2 n+(1 / 2)}\right)$ | $\left(\frac{\|\lambda\|^{2}\left(8+4 c_{1}\right)+2}{\left(\|\lambda\|^{2}\left(2+c_{1}\right)+1\right)^{2}}\right)$ |
| $\Delta_{K F 2(V S)}$ | $(2 / 3)$ | $\left(\frac{n^{2}+n+1}{3 n^{2}+3 n+(3 / 2)}\right)$ | $\left(\frac{\|\lambda\|^{2}\left(8+4 c_{1}\right)+2}{\|\lambda\|^{4}\left(6+4 c_{1}+c_{2}\right)+\|\lambda\|^{2}\left(12+6 c_{1}\right)+3}\right)$ |

The expressions for the different $\Delta_{K F 1}$ and $\Delta_{K F 2}$ in the two limits of interest, viz. $\beta \rightarrow 0$ and $\beta \rightarrow \infty$, are summarized in tables 4.3 and 4.4. The quantities $c_{1}$ and $c_{2}$ in the table 4.4 are the same as those that appear in table 4.2.

It is clear from table 4.3 that for the minisuperspace model we are considering here the quantities $\Delta_{K F 1}$ and $\Delta_{K F 2}$ do not in the adiabatic limit. In fact, as $\beta \rightarrow 0$, they are of order unity for the vacuum and $n$-particle states thus suggesting a breakdown of the semiclassical theory. Even in the complete field theoretic case, the same is bound to happen when the quantities $\Delta_{K F 1}$ and $\Delta_{K F 2}$ are evaluated with regularised expectation values. But we do know that the semiclassical theory is perfectly valid in the adiabatic limit [144]. In field theory, the expectation values have to be regularized. So, to check the validity of the semiclassical theory in the field theoretic case, it would be advisable to monitor the magnitude of the fluctuations in the adiabtaic limit rather than depend on $\Delta_{K F 1}$ or $\Delta_{K F 2}$. Whereas, when $\beta \rightarrow \infty$, we find that both $\Delta_{S C}$ and $\Delta_{K F}$ give identical results for our minisuperspace model. And, in the field theoretic calculations where only $\Delta_{K F 1}$ or $\Delta_{K F 2}$ can be evaluated, we can expect these quantities to give reliable results to help us draw the limits on the validity of the semiclassical theory.

### 4.5 Implications

The results of section 4.3 quite clearly prove that the semiclassical theory we had considered for our minisuperspace model can be relied upon, during all stages of the evolution, only if the quantum system, viz. the scalar field mode is in states like coherent states. It is quite likely that these results we have obtained for our minisuperspace model will hold good even in complete field theory. After
all, regularization procedures in quantum field theory only attempt to subtract the contribution due to the vacuum state for each mode of the quantum field. Therefore, if the semiclassical theory proves to be of a limited validity for a single mode of the quantum field (which is basically the minisuperspace model we have considered here), it is plausible that the same would be true in the field theoretic case too. Hence, if the backreaction problem has to be studied in those states of the quantum field, which do not possess a 'coherent' nature, the semiclassical theory based on equations (4.1) and (4.2) is bound to prove rather inadequate. In such a situation, the fluctuations in the energy-momentum densities of the quantum field have to be systematically taken into account. When done so, the semiclassical Einstein's equation (4.1) can be expected to be described by an equation similar in form to the Langevin equation [145].

## Chapter 5

## Analogues of quantum effects in classical field theory

Earlier, in section 1.1, we had seen that the quantization of a field in Minkowski and Rindler coordinates are not equivalent. In fact, we had found that the Minkowski vacuum state was populated by a thermal distribution number of Rindler particles (cf. equation (1.47)). Also, in subsection 1.3.2, we found that the response of a uniformly accelerating Unruh-DeWitt detector in the Minkowski vacuum turned out to be a thermal spectrum (cf. equation (1.99)). In both these situations, one obtains the thermal spectrum in the strict sense of the word: Not only that the mean occupation number in any mode is Planckian, but the fluctuations around the mean is also characterized by the standard thermal noise. These results suggest that the quantum fluctuations in the vacuum appear as thermal fluctuations in the uniformly accelerated frame.

In contrast to quantum theory, classical field theory does not admit any intrinsic fluctuations. The absence of concepts such as vacuum and fluctuations in classical field theory may lead us to believe that non-trivial phenomena as the one mentioned in the above paragraph will not have any classical analogue. We
shall show, however, that such is not the case.

In this chapter, we discuss a fairly non-trivial and interesting effect that arises purely in the context of classical field theory, which has a formal similarity with the quantum mechanical results mentioned above. We find that, when a real, monochromatic mode of a classical scalar field is Fourier transformed with respect to the proper time of a uniformly accelerating observer, the resulting power spectrum consists of three terms none of which have a simple physical interpretation in terms of classical concepts [146, 147]. However, they closely resemble terms that have a definite quantum mechanical interpretation. More specifically, we show that the three terms which arise are: (i) a factor (1/2) that is typical of the ground state energy of a quantum oscillator, (ii) a Planckian distribution $N(\Omega)$ and (iii) a term proportional to $\sqrt{N(N+1)}$, which is the root mean square fluctuations about the Planckian distribution in a quantum mechanical context. While one could have anticipated the second term $N$ based on earlier results, the first and the third terms could not have been guessed from any previously known result. It is interesting-to say the least-that such terms arise in a situation where there is no genuine thermal phenomena, statistical steady state, thermal or quantum fluctuations etc. The power spectrum has only the form of a thermal spectrum. Similar results are obtained when we consider a real, monochromatic, plane electromagnetic wave. We also find that such a Planckian ambience also proves to be a feature of observers stationed at a constant radius in Schwarzschild and de-Sitter spacetimes.

This chapter is organized as follows. In section 5.1, we evaluate the power spectrum of a real, monochromatic mode of a scalar field as well as that of a plane electromagnetic wave in the frame of a uniformly accelerated observer. In
section 5.2, we generalize our result to different field configurations. In section 5.3, we outline as to how a power spectrum with a Planckian nature proves to be a feature of observers stationed at a constant radius in Schwarzschild and de-Sitter spacetimes. Finally, in section 5.4, we present a model of a detector which responds to the power spectrum of the field with respect to its proper time and also discuss the possible implications of our analysis.

### 5.1 Power spectrum of a real, monochromatic wave in a uniformly accelerated frame

In the following two subsections we shall evaluate the power spectrum of a real, monochromatic plane wave mode of scalar and electromagnetic fields in the frame of a uniformly accelerating observer.

### 5.1.1 Power spectrum of a scalar field mode

Consider a massless, minimally coupled, scalar field which satisfies the KleinGordon equation

$$
\begin{equation*}
\square \Phi \equiv \frac{1}{\sqrt{-g}} \partial_{\mu}\left(\sqrt{-g} g^{\mu \nu} \partial_{\nu}\right) \Phi=0 . \tag{5.1}
\end{equation*}
$$

In flat spacetime, the basis solutions to the above Klein-Gordon equation in the Minkowski coordinates $(t, \mathrm{x})$ can be taken to be plane waves labeled by the wave vector k :

$$
\begin{equation*}
\Phi(t, \mathbf{x})=\cos (\omega t-\mathbf{k} \cdot \mathbf{x}) \tag{5.2}
\end{equation*}
$$

where $\omega=|\mathrm{k}|$. We now ask: Consider an observer who is moving on an arbitrary trajectory $(t(\tau), \mathrm{x}(\tau))$, parametrized by the proper time $\tau$. How will this observer view the above Minkowski plane wave mode?

The moving observer will see the scalar field varying with respect to her proper time in a manner determined by the function $\Phi[t(\tau), \mathrm{x}(\tau)]$. If the observer is in inertial motion then the monochromatic wave will appear to be another monochromatic wave with a Doppler shifted frequency. But, in general, for noninertial trajectories, the wave will not appear to be monochromatic for the moving observer but will prove to be a superposition of waves with different frequencies. To determine the exact decomposition of the wave, we should Fourier analyze the Minkowski mode in the frame of the observer. The Fourier transform of the Minkowski plane wave with respect to the proper time $\tau$ of the observer in motion is described by the integral

$$
\begin{equation*}
\tilde{\Phi}(\Omega)=\int_{-\infty}^{\infty} d \tau e^{-i \Omega \tau} \Phi[t(\tau), \mathbf{x}(\tau)] \tag{5.3}
\end{equation*}
$$

This expression gives the amplitude of a component with frequency $\Omega$ (as defined by the moving observer) present in the original monochromatic wave. Given a particular plane wave, we can always align the coordinates such that the wave is traveling along the $x$-axis, i.e. the wave vector is given by $\mathrm{k}=(k, 0,0)$. Then the plane wave mode (5.2) reduces to

$$
\begin{equation*}
\Phi(t, \mathbf{x})=\cos (\omega t-k x) \tag{5.4}
\end{equation*}
$$

and its Fourier transform is given by the integral

$$
\begin{equation*}
\tilde{\Phi}(\Omega)=\int_{-\infty}^{\infty} d \tau e^{-i \Omega \tau} \cos [\omega t(\tau)-k x(\tau)] . \tag{5.5}
\end{equation*}
$$

We shall now specialize to the case of an observer who is accelerating uniformly with respect to the Minkowski coordinates. We shall assume that the observer is accelerating along the $x$-axis. Let us also assume that the observer is moving with a proper acceleration $g$. The world line of such an observer in the

