PH5875

ADVANCED GENERAL RELATIVITY

January–May 2022

Lecture schedule and meeting hours

- The course will consist of about 42 lectures, including about 8–10 tutorial sessions. To the extent possible, I will try to integrate the tutorial sessions with the lectures. In case I am unable to complete the discussions within the lectures, I will request my teaching assistant to conduct a few separate tutorial sessions.
- The duration of each lecture will be 50 minutes. We will be meeting on Google Meet. I will share the link over smail.
- The first lecture will be on Wednesday, January 19, and the last one will be on Wednesday, April 27.
- We will meet thrice a week. We shall meet during the following hours: 11:00–11:50 AM on Wednesdays, 9:00–9:50 AM on Thursdays, and 8:00–8:50 AM on Fridays.
- We may also meet during 5:00–5:50 PM on Tuesdays for either the quizzes or to make up for lectures that I may have to miss due to other unavoidable commitments. Changes in schedule, if any, will be notified sufficiently in advance.
- If you would like to discuss with me about the course outside the lecture hours, please send me an e-mail at `sriram@physics.iitm.ac.in`. We can converge on a mutually convenient time to meet and discuss online. I would request you to write to me from your smail addresses with the subject line containing the name of the course, i.e. PH5875: Advanced General Relativity.

Information about the course

- All the information regarding the course such as the schedule of the lectures, the structure and the syllabus of the course, suitable textbooks and additional references will be available on the course's page on Moodle at the following URL:

  `https://courses.iitm.ac.in/`
- The exercise sheets and other additional material will also be made available on Moodle.
- A PDF file containing these information as well as completed quizzes will also be available at the link on this course at the following URL:

  `http://physics.iitm.ac.in/~sriram/professional/teaching/teaching.html`

  I will keep updating this file and the course's page on Moodle as we make progress.

Quizzes, end-of-semester exam and grading

- The grading will be based on three scheduled quizzes and an end-of-semester exam.
- I will consider the best two quizzes for grading, and the two will carry 25% weight each.
- The three quizzes will be held on February 18, April 12 and April 26. The first of these three dates is a Friday and the remaining two are Tuesdays. The quizzes will be held during 5:00–6:30 PM on these dates.
- The end-of-semester exam will be held during 9:00 AM–12:00 NOON on Thursday, May 12, and the exam will carry 50% weight.

Syllabus and structure

1. **Describing curved spacetimes** [$\sim$ 4 lectures]

   (a) Manifolds and coordinates – The metric
   (b) Covariant differentiation and the affine connection – Geodesics
   (c) Isometries – The Killing equation and conserved quantities
   (d) The Riemann tensor – The equation of geodesic deviation
   (e) The curvature and the Weyl tensors

Exercise sheets 1, 2, and 3

2. **Field equations of general relativity** [$\sim$ 6 lectures]

   (a) The equivalence principle – The principle of general covariance – The principle of minimal gravitational coupling
   (b) The vacuum Einstein's equations
   (c) Derivation of vacuum Einstein's equations from the action – The Bianchi identities
   (d) The stress-energy tensor – The cases of perfect fluid, scalar and electromagnetic fields
   (e) Non-canonical scalar fields – Relation to relativistic fluids
   (f) The structure of the Einstein's equations

Exercise sheets 4 and 5

Quiz I

3. **Schwarzschild geometry and tests of general relativity** [$\sim$ 6 lectures]

   (a) The general static isotropic metric – The Schwarzschild solution
   (b) Motion of particles and photons in the Schwarzschild metric
   (c) Precession of the perihelion of Mercury – Bending of light – Gravitational redshift

Exercise sheets 6 and 7

Additional exercises I

4. **Static and stationary black holes** [$\sim$ 6 lectures]

   (a) Schwarzschild black hole – Event horizon – Singularities
   (b) The Kruskal extension – Penrose diagrams
   (c) The Reissner-Nordstrom solution
   (d) The Kerr solution – Frame dragging — Ergosphere – Penrose process
   (e) Black hole thermodynamics – Hawking radiation

Exercise sheets 8 and 9

Quiz II

5. **Cosmology** [$\sim$ 12 lectures]

   (a) Homogeneity and isotropy – The Friedmann-Lemaître-Robertson-Walker line-element
   (b) Geodesics – Cosmological red-shift – Luminosity and angular diameter distances
   (c) Friedmann equations – Solutions with different types of matter

(d) Cosmological parameters – Age of the universe – Supernovae and late time acceleration

(e) Cosmic microwave background radiation – Thermal history – Big bang nucleosynthesis

(f) Horizon problem – Inflationary scenario – Generation of perturbations in the early universe

(g) Evolution of perturbations – Anisotropies in the cosmic microwave background – Recent constraints

Exercise sheets 10, 11 and 12

Additional exercises II

Quiz III

6. **Gravitational waves** [$\sim$ 8 lectures]

(a) Linearized Einstein's equations – Transverse-traceless gauge – Solutions to the wave equation

(b) Polarization of gravitational waves – Effects of gravitational waves on a ring of masses

(c) Generation of gravitational waves – The quadrupole formula for the energy loss – Hulse-Taylor binary pulsar

(d) Gravitational waves from binary systems – Interferometric detectors – Detection of gravitational waves

(e) Observed events – Gravitational wave astronomy – Implications for astrophysics of compact sources and theories of gravity

(f) The stochastic gravitational wave background – Current constraints – Implications for the physics of the early universe

Exercise sheets 13, 14 and 15

End-of-semester exam

Advanced problems

Basic textbooks

1. L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Course of Theoretical Physics, Volume 2), Fourth Edition (Pergamon Press, New York, 1975).
2. B. F. Schutz, *A First Course in General Relativity* (Cambridge University Press, Cambridge, 1990).
3. E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley, Redwood City, California, 1990).
4. R. d'Inverno, *Introducing Einstein's Relativity* (Oxford University Press, Oxford, 1992).
5. J. B. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (Pearson Education, Delhi, 2003).
6. S. Carroll, *Spacetime and Geometry* (Addison Wesley, New York, 2004).
7. M. P. Hobson, G. P. Efstathiou and A. N. Lasenby, *General Relativity: An Introduction for Physicists* (Cambridge University Press, Cambridge, 2006).
8. S. Weinberg, *Cosmology* (Oxford University Press, Oxford, England, 2008).

Additional references

1. S. Weinberg, *Gravitation and Cosmology* (John Wiley, New York, 1972).
2. A. P. Lightman, W. H. Press, R. H. Price and S. A. Teukolsky, *Problem Book in Relativity and Gravitation* (Princeton University Press, New Jersey, 1975).
3. S. Dodelson, *Modern Cosmology* (Academic Press, SanDiego, U.S.A., 2003).
4. V. F. Mukhanov, *Physical Foundations of Cosmology* (Cambridge University Press, Cambridge, England, 2005).
5. M. Maggiore, *Gravitational Waves: Volume 1: Theory and Experiments* (Oxford University Press, Oxford, England, 2007).
6. T. Padmanabhan, *Gravitation: Foundation and Frontiers* (Cambridge University Press, Cambridge, 2010).
7. A. Zee, *Einstein Gravity in a Nutshell* (Princeton University Press, Princeton, New Jersey, 2013).

Advanced texts

1. S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge University Press, Cambridge, 1973).
2. C. W. Misner, K. S. Thorne and J. W. Wheeler, *Gravitation* (W. H. Freeman and Company, San Francisco, 1973).
3. R. M. Wald, *General Relativity* (The University of Chicago Press, Chicago, 1984).
4. E. Poisson, *A Relativist's Toolkit* (Cambridge University Press, Cambridge, 2004).
5. M. Maggiore, *Gravitational Waves: Volume 2: Astrophysics and Cosmology* (Oxford University Press, Oxford, England, 2018).

Exercise sheet 1

Covariant differentiation, Christoffel symbols, geodesics and Killing vectors

1. *Geodesics on a plane:* Working in the polar coordinates, arrive at the geodesic equations on the plane. Solve the equations to show that the geodesics are straight lines.

   Note: You are expected to solve the geodesic equation involving the Christoffel symbols, and not the more familiar variation of the problem!

2. *Geodesics on a cone:* Consider a cone with a semi-vertical angle α.

