GENERAL THEORY OF RELATIVITY

Lecture schedule

- The course will consist of about 30 lectures and 5 tutorial sessions. The duration of each lecture and tutorial session will be 1.5 hours.
- We will meet thrice a week. The lectures are scheduled for 11:00 AM–12:30 PM on Mondays, Tuesdays and Wednesdays. We will be meeting in the lecture hall (right above the Director's office) in the top floor of the main building.
- The first lecture will be on August 3, 2009 and the last lecture will be on December 7, 2009.
- We will not be meeting during September 23–30, 2009 due to the mid-semester break.
- Changes in schedule, if any, will be notified sufficiently in advance.

Problem sheets, exams and grading

- The grading will be based on 4 problem sheets (each containing about 5-10 problems), a mid-semester and an end-of-semester exam.
- The 4 problem sheets, in total, will carry 30% weight.
- The mid-semester exam will be on September 21, 2009. The exam will be for a duration of 1.5 hours and will carry 30% weight.
- The end-of-semester exam will be on December 9, 2009. The exam will be of 3 hours duration and will carry 40% weight.
- You will require a total of 50% to pass the course.

Syllabus and structure

1. **Introduction** [$\sim$ 2 Lectures]

   (a) The scope of the general theory of relativity
   (b) Geometry and physics
   (c) Space, time and gravity in Newtonian physics

2. **Special relativity** [$\sim$ 3 Lectures]

   (a) Principles of special relativity – k-calculus – Composition law for velocities
   (b) Lorentz transformations – The relativity of simultaneity – Length contraction and time dilation
   (c) Transformation of velocities and acceleration – Uniform acceleration – Doppler effect
   (d) Action for the relativistic free particle – Charges in an electromagnetic field and the Lorentz force law
   (e) Conservation of relativistic energy and momentum

Problem sheet 1

3. **Tensor algebra and tensor calculus** [$\sim$ 7 Lectures]

   (a) Manifolds and coordinates – Curves and surfaces
   (b) Transformation of coordinates – Contravariant, covariant and mixed tensors – Elementary operations with tensors
   (c) The partial and Lie derivatives of a tensor – The affine connection and covariant differentiation – Affine geodesics
   (d) The Riemann tensor – Geodesic coordinates – Affine flatness
   (e) The metric – Metric geodesics – The metric connection – Metric flatness
   (f) The curvature and the Weyl tensors
   (g) Isometries – The Killing equation and conserved quantities

Problem sheet 2

4. **The principles of general relativity** [$\sim$ 2 Lectures]

   (a) Mach's principle – The equivalence principle – The principle of general covariance – The principle of minimal gravitational coupling

5. **The field equations of general relativity** [$\sim$ 4 Lectures]

   (a) The equation of geodesic deviation – The vacuum Einstein equations
   (b) Derivation of vacuum Einstein equations from the action – The Palatini approach – The Bianchi identities
   (c) The stress-energy tensor – The cases of perfect fluid, scalar and electromagnetic fields
   (d) The structure of the Einstein equations

Mid semester exam

6. **The Schwarzschild solution, and black holes** [$\sim$ 6 Lectures]

   (a) The Schwarzchild solution – Properties of the metric – Symmetries and conserved quantities

(b) Motion of particles in the Schwarzschild metric – Precession of the perihelion – Bending of light
(c) Black holes – Event horizon, its properties and significance – Singularities
(d) The Kruskal extension – Penrose diagrams

Problem sheet 3

7. **The Friedmann-Lemaitre-Robertson-Walker cosmology** [$\sim$ 6 Lectures]

    (a) Homogeneity and isotropy – The Friedmann line-element
    (b) Friedmann equations – Solutions with different types of matter
    (c) Red-shift – Luminosity and angular diameter distances
    (d) The horizon problem – The inflationary scenario

Problem sheet 4

8. **Gravitational waves** [$\sim$ 3 Lectures]

    (a) The linearized Einstein equations – Solutions to the wave equation – Production of weak gravitational waves
    (b) Gravitational radiation from binary stars – The quadrupole formula for the energy loss – Effects of gravitational radiation detected in a binary pulsar

End-of-semester exam

Essential texts

1. J. B. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (Pearson Education, Delhi, 2003).
2. R. d'Inverno, *Introducing Einstein's Relativity* (Oxford University Press, Oxford, 1992).
3. L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Course of Theoretical Physics, Volume 2), Fourth Edition (Pergamon Press, New York, 1975).
4. B. F. Schutz, *A First Course in General Relativity* (Cambridge University Press, Cambridge, 1990).

Additional references

1. S. Carroll, *Spacetime and Geometry* (Addison Wesley, New York, 2004).
2. M. P. Hobson, G. P. Efstathiou and A. N. Lasenby, *General Relativity: An Introduction for Physicists* (Cambridge University Press, Cambridge, 2006).
3. W. Rindler, *Relativity: Special, General and Cosmological* (Oxford University Press, Oxford, 2006).
4. S. Weinberg, *Gravitation and Cosmology* (John Wiley, New York, 1972).

Advanced texts

1. S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge University Press, Cambridge, 1973).

2. C. W. Misner, K. S. Thorne and J. W. Wheeler, *Gravitation* (W. H. Freeman and Company, San Francisco, 1973).

3. E. Poisson, *A Relativist's Toolkit* (Cambridge University Press, Cambridge, 2004).

4. R. M. Wald, *General Relativity* (The University of Chicago Press, Chicago, 1984).

Problem sheet 1

Special relativity

1. Consider a blob of plasma that is moving at a speed v along a direction that makes an angle θ with respect to the line of sight. Show that the *apparent* transverse speed of the source, projected on the sky, will be related to the actual speed v by the relation
$$v_{\rm app} = \left(\frac{v\,\sin\theta}{1-(v/c)\cos\theta}\right).$$
From this expression conclude that the apparent speed $v_{\rm app}$ can exceed the speed of light.

2. Let $\Lambda(\mathbf{v})$ denote the matrix corresponding to a Lorentz boost along the direction $\hat{\mathbf{v}}$ with the speed v. Show that
$$\Lambda(\mathbf{v}_1)\,\Lambda(\mathbf{v}_2) = R(\theta\,\hat{\mathbf{n}})\,\Lambda(\mathbf{v}_3),$$
where $R(\theta\,\hat{\mathbf{n}})$ is the matrix describing spatial rotation by the angle θ about an axis $\hat{\mathbf{n}}$.

