CLASSICAL ELECTRODYNAMICS

Lecture schedule

- The course will consist of about 30 lectures and 5 tutorial sessions. The duration of each lecture and tutorial session will be 1.5 hours.
- We will meet thrice a week. The lectures are scheduled for 9:30–11:00 AM on Wednesdays, Thursdays and Fridays. We will be meeting in the discussion room on the top floor of the main building.
- The first lecture will be on August 13, 2010 and the last lecture will be on December 2, 2010.
- We will not be meeting during the week of October 11–15, 2010, due to the mid-semester break.
- Changes in schedule, if any, will be notified sufficiently in advance.

Problem sheets, exams and grading

- The grading will be based on 4 problem sheets (each containing about 5-10 problems), a mid-semester and an end-of-semester exam.
- The 4 problem sheets, in total, will carry 30% weight.
- The mid-semester exam will be on October 1, 2010. The exam will be for a duration of 1.5 hours and will carry 30% weight.
- The end-of-semester exam will be on December 3, 2010. The exam will be of 3 hours duration and will carry 40% weight.
- You will require a total of 50% to pass the course.

Syllabus and structure

1. **A historical perspective** [$\sim$ 1 Lecture]
   (a) Electricity, magnetism, and light
   (b) Light, ether, and relativity

2. **Special relativity** [$\sim$ 2 Lectures]
   (a) Principles of special relativity – k-calculus – Composition law for velocities
   (b) Lorentz transformations – The relativity of simultaneity – Length contraction and time dilation
   (c) Transformation of velocities and acceleration – Uniform acceleration – Doppler effect

3. **Four vectors, tensors, and relativity** [$\sim$ 2 Lectures]
   (a) Transformation of coordinates – Four vectors
   (b) Contravariant, covariant and mixed tensors – Elementary operations with tensors
   (c) The Minkowski line-element – Action for the relativistic free particle
   (d) Conservation of relativistic energy and momentum – Elastic collisions of particles

Problem sheet 1

4. **Charges in electromagnetic fields** [$\sim$ 4 Lectures]
   (a) Elementary particles in the theory of relativity
   (b) Four potential – Equation of motion of a charge in an electromagnetic field
   (c) Gauge invariance
   (d) Motion in constant, uniform electric and magnetic fields
   (e) The electromagnetic field tensor – Lorentz transformation of the field – Invariants of the field

5. **The electromagnetic field equations** [$\sim$ 4 Lectures]
   (a) The first pair of Maxwell's equations
   (b) The action for the electromagnetic field
   (c) The four dimensional current vector and the equation of continuity
   (d) The second pair of Maxwell's equations
   (e) Energy density and energy flux – The energy-momentum tensor – The energy momentum tensor of the electromagnetic field
   (f) The virial theorem

Problem sheet 2

6. **Constant electromagnetic fields** [$\sim$ 4 Lectures]
   (a) Coulomb's law – Electrostatic energy of charges
   (b) The field of a uniformly moving charge
   (c) Motion in the Coulomb field
   (d) The dipole moment – Multipole moments
   (e) System of charges in an external field

   (f) Constant magnetic field – Magnetic moments
   (g) Larmor's theorem

Mid-semester exam

7. **Electrostatic of conductors and dielectrics** [$\sim$ 2 Lectures]
   (a) The electrostatic field of conductors – The energy associated with the field
   (b) Methods of solving problems in electrostatics
   (c) The electric field in dielectrics
8. **Steady currents and static magnetic fields** [$\sim$ 1 Lecture]
   (a) The current density and conductivity
   (b) The magnetic field of a steady current
   (c) Forces in a magnetic field

Problem sheet 3

9. **Electromagnetic waves and the propagation of light** [$\sim$ 2 Lectures]
   (a) The wave equation – Plane waves – Monochromatic plane waves
   (b) Spectral resolution – Partially polarized light
   (c) The Fourier resolution of the electrostatic field
10. **Propagation of light** [$\sim$ 3 Lectures]
    (a) Geometrical optics – Intensity
    (b) The angular eikonal – Narrow bundles of rays
    (c) Image formation with broad bundles of rays
    (d) The limits of geometrical optics
    (e) Diffraction – Fresnel and Fraunhofer diffraction

Problem sheet 4

11. **The field of moving charges** [$\sim$ 3 Lectures]
    (a) The retarded potentials – The Lienard-Wiechart potentials
    (b) Spectral resolution of the retarded potentials
12. **Radiation of electromagnetic waves** [$\sim$ 4 Lectures]
    (a) The field of a system of charges at large distances
    (b) Dipole radiation – Dipole radiation during collisions
    (c) Radiation in the case of Coulomb interaction
    (d) Quadrapole and magnetic dipole radiation
    (e) Radiation from a rapidly moving charge
    (f) Synchrotron radiation
    (g) Radiation damping – Radiation damping in the relativistic case

End-of-semester exam

Essential texts

1. L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Course of Theoretical Physics, Volume 2), Fourth Edition (Pergamon Press, New York, 1975).
2. L. D. Landau, E. M. Lifshitz and L. P. Pitaevskii, *Electrodynamics of Continuous Media* (Course of Theoretical Physics, Volume 8), Second Edition (Pergamon Press, New York, 1984).
3. D. J. Griffiths, *Introduction to Electrodynamics*, Third Edition (Prentice Hall of India, New Delhi, 1999).
4. J. D. Jackson, *Classical Electrodynamics*, Third Edition (John Wiley and Sons, Singapore, 1999).