   (a) Determine the line element on the cone.

   (b) Obtain the equations governing the geodesics on the cone.

   (c) Solve the equations to arrive at the geodesics.

3. *Geodesics in a Poincaré half plane:* Consider the so-called Poincaré half plane described by the line-element

   $$\mathrm{d}l^2 = \frac{a^2}{y^2}\,\left(\mathrm{d}x^2 + \mathrm{d}y^2\right),$$

   where $-\infty < x < \infty$, while $0 < y < \infty$. Determine the trajectory $y(x)$ of geodesics in this geometry.

4. *Energy of photons in a Friedmann universe:* The spatially flat Friedmann universe is described by the line element

   $$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\,\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right),$$

   where $a(t)$ is known as the scale factor that characterizes the expansion of the universe.

   (a) Obtain the Christoffel symbols corresponding to this line element.

   (b) Explicitly write down the time and the spatial components of the geodesic equation governing a photon in the spatially flat Friedmann universe.

   (c) Solve the geodesic equation and show that the energy, say, E, of the photon behaves as $E \propto 1/a$.

   Note: Recall that the time component of the four momentum of a particle represents its energy. The above result implies that the energy of photons constantly decreases as the universe expands, a phenomenon that is known as cosmological redshift.

5. *Killing vectors of a plane in polar coordinates:* Consider the two dimensional Euclidean plane described in terms of the polar coordinates.

   (a) What is the line element of the Euclidean plane in terms of the polar coordinates?

   (b) Evaluate all the Christoffel symbols associated with the line element.

   (c) Write down the equations describing the Killing vectors in the polar coordinates.

   (d) Obtain all the Killing vectors by solving the equations and interpret the solutions.

Exercise sheet 2

Curvature and geodesic deviation

1. *Rewriting the Riemann tensor:* Recall that, the Riemann tensor is defined as

$$R_{abcd} = g_{ae}\, R^{e}_{\ bcd} = g_{ae}\, \left(\Gamma^{e}_{bd,c} - \Gamma^{e}_{bc,d} + \Gamma^{e}_{fc}\,\Gamma^{f}_{bd} - \Gamma^{e}_{fd}\,\Gamma^{f}_{bc}\right).$$

Show that this can be rewritten as

$$R_{abcd} = \frac{1}{2}\,\left(g_{ad,bc} + g_{bc,ad} - g_{ac,bd} - g_{bd,ac}\right) + g_{ef}\,\left(\Gamma^{e}_{bc}\,\Gamma^{f}_{ad} - \Gamma^{e}_{bd}\,\Gamma^{f}_{ac}\right),$$

an expression which reflects the symmetries of the Riemann tensor more easily.

2. *Is the spacetime curved?* Recall that the FLRW universe is described by the line-element

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\,\left[\frac{\mathrm{d}r^2}{1-\kappa\, r^2} + r^2\,\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right],$$

where the function $a(t)$ is referred to as the scale factor and $\kappa = 0, \pm 1$.

   (a) Consider the case wherein $a(t) = c\,t$ and $\kappa = -1$. What does the metric describe? Is it a curved spacetime?

   Note: Recall that, for instance, the non-zero components of the Ricci tensor and scalar curvature are given by

$$\begin{aligned} R^{0}_{0} &= -\frac{3\,\ddot{a}}{c^2\,a}, \\ R^{i}_{j} &= -\left[\frac{\ddot{a}}{c^2\,a} + 2\left(\frac{\dot{a}}{c\,a}\right)^2 + \frac{2\,\kappa}{a^2}\right]\,\delta^{i}_{j}, \\ R &= -6\left[\frac{\ddot{a}}{c^2\,a} + \left(\frac{\dot{a}}{c\,a}\right)^2 + \frac{\kappa}{a^2}\right]. \end{aligned}$$

   (b) Can you construct a coordinate transformation that reduces the FLRW line-element with $a(t) = c\,t$ and $\kappa = -1$ to the Minkowskian form?

3. *Geodesic deviation:* Consider two nearby geodesics, say, $x^a(\lambda)$ and $\bar{x}^a(\lambda)$, where λ is an affine parameter. Let $\xi^a(\lambda)$ denote a 'small vector' that connects these two geodesics. Working in the locally geodesic coordinates, show that ξ^a satisfies the differential equation

$$\frac{\mathrm{D}^2\xi^a}{\mathrm{D}\lambda^2} + R^{a}_{bcd}\,\dot{x}^b\,\xi^c\,\dot{x}^d = 0,$$

where

$$\frac{\mathrm{D}^2\xi^a}{\mathrm{D}\lambda^2} \equiv \left(\dot{\xi}^a + \Gamma^{a}_{bc}\,\xi^b\,\dot{x}^c\right)^{\boldsymbol{\cdot}},$$

while the overdots denote differentiation with respect to λ.

Note: This implies that a non-zero Riemann tensor R^{a}_{bcd} will lead to a situation where geodesics, in general, will not remain parallel as, for instance, on the surface of the two sphere $\mathbb{S}^2$.

4. *Scalar curvature in two dimensions:* Consider the following $(1+1)$-dimensional line element:

$$\mathrm{d}s^2 = f^2(\eta,\xi)\,\left(\mathrm{d}\eta^2 - d\xi^2\right),$$

where $f(\eta, \xi)$ is an arbitrary function of the coordinates η and ξ. Show that the scalar curvature associated with this line-element can be expressed as

$$R = -\nabla_\mu \nabla^\mu \ln f^2 = -\Box \ln f^2.$$

Note: In $(1+1)$-dimensions, any metric can be reduced to the above, so-called conformally flat form.

5. *Conformal transformations:* Show that, under the conformal transformation,

$$g_{ab}(x^c) \to \Omega^2(x^c)\, g_{ab}(x^c),$$

the Christoffel symbols Γ^a_{bc}, the Ricci tensor R^a_b, and the scalar curvature R of a n-dimensional manifold are modified as follows:

$$\begin{aligned}
\Gamma^a_{bc} &\to \Gamma^a_{bc} + \Omega^{-1} \left(\delta^a_b\, \Omega_{;c} + \delta^a_c\, \Omega_{;b} - g_{bc}\, g^{ad}\, \Omega_{;d}\right), \\
R^a_b &\to \Omega^{-2} R^a_b - (n-2)\, \Omega^{-1}\, g^{ac}\, \left(\Omega^{-1}\right)_{;bc} + \frac{1}{n-2}\, \Omega^{-n}\, \delta^a_b\, g^{cd}\, \left[\Omega^{(n-2)}\right]_{;cd}, \\
R &\to \Omega^{-2} R + 2\,(n-1)\, \Omega^{-3}\, g^{ab}\, \Omega_{;ab} + (n-1)\,(n-4)\, \Omega^{-4}\, g^{ab}\, \Omega_{;a}\, \Omega_{;b}.
\end{aligned}$$

Exercise sheet 3

Fields in curved spacetime

1. *Klein-Gordon equation in a curved spacetime:* Consider the following action that describes a scalar field, say, ϕ, in a generic curved spacetime:

$$S[\phi] = \frac{1}{c}\int \mathrm{d}^4\tilde{x}\,\sqrt{-g}\,\left(\frac{1}{2}\,g_{\mu\nu}\,\partial^\mu\phi\,\partial^\nu\phi - \frac{1}{2}\,\sigma^2\,\phi^2\right),$$

where $g_{\mu\nu}$ is the metric tensor that describes the curved spacetime, while the quantity σ is related to the mass of the field. Also, the quantity g denotes the determinant of the metric tensor $g_{\mu\nu}$.

   (a) Vary the above action to arrive at the equation of motion for the scalar field.

   (b) Show that equation of motion of the scalar field can be written as

$$\nabla_\mu\nabla^\mu\phi + \sigma^2\phi \equiv \phi^{;\mu}{}_{;\mu} + \sigma^2\,\phi = 0.$$

2. *Tachyons:* Consider a scalar field T that is described by the action

$$S[T] = -\frac{1}{c}\int \mathrm{d}^4\tilde{x}\,\sqrt{-g}\,V(T)\,\sqrt{1-\alpha^2\,\partial_\mu T\,\partial^\mu T},$$

where α is a constant of suitable dimensions.

Note: The field T is often referred to as the tachyon.