3. Consider two inertial frames S and S', with the frame S' moving along the x-axis with a velocity v with respect to the frame S. Let the velocity of a particle in the frames S and S' be $\mathbf{u}$ and $\mathbf{u}'$, and let θ and θ' be the angles subtended by the velocity vectors with respect to the common x-axis, respectively.

   (a) Show that
   $$\tan\theta = \left(\frac{u'\sin\theta'}{\gamma\,[u'\cos\theta' + v]}\right),$$
   where $\gamma = \left[1-(v/c)^2\right]^{-1/2}$.

   (b) For $u = u' = c$, show that
   $$\cos\theta = \left(\frac{\cos\theta' + (v/c)}{1+(v/c)\cos\theta'}\right)$$
   and
   $$\sin\theta = \left(\frac{\sin\theta'}{\gamma\;[1+(v/c)\cos\theta']}\right).$$

   (c) For $(v/c) \ll 1$, show that
   $$\Delta\theta = (v/c)\,\sin\theta',$$
   where $\Delta\theta = (\theta' - \theta)$.

   Note: This phenomenon is known as aberration of light.

4. Consider the scattering between a photon of frequency ω and a relativistic electron with velocity $\mathbf{v}$ leading to a photon of frequency ω' and electron with velocity $\mathbf{v}'$. Let α be the angle between the incident and the scattered photon and θ and θ' be the angles between the direction of propagation of photon and the velocity vector of the electron before and after the collision.

   (a) Using the conservation of energy and momentum in the scattering, show that
   $$\left(\frac{\omega'}{\omega}\right) = \left(\frac{1-(v/c)\cos\theta}{1-(v/c)\cos\theta' + (\hbar\,\omega/\gamma\,m_{\rm e}\,c^2)\,(1-\cos\alpha)}\right),$$
   where $\gamma = \left[1-(v/c)^2\right]^{-1/2}$ and $m_{\rm e}$ is the mass of the electron.

   (b) When $(\hbar\,\omega) \ll \left(\gamma\,m_{\rm e}\,c^2\right)$, show that the frequency shift of the photon can be written as
   $$\left(\frac{\Delta\omega}{\omega}\right) = \left[\frac{(v/c)\;(\cos\theta - \cos\theta')}{1-(v/c)\cos\theta'}\right],$$
   where $\Delta\omega = (\omega' - \omega)$.

(c) The above expression implies that, if θ and θ' are randomly distributed, then there is no net increase in the photon energy to the first order in (v/c). Use the exact expression and compute the net energy change of the photon to the order $(v/c)^2$ when θ and θ' are randomly distributed.

5. A purely relativistic process corresponds to the production of electron-positron pairs in a collision of two high energy gamma ray photons. If the energies of the photons are ϵ_1 and ϵ_2 and the relative angle between them is θ, then, by using the conservation of energy and momentum, show that the process can occur only if
$$(\epsilon_1\,\epsilon_2) > \left(\frac{2\,m_{\mathrm{e}}^2\,c^4}{1-\cos\theta}\right),$$
where m_{e} is the mass of the electron.

6. A mirror moves with the velocity v in a direction perpendicular its plane. A ray of light of frequency ν_1 is incident on the mirror at an angle of incidence θ, and is reflected at an angle of reflection ϕ and frequency ν_2.

   (a) Show that
$$\left(\frac{\tan(\theta/2)}{\tan(\phi/2)}\right) = \left(\frac{c+v}{c-v}\right) \qquad \text{and} \qquad \left(\frac{\nu_2}{\nu_1}\right) = \left(\frac{c+v\cos\theta}{c-v\cos\phi}\right).$$

   (b) What happens if the mirror was moving parallel to its plane?

This sheet has to be returned to Dhiraj Hazra by 2:00 PM on August 24, 2009.

Problem sheet 2

Tensor algebra and tensor calculus

1. (a) Write down the transformations from the Cartesian coordinates $x^a = (x, y, z)$ to the spherical polar coordinates $x'^a = (r, \theta, \phi)$ in $\mathbb{R}^3$.

   (b) Express the transformation matrices $[\partial x^a / \partial x'^b]$ and $[\partial x'^a / \partial x^b]$ in terms of the spherical polar coordinates.

   (c) Evaluate the corresponding Jacobians J and J'. Where is J' zero or infinite?

2. (a) Write down the transformations from the Cartesian coordinates $x^a = (x, y)$ to the plane polar coordinates $x'^a = (r, \phi)$ in $\mathbb{R}^2$.

   (b) Express the transformation matrix $[\partial x'^a / \partial x^b]$ in terms of the polar coordinates.

   (c) Consider the tangent vector to a circle of radius, say, a, that is centered at the origin. Find the components of the tangent vector in one of the two coordinate system, and use the transformation property of the vector to obtain the components in the other coordinate system.

3. Consider a scalar quantity ϕ. Show that, while the quantity $(\partial\phi/\partial x^a)$ is a vector, the quantity $\left(\partial^2\phi/\partial x^a\, \partial x^b\right)$ is *not* a tensor.

4. If $X^a{}_{bc}$ is a mixed tensor of rank $(1, 2)$, show that the contracted quantity $Y_c = X^a{}_{ac}$ is a covariant vector.