Additional references

1. R. d'Inverno, *Introducing Einstein's Relativity* (Oxford University Press, Oxford, 1992).
2. W. Greiner, *Classical Electrodynamics* (Springer-Verlag, New York, 1998).
3. J. B. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (Pearson Education, Delhi, 2003).
4. M. Longair, *Theoretical Concepts in Physics*, Second Edition (Cambridge University Press, Cambridge, England, 2003).

Problem sheet 1

Special relativity and tensors

1. Consider two inertial frames that move with the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ with respect to, say, the laboratory frame. Show that the relative velocity v between the two frames can be expressed as

$$v^2 = \left[\frac{(\mathbf{v}_1 - \mathbf{v}_2)^2 - (\mathbf{v}_1 \times \mathbf{v}_2)^2}{(1 - \mathbf{v}_1 \cdot \mathbf{v}_2)^2}\right].$$

2. Consider a blob of plasma that is moving at a speed v along a direction that makes an angle θ with respect to the line of sight. Show that the *apparent* transverse speed of the source, projected on the sky, will be related to the actual speed v by the relation

$$v_{\rm app} = \left(\frac{v \sin\theta}{1 - (v/c)\cos\theta}\right).$$

From this expression conclude that the apparent speed $v_{\rm app}$ can exceed the speed of light.

3. Let $\Lambda(\mathbf{v})$ denote the matrix corresponding to a Lorentz boost along the direction $\hat{\mathbf{v}}$ with the speed v. Show that

$$\Lambda(v_1\,\hat{\mathbf{j}})\;\Lambda(v_1\,\hat{\mathbf{i}}) = R(\theta\,\hat{\mathbf{k}})\;\Lambda(v_2\,\hat{\mathbf{n}}),$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are the unit vectors along the x, y, and z axes, respectively, and $R(\theta\,\hat{\mathbf{k}})$ is a matrix that describes spatial rotation by the angle θ about the z-axis. What is $\hat{\mathbf{n}}$? Also, determine v_2 and θ in terms of v_1.

4. Consider two inertial frames S and S', with the frame S' moving along the positive x direction with a velocity v with respect to the frame S. Let the velocity of a particle in the frames S and S' be $\mathbf{u}$ and $\mathbf{u}'$, and let θ and θ' be the angles subtended by the velocity vectors with respect to the common x-axis, respectively.

   (a) Show that

   $$\tan\theta = \left(\frac{u'\sin\theta'}{\gamma\,[u'\cos\theta' + v]}\right),$$

   where $\gamma = \left[1 - (v/c)^2\right]^{-1/2}$.

   (b) For $u = u' = c$, show that

   $$\cos\theta = \left(\frac{\cos\theta' + (v/c)}{1 + (v/c)\cos\theta'}\right)$$

   and

   $$\sin\theta = \left(\frac{\sin\theta'}{\gamma\;[1 + (v/c)\cos\theta']}\right).$$

   (c) For $(v/c) \ll 1$, show that

   $$\Delta\theta = (v/c)\,\sin\theta',$$

   where $\Delta\theta = (\theta' - \theta)$.

   Note: This phenomenon is known as aberration of light.

5. Consider the scattering between a photon of frequency ω and a relativistic electron with velocity $\mathbf{v}$ leading to a photon of frequency ω' and electron with velocity $\mathbf{v}'$. Let α be the angle between the incident and the scattered photon and θ and θ' be the angles between the direction of propagation of photon and the velocity vector of the electron before and after the collision.

(a) Using the conservation of energy and momentum in the scattering, show that

$$\left(\frac{\omega'}{\omega}\right) = \left(\frac{1-(v/c)\cos\theta}{1-(v/c)\cos\theta' + (\hbar\,\omega/\gamma\, m_{\rm e}\, c^2)\,(1-\cos\alpha)}\right),$$

where $\gamma = \left[1-(v/c)^2\right]^{-1/2}$ and $m_{\rm e}$ is the mass of the electron.

(b) When $(\hbar\,\omega) \ll \left(\gamma\, m_{\rm e}\, c^2\right)$, show that the frequency shift of the photon can be written as

$$\left(\frac{\Delta\omega}{\omega}\right) = \left[\frac{(v/c)\;(\cos\theta - \cos\theta')}{1-(v/c)\cos\theta'}\right],$$

where $\Delta\omega = (\omega' - \omega)$.