   (a) Vary the action with respect to the scalar field T to arrive at the equation of motion governing the field in a curved spacetime.

   (b) Construct the stress-energy tensor associated with the field T.

   (c) Show that the conservation of the stress-energy tensor leads to the equation of motion governing the field T.

3. *Generic scalar fields:* Consider a generic scalar field ϕ that is described by the action

$$S[\phi] = \frac{1}{c}\int \mathrm{d}^4\tilde{x}\,\sqrt{-g}\,\mathcal{L}(X,\phi),$$

where X denotes the kinetic energy of the scalar field and is given by

$$X = \frac{1}{2}\,g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi.$$

   (a) Let the Lagrangian density $\mathcal{L}$ be an arbitrary function of the kinetic term X and the field ϕ. Vary the above action with respect to the metric tensor and obtain the corresponding stress-energy tensor.

   Note: Such scalar fields are often referred to as k-essence.

   (b) Assuming $\mathcal{L} = X - V(\phi)$, where $V(\phi)$ is the potential describing the scalar field, determine the corresponding stress-energy tensor. From the conservation of the stress-energy tensor, arrive at the equation of motion governing the scalar field for the case wherein $V(\phi) = \sigma^2\,\phi^2/2$.

4. *Maxwell's equations in curved spacetime:* Typically, the equations governing fields in a curved spacetime can be arrived at by replacing the partial derivatives encountered in the Minkowski spacetime by the corresponding covariant derivatives.

   (a) Show that

$$F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu} = A_{\nu,\mu} - A_{\mu,\nu}.$$

(b) Establish that the first pair of Maxwell's equations in a curved spacetime, viz.

$$F_{\mu\nu;\lambda} + F_{\lambda\mu;\nu} + F_{\nu\lambda;\mu} = 0,$$

actually reduce to

$$F_{\mu\nu,\lambda} + F_{\lambda\mu,\nu} + F_{\nu\lambda,\mu} = 0.$$

(c) Show that the second pair of Maxwell's equations in a curved spacetime, viz.

$$F^{\mu\nu}{}_{;\nu} = \frac{4\,\pi}{c}\, j^{\mu},$$

can be written as

$$\frac{1}{\sqrt{-g}}\,\partial_{\nu}\left(\sqrt{-g}\;F^{\mu\nu}\right) = \frac{4\,\pi}{c}\, j^{\mu}.$$

5. *Conformal invariance of the electromagnetic action:* Recall that, in a curved spacetime, the dynamics of the source free electromagnetic field is governed by the action

$$S[A^{\mu}] = -\frac{1}{16\,\pi\,c}\int \mathrm{d}^4\tilde{x}\;\sqrt{-g}\;F_{\mu\nu}\,F^{\mu\nu},$$

where

$$F_{\mu\nu} = A_{\mu;\nu} - A_{\nu;\mu} = A_{\mu,\nu} - A_{\nu,\mu},$$

while the commas and semi-colons, as usual, represent partial and covariant differentiation, respectively. Show that this action is invariant under the following conformal transformation:

$$x^{\mu} \to x^{\mu}, \quad A_{\mu} \to A_{\mu} \quad \text{and} \quad g_{\mu\nu} \to \Omega^2\, g_{\mu\nu}.$$

Exercise sheet 4

Einstein's equations

1. _The Palatini relation:_ Show that

$$\delta R^{a}{}_{bcd} = \nabla_c\left(\delta\Gamma^{a}_{bd}\right) - \nabla_d\left(\delta\Gamma^{a}_{bc}\right)$$

and, hence,

$$\delta R_{ab} = \nabla_c\left(\delta\Gamma^{c}_{ab}\right) - \nabla_b\left(\delta\Gamma^{c}_{ac}\right).$$

2. _Einstein tensor:_ Recall that we can write

$$\sqrt{-g}\,R = \sqrt{-g}\,\left[G + \frac{1}{\sqrt{-g}}\,\partial_\alpha\left(\sqrt{-g}\,w^\alpha\right)\right],$$

where

$$G = g^{ab}\left(\Gamma^{c}_{ad}\,\Gamma^{d}_{bc} - \Gamma^{c}_{cd}\,\Gamma^{d}_{ab}\right) \quad \text{and} \quad w^a = g^{bc}\,\Gamma^{a}_{bc} - g^{ab}\,\Gamma^{c}_{bc}.$$

Show that the Einstein tensor can be written as

$$G_{ab} = R_{ab} - \frac{1}{2}\,g_{ab}\,R = \frac{1}{\sqrt{-g}}\left\{\frac{\partial(\sqrt{-g}\,G)}{\partial g^{ab}} - \partial_c\left[\frac{\partial(\sqrt{-g}\,G)}{\partial\left(\partial_c g^{ab}\right)}\right]\right\}.$$

3. _Newtonian limit and Poisson equation:_ Recall that, in the non-relativistic limit, the metric corresponding to the Newtonian potential ϕ is given by

$$\mathrm{d}s^2 = c^2\left[1 + \frac{2\,\phi(\boldsymbol{x})}{c^2}\right]\mathrm{d}t^2 - \mathrm{d}\boldsymbol{x}^2.$$

Let the energy density of the matter field that is giving rise to the Newtonian potential ϕ be $\rho\,c^2$. Show that, in such a case, the time-time component of the Einstein's equations reduces to the conventional Poisson equation in the limit of large c.

Note: It is this Newtonian limit that determines the overall constant in the Einstein-Hilbert action.

4. _The Bianchi identity:_ Recall that, the Riemann tensor is defined as

$$R_{abcd} = g_{ae}\,R^{e}{}_{bcd} = g_{ae}\,\left(\Gamma^{e}_{bd,c} - \Gamma^{e}_{bc,d} + \Gamma^{e}_{fc}\,\Gamma^{f}_{bd} - \Gamma^{e}_{fd}\,\Gamma^{f}_{bc}\right).$$

Also, note that, given the Riemann tensor $R^{a}{}_{bcd}$, the Ricci tensor R_{ab} and the Ricci scalar R are defined as

$$R_{ab} = R^{c}{}_{acb} \qquad \text{and} \qquad R = g^{ab}\,R_{ab}.$$

Further, the Einstein tensor is given by

$$G_{ab} = R_{ab} - \frac{1}{2}\,R\,g_{ab}.$$

(a) Using the expression for the Riemann tensor, establish the following Bianchi identity:

$$\nabla_e\,R_{abcd} + \nabla_d\,R_{abec} + \nabla_c\,R_{abde} = 0.$$

Note: It will be convenient to work in the so-called local coordinates, where the Christoffel symbols vanish, but their derivatives do not.

(b) Using the above identity, show that

$$\nabla_b\,G^{b}_{a} = 0.$$

(c) Using the Bianchi identity, show that the covariant derivative of the Ricci tensor R^a_b can be expressed in terms of the partial derivative of the Ricci scalar R as follows:

$$\nabla_a R^a_b = \frac{1}{2}\,\partial_b R.$$

5. *Structure of the Einstein's equations:* Establish the following properties of the Einstein's equations.

(a) Show that the G^0_0 and G^0_i (i.e. the time-time and time-space) components of the Einstein tensor do not depend on $\dot{g}_{00}$ and $\dot{g}_{0i}$, where the overdots denote differentiation with respect to time. Also, show that these components depend only on $\dot{g}_{ij}$.

(b) Moreover, show that it is only the G^i_j (i.e. the space-space) components of the Einstein tensor that depend on $\ddot{g}_{ij}$.

Note: These imply that the time-time and time-space components of the Einstein's equations are constraints and it is the spatial components of the equations that govern the dynamics of the metric components g_{ij}.

Exercise sheet 5

Stress-energy tensor

1. *Conservation of the stress-energy tensor:* Let the action describing a matter field be given by

$$S = \frac{1}{c}\int \mathrm{d}^4\tilde{x}\,\sqrt{-g}\,\mathcal{L},$$

where $\mathcal{L}$ is the Lagrangian density describing the matter field. Consider the following variation of the spacetimes coordinates: $x^\mu \to x'^\mu = x^\mu + \xi^\mu$, where ξ^μ is an infinitesimal quantity.