5. Evaluate the quantities δ^a_a and $\left(\delta^a_b\, \delta^b_a\right)$ on a n-dimensionsal manifold.

6. Show that the affine connection Γ^a_{bc} transforms as follows:.

$$\Gamma'^a_{bc} = \left(\frac{\partial x'^a}{\partial x^d}\right)\left(\frac{\partial x^e}{\partial x'^b}\right)\left(\frac{\partial x^f}{\partial x'^c}\right)\Gamma^d_{ef} - \left(\frac{\partial x^d}{\partial x'^b}\right)\left(\frac{\partial x^e}{\partial x'^c}\right)\left(\frac{\partial^2 x'^a}{\partial x^d\, \partial x^e}\right).$$

7. (a) In Minkowski spacetime, the action for a relativistic particle that is interacting with the electromagnetic field is given by

$$S[x^\mu] = -m \int \mathrm{d}s - e \int A^\mu\, \mathrm{d}x_\mu,$$

   where $\mathrm{d}s$ is the infinitesimal line interval, m is the mass of the particle, e its electronic charge, and A^μ is the four vector potential describing the electromagnetic field. Vary this action with respect to x^μ to arrive at the following Lorentz force law:

$$m \left(\frac{\mathrm{d}u^\mu}{\mathrm{d}s}\right) = e\, F^{\mu\nu}\, u_\nu,$$

   where $u^\mu = (\mathrm{d}x^\mu/\mathrm{d}s)$ is the four velocity of the particle and the electromagentic field tensor $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} = \left(\partial_\mu\, A_\nu - \partial_\nu\, A_\mu\right).$$

   (b) Show that the above definition of $F_{\mu\nu}$ leads to the following Maxwell's equations in flat spacetime:

$$\left(\frac{\partial F_{\mu\nu}}{\partial x^\lambda}\right) + \left(\frac{\partial F_{\nu\lambda}}{\partial x^\mu}\right) + \left(\frac{\partial F_{\lambda\mu}}{\partial x^\nu}\right) = 0.$$

   Also, show that these equations correspond to the following two source free Maxwell's equations:

$$(\nabla \times \mathbf{E}) = -\left(\frac{\partial \mathbf{B}}{\partial t}\right) \qquad \text{and} \qquad (\nabla \cdot \mathbf{B}) = 0,$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields that are related to the compontents of the four vector potential $A^\mu = (\phi, \mathbf{A})$ as follows:

$$\mathbf{E} = -\nabla\phi - \left(\frac{\partial \mathbf{A}}{\partial t}\right) \qquad \text{and} \qquad \mathbf{B} = \nabla \times \mathbf{A}.$$

(c) Show that the spacetime volume $\mathrm{d}^4x = (\mathrm{d}t\, \mathrm{d}^3\mathbf{x})$ is a Lorentz invariant quantity.

(d) In Minkowski spacetime, the action describing the electromagnetic field that is sourced by the four current j^μ is given by

$$S[A^\mu] = -\int \mathrm{d}^4x\ (A^\mu\, j_\mu) - \left(\frac{1}{16\,\pi}\right) \int \mathrm{d}^4x\ (F^{\mu\nu}\, F_{\mu\nu})\,.$$

Vary this action with respect to the vector potential A^μ and arrive at the following Maxwell's equations:

$$(\partial_\nu\, F^{\mu\nu}) = -(4\,\pi)\ j^\mu.$$

Show that these equations correspond to the following two Maxwell's equations with sources:

$$(\nabla\cdot\mathbf{E}) = (4\,\pi)\,\rho \qquad \text{and} \qquad (\nabla\times\mathbf{B}) = (4\,\pi)\,\mathbf{j} + \left(\frac{\partial\mathbf{E}}{\partial t}\right),$$

where ρ and $\mathbf{j}$ are the time and the spatial components of the four current j^μ, i.e. $j^\mu = (\rho, \mathbf{j})$.

8. Recall that, an observer who has a constant acceleration in the comoving frame is referred to as a uniformly accelerated observer. The transformations from the Minkowski coordinates (t, x) to the coordinates, say, (τ, ξ), of a collection of uniformly accelerated observers moving along the positive x-direction are given by

$$t = \xi\, \sinh(g\,\tau) \qquad \text{and} \qquad x = \xi\, \cosh(g\,\tau).$$

(a) Draw the coordinates associated with these accelerated observers in the t-x plane. How do the coordinates behave on the light cone $x = |t|$?

(b) Express the Minkowski line element in terms of the coordinates (τ, ξ). of the accelerated observers.

(c) Evaluate the metric connection associated with the line element in the accelerated frame.

(d) Now, consider an accelerated observer who is restricted to the domain $x < 0$ of the Minkowski spacetime. Construct the transformations that lead to the coordinates of such a class of observers.

Note: The coordinates (τ, ξ) are often called the Rindler coordinates, and the quadrants $x > |t|$ and $x < |t|$ are known as Rindler space. As we shall see later, the light cones $x = \pm t$ bear some resemblance to the black hole horizons.

9. Show that:

(a) $(\nabla_c\, \nabla_d\, X^a_{\ b} - \nabla_d\, \nabla_c\, X^a_{\ b}) = (R^a_{\ ecd}\, X^e_{\ b} - R^e_{\ bcd}\, X^a_{\ e})$.

(b) $\left[\nabla_X\ (\nabla_Y\, Z^a) - \nabla_Y\ (\nabla_X\, Z^a) - \nabla_{[X,Y]}\, Z^a\right] = \left(R^a_{\ bcd}\, Z^b\, X^c\, Y^d\right)$,

where

$$\nabla_{[X,Y]} = \left(\frac{1}{2}\right)\ (\nabla_X\, \nabla_Y - \nabla_Y\, \nabla_X)\,.$$

10. The three sphere $\mathbb{S}^3$ is the three-dimensional surface in a four-dimensional Euclidean space $\mathbb{R}^4$. Let (x, y, z, w) be the coordinates of $\mathbb{R}^4$, while R is the radius of the three sphere. The three sphere is then described by the equation

$$\left(x^2 + y^2 + z^2 + w^2\right) = R^2.$$

(a) Define new coordinates (r, θ, ϕ, χ) that are related to the coordinates (x, y, z, w) by the following relations:

$$x = r\,\sin\chi\,\sin\theta\,\cos\phi,\quad y = r\,\sin\chi\,\sin\theta\,\sin\phi,\quad z = r\,\sin\chi\,\cos\theta \quad \text{and} \quad w = r\,\cos\chi.$$

Note: These generalize the familiar spherical polar coordinates to a higher dimension.

(b) Evaluate the metric of the three sphere in terms of the coordinates (r, θ, ϕ, χ).