(c) The above expression implies that, if θ and θ' are randomly distributed, then there is no net increase in the photon energy to the first order in (v/c). Use the exact expression and compute the net energy change of the photon to the order $(v/c)^2$ when θ and θ' are randomly distributed.

6. A purely relativistic process corresponds to the production of electron-positron pairs in a collision of two high energy gamma ray photons. If the energies of the photons are ϵ_1 and ϵ_2 and the relative angle between them is θ, then, by using the conservation of energy and momentum, show that the process can occur only if

$$(\epsilon_1\,\epsilon_2) > \left(\frac{2\,m_{\rm e}^2\,c^4}{1-\cos\theta}\right),$$

where $m_{\rm e}$ is the mass of the electron.

7. A mirror moves with the velocity v in a direction perpendicular its plane. A ray of light of frequency ω_1 is incident on the mirror at an angle of incidence θ, and is reflected at an angle of reflection ϕ and frequency ω_2.

(a) Show that

$$\left(\frac{\tan(\theta/2)}{\tan(\phi/2)}\right) = \left(\frac{c+v}{c-v}\right) \qquad \text{and} \qquad \left(\frac{\omega_2}{\omega_1}\right) = \left(\frac{c+v\cos\theta}{c-v\cos\phi}\right).$$

(b) What happens if the mirror was moving parallel to its plane?

8. (a) Show that the following Minkowski line-element

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}\mathbf{x}^2$$

is invariant under the Lorentz transformations.

(b) Show that the four volume $\mathrm{d}^4x \equiv (c\,\mathrm{d}t\;\mathrm{d}^3\mathbf{x})$ is invariant under the Lorentz transformations.

Note: Recall that the coordinates, say, $(c\,t', x', y', z')$, of an inertial system that is moving along the positive x direction at a velocity v with respect to, say, the coordinates $(c\,t, x, y, z)$ of the laboratory are given by

$$c\,t' = \gamma\;[c\,t - (v\,x/c)]\,,\quad x' = \gamma\;(x - v\,t)\,,\quad y' = y,\quad z = z',$$

where $\gamma = \left[1-(v/c)^2\right]^{-1/2}$.

9. (a) Consider a scalar quantity ϕ. Show that, while the quantity $(\partial\phi/\partial x^\mu)$ is a vector under arbitrary coordinate transformations, the quantity $\left(\partial^2\phi/\partial x^\mu\,\partial x^\nu\right)$ is a tensor *only* under linear coordinate transformation such as the Lorentz transformations.

(b) If $X^{\mu}_{\ \nu\lambda}$ is a mixed tensor of rank $(1,2)$, show that the contracted quantity $Y_\mu = X^{\lambda}_{\ \mu\lambda}$ is a covariant vector.

10. (a) Given that $F_{\mu\nu}$ is an anti-symmetric tensor in the laboratory frame, evaluate the components of the tensor in a frame that is moving along the positive x direction with a velocity v with respect to the laboratory.

    (b) Evaluate the quantities δ^{μ}_{μ} and $\left(\delta^{\mu}_{\nu}\,\delta^{\nu}_{\mu}\right)$.

This sheet has to be returned to Dhiraj Hazra by 4:00 PM on September 8, 2010.

Problem sheet 2

Charges in electromagnetic fields and the electromagnetic field equations

1. Determine the frequency of a charged, three-dimensional oscillator that is in a uniform magnetic field oriented along, say, the positive z-axis.

2. (a) Express the three acceleration (i.e. $\dot{\mathbf{v}}$) of a charged particle in terms of the electric and the magnetic fields $\mathbf{E}$ and $\mathbf{H}$.
   Note: The overdot denotes differentiation with respect to the coordinate time t.
   (b) Determine the motion of a charge that is moving relativistically in parallel, uniform, electric and magnetic fields.
   (c) Determine the motion of a charge that is moving relativistically in uniform electric and magnetic fields that are perpendicular to each other.

3. (a) Evaluate how the electromagnetic field tensor $F_{\mu\nu}$ transforms under a Lorentz boost. From the result, arrive at how the electric and magnetic fields transform under the Lorentz transformation.
   (b) Given that the uniform electric and magnetic fields are constant and uniform, determine the velocity of the reference frame in which the two fields are parallel.