Minkowski coordinates $(t, x, y, z)$ is given by the relations (cf. equation (1.92))

$$
\begin{equation*}
t=t_{0}+g^{-1} \sinh (g \tau) \quad ; \quad x=x_{0}+g^{-1} \cosh (g \tau) \quad ; \quad y=y \quad \text { and } \quad z=z \tag{5.6}
\end{equation*}
$$

where $t_{0}$ and $x_{0}$ are constants and $\tau$ is the proper time as measured by a clock in the accelerated frame. (Note that the transformations (1.92) corresponds to the case when $t_{0}=x_{0}=0$.) As we have noted earlier in subsection 1.3.2, the world line of such a uniformly accelerating observer is a hyperbola in the $(t, x)$ plane. The asymptotes of this hyperbola are the past and the future light cones that intersect at the point $\left(t_{0}, x_{0}\right)$. To see how the plane wave (5.4) will be viewed by such an observer, we substitute the coordinate transformations (5.6) in the Fourier integral (5.5), and obtain that [77]

$$
\begin{align*}
\tilde{\Phi}(\Omega) & =\int_{-\infty}^{\infty} d \tau e^{-i \Omega \tau} \cos \left(\omega\left[t_{0}-x_{0}+g^{-1} \sinh (g \tau)-g^{-1} \cosh (g \tau)\right]\right) \\
& =\int_{-\infty}^{\infty} d \tau e^{-i \Omega \tau} \cos \left(\omega g^{-1} e^{-g \tau}-\beta\right) \\
& =\left(\frac{1}{2 g}\right) e^{-i \phi}\left(e^{-\left(\Omega / 4 \Omega_{0}\right)} e^{-i \beta}+e^{\left(\Omega / 4 \Omega_{0}\right)} e^{i \beta}\right) \Gamma\left(i \Omega g^{-1}\right), \tag{5.7}
\end{align*}
$$

where $\Gamma(z)$ is the gamma function,

$$
\begin{equation*}
\phi=\Omega g^{-1} \ln \left(\omega g^{-1}\right) \quad ; \quad \Omega_{0}=g / 2 \pi \quad \text { and } \quad \beta=\omega\left(t_{0}-x_{0}\right) . \tag{5.8}
\end{equation*}
$$

In the above integral we have assumed that the plane wave is traveling to the right so that $k=\omega$. The resulting power spectrum per logarithmic interval in frequency is given by $\mathcal{P}(\Omega) \equiv\left(\Omega|\tilde{\Phi}(\Omega)|^{2}\right)$ and can be written in a remarkable form:

$$
\begin{align*}
\mathcal{P}(\Omega) \equiv \Omega|\tilde{\Phi}(\Omega)|^{2} & =\left(\frac{\pi}{2 g}\right)\left(\operatorname{coth}\left(\Omega / 2 \Omega_{0}\right)+\operatorname{csch}\left(\Omega / 2 \Omega_{0}\right) \cos (2 \beta)\right) \\
& =\left(\frac{\pi}{g}\right)\left\{\frac{1}{2}+N+\sqrt{N(N+1)} \cos (2 \beta)\right\} \tag{5.9}
\end{align*}
$$

where

$$
\begin{equation*}
N(\Omega)=\left(\frac{1}{\exp \left(\Omega / \Omega_{0}\right)-1}\right) \tag{5.10}
\end{equation*}
$$

We shall now consider various features of this result.

To begin with we note that this result is a purely classical one and hence $\hbar$ does not appear anywhere. In ordinary units, $\Omega_{0}=(g / 2 \pi c)$ has the correct dimensions (viz. per second) for a frequency. The quantity $N(\Omega)$ is a Planckian in terms of frequencies and is again independent of $\hbar$. Usually, one tries to express the Planckian distribution in terms of energies of the 'quanta' labeled by frequency $\Omega$ and in such a case we need to write frequencies as, say $\Omega=(E / \hbar)$, thereby artificially introducing $\hbar$; but the result, stated as a power spectrum in frequency space, makes perfect conceptual sense as it stands. For example, radio astronomers measure the power spectrum in frequency space and may not think in terms of photons. Of course, to obtain a quantity with the dimension of temperature we again need to introduce a $\hbar$ into the quantity $\Omega_{0}$. Since there is no real concept of a temperature in the situation we are considering here, we will not introduce $\hbar$.

The analysis done above could have been carried out even in the days before quantum theory-it uses only classical relativity. Had it been done, there would have been no simple way of understanding the terms which arise in (5.9). But, it is our knowledge of quantum theory that allows a suggestive interpretation of the three terms in the power spectrum: The first term—viz. the factor $(1 / 2)$-is typical of the ground state energy of a quantum oscillator. The second term $N$ is a Planckian distribution in $\Omega$, as already mentioned. Note that these two terms are totally independent of the original frequency $\omega$ of the plane wave!

The third term is still more remarkable. When we vary the constants $t_{0}$
and $x_{0}$ this term varies between $-\sqrt{N(N+1)}$ and $+\sqrt{N(N+1)}$. The magnitude of this variation (which is the root mean square deviation about the mean value) is exactly what one would have obtained for a strictly thermal distribution of massless bosonic quanta in quantum field theory. Thus, a classical plane wave, viewed in the accelerated frame, has a power spectrum reminiscent of Planck spectrum with associated thermal fluctuations.

To avoid possible misunderstanding, we stress here the following fact: The system we are considering has no fluctuations or temperature in the sense of statistical physics. Being a purely classical system, it does not have any quantum fluctuations either. But the terms which we get in the accelerated frame have the most natural interpretation in terms of notions like thermal spectrum and its fluctuations.

The quantity $\beta$ is related to $t_{0}$ and $x_{0}$ by equation (5.8). If the original plane wave had an extra phase $\delta$, then the argument of the cosine term will pick up $2 \delta$ additionally. For a specific choice of the constants $\delta, t_{0}$ and $x_{0}$, it possible to kill the fluctuations in the power spectrum. It is also easy to verify that one cannot choose the constants to cancel the first two terms as well. But-in general-all the three terms are present in the power spectrum. We shall now comment on the related aspects of this result.

It may be noted that the existence of the three terms is a direct consequence of our choosing a real plane wave which-in classical field theory-is mandatory. If the same analysis is repeated for a complex mode for the scalar field, say $\Phi(t, x)=$ $\exp -i(\omega t-k x)$, then the resultant power spectrum per logarithmic frequency
interval is

$$
\begin{equation*}
\mathcal{P}(\Omega)=\left(\frac{2 \pi}{g}\right) \quad N(\Omega) \tag{5.11}
\end{equation*}
$$

where $N$ is given by (5.10). We do not get the zero-point term or the fluctuations. Of course, in classical field theory, one must use real modes and that is exactly what we have done here.

Finally, let us consider the limit of $\omega \rightarrow 0$. In this limit, the field in the inertial frame reduces to an unimportant constant-which could be thought of as closest to the concept of a 'vacuum' in the classical theory. The Fourier integral as well as the phase $\phi$ in equation (5.8) diverges when $\omega \rightarrow 0$; but the power spectrum-which is the squared modulus of the amplitude-is well defined:

$$
\begin{equation*}
\left.\mathcal{P}(\Omega)\right|_{\omega \rightarrow 0}=\left(\frac{\pi}{g}\right)\left\{\frac{1}{2}+N+\sqrt{N(N+1)}\right\} . \tag{5.12}
\end{equation*}
$$

However, as long as $\omega$ is treated as a 'regulator' one can say that the accelerated observer will see these terms even in the limit of $\omega \rightarrow 0$. This is very reminiscent of the Minkowski vacuum state appearing as a Planckian spectrum to a uniformly accelerated observer in a manner which is completely independent of the original wave mode. Mathematically, this result arises because our limiting procedure does not commute with that of Fourier transforming the mode. If we consider the $\omega \rightarrow 0$ limit first and then evaluate the Fourier transform, we will-of course-get the square of the Dirac delta function as the power spectrum. But, when we compute the power spectrum first and then take the limit of $\omega \rightarrow 0$ we get a different-and finite-result. Once again, the situation is reminiscent of regularization procedures (like the ' $i \epsilon$ prescription') in quantum theory in which the order of operations matter. In a way, this limiting value turns out to be a more generic feature. (In the above discussion we have assumed that the wave
and the observer are moving along same direction, viz. the $x$-axis. But the result for $\omega \rightarrow 0$ should hold irrespective of this condition. Later, in section 5.2 , we shall show that this is indeed the case.)

### 5.1.2 Power spectrum of a plane electromagnetic wave

The analysis we have carried out for a real, monochromatic scalar field mode can analogously be carried out for a plane electromagnetic wave. Given a vector potential $A^{\mu}$ the electromagnetic field tensor is defined as [42]

$$
\begin{equation*}
F_{\mu \nu}=\left(\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu}\right) . \tag{5.13}
\end{equation*}
$$

The components of the field tensor are then given by

$$
F_{\mu \nu}=\left(\begin{array}{cccc}
0 & E_{x} & E_{y} & E_{z}  \tag{5.14}\\
-E_{x} & 0 & -B_{z} & B_{y} \\
-E_{y} & B_{z} & 0 & -B_{x} \\
-E_{z} & -B_{y} & B_{x} & 0
\end{array}\right),
$$

where $\mathbf{E}=\left(E_{x}, E_{y}, E_{z}\right)$ and $\mathbf{B}=\left(B_{x}, B_{y}, B_{z}\right)$ are the electric and magnetic field vectors respectively.