   (a) Show that, under such a transformation, the contravariant and covariant metric tensors transform as follows:

$$g^{\mu\nu} \to g'^{\mu\nu} = g^{\mu\nu} + \delta g^{\mu\nu} \quad \text{and} \quad g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu},$$

   where $\delta g^{\mu\nu}$ and $\delta g_{\mu\nu}$ are given by

$$\delta g^{\mu\nu} = \xi^{\mu;\nu} + \xi^{\nu;\mu} \quad \text{and} \quad \delta g_{\mu\nu} = -\xi_{\mu;\nu} - \xi_{\nu;\mu}.$$

   (b) Show that, under such a variation, the corresponding variation in the action can be written as

$$\delta S = -\frac{1}{c}\int \mathrm{d}^4\tilde{x}\,\sqrt{-g}\,\left(\nabla_\mu T^\mu_\nu\right)\,\xi^\nu,$$

   where T^μ_ν denotes the stress-energy tensor of the matter field defined through the relation

$$\frac{1}{2}\,\sqrt{-g}\,T_{\mu\nu} = \frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial g^{\mu\nu}} - \partial_\lambda\left[\frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial\left(\partial_\lambda g^{\mu\nu}\right)}\right].$$

   (c) Argue that, invariance of the action under the transformation requires that

$$\nabla_\mu T^\mu_\nu = 0,$$

   i.e. the stress-energy tensor of the matter field is covariantly conserved.

2. *The stress-energy tensor of an ideal fluid:* Consider an ideal fluid described by the energy density $\rho\,c^2$ (with ρ being the mass density) and pressure p. Further, assume that the fluid does not possess any anisotropic stress.

   (a) Argue that, in the comoving frame, the stress energy tensor of the fluid is given by

$$T^\mu_\nu = \text{diag.}\,\left(\rho\,c^2, -p, -p, -p\right).$$

   (b) Further, show that, in a general frame, the stress energy tensor of the fluid can be written as

$$T^\mu_\nu = (\rho\,c^2 + p)\,u^\mu\,u_\nu - p\,\delta^\mu_\nu,$$

   where u^μ is the four velocity of the fluid.

   (c) Using the law governing the conservation of the stress energy tensor, arrive at the equations of motion that describe an ideal fluid in Minkowski spacetime.

3. *The stress-energy tensor of a scalar field:* Recall that, given an action that describes a matter field, the stress-energy tensor associated with the matter field is given by the variation of the action with respect to the metric tensor as follows:

$$\delta S = \frac{1}{2\,c}\int \mathrm{d}^4\tilde{x}\,\sqrt{-g}\,T_{\mu\nu}\,\delta g^{\mu\nu} = -\frac{1}{2\,c}\int \mathrm{d}^4\tilde{x}\,\sqrt{-g}\,T^{\mu\nu}\,\delta g_{\mu\nu}.$$

Consider a scalar field ϕ that is governed by the following action:

$$S[\phi] = \frac{1}{c}\int \mathrm{d}^4x\,\sqrt{-g}\,\left[\frac{1}{2}\,g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right],$$

where $V(\phi)$ is the potential describing the scalar field.

(a) Upon varying this action with respect to the metric tensor, arrive at the stress energy tensor of the scalar field.

(b) Show that the conservation of the stress-energy tensor leads to the equation of motion of the scalar field.

4. *The stress-energy tensor of the electromagnetic field:* In a curved spacetime, the action describing the electromagnetic field is given by

$$S[A^\mu] = -\frac{1}{16\,\pi\,c}\int \mathrm{d}^4\tilde{x}\,\sqrt{-g}\,F_{\mu\nu}\,F^{\mu\nu},$$

where

$$F_{\mu\nu} = A_{\mu;\nu} - A_{\nu;\mu} = A_{\mu,\nu} - A_{\nu,\mu}.$$

(a) Construct the stress-energy tensor associated with the electromagnetic field.

(b) What are the time-time and the time-space components of the stress energy tensor of the electromagnetic field in flat spacetime?

5. *Energy conditions:* There are five so-called energy conditions — viz. weak energy condition (WEC), null energy condition (NEC), dominant energy condition (DEC), dominant null energy condition (DNEC) and the strong energy condition (SEC) — which are often used to characterize the stress-energy tensor of matter fields. Let t^μ and l^μ denote timelike and lightlike vectors, i.e. $t_\mu\,t^\mu > 0$ and $l_\mu\,l^\mu = 0$. The energy conditions are defined in terms of the vectors t^μ and l^μ as follows:

- WEC: $T_{\mu\nu}\,t^\mu\,t^\nu \geq 0$,
- NEC: $T_{\mu\nu}\,l^\mu\,l^\nu \geq 0$,
- DNEC: $T_{\mu\nu}\,t^\mu\,t^\nu \geq 0$ and $T_{\mu\nu}\,t^\mu$ is a non-spacelike vector, i.e. $T_{\mu\nu}\,T^\nu_\lambda\,t^\mu\,t^\lambda \geq 0$,
- NDEC: $T_{\mu\nu}\,l^\mu\,l^\nu \geq 0$ and $T_{\mu\nu}\,l^\mu$ is a non-spacelike vector, i.e. $T_{\mu\nu}\,T^\nu_\lambda\,l^\mu\,l^\lambda \geq 0$,
- SEC: $T_{\mu\nu}\,t^\mu\,t^\nu \geq \frac{1}{2}\,T^\mu_\mu\,t^\nu\,t_\nu$.

Recall that the stress-energy tensor of an ideal fluid is given by

$$T^\mu_\nu = (\rho\,c^2 + p)\,u^\mu\,u_\nu - p\,\delta^\mu_\nu,$$

where ρ, p and u^μ denote the mass density, pressure and four velocity of the fluid. Determine the conditions on ρ and p of an ideal fluid that correspond to the different energy conditions.

Quiz I

From describing curved spacetimes to Einstein's equations

1. *Scalar curvature in two spacetime dimensions:* Consider the following $(1+1)$-dimensional line element:

$$\mathrm{d}s^2 = f^2(\eta,\xi)\,\left(\mathrm{d}\eta^2 - d\xi^2\right),$$

where $f(\eta,\xi)$ is an arbitrary function of the coordinates η and ξ. Show that the scalar curvature associated with this line-element can be expressed as **10 marks**

$$R = -\nabla_\mu \nabla^\mu \ln f^2 = -\Box \ln f^2.$$

2. *Gravitation in two spacetime dimensions:* Consider gravitation in two spacetime dimensions.

   (a) Determine the number of independent components of the Riemann tensor in such a case. Which are these components? Express the scalar curvature in terms of these components **3 marks**

   (b) Using the above result and the properties of the Riemann tensor under the interchange of its indices, express the Riemann tensor purely in terms of the scalar curvature and the metric tensor. **4 marks**

   (c) Determine the corresponding Ricci and Einstein tensors. **3 marks**

3. *Useful identities:* Establish the following identities.

   (a) Show that **3 marks**

$$\frac{\partial g^{cd}}{\partial g_{ab}} = -\frac{1}{2}\left(g^{ac}\,g^{bd} + g^{bc}\,g^{ad}\right).$$

   (b) If $\bar{g}_{ab} = \sqrt{-g}\,g_{ab}$, show that **4 marks**

$$\bar{g}^{ab}{}_{,c} = \Gamma^d_{dc}\,\bar{g}^{ab} - \Gamma^a_{dc}\,\bar{g}^{db} - \Gamma^b_{dc}\,\bar{g}^{ad}.$$

   (c) Show that **3 marks**

$$\frac{\partial\left(\sqrt{-g}\,R\right)}{\partial g_{ab,cd}} = \sqrt{-g}\,\left[\frac{1}{2}\left(g^{ac}\,g^{bd} + g^{ad}\,g^{bc}\right) - g^{ab}\,g^{cd}\right].$$

4. *Another identity:* For any tensor A^{ab}, show that **10 marks**

$$\nabla_a\,\nabla_b\,A^{ab} = \nabla_b\,\nabla_a\,A^{ab}.$$

5. *Non-canonical scalar field:* Consider a scalar field, say, ϕ, that is governed by the following action:

$$S[\phi(\tilde{x})] = \frac{1}{c}\int \mathrm{d}^4\tilde{x}\,\sqrt{-g}\,P(X,\phi),$$

where $\mathrm{d}^4\tilde{x} = c\,\mathrm{d}t\,\mathrm{d}^3\boldsymbol{x}$, the quantity X is given by

$$X = \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi,$$

while $P(X,\phi)$ is an arbitrary function of X and ϕ.