11. Establish the following identities for a general metric tensor $g_{\mu\nu}$ in a general coordinate system:
    (a) $\Gamma^{\mu}_{\mu\nu} = (1/2)\,(\ln|g|)_{,\nu}$,
    (b) $\left(g^{\mu\nu}\,\Gamma^{\alpha}_{\mu\nu}\right) = -\,(1/\sqrt{-g})\,\left(\sqrt{-g}\,g^{\alpha\beta}\right)_{,\beta}$,
    (c) For an anti-symmetric tensor $F^{\mu\nu}$, $F^{\mu\nu}{}_{;\nu} = (1/\sqrt{-g})\,(\sqrt{-g}\,\,F^{\mu\nu})_{,\nu}$,
    (d) $(g^{\mu\nu}\,g_{\nu\alpha,\beta}) = -\left(g^{\mu\nu}{}_{,\beta}\,g_{\nu\alpha}\right)$,
    (e) $g^{\mu\nu}{}_{,\alpha} = -\left(\Gamma^{\mu}_{\alpha\beta}\,g^{\beta\nu} + \Gamma^{\nu}_{\alpha\beta}\,g^{\mu\beta}\right)$,

    where the commas denote partial derivatives and the $\Gamma^{\alpha}_{\mu\nu}$ is the metric connection given by

$$\Gamma^{\alpha}_{\mu\nu} = \left(\frac{1}{2}\right)\,g^{\alpha\beta}\,\left(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}\right).$$

12. Consider the following line element that describes a spherically symmetric spacetimes in $(3+1)$-dimensions:

$$\mathrm{d}s^2 = e^{\Phi(t,r)}\,\mathrm{d}t^2 - e^{\Psi(t,r)}\,\mathrm{d}r^2 - r^2\,\left(\mathrm{d}\theta^2 + \sin^2\!\theta\,\mathrm{d}\phi^2\right),$$

    where $\Phi(t,r)$ and $\Psi(t,r)$ are arbitrary functions of the coordinates t and r.
    (a) Find $g_{\mu\nu}$, $g^{\mu\nu}$, and g corresponding to this line element.
    (b) Evaluate the corresponding $\Gamma^{\alpha}_{\mu\nu}$.
    (c) Obtain $R^{\alpha}_{\beta\mu\nu}$.
    (d) Calculate $R_{\mu\nu}$, R and $G_{\mu\nu}$.
    (e) Compute G^{μ}_{ν}.

13. Show that the spacetime volume $[\sqrt{-g}\,\mathrm{d}^4x]$, where $g = \text{det.}\ (g_{\mu\nu})$, is invariant under arbitrary coordinate transformations.

14. In a curved spacetime, the dynamics of the source free electromagnetic field is described by the action

$$\mathcal{S}[A^{\mu}] = -\left(\frac{1}{16\,\pi}\right)\int \mathrm{d}^4x\,\sqrt{-g}\,\left(F^{\mu\nu}\,F_{\mu\nu}\right),$$

    where

$$F_{\mu\nu} = \left(A_{\nu;\mu} - A_{\mu;\nu}\right) = \left(A_{\nu,\mu} - A_{\mu,\nu}\right)$$

    and the semi-colons denote covariant differentiation, while, as before, the commas denote partical derivatives. Show that this action is invariant under the following conformal transformation:

$$A_{\mu} \to A_{\mu},\quad x^{\mu} \to x^{\mu},\quad g_{\mu\nu} \to \left(\Omega^2(x^{\lambda})\,g_{\mu\nu}\right).$$

This sheet has to be returned to Dhiraj Hazra by 2:00 PM on September 15, 2009.

Mid-semester exam

From special relativity to the Schwarzschild spacetime

1. Consider a charge e that is moving with a velocity v and acceleration a along the positive x-axis. The energy of the radiation emitted by the charge per unit time and per unit solid angle is given by (in units wherein $c = 1$)
$$\left(\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t\,\mathrm{d}\Omega}\right) = \left(\frac{e^2\,a^2}{4\,\pi}\right)\,\left[\frac{1}{(1 - v\cos\theta)^6}\right]\,\sin^2\theta,$$
where θ is the angle between the direction of propagation of the emitted radiation and the x-axis. Draw the angular distribution of the emitted radiation (i.e. as a function of θ) when $v \ll 1$ and when $v \simeq 1$.

   Note: The effect when $v \simeq 1$ is known as relativistic beaming.

2. Consider a vector field, say, A^a, in two dimensionsal Euclidean space that is described by the line-element in terms of the polar coordinates ρ and ϕ as follows:
$$\mathrm{d}s^2 = \left(\mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\phi^2\right).$$
In the polar coordinates, the vector A^a has the components $(1, 0)$ at the point $\phi = 0$ on the unit circle about the origin. If the vector is *parallel transported* along the unit circle, determine the components of the vector (in the polar coordinates) when $\phi = (\pi/2)$ on the unit circle?

3. Calculate the Riemann tensor of a cylinder of constant radius, say, R, in three-dimensionsal Euclidean space. What does the result you find imply?

   Note: The surface of the cylinder is actually two-dimensionsal.

4. A scalar field of mass m satisfies the following Klein Gordon equation in a curved spacetime:
$$\left(\phi^{;a}_{\ ;a} + m^2\,\phi\right) = 0.$$
Show that this equation can be written as
$$\left[\left(\frac{1}{\sqrt{-g}}\right)\,\partial_a\left(\sqrt{-g}\,g^{ab}\,\partial_b\right)\,\phi + m^2\,\phi\right] = 0.$$

5. (a) Consider a relativistic particle that is moving with the four momentum p^μ in a given curved spacetime. If the spacetime admits a Killing vector ξ^μ, show that $(p_\mu\,\xi^\mu)$ is conserved (i.e. it is a constant) along a geodesic.

   (b) The spacetime around a cosmic string is described by the line-element
$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}\rho^2 - \alpha^2\,\rho^2\,\mathrm{d}\phi^2 - \mathrm{d}z^2,$$
where α is a constant that is called the deficit angle. List the components of the momentum of a relativistic particle on geodesic motion in this spacetime that are conserved.

   (c) Consider a particle of mass m that is moving along a time-like geodesic in the spacetime of a cosmic string. Using the relation $(p^\mu\,p_\mu) = m^2$ and the conserved momenta, obtain the (first order) differential equation for $(\mathrm{d}\rho/\mathrm{d}t)$ of the particle in terms of *all* the conserved components of its momenta.

6. The spacetime of a worm hole is described by the line-element
$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}r^2 - \left(b^2 + r^2\right)\,\left(\mathrm{d}\theta^2 + \sin^2\!\theta\,\mathrm{d}\phi^2\right),$$
where b is a constant with the dimensions of length that describes the size of the 'traversable' region of the worm hole. Show that the energy density of matter has to be negative to sustain such a spacetime.