A real and monochromatic, plane electromagnetic wave traveling along the $x$-axis can be described by the following vector potential:

$$
\begin{equation*}
A^{\mu}=(0,0,1,1) \cos (\omega t-k x), \tag{5.15}
\end{equation*}
$$

where $\omega=|k|$. The electromagnetic field tensor corresponding to such a vector potential is then given by

$$
F_{\mu \nu}=\left(\begin{array}{cccc}
0 & 0 & \omega & \omega  \tag{5.16}\\
0 & 0 & -k & -k \\
-\omega & k & 0 & 0 \\
-\omega & k & 0 & 0
\end{array}\right) \times \cos (\omega t-k x) .
$$

In the frame of a uniformly accelerating observer whose world line is given by the transformations (5.6), the electromagnetic field tensor transforms to

$$
\bar{F}_{\mu \nu}=\left(\begin{array}{cccc}
0 & 0 & \omega e^{-g \tau} & \omega e^{-g \tau}  \tag{5.17}\\
0 & 0 & k e^{-g \tau} & -k e^{-g \tau} \\
-\omega e^{-g \tau} & -k e^{-g \tau} & 0 & 0 \\
-\omega e^{-g \tau} & k e^{-g \tau} & 0 & 0
\end{array}\right) \times \cos (\omega t(\tau)-k x(\tau)) .
$$

Notice that the acceleration of the observer is along the same axis as that of the direction of propagation of the wave. Let us now assume that the electromagnetic wave is traveling to the right, i.e. $k=\omega$. Then, from the above equation it can be easily seen that the all the transformed components of the field tensor are of the following form:

$$
\begin{equation*}
F(\tau)= \pm \omega e^{-g \tau} \cos (\omega t(\tau)-k x(\tau)) \tag{5.18}
\end{equation*}
$$

Fourier transforming $F(\tau)$ with respect to the proper time of the uniformly accelerating observer, we obtain

$$
\begin{equation*}
\tilde{F}(\Omega)= \pm\left(\frac{\Omega}{2 g}\right) e^{-i \phi}\left(e^{-\left(\Omega / 4 \Omega_{0}\right)} e^{-i \beta}-e^{\left(\Omega / 4 \Omega_{0}\right)} e^{i \beta}\right) \Gamma\left(i \Omega g^{-1}\right) \tag{5.19}
\end{equation*}
$$

where $\phi, \Omega_{0}$ and $\beta$ are given by equation (5.8). The resulting power spectrum per unit logarithmic interval in frequency $\mathcal{P}(\Omega) \equiv\left(\Omega|\tilde{F}(\Omega)|^{2}\right)$ is then given by

$$
\begin{equation*}
\mathcal{P}(\Omega)=\left(\frac{\pi}{g}\right) \Omega^{2}\left\{\frac{1}{2}+N-\sqrt{N(N+1)} \cos (2 \beta)\right\}, \tag{5.20}
\end{equation*}
$$

where $N$ is given by equation (5.10). In the limit of $\omega \rightarrow 0$ this power spectrum reduces to

$$
\begin{equation*}
\mathcal{P}(\Omega)=\left(\frac{\pi}{g}\right) \Omega^{2}\left\{\frac{1}{2}+N-\sqrt{N(N+1)}\right\} \tag{5.21}
\end{equation*}
$$

Thus, even in the case of the electromagnetic field, the power spectrum is well defined in the limit of $\omega \rightarrow 0$.

The power spectrum per unit logarithmic frequency interval obtained above has an extra factor $\Omega^{2}$ multiplying the expression in the braces which was absent in the power spectrum (5.9) for the scalar field. This extra factor has a simple explanation. Consider the power spectrum of the scalar field mode (5.2) in the Minkowski coordinates. The resultant power spectrum would be proportional to the square of a delta function and the proportionality constant would be independent of $\omega$. In the Rindler frame too, the power spectrum of the scalar field mode as given by equation (5.9) has no term dependent on $\Omega$ multiplying the expression in the braces. In contrast, consider the power spectrum of the electromagnetic wave (5.18) in the Minkowski frame. It would again be proportional to the square of a delta function, but, in this case, the proportionality factor would be of the form $\omega^{2}$. Extrapolating the result for the scalar field, one would expect that the power spectrum (per unit logarithmic frequency interval) of the electromagnetic wave would have a $\Omega^{2}$ factor in the uniformly accelerated frame. This is exactly the result we have obtained in equation (5.20).

### 5.2 Generalization to other field configurations

In the last section, we have carried out our analysis for real Minkowski modes that were traveling to the right. It is straight forward to verify that the same power spectrum can be obtained for left moving waves, i.e when $k=-\omega$.

A more general case is as follows. Consider a function of $\Phi(t-x)$ that satisfies the Klein-Gordon equation and is either odd or even in $(t-x)$. Such a function $\Phi(t-x)$, which will represent a wave packet that is traveling along the
$x$-axis, can be Fourier decomposed into the following form

$$
\begin{equation*}
\Phi(t-x)=\int_{-\infty}^{\infty} d \alpha f(\alpha) \exp i \alpha(t-x) \tag{5.22}
\end{equation*}
$$

The function $f(\alpha)$ will prove to be odd or even depending on whether $\Phi(t-x)$ is odd or even. Substituting the transformation equations (5.6) in (5.22) and Fourier transforming, as before, with respect to the proper time of the Rindler observer, we obtain that

$$
\begin{equation*}
\tilde{\Phi}(\Omega)=g^{-1} \Gamma\left(i \Omega g^{-1}\right)\left(e^{\left(\Omega / 4 \Omega_{0}\right)} F_{1}(\Omega) \pm e^{-\left(\Omega / 4 \Omega_{0}\right)} F_{2}(\Omega)\right) \tag{5.23}
\end{equation*}
$$

where the plus sign is to be chosen if $\Phi(t-x)$ is an even function and the minus sign if $\Phi(t-x)$ is an odd function and $\Omega_{0}$ is given by (5.8). The distributions $F_{1}(\Omega)$ and $F_{2}(\Omega)$ are described by the integrals

$$
\begin{equation*}
F_{1}(\Omega)=\int_{0}^{\infty} d \alpha f(\alpha) e^{i \alpha\left(t_{0}-x_{0}\right)} \exp -\left(i \Omega g^{-1} \ln \left(g^{-1} \alpha\right)\right) \tag{5.24}
\end{equation*}
$$

and

$$
\begin{equation*}
F_{2}(\Omega)=\int_{0}^{\infty} d \alpha f(\alpha) e^{-i \alpha\left(t_{0}-x_{0}\right)} \exp -\left(i \Omega g^{-1} \ln \left(g^{-1} \alpha\right)\right) \tag{5.25}
\end{equation*}
$$

We then obtain that

$$
\begin{align*}
& \mathcal{P}(\Omega) \equiv \Omega|\tilde{\Phi}(\Omega)|^{2} \\
&=\left(\frac{\pi}{g \sinh \left(\Omega / 2 \Omega_{0}\right)}\right)\left\{\begin{array}{l}
\left(\Omega / 2 \Omega_{0}\right)\left|F_{1}(\Omega)\right|^{2}+e^{-\left(\Omega / 2 \Omega_{0}\right)}\left|F_{2}(\Omega)\right|^{2} \\
\\
\left.\quad \pm\left(F_{1}^{*}(\Omega) F_{2}(\Omega)+F_{1}(\Omega) F_{2}^{*}(\Omega)\right)\right\}
\end{array}\right.
\end{align*}
$$

This spectrum, of course, does not have a thermal nature since it depends explicitly on the form of $f(\alpha)$.

But a simplification occurs if we treat $f(\alpha)$ as a stochastic variable so that when averaged over an ensemble of realizations, it satisfies the relation

$$
\begin{equation*}
\left\langle f(\alpha) f^{*}\left(\alpha^{\prime}\right)\right\rangle=P(\alpha) \delta\left(\alpha-\alpha^{\prime}\right) \tag{5.27}
\end{equation*}
$$

with some power spectrum $P(\alpha)$, such that $\int_{-\infty}^{\infty} d \alpha P(\alpha)=2 C$. In such a case, when $\left|F_{1}(\Omega)\right|^{2}$ and $\left|F_{2}(\Omega)\right|^{2}$ are averaged over the stochastic variable $f(\alpha)$, both reduce to a constant independent of $\Omega$, i.e.

$$
\begin{equation*}
\left.\left.\left.\langle | F_{1}(\Omega)\right|^{2}\right\rangle=\left.\langle | F_{2}(\Omega)\right|^{2}\right\rangle=\int_{0}^{\infty} d \alpha P(\alpha)=C \tag{5.28}
\end{equation*}
$$

The power spectrum (5.26) when it is averaged over the stochastic variable $f(\alpha)$ is given by

$$
\begin{equation*}
\langle\mathcal{P}(\Omega)\rangle=\left(\frac{4 \pi C}{g}\right)\left\{\frac{1}{2}+N \pm \sqrt{N(N+1)} \cos \left(2 \beta^{\prime}\right)\right\} \tag{5.29}
\end{equation*}
$$

where $\beta^{\prime}$ is a function of $\left(t_{0}-x_{0}\right)$ and is defined by the relation

$$
\begin{align*}
\cos \left(2 \beta^{\prime}\right) & =\left(\frac{1}{2 C}\right)\left\langle F_{1}^{*}(\Omega) F_{2}(\Omega)+F_{1}(\Omega) F_{2}^{*}(\Omega)\right\rangle \\
& =\left(\frac{1}{C}\right) \int_{0}^{\infty} d \alpha P(\alpha) \cos \left[2 \alpha\left(t_{0}-x_{0}\right)\right] \tag{5.30}
\end{align*}
$$

So a stochastic wave field in the Minkowski frame will also reproduce all the three terms in the power spectrum obtained earlier.