   (a) Vary the above action with respect to the scalar field ϕ to arrive at the equation of motion governing the field. **3 marks**

   (b) Determine the stress-energy tensor $T_{\mu\nu}$ associated with the scalar field ϕ by varying the above action with respect to the metric tensor $g^{\mu\nu}$. **7 marks**

Exercise sheet 6

Schwarzschild spacetime

1. *Spherically symmetric spacetimes:* Consider the following line element that describes spherically symmetric spacetimes in $(3+1)$-dimensions:

$$\mathrm{d}s^2 = c^2\, e^{\Phi(t,r)}\, \mathrm{d}t^2 - e^{\Psi(t,r)}\, \mathrm{d}r^2 - r^2\, \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right),$$

where $\Phi(t,r)$ and $\Psi(t,r)$ are arbitrary functions of the coordinates t and r.

   (a) Find $g_{\mu\nu}$ and $g^{\mu\nu}$ corresponding to this line element.

   (b) Evaluate the resulting $\Gamma^{\alpha}_{\mu\nu}$.

   (c) Also, calculate the corresponding $R_{\mu\nu}$ and R.

2. *Utilizing the Bianchi identities:* Compute the Einstein tensor corresponding to the above line element and show that its non-zero components are given by

$$\begin{aligned}
G^t_t &= \left(\frac{\Psi'}{r} - \frac{1}{r^2}\right)\, \mathrm{e}^{-\Psi} + \frac{1}{r^2},\\
G^r_t &= -\frac{\dot{\Psi}}{c\, r\, \mathrm{e}^{\Psi}} = -G^t_r\, \mathrm{e}^{(\Psi-\Phi)},\\
G^r_r &= -\left(\frac{\Phi'}{r} + \frac{1}{r^2}\right)\, \mathrm{e}^{-\Psi} + \frac{1}{r^2},\\
G^\theta_\theta &= G^\phi_\phi = \frac{1}{2}\, \left(\frac{\Phi'\, \Psi'}{2} + \frac{\Psi'}{r} - \frac{\Phi'}{r} - \frac{\Phi'^2}{2} - \Phi''\right)\, \mathrm{e}^{-\Psi} + \frac{1}{2\, c^2}\, \left(\ddot{\Psi} + \frac{\dot{\Psi}^2}{2} - \frac{\dot{\Phi}\, \dot{\Psi}}{2}\right)\, \mathrm{e}^{-\Phi},
\end{aligned}$$

where the overdots and the overprimes denote differentiation with respect to t and r, respectively. Show that the contracted Bianchi identities, viz. $\nabla_\mu\, G^\mu_\nu = 0$, imply that the last of the above equations vanishes, if the remaining three equations vanish.

3. *Spherically symmetric vacuum solution of the Einstein's equations:* In the absence of any sources, the above components of the Einstein tensor should vanish. Integrate the equations to arrive at the following Schwarzschild line element:

$$\mathrm{d}s^2 = c^2\, \left(1 - \frac{2\, G\, M}{c^2\, r}\right)\, \mathrm{d}t^2 - \left(1 - \frac{2\, G\, M}{c^2\, r}\right)^{-1}\, \mathrm{d}r^2 - r^2\, \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right),$$

where M is a constant of integration that denotes the mass of the central object that is responsible for the gravitational field.

4. *Isotropic coordinates:* Consider a transformation such that the coordinates (t, θ, ϕ) of the Schwarzschild line element remain unchanged, while the radial coordinate is transformed to a new coordinate, i.e. $r \to \rho = \rho(r)$.

   (a) If the above Schwarzschild line element can be expressed as

$$\mathrm{d}s^2 = c^2\, \left(1 - \frac{2\, G\, M}{c^2\, r}\right)\, \mathrm{d}t^2 - \lambda^2(\rho)\, \left[\mathrm{d}\rho^2 + \rho^2\, \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right)\right],$$

   determine the function $\lambda(\rho)$.

   (b) Also, express the complete line element in terms of the new coordinates (t, ρ, θ, ϕ).

Note: The new set of coordinates (t, ρ, θ, ϕ) are known as the isotropic coordinates.

5. *The Schwarzschild singularity:* Evaluate the so-called Kretschmann scalar, viz. $R_{\alpha\beta\gamma\delta}\, R^{\alpha\beta\gamma\delta}$, for the case of the Schwarzschild metric. Show that, whereas the quantity is finite at the Schwarzschild radius $r_{\rm S} = 2\, G\, M/c^2$, it diverges at the origin.

   Note: This implies that, while the Schwarzschild radius is a coordinate singularity (which can be avoided with a better choice of coordinates to describe the spacetime), the singularity at the origin is an unavoidable, physical one.

Exercise sheet 7

Classical tests of general relativity

1. *Gravitational redshift:* Consider two observers located at radii r_1 and r_2 in the static and spherically symmetric gravitational field around an object of mass M. Let the infinitesimal proper times in the frames of the two observers at r_1 and r_2 be $\mathrm{d}\tau_1$ and $\mathrm{d}\tau_2$, respectively.

   (a) Assuming that the static spacetime around the central object of mass M is described by the Schwarzschild line element, arrive at the relation between the proper times $\mathrm{d}\tau_1$ and $\mathrm{d}\tau_2$.

   (b) If the proper times denote the characteristic frequencies of atomic clocks located at the two radii, determine the relation between the corresponding angular frequencies, say, ω_1 and ω_2, or, equivalently, the associated energies E_1 and E_2.

   (c) If the energies E_1 and E_2 are associated with photons that climb out of the potential well, show that photons lose energy or, equivalently, their wavelengths grow longer.

   (d) Consider a photon that is emitted at the surface of the Sun, which reaches the Earth. If λ_{E} and λ_{O} denote the emitted and observed wavelengths, determine the extent to which the wavelength of the photon shifts as it reaches the Earth, i.e. calculate $(\lambda_{\mathrm{E}} - \lambda_{\mathrm{O}})/\lambda_{\mathrm{E}}$.

   Note: The radius of the Sun is $6.96 \times 10^8\,\mathrm{m}$, while the radius of Earth's orbit is $1.5 \times 10^{11}\,\mathrm{m}$.

   Note: This phenomenon where the wavelengths of the photons exhibit a shift towards the red end of the electromagnetic spectrum, as they climb out of a potential well, is known as gravitational redshift.

2. *Effective potential governing the motion of massive particles in Schwarzschild spacetime:* Consider a particle of mass m that is moving in the Schwarzschild spacetime.

   (a) Using the relation $p^\mu\, p_\mu = m^2\, c^2$ and the conserved energy, say, E, and the conserved angular momentum, say, L, show that the motion of the particle is described by the equation

   $$\left(\frac{\mathrm{d}r}{c\,\mathrm{d}\tau}\right)^2 = \tilde{E}^2 - \tilde{V}_{\mathrm{eff}}^2(r),$$

   where the effective potential $\tilde{V}_{\mathrm{eff}}^2(r)$ is given by

   $$\tilde{V}_{\mathrm{eff}}^2(r) = \left(1 - \frac{2\,G\,M}{c^2\,r}\right)\,\left(1 + \frac{\tilde{L}^2}{2\,r^2}\right),$$

   while $\tilde{E} = E/(m\,c^2)$ and $\tilde{L} = L/(m\,c)$.

   (b) Plot the effective potential $\tilde{V}_{\mathrm{eff}}^2(r)$ as a function of r/r_{S}, where $r_{\mathrm{S}} = 2\,G\,M/c^2$ is the Schwarzschild radius.

   (c) What are the radii at which circular orbits arise? Do these correspond to stable or unstable circular orbits?

   (d) Determine the value of $\tilde{L}$ below which no circular orbits are possible.

3. *Effective potential governing the motion of photons in Schwarzschild spacetime:* Consider a photon that is moving in the Schwarzschild spacetime. Let λ be the affine parameter that describes the trajectory of the photon.