Problem sheet 3

The field equations of general relativity and the Schwarzschild metric

1. Solve the Killing equation in flat spacetime and construct all the independent Killing vectors.

2. Recall that the Riemann tensor is defined as

$$R^a_{bcd} = (\partial_c\, \Gamma^a_{bd} - \partial_d\, \Gamma^a_{bc}) + (\Gamma^e_{bd}\, \Gamma^a_{ec} - \Gamma^e_{bc}\, \Gamma^a_{ed}),$$

while the Ricci tensor R_{ab} and the Ricci scalar R are given by

$$R_{ab} = R^c_{acb} \qquad \text{and} \qquad R = \left(g^{ab}\, R_{ab}\right).$$

Also, the Einstein tensor is defined as

$$G_{ab} = R_{ab} - \left(\frac{1}{2}\right) R\, g_{ab}.$$

   (a) Using the above expression for the Riemann tensor, establish the following Bianchi identity:

$$(\nabla_a\, R_{debc} + \nabla_c\, R_{deab} + \nabla_b\, R_{deca}) = 0.$$

   (b) Using this identity, show that

$$\left(\nabla_b\, G^b_a\right) = 0.$$

3. Show that the Euler-Lagrange equation corresponding to the integral

$$J = \int_{x_1}^{x_2} \mathrm{d}x\, f(y, y_x, y_{xx}, x),$$

where $y_x \equiv (\mathrm{d}y/\mathrm{d}x)$ and $y_{xx} \equiv \left(\mathrm{d}^2y/\mathrm{d}x^2\right)$ is given by

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial f}{\partial y_{xx}}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y_x}\right) + \left(\frac{\partial f}{\partial y}\right) = 0.$$

Note: In order to obtain this equation, the variation as well its first derivative need to be set to zero at the end points.

4. Establish the following relations:

$$\begin{aligned}\left(\frac{\partial\, g^{cd}}{\partial\, g_{ab}}\right) &= -\left(\frac{1}{2}\right)\left(g^{ac}\, g^{bd} + g^{ad}\, g^{bc}\right),\\ \left(\frac{\partial\, (\sqrt{-g}\, R)}{\partial\, g_{ab,cd}}\right) &= \sqrt{-g}\,\left[\left(\frac{1}{2}\right)\left(g^{ac}\, g^{bd} + g^{ad}\, g^{bc}\right) - g^{ab}\, g^{cd}\right].\end{aligned}$$

5. Use the variational principle to arrive at the equations of motion of a theory that is described by the following action:

$$S[g_{ab}] = \int d\Omega\, \sqrt{-g}\, \left(R^{abcd}\, R_{abcd}\right),$$

where $d\Omega$ is the volume element in a n-dimensional manifold, and R_{abcd} is the Riemann tensor.

6. Consider a generic scalar field ϕ that is described by the action

$$S[\phi] = \int \mathrm{d}\Omega\, \sqrt{-g}\; \mathcal{L}(X, \phi),$$

where $\mathrm{d}\Omega$ is again the volume element in a n-dimensional manifold, while X is a term that describes the kinetic energy of the scalar field, and is defined as

$$X = \left(\frac{1}{2}\right)\, \partial_c\phi\, \partial^c\phi,$$

Let the Lagrangian density $\mathcal{L}$ be an arbitrary function of the kinetic term X and the field ϕ. Vary the above action with respect to the metric tensor, and show that the corresponding stress-energy tensor can be written as

$$T^a_b = \left(\frac{\partial \mathcal{L}}{\partial X}\right)\, (\partial^a\phi\, \partial_b\phi) - \delta^a_b\, \mathcal{L}.$$

Note: Such scalar fields are often referred to as k-essence.

7. Consider the following line element that describes a spherically symmetric spacetimes in $(3+1)$-dimensions:

$$\mathrm{d}s^2 = e^{\Phi(t,r)}\, \mathrm{d}t^2 - e^{\Psi(t,r)}\, \mathrm{d}r^2 - r^2\, \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right),$$

where $\Phi(t,r)$ and $\Psi(t,r)$ are arbitrary functions of the coordinates t and r. Earlier, you would have arrived at the following expressions for the non-zero components of the Einstein tensor corresponding to this line-element:

$$\begin{aligned}
G^t_t &= \left[\left(\frac{\Psi'}{r}\right) - \left(\frac{1}{r^2}\right)\right]\, e^{-\Psi} + \left(\frac{1}{r^2}\right), \\
G^r_t &= -\left(\frac{\dot{\Psi}}{r\, e^{\Psi}}\right) = -G^t_r\, e^{(\Psi-\Phi)}, \\
G^r_r &= -\left[\left(\frac{\Phi'}{r}\right) + \left(\frac{1}{r^2}\right)\right]\, e^{-\Psi} + \left(\frac{1}{r^2}\right), \\
G^\theta_\theta &= G^\phi_\phi = \left(\frac{1}{2}\right)\, \left[\left(\frac{\Phi'\,\Psi'}{2}\right) + \left(\frac{\Psi'}{r}\right) - \left(\frac{\Phi'}{r}\right) - \left(\frac{\Phi'^2}{2}\right) - \Phi''\right]\, e^{-\Psi} \\
&\quad + \left(\frac{1}{2}\right)\, \left[\ddot{\Psi} + \left(\frac{\dot{\Psi}^2}{2}\right) - \left(\frac{\dot{\Phi}\,\dot{\Psi}}{2}\right)\right]\, e^{-\Phi},
\end{aligned}$$

where the dots and primes denote differentiation with respect to t and r, respectively. Show that the contracted Bianchi identities, viz. $(\nabla_b\, G^{ab}) = 0$, imply that the last of the above equations vanishes, if the remaining three equations vanish.

8. Consider the action

$$S[x(u)] = \int_{u_1}^{u_2} du\; K = \int_{u_1}^{u_2} du\; \left(\frac{1}{2}\right)\, \left(g_{ab}\, \dot{x}^a\, \dot{x}^b\right),$$

where u is an affine parameter, and the dots denote differentiation with respect to u. Show that the following Euler-Lagrange equation corresponding to this action:

$$\frac{d}{du}\left(\frac{\partial K}{\partial \dot{x}^a}\right) - \left(\frac{\partial K}{\partial x^a}\right) = 0$$

leads to the standard geodesic equation.