The wave field described above did not have explicit random phases. It is possible to define a different random field in the following way. Consider a random superposition of real modes for the scalar field:

$$
\begin{equation*}
\Phi(t, x)=\int_{-\infty}^{\infty} d \omega A(\omega) \cos [\omega(t-x)+\theta(\omega)] \tag{5.31}
\end{equation*}
$$

where $A(\omega)$ is a stochastic variable satisfying the relation

$$
\begin{equation*}
\left\langle A(\omega) A\left(\omega^{\prime}\right)\right\rangle=\bar{P}(\omega) \delta\left(\omega-\omega^{\prime}\right) \tag{5.32}
\end{equation*}
$$

and $\bar{P}(\omega)$ is an arbitrary function of $\omega$ such that $\bar{C}=\int_{-\infty}^{\infty} d \omega \bar{P}(\omega)$ is a finite constant. Further, we shall assume that $\theta(\omega)$ is a random phase factor distributed uniformly in the range $(0,2 \pi)$. We can now set $t_{0}=x_{0}=0$ in (5.6) without any
loss of generality. Substituting the coordinate transformations (5.6) in the scalar field configuration given by (5.31) and Fourier transforming the same with respect to the proper time of the uniformly accelerated observer, we obtain

$$
\begin{align*}
\tilde{\Phi}(\Omega)= & \int_{-\infty}^{\infty} d \tau \int_{-\infty}^{\infty} d \omega A(\omega) \cos (\omega[t(\tau)-x(\tau)]+\theta(\omega)) e^{-i \Omega \tau} \\
= & \int_{-\infty}^{\infty} d \omega A(\omega) \int_{-\infty}^{\infty} d \tau \cos \left(\omega g^{-1} e^{-g \tau}-\theta(\omega)\right) e^{-i \Omega \tau} \\
= & \left(\frac{1}{2 g}\right) \Gamma\left(i \Omega g^{-1}\right) \int_{-\infty}^{\infty} d \omega A(\omega) e^{-i \phi} \\
& \times\left(e^{-\left(\Omega / 4 \Omega_{0}\right)} e^{-i \theta(\omega)}+e^{\left(\Omega / 4 \Omega_{0}\right)} e^{i \theta(\omega)}\right), \tag{5.33}
\end{align*}
$$

where $\phi$ and $\Omega_{0}$ are given by (5.8). The power spectrum per logarithmic frequency interval, viz. the quantity $\left(\Omega|\tilde{\Phi}(\Omega)|^{2}\right)$ when averaged over the stochastic variables $A(\omega)$ and $\theta(\omega)$ then reduces to

$$
\begin{equation*}
\langle\mathcal{P}(\Omega)\rangle=\left(\frac{\pi \bar{C}}{g}\right)\left\{\frac{1}{2}+N\right\} . \tag{5.34}
\end{equation*}
$$

In this case, the random phases have averaged out the fluctuation term, viz. the factor $\sqrt{N(N+1)}$ that had appeared in the power spectrum (5.9). A somewhat similar result was obtained earlier by Boyer [148]. He had modeled the zero-point fluctuations as due to random superposition of Minkowski plane wave modes, and used it as a basis for investigating the 'spectrum' observed by a uniformly accelerating observer. He showed that the correlation function of an accelerating observer 'in a random classical scalar zero-point radiation' exactly matches the correlation function of an inertial observer in a thermal background. Our analysis here shows that the effect reported by Boyer arises when a random superposition of Minkowski real modes are simply Fourier analyzed in the frame of a uniformly accelerating observer (cf. equation (5.34)). But notice that, such an approach has killed a very interesting $\sqrt{N(N+1)}$ term which was originally present.

Finally, we shall discuss a case in which the wave and the observer are not moving along the same direction. Let us now assume that the plane wave mode (5.2) is traveling in an arbitrary direction described by the wave vector $\mathbf{k}=$ ( $k_{x}, \mathbf{k}_{\perp}$ ). Substituting the mode (5.2) and the coordinate transformations (5.6) in the Fourier transform (5.3), we obtain that

$$
\begin{array}{r}
\tilde{\Phi}(\Omega)=\int_{-\infty}^{\infty} d \tau \cos \left[\left(\omega t_{0}-k_{x} x_{0}-\mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}\right)+g^{-1}\left(\omega \sinh (g \tau)-k_{x} \cosh (g \tau)\right)\right] \\
\times \exp -i(\Omega \tau) \\
=g^{-1} e^{-\left(i \Omega \alpha g^{-1}\right)} K_{i \Omega g^{-1}}\left(\left|\mathbf{k}_{\perp}\right| g^{-1}\right)\left(e^{\left(\Omega / 4 \Omega_{0}\right)} e^{i \bar{\beta}}+e^{-\left(\Omega / 4 \Omega_{0}\right)} e^{-i \bar{\beta}}\right), \tag{5.35}
\end{array}
$$

where $\bar{\beta}=\left(\omega t_{0}-k_{x} x_{0}-\mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}\right), \alpha=\operatorname{arctanh}\left(k_{x} / \omega\right), K_{i \Omega g^{-1}}\left(\left|\mathbf{k}_{\perp}\right| g^{-1}\right)$ is the Macdonald function (a Bessel function of imaginary order and argument) and $\Omega_{0}$ is given by (5.8). The resultant power spectrum per logarithmic frequency interval is then given by

$$
\begin{align*}
& \mathcal{P}(\Omega)=4 \Omega g^{-2} \sinh \left(\Omega / 2 \Omega_{0}\right)\left|K_{i \Omega g^{-1}}\left(\left|\mathbf{k}_{\perp}\right| g^{-1}\right)\right|^{2} \\
& \times\left\{\frac{1}{2}+N(\Omega)+\sqrt{N(N+1)} \cos (2 \bar{\beta})\right\} . \tag{5.36}
\end{align*}
$$

This power spectrum does not have a Planckian nature because of the terms that multiply the expression in the curly brackets. We can therefore conclude that a Planckian ambience arises only for observers whose acceleration is along the same axis as the direction of propagation.

It is however interesting to ask: What happens to the power spectrum (5.36) in the limit of $\omega \rightarrow 0$ ? In the limit of $\omega \rightarrow 0$ the wave field (5.2) is a constant and therefore any relative direction of motion between the wave and the observer should be equivalent. Hence we expect a Planckian spectrum in this limit even for the mode (5.2) and this indeed happens to be the case. In the limit of $\mathbf{k}_{\perp} \rightarrow 0$,
we have

$$
\begin{equation*}
\left|K_{i \Omega g^{-1}}\left(\left|\mathbf{k}_{\perp}\right| g^{-1}\right)\right|_{\mathbf{k}_{\perp} \rightarrow 0}^{2}=\left(\frac{\pi}{4 \Omega g^{-1} \sinh \left(\pi \Omega g^{-1}\right)}\right) . \tag{5.37}
\end{equation*}
$$

Setting $k_{x}=0$ and substituting the above approximation for $\left|K_{i \Omega g^{-1}}\right|^{2}$ in (5.36), we recover the result we had obtained earlier in (5.12).

### 5.3 Planckian ambience in Schwarzschild and de-Sitter spacetimes

In this section, we shall briefly outline as to how the results we have obtained above can be extended to Schwarzschild and de-Sitter spacetimes. The solution to the Klein-Gordon equation in these spacetimes cannot be expressed in terms of simple functions in $(3+1)$ dimensions and hence we shall work in $(1+1)$ dimensions.

In $(1+1)$ dimensions, the Schwarzschild spacetime is described by the lineelement (see subsection 1.3.3)

$$
\begin{equation*}
d s^{2}=\left(1-\frac{2 M}{r}\right) d t^{2}-\left(1-\frac{2 M}{r}\right)^{-1} d r^{2} . \tag{5.38}
\end{equation*}
$$

In terms of the Regge-Wheeler coordinates $\left(t, r^{*}\right)$, where

$$
\begin{equation*}
r^{*}=r+2 M \ln \left(\frac{r}{2 M}-1\right), \tag{5.39}
\end{equation*}
$$

the Schwarzschild line element turns out to be conformal to the flat space metric, i.e.

$$
\begin{equation*}
d s^{2}=\left(1-\frac{2 M}{r}\right)\left(d t^{2}-d r^{* 2}\right) . \tag{5.40}
\end{equation*}
$$

And, in terms of the Kruskal-Szekeres coordinates $(v, u)$, which are related to the Regge-Wheeler coordinates $\left(t, r^{*}\right)$ by the transformations

$$
\begin{equation*}
v=v_{0}+e^{r^{*} / 4 M} \sinh (t / 4 M) \quad \text { and } \quad u=u_{0}+e^{r^{*} / 4 M} \cosh (t / 4 M), \tag{5.41}
\end{equation*}
$$

(where $u_{0}$ and $v_{0}$ are arbitrary constants) the Schwarzschild line-element reduces to

$$
\begin{equation*}
d s^{2}=\left(\frac{32 M^{3}}{r}\right) e^{-(r / 2 M)}\left(d v^{2}-d u^{2}\right) \tag{5.42}
\end{equation*}
$$

(Note that the transformations (1.103) correspond to the special case when $v_{0}$ and $u_{0}$ have been set to zero.) The proper time $\tau$ of an observer stationed at a constant $r$ is then related to the Schwarzschild time coordinate $t$ by the equation

$$
\begin{equation*}
\tau=\lambda(r) t \quad \text { where } \quad \lambda(r)=\left(1-\frac{2 M}{r}\right)^{1 / 2} \tag{5.43}
\end{equation*}
$$

Just as the trajectory of a uniformly accelerating observer is a hyperbola in the plane of the Minkowski coordinates, the world line of an observer stationed at a constant $r$ is a hyperbola in the $(v, u)$ plane. And, the asymptotes of this hyperbola are the past and the future horizons of the Schwarzschild spacetime that intersect at the point $\left(v_{0}, u_{0}\right)$.