4. Show that:
   (a) $(F^{\mu\nu}\, F_{\mu\nu}) = -2\,\left(\mathbf{E}^2 - \mathbf{H}^2\right)$.
   (b) $\left(e^{\alpha\beta\gamma\delta}\, F_{\alpha\beta}\, F_{\gamma\delta}\right) = -8\,(\mathbf{E}\cdot\mathbf{H})$.
   Note: The quantity $e^{\alpha\beta\gamma\delta}$ is a tensor that is anti-symmetric in all its indices, and has a magnitude of unity. We work with the convention wherein $e^{0123} = 1$.
   (c) $\left(e^{\alpha\beta\gamma\delta}\, F_{\alpha\beta}\, F_{\gamma\delta}\right)$ is a total four-divergence.
   (d) The source-free Maxwell's equations can be written as $\left[e^{\alpha\beta\gamma\delta}\ (\partial_\beta\, F_{\gamma\delta})\right] = 0$.
   Note: The source-free Maxwell's equations were originally obtained to be:

$$(\partial_\lambda\, F_{\mu\nu} + \partial_\nu\, F_{\lambda\mu} + \partial_\mu\, F_{\nu\lambda}) = 0.$$

5. Consider the following action that describes a scalar field, say, ϕ:

$$S[\phi] = \int c\, \mathrm{d}t\, \mathrm{d}^3x\, \left(\frac{1}{2}\right)\, \left(\partial_\mu\phi\, \partial^\mu\phi - m^2\, \phi^2\right),$$

   where m is referred to as the mass of the scalar field.
   (a) Vary the action to arrive at the equation of motion of the scalar field.
   (b) Evaluate the energy-momentum tensor $T_{\mu\nu}$ associated with the field.

6. Given the energy-momentum tensor $T_{\mu\nu}$ associated with the electromagnetic field,
   (a) Show that energy-momentum tensor possesses no trace.
   (b) Show that the energy-momentum tensor is divergence-free in the absence of sources.
   (c) Evaluate $\left(T^{\mu\lambda}\, T_{\lambda\nu}\right)$.

This sheet has to be returned to Dhiraj Hazra by 4:00 PM on September 23, 2009.

Mid-semester exam

From special relativity to constant electromagnetic fields

1. (a) Show that the phase space volume $\left(\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p}\right)$ is a Lorentz invariant quantity.
   (b) The specific intensity of radiation I_ν measures the intensity of the radiation at a particular frequency ν in a particular direction. It is defined as the flux per unit frequency interval, per unit solid angle. Show that $\left(I_\nu/\nu^3\right)$ is Lorentz invariant.

2. Consider a frame S' that is moving with a velocity $\mathbf{v}$ with respect to the frame S. A rod in the frame S' makes an angle θ' with respect to the forward direction of motion. What is the angle θ as measured in S?

3. Determine the magnetic field $\mathbf{H}$ due to a current $\mathbf{I}$ being carried in an infinitely long straight wire, by an appropriate Lorentz transformation of the electric field of an infinitely long and straight charge distribution.

4. A particular electromagnetic field has its electric field $\mathbf{E}$ at an angle θ to the magnetic field $\mathbf{H}$. All inertial frames find the value of the angle θ to be the same. What is the value of θ? Are you aware of an example of such a field configuration?

5. Consider a conducting element with conductivity σ that is carrying a current due to an imposed electric field. Let the current and electric field in the frame of the conductor be given by the three dimensional vectors $\mathbf{j}$ and $\mathbf{E}$, respectively. Then, according to the Ohm's law, the current and the imposed electric field are related through the conductivity of the medium as follows: $\mathbf{j} = (\sigma\,\mathbf{E})$. Express the Ohm's law in a covariant form in terms of the four current j^μ, the field tensor $F^{\mu\nu}$, and the four velocity u^μ of the conducting element.

   Hint: The Ohm's law in the form expressed above is valid in the frame of the conductor wherein $u^\mu = (1,0,0,0)$. Attempt to write the Ohm's law in terms of the four velocity u^μ. Once expressed in a covariant form, if it is valid in one frame, it will be valid in all frames.

6. A small test particle of positive charge q and mass m makes circular orbits around a fixed, very massive body of positive charge Q. A uniform magnetic field $\mathbf{H}$ perpendicular to the orbital plane helps to keep the particle in orbit at a radius, say, R. In the inertial frame in which the central, massive body is at rest, the test charge is seen to circle in the plane perpendicular to the magnetic field $\mathbf{H}$ with an angular frequency ω. Show that the charge to mass ratio (q/m) of the test particle can be expressed in terms of ω, Q and R as follows:

$$\left(\frac{q}{m}\right) = \left(\frac{\omega^2}{\left[1-(\omega\,R/c)^2\right]^{1/2}\,\left[(\omega\,H/c)-(Q/R^3)\right]}\right).$$

Mid-semester exam II

From special relativity to constant electromagnetic fields

1. (a) Show that the phase space volume $\left(\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p}\right)$ is a Lorentz invariant quantity.

   (b) The specific intensity of radiation I_ν measures the intensity of the radiation at a particular frequency ν in a particular direction. It is defined as the flux per unit frequency interval, per unit solid angle. Show that $\left(I_\nu/\nu^3\right)$ is Lorentz invariant.