   (a) Using the relation $p^\mu\, p_\mu = 0$ and the conserved energy E and angular momentum L, show that the motion of the photon is described by the equation

   $$\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 = \frac{E^2}{c^2} - B_{\mathrm{eff}}^2(r),$$

where the 'effective potential' $B_{\rm eff}^2(r)$ is given by

$$B_{\rm eff}^2(r) = \left(1 - \frac{2\,G\,M}{c^2\,r}\right)\,\frac{L^2}{r^2}.$$

(b) Plot the 'effective potential' $B_{\rm eff}^2(r)$ as a function of $r/r_{\rm S}$, and determine the location of the circular orbit. Is the circular orbit stable or unstable?

(c) Are stable orbits possible for any value of L?

4. *Precession of the perihelion of Mercury:* Consider a particle of mass m moving in the Schwarzschild spacetime.

   (a) From the equation of motion obtained above, upon setting $u = 1/r$, arrive at the following differential equation describing the orbital motion of massive particles:

   $$\frac{{\rm d}^2\,u}{{\rm d}\phi^2} + u = \frac{G\,M}{\tilde{L}^2} + \frac{3\,G\,M}{c^2}\,u^2.$$

   (b) The second term on the right hand side of the above equation would have been absent in the case of the conventional, non-relativistic, Kepler problem. Treating the term as a small perturbation, show that the orbits are no more closed, and the perihelion precesses by the angle

   $$\Delta\phi \simeq \frac{6\,\pi\,(G\,M)^2}{\tilde{L}^2\,c^2} = \frac{6\,\pi\,G\,M}{a\,(1-e^2)\,c^2}\;\;\text{radians/revolution},$$

   where e and a are the eccentricity and the semi-major axis of the original closed, Keplerian elliptical orbit.

   (c) For the case of the planet Mercury, $a = 5.8 \times 10^{10}$ m, while $e = 0.2$. Also, the period of the Mercury's orbit around the Sun is 88 days. Further, the mass of the Sun is $M_\odot = 2 \times 10^{30}$ kg. Use these information to determine the angle by which the perihelion of Mercury would have shifted in a century.

   Note: The measured precession of the perihelion of the planet Mercury proves to be $5599''.7 \pm 0''.4$ per century, but a large part of it is caused due to the influences of the other planets. When the other contributions have been subtracted, the precession of the perihelion of the planet Mercury due to the purely relativistic effects amounts to 43.1 ± 0.5 seconds of arc per century.

5. *Gravitational bending of light:* Consider the propagation of photons in the Schwarzschild spacetime.

   (a) From the equation of motion obtained above, upon setting $u = 1/r$, arrive at the following differential equation describing the orbital motion of massless particles:

   $$\frac{{\rm d}^2\,u}{{\rm d}\phi^2} + u = \frac{3\,G\,M}{c^2}\,u^2,$$

   (b) Establish that, in the absence of the term on the right hand side, the photons will travel in straight lines.

   (c) As in the previous exercise, treating the term on the right hand side as a small perturbation, show that it leads to a deflection of a photon's trajectory by the angle

   $$\Delta\phi \simeq \frac{4\,G\,M}{c^2\,b},$$

   where $b = E/(c\,L)$ is the impact parameter of the photon (i.e. the distance of the closest approach of the photon to the central mass).

(d) Given that the radius of the Sun is 6.96×10^8 m, determine the deflection angle $\Delta\phi$ for a ray of light that grazes the Sun.

Note: The famous 1919 eclipse expedition led by Eddington led to two sets of results, viz.

$$\Delta\phi = 1''.98 \pm 0''.16 \quad \text{and} \quad \Delta\phi = 1''.61 \pm 0''.4,$$

both of which happen to be consistent with the theory.

Additional exercises I

Tensor algebra, calculus and general relativity

1. *Derivative of the determinant of the metric tensor:* Note that the determinant g of the metric tensor g_{ab} can be expressed as

$$g = \sum_b (-1)^{a+b}\, g_{ab}\, |m_{ab}|,$$

where we have explicitly written the sum over the repeated index b and we should mention that there is no sum over the index a. Also, the quantity $|m_{ab}|$ denotes the minor of the matrix element g_{ab}.

   (a) Argue that the quantity

$$\sum_b (-1)^{a+b}\, g_{cb}\, |m_{ab}|$$

   vanishes when $c \neq a$ so that we can write

$$\sum_b (-1)^{a+b}\, g_{cb}\, |m_{ab}| = \delta_{ca}\, g.$$

   (b) Since the inverse g^{ab} of the metric tensor g_{ab} is defined as

$$g^{ba} = \frac{1}{g}\,(-1)^{a+b}\,|m_{ab}|,$$

   show that we can write

$$\delta g = g\, g^{ba}\, \delta g_{ab}.$$

   (c) Therefore, establish that

$$\partial_c g = g\, g^{ab}\, g_{ab,c} = -g\, {g^{ab}}_{,c}\, g_{ab}.$$

2. *Properties of Killing vectors:* If ξ^a is a Killing vector, show that

   (a) $\xi_{a;b;c} = R_{dcba}\, \xi^d$,

   (b) ${\xi_{a;b}}^{;b} + R_{ac}\, \xi^c = 0$.

3. *Weyl tensor:* In $(3+1)$-spacetime dimensions, the Weyl tensor $C_{\alpha\beta\gamma\delta}$ is defined as follows:

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + \frac{1}{2}\,(g_{\alpha\delta}\, R_{\beta\gamma} + g_{\beta\gamma}\, R_{\alpha\delta} - g_{\alpha\gamma}\, R_{\beta\delta} - g_{\beta\delta}\, R_{\alpha\gamma}) + \frac{1}{6}\,(g_{\alpha\gamma}\, g_{\delta\beta} - g_{\alpha\delta}\, g_{\gamma\beta})\, R.$$

Show that the Weyl tensor is invariant under conformal transformations.

Note: This implies that a spacetime is conformally flat if the Weyl tensor vanishes.

4. *Actions involving higher derivatives:* There is another approach (apart from one I had discussed in the lectures) to arrive at the Einstein's equations. Recall that, in the introductory course, we had considered a scalar field ϕ that is described by the action

$$S[\phi(\tilde{x})] = \frac{1}{c}\int \mathrm{d}^4\tilde{x}\, \mathcal{L}(\phi, \partial_\mu\phi, \partial_\mu\partial_\nu\phi).$$

Note that, in addition to the usual dependence on the scalar field ϕ and its first derivative $\partial_\mu\phi$, the Lagrangian density $\mathcal{L}$ depends on the second derivatives $\partial_\mu\partial_\nu\phi$.

   (a) Derive the Euler-Lagrange equation for the system.

   Note: In order to obtain the Euler-Lagrange equation, the variation of the field *as well as* its first derivative need to be set to zero at the initial and final hypersurfaces.

(b) Consider an action of the following form that describes a theory of gravitation:

$$S[g_{\mu\nu}(\tilde{x})] = \int \mathrm{d}^4\tilde{x}\,\sqrt{-g}\,\mathcal{L}_{\mathrm{G}}(g_{\mu\nu}, \partial_\alpha g_{\mu\nu}, \partial_\alpha\partial_\beta g_{\mu\nu}).$$

Note that, apart from the metric tensor $g_{\mu\nu}$ and its first derivative $g_{\mu\nu,\alpha}$, the above action depends on the second derivative of the metric tensor, viz. $g_{\mu\nu,\alpha\beta}$, as well. In a manner similar to that of the scalar field that we considered above, vary the above action with respect to the metric tensor to arrive at the following equation of motion:

$$\partial_\alpha\partial_\beta\left(\frac{\partial\mathcal{L}_{\mathrm{G}}}{\partial g_{\mu\nu,\alpha\beta}}\right) - \partial_\alpha\left(\frac{\partial\mathcal{L}_{\mathrm{G}}}{\partial g_{\mu\nu,\alpha}}\right) + \frac{\partial\mathcal{L}_{\mathrm{G}}}{\partial g_{\mu\nu}} = 0.$$

(c) Assuming that gravitation is described by the standard Einstein-Hilbert action, i.e. with the Lagrangian density being given by

$$\mathcal{L}_{\mathrm{G}} = -\frac{c^3}{16\,\pi\,G}\,R,$$

use the above Euler-Lagrange equation governing the metric tensor to arrive at the Einstein's equations in vacuum.