As we have noted in subsection 1.1.2, the action for a massless, minimally coupled scalar field is invariant under conformal transformations in $(1+1)$ dimensions. Hence the normal modes of such a scalar field in conformally flat metrics are just plane waves. So, the normal mode solutions of the Schwarzschild spacetime in the Kruskal-Szekeres coordinates $(v, u)$ are just plane waves. Consider a single real mode described by the equation

$$
\begin{equation*}
\Phi(v, u)=\cos (\omega v-k u) . \tag{5.44}
\end{equation*}
$$

We would like to know how an observer located at constant Schwarzschild radial coordinate $r$ will describe this mode. Assuming that the plane wave is traveling to the right (i.e. $k=\omega$ ) and Fourier tranforming the monochromatic wave given in equation (5.44) with respect to the proper time $\tau$ of an observer stationed at a
constant $r$, we obtain that

$$
\begin{align*}
\tilde{\Phi}(\Omega) & =\int_{-\infty}^{\infty} d \tau \Phi[v(\tau), u(\tau)] e^{-i \Omega \tau} \\
& =\lambda \int_{-\infty}^{\infty} d t \cos \left(\omega e^{\left(r^{*}-t\right) / 4 M}-\beta\right) e^{-i \Omega \lambda t} \\
& =2 M \lambda e^{-i \mu}\left(e^{-2 \pi \Omega M \lambda} e^{-i \beta}+e^{2 \pi \Omega M \lambda} e^{i \beta}\right) \Gamma(4 i \Omega M \lambda) \tag{5.45}
\end{align*}
$$

where

$$
\begin{equation*}
\mu=4 \Omega M \lambda \ln \left(\omega e^{r^{*} / 4 M}\right) \tag{5.46}
\end{equation*}
$$

and $\beta$ is now defined as

$$
\begin{equation*}
\beta=\omega\left(v_{0}-u_{0}\right) . \tag{5.47}
\end{equation*}
$$

The resulting power spectrum per logarithmic frequency interval is then given by

$$
\begin{equation*}
\mathcal{P}(\Omega) \equiv \Omega|\tilde{\Phi}(\Omega)|^{2}=(4 \pi M \lambda)\left\{\frac{1}{2}+N+\sqrt{N(N+1)} \cos (2 \beta)\right\} \tag{5.48}
\end{equation*}
$$

where

$$
\begin{equation*}
N(\Omega)=\left(\frac{1}{\exp (8 \pi M \Omega \lambda)-1}\right) \tag{5.49}
\end{equation*}
$$

We once again obtain the three terms discussed before.

The analysis for the de-Sitter spacetime is similar. The line element that describes the de-Sitter spacetime is given by (see subsection 1.3.3)

$$
\begin{equation*}
d s^{2}=\left(1-H^{2} r^{2}\right) d t^{2}-\left(1-H^{2} r^{2}\right)^{-1} d r^{2} \tag{5.50}
\end{equation*}
$$

In terms of the 'Regge-Wheeler' coordinates $\left(t, r^{*}\right)$ corresponding to the de-Sitter spacetime, where

$$
\begin{equation*}
r^{*}=H^{-1} \operatorname{arctanh}(H r) . \tag{5.51}
\end{equation*}
$$

the de-Sitter line-element turns out to be

$$
\begin{equation*}
d s^{2}=\left(1-H^{2} r^{2}\right)\left(d t^{2}-d r^{* 2}\right) \tag{5.52}
\end{equation*}
$$

The 'Kruskal-Szekeres' coordinates ( $v, u$ ) corresponding to the de-Sitter spacetime are related to the coordinates 'Regge-Wheeler' coordinates ( $t, r^{*}$ ) by the equations

$$
\begin{equation*}
v=v_{0}+e^{H r^{*}} \sinh (H t) \quad \text { and } \quad u=u_{0}+e^{H r^{*}} \cosh (H t) . \tag{5.53}
\end{equation*}
$$

(Again, note that the transformations (1.114) correspond to the particular case: $v_{0}=u_{0}=0$.) The de-Sitter line-element in terms of the coordinates $(v, u)$ then reduces to

$$
\begin{equation*}
d s^{2}=H^{-2}(1-H r)^{2}\left(d v^{2}-d u^{2}\right) . \tag{5.54}
\end{equation*}
$$

Consider an observer who is stationed at a constant $r$ in de-Sitter spacetime. The world line of such an observer, just as in the Schwarzschild case, is a hyperbola in the $(v, u)$ plane whose asymptotes are the past and the future horizons of the de-Sitter spacetime that intersect at the point $\left(v_{0}, u_{0}\right)$. The proper time $\tau$ of this observer is related to the de-Sitter time coordinate $t$ as follows

$$
\begin{equation*}
\tau=\lambda t, \quad \text { where now } \quad \lambda=\left(1-H^{2} r^{2}\right)^{1 / 2} . \tag{5.55}
\end{equation*}
$$

For the case of a real wave as given in (5.44), where the coordinates $v$ and $u$ are now related to de-Sitter coordinates $t$ and $r$ by the equations (5.53) and (5.51), the power spectrum per logarithmic frequency interval as seen by the observer stationed at a constant $r$ is given by

$$
\begin{equation*}
\mathcal{P}(\Omega) \equiv \Omega|\tilde{\Phi}(\Omega)|^{2}=\left(\pi H^{-1} \lambda\right)\left\{\frac{1}{2}+N+\sqrt{N(N+1)} \cos (2 \beta)\right\}, \tag{5.56}
\end{equation*}
$$

where

$$
\begin{equation*}
N(\Omega)=\left(\frac{1}{\exp \left(2 \pi \Omega H^{-1} \lambda\right)-1}\right) . \tag{5.57}
\end{equation*}
$$

In evaluating the power spectrum above, it has been assumed that $k=\omega$, so that $\beta=\omega\left(v_{0}-u_{0}\right)$. The similarity to the previous results are obvious.

### 5.4 Discussion

It will be interesting to investigate whether the power spectrum we have evaluated in the last three sections can, in principle, be measured. We shall present here a model of a detector that is capable of measuring the Fourier spectrum of the classical field with respect to its proper time.

By a detector we have in mind a pointlike object which nevertheless has internal degrees of freedom. We shall also assume that the world line of the detector is given to us a priori and does not form a part of the dynamics. One such detector would be a simple harmonic oscillator that is coupled directly to the components of the classical field through a linear coupling. If the internal degree of freedom of the oscillator is $q$, then the interaction Lagrangian between the field and the detector would be of the form $q F$, where $F$ is one of the components of the classical field. (Notice the similarity of this classical detector with the UnruhDeWitt detector we had discussed in subsection 1.3.1.) Varying the total action of the detector and the field with respect to the degree of freedom $q$, we find that the equation of motion satisfied by the harmonic oscillator $q$ is given by

$$
\begin{equation*}
\frac{d^{2} q}{d \tau^{2}}+\alpha^{2} q=\bar{F}(\tau) \tag{5.58}
\end{equation*}
$$

where $\alpha$ is the frequency of the oscillator and $\bar{F}(\tau)$ is the component of the classical field in the frame of the oscillator. The total energy gained by any forced harmonic oscillator is proportional to the modulus square of the Fourier transform of the driving force. Therefore, the total energy $\varepsilon$ absorbed by the harmonic oscillator that is coupled to the field $F$ is then given by

$$
\begin{equation*}
\varepsilon(\alpha)=\left|\int_{-\infty}^{\infty} d \tau \bar{F}(\tau) e^{-i \alpha \tau}\right|^{2} \tag{5.59}
\end{equation*}
$$

Consider, for instance, a simple harmonic oscillator, say, a bound electric charge, that is coupled to the $y$-component of the electric field. Let us assume that the electric field is the plane electromagnetic wave we have considered in subsection 5.1.2. Let us also assume that the harmonic oscillator is accelerating uniformly described by worldline (5.6). We saw in subsection 5.1.2 that in the frame of the uniformly accelerating observer the $y$-component of the electromagnetic wave is given by

$$
\begin{equation*}
\bar{E}_{y}(\tau)=\omega e^{-g \tau} E_{y}(\tau)=\omega e^{-g \tau} \cos [\omega(t(\tau)-x(\tau))] . \tag{5.60}
\end{equation*}
$$

(We have assumed here that the plane electromagnetic wave is moving to the right. Hence, $k=\omega$.) The energy gained by such an oscillator due to its interaction with the plane electromagnetic wave is then given by

$$
\begin{align*}
\varepsilon(\alpha) & =\left|\int_{-\infty}^{\infty} d \tau \bar{E}_{y}(\tau) e^{-i \alpha \tau}\right|^{2} \\
& =\left(\frac{\pi \alpha}{g}\right)\left\{\frac{1}{2}+N+\sqrt{N(N+1)} \cos (2 \beta)\right\} \tag{5.61}
\end{align*}
$$

where $N$ is given by the equation

$$
\begin{equation*}
N(\alpha)=\left(\frac{1}{\exp \left(\alpha / \Omega_{0}\right)-1}\right) \tag{5.62}
\end{equation*}
$$

and $\Omega_{0}$ and $\beta$ are as in equation (5.8). Therefore, the power spectra we have evaluated in this chapter can, in principle, be measured physically.

In conclusion, we would like to stress those aspects of our results which are unexpected and contrast them with those which could have been anticipated with some hindsight.