2. Consider a frame S' that is moving with a velocity $\mathbf{v}$ with respect to the frame S. A particle in the frame S' is moving with a velocity $\mathbf{v}'$ at an angle θ' with respect to the forward direction of motion. What is the angle θ as measured in S?

3. Determine the magnetic field $\mathbf{H}$ due to a current $\mathbf{I}$ being carried in an infinitely long straight wire, by an appropriate Lorentz transformation of the electric field of an infinitely long and straight charge distribution.

4. Consider the following alternative action to describe the electromagnetic field:

$$S[A_\mu] = \int c\,\mathrm{d}t\,\mathrm{d}^3x\,\left[-\left(\frac{1}{8\,\pi}\right)(\partial_\mu A_\nu)\,(\partial^\mu A^\nu) - \left(\frac{1}{c}\right) j_\mu\,A^\mu\right],$$

   where A_μ and j_μ are the usual four potential and current. Derive the corresponding equations of motion. Are they the Maxwell's equations? If they are, under what assumptions are they so?

5. If $B_{\mu\nu}$ is an anti-symmetric tensor in one inertial frame, establish that it is so in all other frames.

Problem sheet 3

Constant electromagnetic fields and electrostatics of conductors and dielectrics

1. Working in the cartesian coordinates, solve the following Poisson's equation using the method of the Green's function:
$$\nabla^2\phi = -\,(4\,\pi)\;\rho.$$
Note: The solution to the above Poisson's equation can be written as
$$\rho\,(\mathbf{x}) = \int \mathrm{d}^3\mathbf{x}'\; G\left(\mathbf{x},\mathbf{x}'\right)\;\rho\left(\mathbf{x}'\right),$$
where the Green's function $G\,(\mathbf{x},\mathbf{x}')$ satisfies the differential equation
$$\nabla^2 G\left(\mathbf{x},\mathbf{x}'\right) = \delta^{(3)}\left(\mathbf{x}-\mathbf{x}'\right).$$

2. In cases which possess cylindrical symmetry, it may be easier to work in the cylindrical polar coordinates. Evaluate the Green's function $G\,(\mathbf{x},\mathbf{x}')$ in the cylindrical polar coordinates.

   Note: Since the Green's function $G\,(\mathbf{x},\mathbf{x}')$ can be expressed in terms of the (spatial) coordinate invariant distance $|\mathbf{x}-\mathbf{x}'|$, the Green's function in the cylindrical polar coordinates can be easily obtained by simply transforming this distance from the cartesian coordinates to the cylindrical polar coordinates. But, I would like you to obtain the Fourier modes in the cylindrical polar coordinates, and sum over them, to arrive at the Green's function.

3. What is the Green's function $G\,(\mathbf{x},\mathbf{x}')$ to the Poisson's equation in one lower dimension, i.e. in two spatial dimensions?

4. We had worked out the potential that arises due to a point charge in the presence of a charged conducting sphere. Draw the lines of force of the resulting electric field.

5. Two charges are moving with the same velocity, say, $\mathbf{v}$, with respect to the laboratory frame. What is the force between them?

   Hint: Determine the electric and magnetic fields produced by one of the charges, and use the Lorentz force law to determine the force on the other charge.

6. Consider a relativistic test charge is moving in the centrally symmetric field of another charge e which generates the electrostatic potential $\phi = (e/r)$. Let the mass of the central charge be so large that it can be considered to be fixed.
   (a) Determine the trajectory of the test charge.
   (b) What are the conditions under which the motion of the particle will be unbounded?
   (c) What are conditions under which the test charge will exhibit bounded motion?
   (d) When the charge is exhibiting bounded motion, is the trajectory of the charge an ellipse as in the non-relativistic case?
   (e) Draw the trajectory of the charge in both the bounded and unbounded cases.

7. Recall that, in the multipole expansion, the potential due to a distribution of charges can be expressed as
$$\phi = \sum_{l=0}^{\infty}\phi^{(l)}.$$
The term $\phi^{(l)}$ is given by
$$\phi^{(l)} = \left(\frac{1}{r_0^{(l+1)}}\right)\sum_{m=-l}^{l}\left(\frac{4\,\pi}{2\,l+1}\right)^{1/2}\,Q_m^{(l)}\;Y_{lm}^{*}(\vartheta,\varphi),$$

where

$$Q_m^{(l)} = \left(\frac{4\,\pi}{2\,l+1}\right)^{1/2} \sum_a e_a\, r_a^l\, Y_{lm}(\theta_a, \phi_a)$$

with the sum being over all the charges in the system. The vectors $\mathbf{r}_0$ and $\mathbf{r}$ denote the point of observation and the location of the charges, while (ϑ, φ) and (θ_a, ϕ_a) are the spherical angles corresponding to these two vectors, respectively. The set of $(2\,l+1)$ quantities $Q_m^{(l)}$ describe the 2^l-pole moment of the system of charges. Relate the components of the dipole and the quadrapole moments of the distribution to the quantities $Q_m^{(l)}$.