9. The Schwarzschild spacetime around an object of mass M is described by the line-element

$$\mathrm{d}s^2 = \left(1 - \frac{2\,M}{r}\right)\,\mathrm{d}t^2 - \left(1 - \frac{2\,M}{r}\right)^{-1}\,\mathrm{d}r^2 - r^2\,\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right).$$

   (a) Identify the Killing vectors and the corresponding conserved momenta in this spacetime.

   (b) Using the relation $(p^\mu\,p_\mu) = m^2$ and the conserved momenta, obtain the first order differential equation describing the orbital motion of massive as well as massless particles propagating in this spacetime.

10. Consider a particle that is falling *radially* onto the central mass M in the Schwarzschild spacetime.

   (a) Integrate the first order equations of motion of the particle, and show that the radial trajectory of the particle can be expressed in terms of the proper time τ as follows:

$$(\tau - \tau_0) = -\left(\frac{2}{3\,(2\,M)^{1/2}}\right)\,\left(r^{3/2} - r_0^{3/2}\right),$$

   where it has been assumed that the particle was located at $r = r_0$ at some initial proper time τ_0.

   (b) Integrate the first order equations of motion of the particle, and show that the radial trajectory of the particle can be expressed in terms of the coordinate time t as

$$\begin{aligned}(t - t_0) \quad = \quad & -\left(\frac{2}{3\,(2\,M)^{1/2}}\right)\,\left[\left(r^{3/2} - r_0^{3/2}\right) + (6\,M)\,\left(r^{1/2} - r_0^{1/2}\right)\right] \\ & + \,(2\,M)\,\ln\left(\frac{\left[r^{1/2} + (2\,M)^{1/2}\right]\,\left[r_0^{1/2} - (2\,M)^{1/2}\right]}{\left[r_0^{1/2} + (2\,M)^{1/2}\right]\,\left[r^{1/2} - (2\,M)^{1/2}\right]}\right),\end{aligned}$$

   where it has been assumed that the particle started at $r = r_0$ at some initial time t_0.

   (c) How much proper and coordinate times does the particle require to reach $r = (2\,M)$?

This sheet has to be returned to Dhiraj Hazra by 2:00 PM on October 31, 2009.

Miscellaneous problems I

From special relativity to the Schwarzschild metric

Special relativity

1. Consider two inertial frames that move with the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ with respect to, say, the laboratory frame. Show that the relative velocity v between the two frames can be expressed as

$$v^2 = \left[\frac{(\mathbf{v}_1 - \mathbf{v}_2)^2 - (\mathbf{v}_1 \times \mathbf{v}_2)^2}{(1 - \mathbf{v}_1 \cdot \mathbf{v}_2)^2}\right].$$

Tensor algebra and tensor calculus

2. Show that the determinant of the metric tensor, viz. $g = \mathrm{det.}\,(g_{ab})$, is *not* a scalar.

3. Show that, on a n-dimensional manifold, the number of independent components of the Riemann tensor R_{abcd} are $\left[\left(n^2/12\right)\,\left(n^2-1\right)\right]$.

4. Show that all two dimensional manifolds are conformally flat.

5. Show that, under the conformal transformation,

$$g_{ab}(x^c) \to \left[\Omega^2(x^c)\; g_{ab}(x^c)\right],$$

   the Christoffel symbols Γ^a_{bc}, the Ricci tensor R^a_b, and the scalar curvature R of a n-dimensional manifold are modified as follows:

$$\begin{aligned}
\Gamma^a_{bc} &\to \Gamma^a_{bc} + \Omega^{-1}\,\left(\delta^a_b\,\Omega_{;c} + \delta^a_c\,\Omega_{;b} - g_{bc}\,g^{ad}\,\Omega_{;d}\right),\\
R^a_b &\to \left(\Omega^{-2}\,R^a_b\right) - (n-2)\,\Omega^{-1}\,g^{ac}\,\left(\Omega^{-1}\right)_{;bc} + \left(\frac{1}{n-2}\right)\,\Omega^{-n}\,\delta^a_b\,g^{cd}\,\left[\Omega^{(n-2)}\right]_{;cd},\\
R &\to \left(\Omega^{-2}\,R\right) + 2\,(n-1)\,\Omega^{-3}\,\left(g^{ab}\,\Omega_{;ab}\right) + (n-1)\,(n-4)\,\Omega^{-4}\,\left(g^{ab}\,\Omega_{;a}\,\Omega_{;b}\right).
\end{aligned}$$

6. If ξ^a is a Killing vector, prove that

$$\xi_{a;bc} = \left(R_{dcba}\,\xi^d\right).$$

7. Show that any Killing vector field ξ^a satisfies the following equation:

$$\left(\xi^{a;b}{}_{;b} + R^a_c\,\xi^c\right) = 0.$$

The Einstein's equations

8. (a) Show that, in two dimensions, the Riemann tensor of an arbitrary manifold that is described by the metric g_{ab} can be expressed as follows:

$$R_{abcd} = K\,\left(g_{ac}\,g_{bd} - g_{ad}\,g_{bc}\right),$$

   where K is a scalar that is, in general, a function of the coordinates.

   (b) Using this result, show that the Einstein tensor vanishes *identically* in two dimensions.

   (c) Consider the following $(1+1)$-dimensional line-element:

$$\mathrm{d}s^2 = f(\eta,\xi)\,\left(\mathrm{d}\eta^2 - d\xi^2\right),$$

   where $f(\eta,\xi)$ is an arbitrary function of the coordinates η and ξ. Show that the scalar curvature associated with this line-element can be expressed as

$$R = -\left(\Box\,\ln f\right).$$

9. Consider a scalar field ϕ of mass m that is described by the action

$$S[\phi] = \int \mathrm{d}\Omega\, \sqrt{-g}\, \left[\left(\frac{1}{2}\right) \partial_c\phi\, \partial^c\phi - \left(m^2 + \xi\, R\right)\, \phi^2\right],$$

where $\mathrm{d}\Omega$ is the volume element in a n-dimensional manifold, R is the scalar curvature associated with the manifold, and ξ is an arbitrary constant.