To begin with, the following fact is well-known: In quantum field theory, the amplitude for transition of an Unruh-DeWitt detector, up to the first order in perturbation theory, is described by an integral that is similar in form
to (5.3) (cf. equation (1.88). When the scalar field is decomposed in terms of the Minkowski modes, the transition probability, per unit proper time, of a uniformly accelerating Unruh-DeWitt detector turns out to be a thermal spectrum (cf. equation 1.99). It might, therefore, seem that when a traveling wave is Fourier transformed with respect to the proper time of a uniformly accelerated observer, the resulting power spectrum will have a thermal nature.

However, there are some subtlities involved. To begin with, the modes of the quantum field are complex while here we are dealing with real plane wave modes. This makes the vital difference. As we have mentioned before, while a complex mode like $\exp -i(\omega t-k x)$ will give a Planckian distribution it will not yield the two other terms we have obtained in our analysis. In this sense, the real wave is quite different from the complex one. We stress the fact that, when a real Minkowski mode is Fourier transformed with respect to the proper time of a uniformly accelerating observer, the resulting power spectrum not only contains a Planckian distribution but also contains the root mean square fluctuations about the Planckian. As mentioned earlier, it is the appearance of these fluctuations that motivates us to attribute a 'thermal' nature to the power spectrum. We know of no simple way to guess at this answer.

Secondly, note that the effect survives in the power spectrum even in the limit of $\omega \rightarrow 0$. This is the closest to what one can call a 'classical vacuum'-and our result shows that such a mode, with infinitesimal frequency, leads to a thermal ambience in the accelerated frame which is totally independent of the properties of the original wave.

A somewhat similar analysis, viz. Fourier analyzing the Minkowski modes
in the frame of an uniformly accelerated observer was carried out earlier by Gerlach [149]. He had constructed a linear superposition of Minkowski modes in $(3+1)$ dimensions such that the modulus square of the amplitude of these modes (which represents the total classical energy of these modes) to be equivalent to that of the ground state energy of a quantum oscillator. Fourier analyzing such a field configuration with respect to the proper time of a uniformly accelerating observer, Gerlach had obtained a power spectrum (in a particular limit) similar in form to equation (5.9). He had presented his result as a 'heuristic derivation of the thermal spectrum' that arises in quantum field theory due to the inequivalent quantization in Minkowski and Rindler coordinates.

Our results and emphasis are different in several ways. To begin with, the effect we are reporting here is a feature of classical field theory and no quantum processes are involved. It is physically motivated in a clear and simple manner and we do not have to resort to any superposition of modes. Secondly, our results are exact for a real, monochromatic plane wave while Gerlach needed to resort to some approximation because of the particular superposition of modes he had chosen. Thirdly, we would like to draw attention to the zero-frequency limit of the wave, when it takes a life of its own in the accelerated frame. This result, as far as we know, has not been noted in the literature before. Finally, Gerlach had offered no explanation for the appearance of the factor $\cos (2 \beta)$ as the coefficient of the fluctuation term. Our analysis clearly shows that it arises due to the shift in the origin of the Minkowski coordinates.

These results we have presented in this chapter suggest that there is a deep connection between plane waves, accelerated frames and thermal fluctuations even at the classical level. This connection could be worth exploring.

## Chapter 6

## Conclusions and outlook

In this final chapter of the thesis, we shall summarize the conclusions that can be drawn from the results we have obtained in the last four chapters and also present an outlook on possible future research.

Consider a particular classical electromagnetic background, say, an electromagnetic field configuration produced by certain distribution of capacitors and solenoids in the laboratory. Given such a background we would like to have an answer for the following two questions: (i) Under what conditions can particle production take place in such a background? and (ii) How many particles will this background produce? In a laboratory, it is not possible to implement a gauge with the help of capacitor plates and solenoids. Hence gauge dependent concepts have no meaning. Therefore, the concept of a particle as well as the criterion for an electromagnetic background to produce particles better be gauge invariant.

These arguments apply equally well for gravitational backgrounds too. Consider a certain distribution of matter fields. The presence of these matter fields can give rise to a non-trivial gravitational background. Only a covariant crite-
rion can help us unambiguously conclude whether the spacetime induced by these matter fields can produce particles or not. It makes no sense physically to talk of coordinate dependent concepts. No coordinate system is more natural than any other coordinate system. Hence, a covariant formulation is mandatory to describe the concept of a particle as well as the phenomenon of particle production.

Apart from the dynamics, the initial conditions also play a role in deciding as to how many particles will a given background produce. A typical example would be that of a Friedmann universe. Given that particle production will take place in a particular Friedmann universe, more particles will be produced in an $n$-particle state of the quantum field than in the vacuum state. This is fairly obvious. In a vacuum state, there will only be spontaneous creation of pairs, whereas an $n$-particle state would lead to spontaneous as well as induced creation of pairs.

The fact that the number of particles produced by a given background will be dependent on the initial state of the quantum field can also be illustrated with the example of Schwarzschild spacetime. In a Schwarzschild spacetime, if we choose the initial state of the quantum field to be the Boulware vacuum the background does not produce particles. On the other hand, if the initial state of the quantum field is considered to be the Unruh vacuum state the Schwarzschild spacetime would give rise to Hawking radiation.

Though the number of particles produced by a given background can be dependent on the initial conditions, the invariant criterion, that decides whether a background is capable of producing particles, can not be dependent on the initial conditions. A typical example to illustrate this feature would be that of
a time independent, but otherwise, arbitrary magnetic field background. A time independent magnetic field does not give rise to an electric field (in a particular Lorentz frame) and a pure magnetic field cannot do any work. Due to this reason, one does not expect such a background to produce particles irrespective of the initial conditions that have been chosen. The gauge invariant criterion for particle production should reflect this feature.

In a curved spacetime, it is the tidal forces of the gravitational field that is responsible for the particle production. Particle production can be said to take place when the geodesics of a particle, anti-particle pair in a virtual quantum loop diverge due to the tidal action of the Riemann curvature tensor. Such a heuristic picture then suggests that if the tidal forces are strong enough then the virtual pairs will be converted into real pairs of particles. There exists a general belief in literature that only a time dependent backgrounds can produce particles, whereas time independent backgrounds cannot (see, for instance, ref. [150]). But, one would expect that even a time independent gravitational background will produce particles if the gravitational field is strong enough to deviate the geodesics of the virtual pairs adequately so that they are converted into real particles. Also, the time coordinate is not a covariant concept and hence a metric that is time dependent in a particular coordinate system may well prove to be independent of the time coordinate in a different coordinate system. These features suggest that just as we define positive frequency components with respect to a time coordinate, we can also define positive 'frequency' components with respect to space coordinates and thus discuss particle production in terms of mixing of space dependent positive and negative 'frequency' components in time independent backgrounds. In effect, this is exactly what has been attempted when a tunneling interpretation
was invoked to explain particle production in time independent electromagnetic backgrounds. There is no reason as to why such an attempt cannot be made for the gravitational case.

Earlier, in chapter 1, we had mentioned that in the absence of a timelike killing vector field particles cannot be defined at all. Only if there exist domains wherein the gravitational field is constant can the energy eigenstates of the quantum field be classified as particle numbers. Essentially, in a time dependent situation it is impossible to set up a state with a definite energy and particle number at a given time. In fact, this is precisely the reason why particle creation takes place at all. We have seen from our discussion in chapter 3 that the effective Lagrangian approach proves to be the most reliable approach available at present to study phenomena such as vacuum polarization and particle production in classical backgrounds. In the most general case, the effective Lagrangian will be dependent on space and time coordinates through invariant (gauge or coordinate invariant) quantities. Let us say that the imaginary part of the effective Lagrangian for a certain classical background is nonzero. Let us also assume that the imaginary of the effective Lagrangian is nonzero in certain regions of spacetimes and vanishes in certain other regions of spacetime. This clearly suggests that particle production takes place in certain regions of spacetime and does not take place in other regions. If this result from the effective Lagrangian approach has to be reproduced from a canonical quantization of the quantum field, a localized definition of the particle in space as well as time is clearly required. Therefore, the very fact that is considered to be a problem in literature can be turned around to provide a localized definition of the particle concept. Also, in section 3.3, we saw that the Klein approach turns out to be inadequate to describe the phenomenon of particle
production in a generic situation because of the very fact that we defined particles only in the asymptotic domains. Whereas, if we define a localized positive 'frequency' modes with respect to space (as we would, to diagonalize the Hamiltonian of a quantum field at any instant of time in a Friedmann universe) we will be able to circumvent this limitation of the Klein approach. These discussions clearly point towards the requirement of a localized definition of the particle concept.