8. Determine the dipole and quadrapole moments of a uniformly charged ellipsoid with respect to its centre.

9. What is the Green's function (in three spatial dimensions) when an infinite planar conductor is located at $x = 0$?

10. Determine the capacitance of a ring of thin conducting wire of radius b and circular cross-section a such that $a \ll b$.

This sheet has to be returned to Dhiraj Hazra by 4:00 PM on November 4, 2010.

Miscellaneous problems

Tensors in space and spacetime

Tensors in Euclidean space – Working in the cartesian coordinates

1. (a) What are the components of the metric tensor that describes the three dimensional Euclidean space in the cartesian coordinates? Is there any difference between the covariant and the contravariant three dimensional vectors in such a coordinate system?

   (b) Is the magnitude of a vector always positive definite in three dimensional Euclidean space? Why is that so?

2. (a) Show that the vector product of two three-dimensional, spatial vectors, say, **A** and **B**, can be written as
$$(\mathbf{A} \times \mathbf{B})_i = (e_{ijk}\, A_j\, B_k)\,,$$
where e_{ijk} is the three dimensional, completely anti-symmetric tensor with the indices (i, j, k) going from 1 to 3.

   (b) Use the anti-symmetric property of e_{ijk} to show that $[\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B})] = 0$.

3. Show that, in three-dimensional Euclidean space,

   (a) $\delta_{ii} = 3$,

   (b) $(\delta_{ij}\, e_{ijk}) = 0$,

   (c) $(e_{ikl}\, e_{jkl}) = (2\, \delta_{ij})$,

   (d) $(e_{ijk}\, e_{ijk}) = 6$,

   (e) $(e_{ijk}\, e_{mnk}) = (\delta_{im}\, \delta_{jn} - \delta_{in}\, \delta_{jm})$.

4. Write $[\nabla \cdot (\nabla \times \mathbf{A})]$ and $[\nabla \times (\nabla \phi)]$ in terms of the anti-symmetric tensor e_{ijk}, and show that they vanish.

5. Using the anti-symmetric tensor e_{ijk}, establish the following relation:
$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}\, (\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}\, (\mathbf{A} \cdot \mathbf{B})\,.$$

Tensors in Euclidean space – Working in the cylindrical and the spherical polar coordinates

6. What are the components of the metric tensor that describes the three dimensional Euclidean space in the cylindrical polar and the spherical polar coordinates? Is there any difference between the covariant and the contravariant vectors in these coordinate systems?

7. Express the three-dimensional Laplacian in the cylindrical polar and the spherical polar coordinates in terms of the metric tensor in these coordinate systems.

Tensors in Minkowski spacetime

8. Performing the Lorentz transformations explicitly, show that the metric tensor in Minkowski spacetime, viz. $\eta_{\mu\nu} = \mathrm{diag.}\ (1, -1, -1, -1)$, is form invariant in any inertial coordinate system.

9. Using the standard spatial transformations that relate the cartesian coordinates to the cylindrical and the spherical polar coordinates, obtain the Minkowski covariant metric tensor in these coordinates. What are the corresponding contravariant metric tensors?

10. Consider two four vectors A^μ and B^μ given by
$$A^\mu = (2, 0, 0, -1) \quad \text{and} \quad B^\mu = (-5, 0, -3, -4)\,.$$

(a) Is A^{μ} timelike, spacelike or null? What about B^{μ}?

(b) Compute $(A^{\mu} - 5\, B^{\mu})$.

(c) What is $(A^{\mu}\, B_{\mu})$?

(d) Evaluate the components of the tensors A^{μ} and B^{μ} in an inertial frame, say, S, that is moving with a speed $v = (0.2\, c)$ along the positive x-axis.

(e) What are the components of the second rank tensor $(A^{\mu}\, B^{\nu})$ in the frame S? What are the components of $(A^{\mu} B_{\nu})$ and $(A_{\mu}\, B_{\nu})$?

11. A particle of rest mass m and four momentum p^{μ} is examined by an observer with a four velocity u^{μ}. Show that:

(a) The energy of the particle as measured by the observer is $E = (p^{\mu}\, u_{\mu})\, c$.

(b) The rest mass that the observer attributes to the particle is $m^2 = \left[(p^{\mu}\, p_{\mu})\,/c^2\right]$.