5. _Palatini formalism:_ According to the Palatini approach, the metric, specifically $\bar{g}_{ab} = \sqrt{-g}\,g_{ab}$, and the metric connection Γ^a_{bc} that it symmetric in its covariant indices are to be treated as separate dynamical variables.

   (a) Vary the Einstein-Hilbert action with respect to $\bar{g}_{ab}$ and show that, in vacuum, the equation of motion governing the gravitational field is given by $R_{ab} = 0$.

   (b) Vary the Einstein-Hilbert action with respect to Γ^a_{bc} and show that it leads to the condition that the covariant derivative of the metric tensor vanishes identically.

   Note: As we have discussed earlier, it is this condition that helps us express the Christoffel symbols in terms of the derivatives of the metric tensor.

6. _Modified theories of gravitation:_ Consider a theory of gravitation that is described by the action

$$S[g_{\mu\nu}(\tilde{x})] = A\int \mathrm{d}^4\tilde{x}\,\sqrt{-g}\,f(R),$$

   where $f(R)$ is an arbitrary function of the scalar curvature R, and A is a constant.

   (a) What is dimension of the constant A?

   (b) Vary the action with respect to the metric tensor $g_{\mu\nu}$ to arrive at the equation of motion.

7. _Eddington's theory of gravitation:_ Consider a theory of gravitation that is described by the action

$$S[g_{\mu\nu}(\tilde{x})] = A\int \mathrm{d}^4\tilde{x}\,\sqrt{-g}\,R_{\alpha\beta\gamma\delta}\,R^{\alpha\beta\gamma\delta},$$

   where A is a constant.

   (a) Determine the dimension of the constant A.

   (b) Vary the action with respect to the metric tensor $g_{\mu\nu}$ to arrive at the equation of motion.

8. _The nature of a worm hole:_ The spacetime of a worm hole is described by the line-element

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}r^2 - \left(b^2 + r^2\right)\,\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right),$$

   where b is a constant with the dimensions of length that reflects the size of the 'traversable' region. Show that the energy density of matter has to be negative to sustain such a spacetime.

9. *Raychaudhuri equation:* Let u^α denote the four velocity of a fluid.

   (a) Show that
$$u_{\alpha;\beta} = \omega_{\alpha\beta} + \sigma_{\alpha\beta} + \frac{1}{3}\,\theta\, P_{\alpha\beta} - a_\alpha\, u_\beta,$$
   where the acceleration a^α and the expansion of the fluid world lines θ are defined as
$$a_\alpha = u_{\alpha;\beta}\, u^\beta, \quad \theta = u^\alpha_{\ ;\alpha},$$
   while the quantities $\omega_{\alpha\beta}$ and $\sigma_{\alpha\beta}$ which describe rotation and shear of the fluid are given by
$$\omega_{\alpha\beta} = \frac{1}{2}\,\left(u_{\alpha;\mu}\, P^\mu_\beta - u_{\beta;\mu}\, P^\mu_\alpha\right), \quad \sigma_{\alpha\beta} = \frac{1}{2}\,\left(u_{\alpha;\mu}\, P^\mu_\beta + u_{\beta;\mu}\, P^\mu_\alpha\right) - \frac{1}{3}\,\theta\, P_{\alpha\beta},$$
   with the quantity $P_{\alpha\beta}$ being the operator that projects a vector onto the three-surface perpendicular to the four velocity u^μ given by
$$P_{\alpha\beta} = g_{\alpha\beta} + u_\alpha\, u_\beta.$$

   (b) Also, show that
$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = a^\alpha_{\ ;\alpha} + 2\,(\omega^2 - \sigma^2) - \frac{1}{3}\,\theta^2 - R_{\alpha\beta}\, u^\alpha\, u^\beta,$$
   where
$$\omega^2 = \frac{1}{2}\,\omega_{\alpha\beta}\,\omega^{\alpha\beta}, \quad \sigma^2 = \frac{1}{2}\,\sigma_{\alpha\beta}\,\sigma^{\alpha\beta}.$$
   Note: The above equation governing the expansion scalar θ is known as the Raychaudhuri equation.

10. *Shapiro time delay:* A fourth classical test of general relativity is to measure the delay in the propagation of light as it travels around a central mass. Consider a situation wherein Earth, one of the inner planets, either Mercury or Venus, and the Sun are nearly aligned. Consider a light ray that travels from Earth to Mercury or Venus, passing the Sun, with D being the distance of the light ray to the Sun at the point of closest approach. Let us assume that the light ray is propagating in the $\theta = \pi/2$ plane in a Schwarzschild spacetime with the Sun as the central mass.

   (a) Using the fact that $\mathrm{d}s^2 = 0$ for a light ray and, considering the situation wherein $\mathrm{d}\theta = 0$, arrive at the differential relation between $\mathrm{d}t$, $\mathrm{d}r$ and $\mathrm{d}\phi$.

   (b) If we work at the order μ/r, where $\mu = G\,M/c^2$, we can assume that the light ray is traveling along a straight line so that $r = D\sin\phi$. Working under such an assumption, arrive at the following equation relating $\mathrm{d}t$ and $\mathrm{d}r$ upto the first order in μ/r:
$$\mathrm{d}t \simeq \pm\frac{r\,\mathrm{d}r}{\sqrt{r^2 - D^2}}\left(1 + \frac{2\,\mu}{r} - \frac{\mu\, D^2}{r^3}\right)^{1/2}.$$

   (c) Integrate this equation to arrive at the following expression for the time taken, say, T, for a light ray to travel from Earth to the planets Mercury or Venus:
$$\begin{aligned} c\,T \;=\; & \sqrt{D_{\mathrm{P}}^2 - D^2} - \sqrt{D_{\mathrm{E}}^2 - D^2} \\ & + 2\,\mu\ln\left[\left(\sqrt{D_{\mathrm{P}}^2 - D^2} + D_{\mathrm{P}}\right)\left(\sqrt{D_{\mathrm{E}}^2 - D^2} + D_{\mathrm{E}}\right)/D^2\right] \\ & - \mu\,\left[\left(\sqrt{D_{\mathrm{P}}^2 - D^2}/D_{\mathrm{P}}\right) + \left(\sqrt{D_{\mathrm{E}}^2 - D^2}/D_{\mathrm{E}}\right)\right], \end{aligned}$$
   where D_{E} and D_{P} denote the average radii of the orbits of Earth and the planets Mercury or Venus.

   Note: The two terms in the first line of the above expression for T is the result in flat spacetime in the absence of the central mass μ.

(d) Using the average radii of the Earth and orbits of the inner planets, estimate the time T.

(e) The delay is experimentally verified by sending pulsed radar signals to Venus and Mercury and using the echoes to determine the travel time as the positions of Earth and the planets change relative to the Sun. In the case of Venus, the delay has been measured to be about $200\,\mu\mathrm{s}$. Compare the observed value with the theoretical estimate.

Exercise sheet 8

Static black holes

1. <u>*Coordinate time versus proper time:*</u> Consider a particle of mass m that is falling radially into a Schwarzschild black hole. Assume that the particle starts at the radius r_0 at the proper time τ_0 and the coordinate time t_0.

   (a) Determine the radial position of the particle as a function of the proper time τ. What is the proper time when the particle reaches the event horizon of the black hole? When does it reach the singularity at $r = 0$?

   (b) Determine the radial position of the particle as a function of the coordinate time t. When does the particle reach the horizon of the black hole in terms of the coordinate time? Does it cross the event horizon of the black hole?

   Note: The Schwarzschild radius $r_{\mathrm{S}} = 2\,G\,M/c^2$ is referred to as the event horizon of the black hole. It is called so since it represents the boundary of all events that can be observed by observers located outside the radius.

2. <u>*Spacetime diagram in the Schwarzschild coordinates:*</u> Consider radially ingoing and outgoing null rays propagating in the Schwarzschild line element. Solve for the trajectories of these rays and plot them for different initial values of the radius r.

3. <u>*Kruskal-Szekeres coordinates and the maximal extension:*</u> Let us focus on the time and radial parts of the Schwarzschild metric.