We had seen in chapter 1 that it is the coefficient of the positive frequency component of the normal modes of a quantum field that is identified to be the annihilation operator (see our comment following equation (1.15)). And, the state annihilated by this operator is defined to be the vacuum state. In a curved spacetime, the normal modes of a quantum field in different coordinate systems are not coordinate transforms of each other. (This can be easily illustrated for the flat spacetime example we had discussed in section 1.1. It is easy to check that when the coordinate transformations (1.20) are substituted in the Minkowski mode (1.10) it does not lead to the Rindler mode (1.31).) Basically, it is this very feature that leads to the coordinate dependence of the particle concept. Similar features can arise when fields are quantized in classical electromagnetic backgrounds. It can be easily shown that the normal modes of a quantum field in different gauges, in general, are not gauge transforms of each other. (This can be shown for the case of a constant electric field background we had discussed in section 1.4. One can convince oneself that this is indeed the case by comparing the normal modes (1.123) and (1.137). It is easy to see that they are not related by the gauge factor $\exp \pm i(q E x t)$. We had, in fact, pointed out this feature, earlier, in section 3.5.) Just as in the gravitational case, this feature can lead to a gauge dependence of the particle concept in electromagnetic backgrounds
(see ref. [38], section 4.6). It has been conjectured in literature that coordinate or gauge dependence of the particle concept can arise due to the fact that only a sub-class of classically allowed (coordinate or gauge) transformations can be implemented unitarily [37]. It is possible that the particle concept would prove to be invariant only under transformations that leaves the asymptotic domains of the background unchanged. This conjecture has been proved to be true at least in one example. The Rindler transformation destroys the nice asymptotic properties of the Minkowski metric and Gerlach has explicitly shown that there exists no unitary transformation relating the Hilbert space of quantum states constructed from the Rindler vacuum and the Hilbert space determined by the Minkowski vacuum [151]. It would be a worthwhile effort to extend Gerlach's analysis to the case of the constant electric field background. That is, it will be interesting to examine whether the Hilbert spaces of, say, a complex scalar field when it quantized in a constant electric field background, in the time dependent and the space dependent gauges $A_{1}^{\mu}$ and $A_{2}^{\mu}$ (given by equations (1.120) and (1.121), respectively), are related by a unitary transformation.

The idea of detectors were developed with the hope of improving our understanding of the concept of a particle in curved spacetimes. But, as we have discussed in section 2.4, the connection between the response of detectors and the canonical formulation of quantum field theory still remains obscure. Objects such as field intensity, energy, etc. that we deal with in the conventional formulation of quantum field theory are not directly measurable quantities. These quantities are reflected through some physical measurements made by a detector. And, any detector that is used measure these quantities can couple to such variables only via the exchange of field quanta. But, we find that the field quanta are not gener-
ally covariant objects. They are defined through the choice of positive frequency components of the mode functions. Granted these facts there seems to be no escape from the conclusion that the operational and formal covariance have different meaning. This gap in our understanding of the relationship between the response of detectors and the canonical formulation of quantum field theory needs to be bridged.

We shall close this thesis with a few remarks on possible quantum gravitational effects in the domain of semiclassical gravity.

We had mentioned in the abstract of this thesis that quantum gravitational effects will become important only at energy scales of the order of Planck energy ( $\sim 10^{19} \mathrm{GeV}$ ) and in the domain between $10^{2}$ and $10^{19} \mathrm{GeV}$ we can study the evolution of quantum fields in classical gravitational backgrounds. But such arguments are, to say the least, naïve. It is not entirely correct to say that, in field theory, quantum mechanical effects are important in a particular range of scales. The divergences that arise in field theory are closely related to the small distance behavior of the Green's functions. If gravitational effects alter this small distance behavior, then the formalism of field theory will become very different from what we are used to. Since quantum field theory 'sums over' virtual states of arbitrary high energies, it is not entirely clear whether any definite energy or length scale can be associated with field theory phenomena.

Added to this consideration is the fact that gravity couples to matter and itself with equal strength. Thus a photon and a graviton (of the same energy) couple to an external gravitational field with equal strength. Creation of photons by a changing gravitational field will have a counterpart of creation of gravitons.

Clearly, the classical background gravitational field itself is experiencing first order perturbations. This is an additional complication that is not present in a linear field theory, say that of the electromagnetic field.

As we have seen in chapter 4, the semiclassical theory has a very limited validity unless the fluctuations in the backreaction term are systematically taken into account. One can possibly look for an Einstein-Langevin equation to describe the backreaction problem more completely. But it is likely that when we take into account these fluctuations the semiclassical approximation breaks down and quantum gravitational effects will become important. One needs to be very careful when one is pushing the domain of semiclassical gravity close to the quantum gravitational regime.

There has been a wide spectrum of opinion on the actual relationship between quantum field theory in curved spacetime and quantum gravity proper. Quantum gravity certainly has nothing to lose from critical investigations of semiclassical gravity.

One of the main problems of quantum field theory are the divergences. In a curved spacetime there exists no covariant formulation of handling these divergences. It may be, as many have speculated, that quantum gravity has its own cut-off-that is it is actually finite. In fact, a recent work by Padmanabhan points exactly in this direction. In his work, Padmanabhan shows that fluctuations in the gravitational background leads to a zero-point length of spacetime and thereby to a high energy cut-off [152]. This work clearly suggests that quantum gravity after all may sweep away the nagging problems of divergences in field theory. Though the result sounds quite plausible, the work by Padmanabhan is
at best a prescription to successfully handle the divergences and can possibly be a first step towards the ultimate theory. There is a very strong chance that the ultimate unification of gravity with the rest of modern physics will look very little like either contemporary general relativity or contemporary quantum field theory. There exist fundamental structural and conceptual mismatches between general relativity and quantum theory. It is possible that the quantum field theory we know of today with its apparatus of Fock spaces, Lagrangians, field equations, commutation relations and S-matrices, will just turn out to be a misguided and naive attempt at a forced marriage of classical field theory with quantum particle mechanics [153]. Radical new ideas may be needed to construct a quantum theory of gravity. The analysis of the semiclassical gravity during the last couple of decades have provided a glimpse of certain features that a quantum theory of gravity is likely to possess; the summit has been glimpsed, but it is yet to be reached. One fondly hopes that the summit will be reached soon.

## Appendix A

## Contour integrals

In this appendix, we shall evaluate the response of inertial and uniformly accelerated Unruh-DeWitt detectors (in the Minkowski vacuum state) when they are switched on for a finite time interval with a rectangular window function. The response of a Unruh-DeWitt detector that is switched on and off with a rectangular window function is described by the following integral (cf. equation (2.97)):

$$
\begin{equation*}
\mathcal{F}(\Omega, T)=\int_{-2 T}^{2 T} d x e^{-i \Omega x} G^{+}(x)(2 T-|x|) \tag{A.1}
\end{equation*}
$$

## A. 1 Response of the inertial detector

For the case of a detector on an inertial trajectory the integrals to be evaluated are (see subsection 2.2.3)

$$
\begin{equation*}
\mathcal{F}_{\text {ine } 1}(\Omega, T)=-\frac{T}{2 \pi^{2}} \int_{-2 T}^{2 T} d x \frac{e^{-i \Omega x}}{(x-i \epsilon)^{2}} \tag{A.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathcal{F}_{\text {ine } 2}(\Omega, T)=\frac{1}{4 \pi^{2}} \int_{-2 T}^{2 T} d x \frac{e^{-i \Omega x}|x|}{(x-i \epsilon)^{2}} \tag{A.3}
\end{equation*}
$$

so that

$$
\begin{equation*}
\mathcal{F}_{\text {ine }}(\Omega, T)=\mathcal{F}_{\text {ine } 1}(\Omega, T)+\mathcal{F}_{\text {ine } 2}(\Omega, T) \tag{A.4}
\end{equation*}
$$



Figure A.1: Contour for $\mathcal{F}_{\text {ine1 }}(\Omega, T)$
The integral for $\mathcal{F}_{\text {ine1 }}$ can be evaluated on a rectangular contour (see figure A.1) in the lower-half of the complex $x$-plane with the vertices given by $A_{i 1}(-2 T, 0)$, $B_{i 1}(2 T, 0), C_{i 1}(2 T,-i \infty)$ and $D_{i 1}(-2 T,-i \infty)$. Since this contour does not enclose the pole, by Cauchy's theorem the integral around this closed contour is zero. The value of the integral over the edge $A_{i 1} B_{i 1}$ can then be expressed in terms of the integrals over the other edges $B_{i 1} C_{i 1}$ and $D_{i 1} A_{i 1}$. Since the integrand vanishes on the edge $C_{i 1} D_{i 1}$ the contribution to the integral from this edge is zero. Thus

$$
\begin{equation*}
\mathcal{F}_{\text {ine } 1}(\Omega, T)=\left(\frac{T}{2 \pi^{2}}\right)\left\{\int_{2 T}^{2 T-i \infty} d x \frac{e^{-i \Omega x}}{(x-i \epsilon)^{2}}+\int_{-2 T-i \infty}^{-2 T} d x \frac{e^{-i \Omega x}}{(x-i \epsilon)^{2}}\right\} \tag{A.5}
\end{equation*}
$$

and after some simple manipulations these integrals can be expressed as follows

$$
\begin{align*}
\mathcal{F}_{\text {ine1 }}(\Omega, T)=\left(\frac{T}{2 \pi^{2}}\right)\left\{i e^{-2 i \Omega T}\right. & \int_{0}^{\infty} d v \frac{e^{-\Omega v}}{(v+\epsilon+2 i T)^{2}} \\
& \left.-i e^{2 i \Omega T} \int_{0}^{\infty} d v \frac{e^{-\Omega v}}{(v+\epsilon-2 i T)^{2}}\right\} \tag{A.6}
\end{align*}
$$



Figure A.2: Contour for $\mathcal{F}_{\text {ine } 2 A}(\Omega, T)$
The integral $\mathcal{F}_{\text {ine } 2}$ in the response function of the inertial detector has a $|x|$ term in the integrand and hence it has to expressed as a sum of the integrals over limits $(-2 T, 0)$ and $(0,2 T)$. Therefore, the intergals that have to be evaluated are

$$
\begin{equation*}
\mathcal{F}_{\text {ine2 }}(\Omega, T)=\frac{1}{4 \pi^{2}}\left\{\int_{0}^{2 T} d x \frac{e^{i \Omega x} x}{(x+i \epsilon)^{2}}+\int_{0}^{2 T} d x \frac{e^{-i \Omega x} x}{(x-i \epsilon)^{2}}\right\} . \tag{A.7}
\end{equation*}
$$