(c) The magnitude of the three momentum of the particle as measured by the observer is

$$p = \left[(p^{\mu}\, u_{\mu})^2 - (p^{\mu}\, p_{\mu})\right]^{1/2}.$$

(d) The magnitude of the three velocity of the particle as measured by the observer is

$$v = \left[1 - \left(\frac{p^{\mu}\, p_{\mu}}{(p^{\mu}\, u_{\mu})^2}\right)\right]^{1/2} c.$$

12. Show that, if a tensor, say, $T^{\alpha\beta\cdots}$ is of rank n, then

(a) $\left(\partial T^{\alpha\beta\cdots}/\partial x^{\gamma}\right)$ is a tensor of rank $(n+1)$,

(b) whereas $\left(\partial T^{\alpha\beta\cdots}/\partial x^{\alpha}\right)$ is a tensor of rank $(n-1)$.

Problem sheet 4

From electromagnetic waves to radiation from moving charges

1. Determine the trajectory of a charge that is in motion in the fields of:

   (a) a plane electromagnetic wave described by the vector potential $\mathbf{A}[t-(x/c)]$,

   (b) a plane and monochromatic, linearly polarized, electromagnetic wave,

   (c) a plane and monochromatic, circularly polarized, electromagnetic wave.

2. Recall that, we had shown that the electromagnetic field tensor $F^{\mu\nu}$ associated with the field of a particle with charge e that is in motion along the trajectory $r^{\mu}(\tau)$ can be written as

$$F^{\mu\nu} = \left(\frac{e}{\bar{u}^{\lambda}\,[x_{\lambda}-r_{\lambda}(\tau)]}\right)\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{[x^{\mu}-r^{\mu}(\tau)]\;\bar{u}^{\nu}\;-[x^{\nu}-r^{\nu}(\tau)]\;\bar{u}^{\mu}}{\bar{u}^{\lambda}\;[x_{\lambda}-r_{\lambda}(\tau)]}\right],$$

   where $u^{\lambda} = (\mathrm{d}r^{\lambda}/d\tau)$, and all the quantities are to be evaluated at the retarded time. Show that the resulting electric and magnetic fields are given by

$$\begin{aligned}
\mathbf{E} &= e\left(\frac{\hat{\mathbf{n}}-(\mathbf{v}/c)}{\gamma^2\;[1-(\mathbf{v}\cdot\hat{\mathbf{n}}/c)]^3\;R^2}\right)+\left(\frac{e}{c}\right)\left(\frac{\hat{\mathbf{n}}\times([\hat{\mathbf{n}}-(\mathbf{v}/c)]\times(\dot{\mathbf{v}}/c))}{[1-(\mathbf{v}\cdot\hat{\mathbf{n}}/c)]^3\;R}\right),\\
\mathbf{H} &= (\hat{\mathbf{n}}\times\mathbf{E}),
\end{aligned}$$

   where R is the spatial distance from the position of the particle at the retarded time to the point of observation, $\hat{\mathbf{n}}$ is unit vector pointing from the former to the latter, $\mathbf{v}$ is the three-velocity of the particle at the retarded time and, as usual, $\gamma = \left[1-(v/c)^2\right]^{-1/2}$.

3. Find the radiation from a dipole $\mathbf{d}$ that is rotating in a plane with a constant angular velocity, say, Ω.

4. Consider two particles with charges (e_1, e_2) and masses (m_1, m_2) that are moving in the mutual Coulomb field. Let E be the total energy of the system (omitting the rest mass energies), μ be its reduced mass, and ε be the eccentricity of the trajectory.

   (a) Recall that, for the case wherein the motion is along an elliptical trajectory (when the particles carry opposite charges), the intensity I_n of the radiation emitted by the system at the frequency $(n\,\omega_0)$ can be obtained to be

$$I_n = \left(\frac{64\,n^2\,E^4}{3\,c^3\,\alpha^2}\right)\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)^2\left[J_n'^{\,2}(n\,\varepsilon)+\left(\frac{1-\varepsilon^2}{\varepsilon^2}\right)\,J_n^2(n\,\varepsilon)\right],$$

   where $J_n(x)$ is the Bessel function, while an overprime denotes differentiation with respect to the argument. Note that, in the above expression, $\alpha = |e_1\,e_2|$, while the fundamental frequency ω_0 is given by $\omega_0 = [8\,|E|^3/(\alpha^2\,\mu)]^{1/2}$.

      i. Plot the above I_n as a function of $(n\,\omega_0)$.

      ii. Calculate the average total intensity (i.e. per period) of the radiation emitted by the charge.

   (b) For the case wherein the system is undergoing hyperbolic motion, the energy emitted by the system per unit frequency was found to be

$$\left(\frac{\mathrm{d}\mathcal{E}_{\omega}}{\mathrm{d}\omega}\right) = \left(\frac{\pi\,\mu^2\,\alpha^2\,\omega^2}{6\,c^3\,E^2}\right)\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)^2\left(\left[H_{i\nu}^{(1)\prime}(i\,\nu\,\varepsilon)\right]^2+\left(\frac{\varepsilon^2-1}{\varepsilon^2}\right)\left[H_{i\nu}^{(1)}(i\,\nu\,\varepsilon)\right]^2\right),$$

   where $\nu = [\omega\,\alpha/(\mu\,v_0^3)]$, and $E = (\mu\,v_0^2/2)$, with v_0 being the relative velocity of the particles at infinity. Plot the above quantity $(\mathrm{d}\mathcal{E}_{\omega}/\mathrm{d}\omega)$ as a function of ω.