(a) Show that the above action is invariant under the conformal transformation $g_{ab}(x^c) \to \left[\Omega^2(x^c)\, g_{ab}(x^c)\right]$, provided $m = 0$ (i.e. the field is massless), the scalar field transforms as

$$\phi \to \Omega^{[(2-n)/2]}\, \phi,$$

and ξ is given by

$$\xi = \left(\frac{n-2}{4\,(n-1)}\right).$$

Note: Such a scalar field is said to be conformally coupled to gravity.

(b) Vary the above action with respect to the scalar field and show that it satisfies the following equation of motion:

$$\left(\Box + m^2 + \xi\, R\right)\, \phi = 0.$$

(c) Vary the above action with respect to the metric tensor, and show that the corresponding stress-energy tensor can be written as

$$\begin{aligned} T_{ab} &= (1 - 2\,\xi)\, (\partial_a\phi\, \partial_b\phi) + \left(\frac{4\,\xi - 1}{2}\right) g_{ab}\, (\partial^c\phi\, \partial_c\phi) - 2\,\xi\, \phi\, \phi_{;ab} \\ &\quad + \left(\frac{2\,\xi}{n}\right) g_{ab}\, (\phi\, \Box\, \phi) - \xi\, \left[G_{ab} + \left(\frac{2\,(n-1)\,\xi}{n}\right) R\, g_{ab}\right] \phi^2 \\ &\quad + \left[\left(\frac{1}{2}\right) - \left(\frac{2\,(n-1)\,\xi}{n}\right)\right] g_{ab}\, \left(m^2\, \phi^2\right). \end{aligned}$$

Black holes

10. Use the given `Mathematica` file to evaluate the metric connections, the Riemann, the Ricci, and the Einstein tensors as well as the Ricci scalar around the Reissner-Nordstrom and the Kerr black holes that are described by the following line-elements:

$$\mathrm{d}s^2 = \left(1 - \frac{2\,M}{r} + \frac{Q^2}{r^2}\right) \mathrm{d}t^2 - \left(1 - \frac{2\,M}{r} + \frac{Q^2}{r^2}\right)^{-1} \mathrm{d}r^2 - r^2\, \left(\mathrm{d}\theta^2 + \sin^2\!\theta\, \mathrm{d}\phi^2\right)$$

and

$$\mathrm{d}s^2 = \left(\frac{\Delta}{\rho^2}\right) \left(\mathrm{d}t - a\, \sin^2\theta\, \mathrm{d}\phi\right)^2 - \left(\frac{\sin^2\theta}{\rho^2}\right) \left[\left(r^2 + a^2\right)\, \mathrm{d}\phi - a\, \mathrm{d}t\right]^2 - \left(\frac{\rho^2}{\Delta}\right) \mathrm{d}r^2 - \rho^2\, \mathrm{d}\theta^2,$$

where

$$\Delta = \left[r^2 - (2\,M\,r) + a^2\right], \quad \rho^2 = \left(r^2 + a^2 \cos^2\theta\right), \quad \text{and} \quad a = (J/M).$$

The quantities M, Q and J are constants that denote the mass, the electric charge and the angular momentum associated with the black holes, respectively.

Problem sheet 4

The Friedmann model

1. Recall that the Friedmann universe is described by the line-element

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left[\left(\frac{\mathrm{d}r^2}{1-\kappa\, r^2}\right) + r^2\,\left(\mathrm{d}\theta^2 + \sin^2\!\theta\,\mathrm{d}\phi^2\right)\right],$$

where $\kappa = 0, \pm 1$.

Arrive at the following expressions for the Ricci tensor R^μ_ν, the scalar curvature R, and the Einstein tensor G^μ_ν for the above Friedmann metric:

$$\begin{aligned}
R^t_t &= -3\left(\frac{\ddot{a}}{a}\right),\\
R^i_j &= -\left[\left(\frac{\ddot{a}}{a}\right) + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\left(\frac{\kappa}{a^2}\right)\right]\delta^i_j,\\
R &= -6\left[\left(\frac{\ddot{a}}{a}\right) + \left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{\kappa}{a^2}\right)\right],\\
G^t_t &= 3\left[\left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{\kappa}{a^2}\right)\right],\\
G^i_j &= \left[2\left(\frac{\ddot{a}}{a}\right) + \left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{\kappa}{a^2}\right)\right]\delta^i_j,
\end{aligned}$$

where the overdots denote differentiation with respect to the cosmic time t.

2. Evaluate the three-curvature associated with the spatial surface of the Friedmann metric.

3. (a) In n-dimensions, the Weyl tensor C_{abcd} is defined as follows:

$$\begin{aligned}
C_{abcd} &= R_{abcd} + \left(\frac{1}{n-2}\right)\left(g_{ad}\,R_{cb} + g_{bc}\,R_{da} - g_{ac}\,R_{db} - g_{bd}\,R_{ca}\right)\\
&\quad + \left(\frac{1}{(n-1)\,(n-2)}\right)\left(g_{ac}\,g_{db} - g_{ad}\,g_{cb}\right)\,R.
\end{aligned}$$

   Show that the Weyl tensor vanishes for the above Friedmann metric.

   (b) Since the Weyl tensor vanishes, the metric ought to be conformally flat. Construct the coordinate systems in which the metrics corresponding to $\kappa = 0, \pm 1$ can be expressed in such a form.

4. Show that, in a Friedmann universe, the time component of the following stress-energy tensor conservation law

$$T^\mu_{\nu;\mu} = 0$$

leads to the equation

$$\dot{\rho} + (3\,H)\,(\rho + p) = 0.$$

5. (a) The Cosmic Microwave Background Radiation (CMBR) is considered to be the dominant contribution to the relativistic energy density in the universe. Given that the temperature of the CMBR today is $T \simeq 2.73\,\mathrm{K}$, show that

$$\left(\Omega_{\mathrm{R}}\,h^2\right) \simeq 2.56\times 10^{-5},$$

   where h is related to the Hubble constant H_0 as follows:

$$H_0 \simeq 100\,h\ \mathrm{km\,s^{-1}\,Mpc^{-1}}.$$

(b) i. Show that the redshift $z_{\rm eq}$ at which the energy density of matter and radiation were equal is given by

$$(1+z_{\rm eq}) = \left(\frac{\Omega_{\rm NR}}{\Omega_{\rm R}}\right) \simeq 3.9\times 10^4\,\left(\Omega_{\rm NR}\,h^2\right).$$

ii. Also, show that the temperature of the radiation at this epoch is given by

$$T_{\rm eq} \simeq 9.24\,\left(\Omega_{\rm NR}\,h^2\right)\ {\rm eV}.$$

(c) Given that $H_0 \simeq 72$ km s^{-1} Mpc^{-1}, estimate the numerical value of the critical density $\rho_{\rm c}$ today.