The first of these integrals can be evaluated on a rectangular contour (see figure A.2) in the upper-half of the complex $x$-plane with the vertices at $A_{i 2}(0,0)$, $B_{i 2}(2 T, 0), C_{i 2}(2 T, i \infty)$ and $D_{i 2}(0, i \infty)$. Similarly, the second integral can be evaluated on another rectangular contour (see figure A.3), this time in the lower-half of the complex $x$-plane with the vertices at $A_{i 2 *}(0,0), B_{i 2 *}(2 T, 0), C_{i 2 *}(2 T,-i \infty)$ and $D_{i 2 *}(0,-i \infty)$. Since neither of these contours enclose any poles, the integral over the edge $A_{i 2} B_{i 2}\left(A_{i 2 *} B_{i 2 *}\right)$ can be expressed in terms of integrals over the edges $B_{i 2} C_{i 2}\left(B_{i 2 *} C_{i 2 *}\right)$ and $D_{i 2} A_{i 2}\left(D_{i 2 *} A_{i 2 *}\right)$. The edge $C_{i 2} D_{i 2}\left(C_{i 2 *} D_{i 2 *}\right)$ does


Figure A.3: Contour for $\mathcal{F}_{\text {ine } 2 B}(\Omega, T)$
not contribute to the integral since the integrand vanishes on this edge. After some simple algebra we obtain that

$$
\begin{align*}
\mathcal{F}_{\text {ine } 2}(\Omega, T)=\frac{1}{4 \pi^{2}}\{ & 2 i T e^{2 i \Omega T} \int_{0}^{\infty} d v \frac{e^{-\Omega v}}{(v+\epsilon-2 i T)^{2}} \\
& -2 i T e^{-2 i \Omega T} \int_{0}^{\infty} d v \frac{e^{-\Omega v}}{(v+\epsilon+2 i T)^{2}} \\
& -e^{2 i \Omega T} \int_{0}^{\infty} d v \frac{e^{-\Omega v} v}{(v+\epsilon-2 i T)^{2}}+2 \int_{0}^{\infty} d v \frac{e^{-\Omega v} v}{(v+\epsilon)^{2}} \\
& \left.-e^{-2 i \Omega T} \int_{0}^{\infty} d v \frac{e^{-\Omega v} v}{(v+\epsilon+2 i T)^{2}}\right\} \tag{A.8}
\end{align*}
$$

The finite time inertial detector response is then given by

$$
\begin{array}{r}
\mathcal{F}_{\text {ine }}(\Omega, T)=\frac{1}{4 \pi^{2}}\left\{2 \int_{0}^{\infty} d v \frac{e^{-\Omega v} v}{(v+\epsilon)^{2}}-e^{2 i \Omega T} \int_{0}^{\infty} d v \frac{e^{-\Omega v} v}{(v+\epsilon-2 i T)^{2}}\right. \\
\left.-e^{-2 i \Omega T} \int_{0}^{\infty} d v \frac{e^{-\Omega v} v}{(v+\epsilon+2 i T)^{2}}\right\} . \tag{A.9}
\end{array}
$$

This is the result quoted in the text.


Figure A.4: Contour for $\mathcal{F}_{\text {acc1n }}(\Omega, T)$

## A. 2 Response of the accelerated detector

The response of the accelerated detector when it is turned on and off with a rectangular window function is described by the integrals (see subsection 2.2.3)

$$
\begin{equation*}
\mathcal{F}_{a c c}(\Omega, T)=\sum_{n=-\infty}^{\infty} \mathcal{F}_{a c c 1 n}(\Omega, T)+\mathcal{F}_{a c c 2 n}(\Omega, T) \tag{A.10}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{F}_{a c c 1 n}(\Omega, T)=-\left(\frac{T}{2 \pi^{2}}\right) \int_{-2 T}^{2 T} d x \frac{e^{-i \Omega x}}{\left(x-i b_{n}\right)^{2}} \tag{A.11}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathcal{F}_{a c c 2 n}(\Omega, T)=\frac{1}{4 \pi^{2}} \int_{-2 T}^{2 T} d x \frac{e^{-i \Omega x}|x|}{\left(x-i b_{n}\right)^{2}} \tag{A.12}
\end{equation*}
$$

$\mathcal{F}_{\text {accin } n}$ can be evaluated on a rectangular contour (see figure A.4) with the vertices at $A_{a 1}(-2 T, 0), B_{a 1}(2 T, 0), C_{a 1}(2 T,-i \infty)$ and $D_{a 1}(-2 T,-i \infty)$. This contour encloses the poles corresponding to the values of $n$ between one and infinity and
the integral for $\mathcal{F}_{\text {accin }}$ can be expressed in terms of the integrals over the edges $B_{a 1} C_{a 1}$ and $D_{a 1} A_{a 1}$ and the residues corresponding to the enclosed poles. After some manipulations we obtain that

$$
\begin{array}{r}
\mathcal{F}_{a c c 1 n}(\Omega, T)=\left(\frac{T}{2 \pi^{2}}\right)\left\{2 \pi \Omega \Theta(n) e^{\Omega b_{n}}+i e^{-2 i \Omega T} \int_{0}^{\infty} d v \frac{e^{-\Omega v}}{\left(v+b_{n}+2 i T\right)^{2}}\right. \\
\left.-i e^{2 i \Omega T} \int_{0}^{\infty} d v \frac{e^{-\Omega v}}{\left(v+b_{n}-2 i T\right)^{2}}\right\}, \tag{A.13}
\end{array}
$$

where $\Theta(n)=1$ for $n>0$ and zero otherwise.
$\mathcal{F}_{\text {acc } 2 n}$ after it has been split into two integrals with the limits ( $-2 T, 0$ ) and $(0,2 T)$, is given by

$$
\begin{equation*}
\mathcal{F}_{a c c 2 n}(\Omega, T)=\frac{1}{4 \pi^{2}}\left\{\int_{0}^{2 T} d x \frac{e^{i \Omega x} x}{\left(x+i b_{n}\right)^{2}}+\int_{0}^{2 T} d x \frac{e^{-i \Omega x} x}{\left(x-i b_{n}\right)^{2}}\right\} \tag{A.14}
\end{equation*}
$$

The first of these integrals can be evaluated on a rectangular contour (see figure A.5) on upper-half of the complex $x$-plane with the vertices at $A_{a 2}(0,0)$, $B_{a 2}(2 T, 0), C_{a 2}(2 T, i \infty)$ and $D_{a 2}(0, i \infty)$. Since the pole in the integrand sits right on the edge $D_{a 2} A_{a 2}$ when $n>0$, to avoid it, we indent the contour in such a way so that the pole is left outside the contour. Similarly, for evaluating the second integral in (A.14) a contour (see figure A.6) with vertices at $A_{a 2 *}(0,0), B_{a 2 *}(2 T, 0)$, $C_{a 2 *}(2 T,-i \infty)$ and $D_{a 2 *}(0,-i \infty)$ can be chosen and the poles that lie on the edge $D_{a 2 *} A_{a 2 *}$ for the values of $n$ between one and infinity can be avoided with an indentation of the contour so that they are left outside. The indentation on the contours contribute a residue corresponding to the infinitesimal semicircle around the pole with the result

$$
\begin{aligned}
\mathcal{F}_{a c c 2 n}(\Omega, T)=\frac{1}{4 \pi^{2}}\{ & 2 i T e^{2 i \Omega T} \int_{0}^{\infty} d v \frac{e^{-\Omega v}}{\left(v+b_{n}-2 i T\right)^{2}} \\
& -2 i T e^{-2 i \Omega T} \int_{0}^{\infty} d v \frac{e^{-\Omega v}}{\left(v+b_{n}+2 i T\right)^{2}}
\end{aligned}
$$



Figure A.5: Contour for $\mathcal{F}_{a c c 2 n A}(\Omega, T)$


Figure A.6: Contour for $\mathcal{F}_{a c c 2 n B}(\Omega, T)$

$$
\begin{align*}
-e^{2 i \Omega T} \int_{0}^{\infty} d v & \frac{e^{-\Omega v} v}{\left(v+b_{n}-2 i T\right)^{2}}+2 \int_{0}^{\infty} d v \frac{e^{-\Omega v} v}{\left(v+b_{n}\right)^{2}} \\
& \left.-e^{-2 i \Omega T} \int_{0}^{\infty} d v \frac{e^{-\Omega v} v}{\left(v+b_{n}-2 i T\right)^{2}}\right\} . \tag{A.15}
\end{align*}
$$

When the pole happens to settle right on the axis of integration in any of the integrals in the above expression the result of the integral over that axis is assumed to be given by the principal value of the integral. The complete accelerated detector response is then given by

$$
\begin{array}{r}
\mathcal{F}_{a c c}(\Omega, T)=\frac{1}{4 \pi^{2}} \sum_{n=-\infty}^{\infty}\left\{4 \pi \Omega T \Theta(n) e^{\Omega b_{n}}+2 \int_{0}^{\infty} d v \frac{e^{-\Omega v} v}{\left(v+b_{n}\right)^{2}}\right. \\
-e^{2 i \Omega T} \int_{0}^{\infty} d v \frac{e^{-\Omega v} v}{\left(v+b_{n}-2 i T\right)^{2}} \\
\left.-e^{-2 i \Omega T} \int_{0}^{\infty} d v \frac{e^{-\Omega v} v}{\left(v+b_{n}+2 i T\right)^{2}}\right\} \tag{A.16}
\end{array}
$$

which is the result quoted in the text.

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