5. Recall that, for a relativistic particle, the intensity of radiation emitted by the particle into a solid angle $\mathrm{d}\Omega$ can be expressed as

$$\left(\frac{\mathrm{d}I}{\mathrm{d}\Omega}\right) = \left(\frac{e^2}{4\,\pi\,c^3}\right)\left[\left(\frac{2\,(\mathbf{n}\cdot\mathbf{w})\,(\mathbf{v}\cdot\mathbf{w})}{c\,\left[1-(\mathbf{v}\cdot\hat{\mathbf{n}}/c)\right]^5}\right) + \left(\frac{\mathbf{w}^2}{\left[1-(\mathbf{v}\cdot\hat{\mathbf{n}}/c)\right]^4}\right) - \left(\frac{(\mathbf{n}\cdot\mathbf{w})^2}{\gamma^2\,\left[1-(\mathbf{v}\cdot\hat{\mathbf{n}}/c)\right]^6}\right)\right],$$

   where $\mathbf{w} = \dot{\mathbf{v}}$ is the three-acceleration of the charge. Assuming that $\mathbf{v}$ and $\mathbf{w}$ are in general directions, plot the above quantity as a function of the two spherical angles θ and ϕ.

6. Find the rate of change of energy with time of a particle that is moving in a circular orbit in a constant magnetic field, and is losing energy due to synchrotron radiation.

This sheet has to be returned to Dhiraj Hazra by 4:00 PM on November 25, 2010.

End-of-semester exam

From special relativity to radiation from moving charges

1. At each event in spacetime, the Cosmic Microwave Background (CMB) has a mean rest frame and, as seen in the mean rest frame, the CMB is isotropic and thermal at the temperature $T_0 = 2.73$ K. Actually, the Earth moves relative to the mean rest frame of the CMB with a speed of about 600 km s^{-1} towards the Hydra-Centaurus region of the sky. Consider an observer on Earth who points his microwave receiver in a direction that makes an angle θ with the direction of that motion, as measured in the Earth's frame.

   (a) Show that the intensity of the radiation received is precisely Planckian in form, but with the Doppler shifted temperature
   $$T = T_0 \left(\frac{\left[1-(v/c)^2\right]^{1/2}}{1+(v/c)\cos\theta}\right).$$
   Note: Recall that, we had shown that $\left(I_\nu/\nu^3\right)$ is a Lorentz invariant quantity, where I_ν is the intensity of radiation at the frequency ν.

   (b) Note that the θ dependence of the temperature corresponds to an anisotropy of the CMB as seen from earth. Show that, because the earth's velocity is small compared to the velocity of light, the anisotropy is dipolar in form.

   (c) What is the magnitude of $(\Delta T/T)$ of the variations between the maximum and minimum CMB temperature on the sky?

2. The time-averaged electrostatic potential of a neutral hydrogen atom is given by
   $$\phi = \left(\frac{e}{r}\right)\left[1+\left(\frac{\alpha\, r}{2}\right)\right] e^{-\alpha\, r},$$
   where e is the magnitude of the electronic charge and $\alpha = (a_0/2)$, with a_0 being the Bohr radius. Find the charge distribution (both continuous and discrete) that will give rise to such a potential.

3. Evaluate the electrostatic potential of a uniformly charged and finite rod of a given length. What are the equipotential surfaces?

4. Compute the quadrapole moment (say, $Q_m^{(l)*}$) of a uniform charge distribution whose surface is a slightly deformed sphere with a radius given by
   $$R = R_0\left[1+\sum_{m=-2}^{2} \alpha_{2m}^{*}\, Y_{2m}(\theta,\phi)\right],$$
   where R_0 is a constant, while $|\alpha_{2m}| \ll 1$.

5. An infinitely long wire, that is bent in the shape of a hair pin, carries a steady current. Determine the magnetic field at the centre of the semi-circle that constitutes the hair pin bend.

6. Obtain an integral expression for the retarded Green's function associated with a massive field that satisfies the Klein-Gordon equation. Can you comment if the Green's function is non-zero beyond the light cone?

7. The Rutherford's model of the atom assumes that electrons move around the nucleus in circular orbits at non-relativistic velocities. Classically, estimate the time within which the electron in the hydrogen atom will fall onto the nucleus.

8. A charged particle exhibiting simple harmonic motion emits radiation. Assuming that the energy lost by the charge due to the emitted radiation is small when compared to the energy of the charge, determine the profile of the emitted spectrum.