6. Show that, for a universe dominated by non-relativistic matter, the Hubble radius and the luminosity distance can be expressed in terms of the red-shift as follows:

$$\begin{aligned} d_{\rm H}(z) &= \left[H_0\,(1+z)\,(1+\Omega_{\rm NR}\,z)^{1/2}\right]^{-1}, \\ d_{\rm L}(z) &= \left(\frac{2}{H_0\,\Omega_{\rm NR}^2}\right)\left(\Omega_{\rm NR}\,z + (\Omega_{\rm NR}-2)\,\left[(1+\Omega_{\rm NR}\,z)^{1/2}-1\right]\right). \end{aligned}$$

7. (a) Integrate the first Friedmann equation for a $\kappa = 0$ universe with matter and radiation to obtain that

$$a(\eta) = \sqrt{\Omega_{\rm R}\,a_0^4}\,\left(H_0\,\eta\right) + \left(\frac{\Omega_{\rm NR}\,a_0^3}{4}\right)\,\left(H_0\,\eta\right)^2,$$

where η is the conformal time coordinate.

Note: In obtaining the above result, it has been assumed that $a=0$ at $\eta=0$.

(b) Integrate the Friedmann equation for a $\kappa = 0$ universe with matter and cosmological constant to obtain that

$$\left(\frac{a(t)}{a_0}\right) = \left(\frac{\Omega_{\rm NR}}{\Omega_\Lambda}\right)\,\sinh^{2/3}\left(\frac{3\,\Omega_\Lambda^{3/2}\,H_0\,t}{2\,\Omega_{\rm NR}}\right).$$

This sheet has to be returned to Dhiraj Hazra by 2:00 PM on December 4, 2009.

Miscellaneous problems II

The Friedmann metric and gravitational waves

The Friedmann model

1. Metric of spacetimes of constant curvature (ESU, dS and AdS).
2. Warp drive spacetime: Hartle, Example 7.4 and problem 22.14.
3. Geodesic equation from the Einstein equations...

Gravitational waves

4. ....

End-of-semester exam

From special relativity to gravitational waves

1. One can introduce a metric on the velocity space of a particle by defining the distance between two nearby velocities as their relative velocity. Show that the metric can be written in the form

$$\mathrm{d}s^2 = \mathrm{d}\chi^2 + \sinh^2\chi\, \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right),$$

   where the magnitude of the velocity is defined as $v = \tanh\chi$.

   Hint: Use the expression for the relative velocity in Problem 1 of Miscellaneous Problems I.

2. A frame is rotating with an angular velocity Ω about the z-axis in Minkowski spacetime. Construct the line-element in the rotating frame.

3. A space purports to be three dimensional, with coordinates x, y and z, and the metric

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - \left[\left(\frac{3}{13}\right)\mathrm{d}x + \left(\frac{4}{13}\right)\mathrm{d}y + \left(\frac{12}{13}\right)\mathrm{d}z\right]^2.$$

   Show that it is actually a two dimensional space.

4. (a) In a curved spacetime, the electromagnetic field tensor is defined as

$$F_{\mu\nu} = \left(A_{\nu;\mu} - A_{\mu;\nu}\right).$$

   Vary the electromagnetic action with the standard interaction term in a curved spacetime, and arrive at the following Maxwell's equations with the source term:

$$\left(\frac{1}{\sqrt{-g}}\right)\partial_\nu\left(\sqrt{-g}\, F^{\mu\nu}\right) = -(4\,\pi)\, j^\mu.$$

   (b) What are the source free Maxwell's equations in a curved spacetime?

5. Show that, in arbitrary dimensions, a space is conformally flat, if the Riemann tensor can be expressed in terms of the metric g_{ab} as follows:

$$R_{abcd} = K\, \left(g_{ac}\, g_{bd} - g_{ad}\, g_{bc}\right),$$

   where K is a quantity that is a function of the coordinates.

6. Let the equation of state describing the matter that is driving the expansion of the Friedmann universe be $p = -\rho$.

   (a) Solve the Friedmann equations for the spatially closed (i.e. $\kappa = 1$) and flat ($\kappa = 0$) cases, and arrive at the resulting line elements.

   (b) Do the two line-elements describe the same spacetime?

7. Recall that the stress-energy tensor of a perfect fluid is given by

$$T^\mu_\nu = (\rho + p)\, u^\mu\, u_\nu - p\, \delta^\mu_\nu,$$

   where ρ, p and u^μ are the energy density, pressure and the four velocity of the perfect fluid, respectively. Using the conservation of the stress-energy tensor, arrive at the following equations that describe the evolution of the relativistic fluid in a curved spacetime:

$$\begin{aligned}(\rho\, u^\mu)_{;\mu} + p\, u^\mu{}_{;\mu} &= 0,\\ (\rho + p)\, u^\mu\, u_{\nu;\mu} &= \left(\delta^\mu_\nu - u^\mu\, u_\nu\right) p_{;\mu}.\end{aligned}$$

8. Consider the metric

$$\mathrm{d}s^2 = [1 + 2\,\Phi(t, \mathbf{x})]\;\mathrm{d}t^2 - [1 - 2\,\Phi(t, \mathbf{x})]\;\mathrm{d}\mathbf{x}^2,$$

where the quantity Φ describes a *small* perturbation about the Minkowski metric. If the stress-energy tensor of the matter field that is causing the perturbation is given by $T^{\mu}_{\nu} = \mathrm{diag.}\,[\rho(\mathbf{x}), 0, 0, 0]$, using the Einstein's equations, obtain the equation of motion describing the quantity Φ, *at the first order in the perturbation*.

Hint: When, say, evaluating the inverse metric, work *only* upto the first order in the perturbation, as we had done when we had arrived at the equation of motion describing the gravitational waves.