   (a) For $r > r_{\mathrm{S}}$, determine the new radial coordinate, say, r^*, in terms of which the Schwarzschild line-element can be expressed in the following form:

   $$\mathrm{d}s^2 = \left(1 - \frac{2\,G\,M}{c^2\,r}\right)\left(c^2\,\mathrm{d}t^2 - \mathrm{d}r^{*2}\right).$$

   Note: The radial coordinate r^* is often referred to as the tortoise coordinate.

   (b) In terms of the null coordinates $u = c\,t - r^*$ and $v = c\,t + r^*$, show that the above line element reduces to

   $$\mathrm{d}s^2 = \left(1 - \frac{2\,G\,M}{c^2\,r}\right)\mathrm{d}u\,\mathrm{d}v.$$

   (c) Let us introduce two new coordinates, say, u' and v', through the relations

   $$u' = -\frac{4\,G\,M}{c^2}\exp\left(\frac{-c^2\,u}{4\,G\,M}\right), \quad v' = \frac{4\,G\,M}{c^2}\exp\left(\frac{c^2\,v}{4\,G\,M}\right).$$

   Express the Schwarzschild metric in terms of the coordinates u' and v'.

   (d) Defining new coordinates t' and $r^{*\prime}$ as

   $$c\,t' = \frac{1}{2}\,(v' + u'), \quad r^{*\prime} = \frac{1}{2}\,(v' - u'),$$

   express the line element in terms of the t' and $r^{*\prime}$.

   (e) Arrive at the relation between the coordinates $(c\,t, r)$ and $(c\,t', r^{*\prime})$.

   (f) In the $c\,t'$-$r^{*\prime}$-plane, draw the curves corresponding to constant t and r.

   (g) Repeat the above exercises for $r < r_{\mathrm{S}}$.

   (h) Extend the above arguments for the domain $r^{*\prime} < 0$.

4. *Penrose diagram of a Schwarzschild black hole:* Construct suitable conformal transformations to arrive at the Penrose diagram of the Schwarzschild black hole in the maximally extended Kruskal-Szekeres coordinates.

5. *Reissner-Nordstrom black holes:* Consider a spherically symmetric black hole of mass M that also carries an electric charge Q. In such a case, to determine the line element describing the spacetime around the black hole, apart from solving the vacuum Einstein's equations, we also need to solve the following Maxwell's equations in the source free region:

$$\nabla_\mu F^{\mu\nu} = 0, \quad \partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0.$$

As in the Schwarzschild case, one can assume that the line element, in general, can be expressed in terms of the functions $\Phi(t, r)$ and $\Psi(t, r)$ as follows:

$$\mathrm{d}s^2 = c^2\, \mathrm{e}^{\Phi(t,r)}\, \mathrm{d}t^2 - \mathrm{e}^{\Psi(t,r)}\, \mathrm{d}r^2 - r^2\, \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right).$$

(a) Assuming the spacetime to be static, show that the electromagnetic field tensor can be expressed as follows:

$$F_{\mu\nu} = E(r) \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(b) Establish that, under these assumptions, the Maxwell's equations lead to

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[r^2\, \mathrm{e}^{-(\Phi+\Psi)/2}\, E\right] = 0.$$

(c) Integrate the equation to determine the electric field $E(r)$ in terms of the charge Q carried by the black hole.

(d) Using the Einstein's and Maxwell's equations, arrive at the following equation governing Φ and Ψ:

$$\frac{\mathrm{d}}{\mathrm{d}r}\, (\Phi + \Psi) = 0.$$

(e) Solve the Einstein's equations to arrive at

$$\mathrm{e}^{\Phi} = 1 - \frac{2\, G\, M}{c^2\, r} + \frac{G\, Q^2}{4\, \pi\, \epsilon_0\, c^4\, r^2}.$$

Note: The resulting line element is of the form

$$\mathrm{d}s^2 = c^2 \left(1 - \frac{2\, G\, M}{c^2\, r} + \frac{G\, Q^2}{4\, \pi\, \epsilon_0\, c^4\, r^2}\right) \mathrm{d}t^2 - \left(1 - \frac{2\, G\, M}{c^2\, r} + \frac{G\, Q^2}{4\, \pi\, \epsilon_0\, c^4\, r^2}\right)^{-1} \mathrm{d}r^2 - r^2\, \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right),$$

which, evidently, reduces to the Schwarzschild metric when $Q = 0$.

Note: Black holes with mass and electric charge are referred to as Reissner-Nordstrom black holes.

Exercise sheet 9

Stationary black holes

1. *Frame dragging:* Consider a particle that is dropped radially (i.e. with zero angular momentum) into a Kerr black hole from infinity.

   (a) Working, say, in the Boyer-Lindquist coordinates, show that the particle picks up a non-zero angular velocity.

   (b) Express the angular velocity in terms of the components of the contravariant metric tensor.

   Note: This effect is known as dragging of inertial frames. In a sense, it is similar to the effect of a spinning charge generating a magnetic field in electromagnetism. For this reason, such effects are referred to as gravitomagnetism.

2. *Event horizons of the Kerr black hole:* As we had discussed, an event horizon is the boundary that separates the spacetime into events that can communicate with distant observers and events that cannot. In the case of the Schwarzschild black hole, it is clear that the event horizon is located at the Schwarzschild radius. This is because of the fact that, once inside the Schwarzschild radius, the null rays (even outgoing ones) are inevitably dragged towards the singularity at the centre of the black hole.

   (a) Mathematically, the location of the event horizon can be said to be the hypersurface of constant radius where the normal to the hypersurface becomes null. Determine the locations of the two horizons in the case of the Kerr black hole.

   (b) Also, evaluate the area of the outer event horizon.

3. *Infinite redshift surfaces:* What is the condition that determines the radius at which photons exhibit infinite redshift with respect to an observer at infinity? Use the condition to determine the infinite redshift surfaces of the Kerr black hole.

   Note: In the case of the Schwarzschild black hole, the location of the event horizon and the infinite redshift surface coincide. This is not so in the case of the Kerr black hole.

4. *Ergosphere of the Kerr black hole:* Consider photons that are moving along the $\pm\phi$-directions in the equatorial plane (i.e. wherein $\theta = \pi/2$) at a fixed radius r around the Kerr black hole.

   (a) Assuming that $\mathrm{d}r = \mathrm{d}\theta = 0$ and using the fact that $\mathrm{d}s^2 = 0$ for photons, express the two solutions for the angular velocity of the photon, viz. $\mathrm{d}\phi/\mathrm{d}t$, in terms of the components of the metric tensor.

   (b) Determine the angular velocities when the metric component g_{00} vanishes.

   Note: The region between the outer event horizon and location wherein the metric component g_{00} vanishes is known as the ergosphere.

   (c) Determine the location of the ergosphere of the Kerr black hole.

   (d) Show that, once inside the ergosphere, photons cannot move in the direction opposite to the rotation of the black hole. Also, argue that, inside the ergosphere, the photons have to co-rotate with the hole, independent of their angular momentum.

   (e) Understand the structure of the ergosphere in relation to the horizon of the Kerr black hole.

5. *Singularities in the Reissner-Nordstrom and Kerr black holes:* Use the given `Mathematica` code to evaluate the Christoffel symbols, the Riemann, the Ricci, and the Einstein tensors as well as the Ricci scalar around the charged Reissner-Nordstrom and the Kerr black holes that are described by the following line elements:

$$\mathrm{d}s^2 = c^2\left(1 - \frac{2\,\mu}{r} + \frac{q^2}{r^2}\right)\mathrm{d}t^2 - \left(1 - \frac{2\,\mu}{r} + \frac{q^2}{r^2}\right)^{-1}\mathrm{d}r^2 - r^2\,\left(\mathrm{d}\theta^2 + \sin^2\!\theta\,\mathrm{d}\phi^2\right),$$

where

$$\mu = \frac{G\,M}{c^2} \quad \text{and} \quad q^2 = \frac{G\,Q^2}{4\,\pi\,\epsilon_0\,c^4},$$

and

$$\mathrm{d}s^2 = c^2\,\frac{\rho^2\,\Delta}{\Sigma^2}\,\mathrm{d}t^2 - \frac{\Sigma^2\sin^2\theta}{\rho^2}\,(\mathrm{d}\phi - \omega\,\mathrm{d}t)^2 - \frac{\rho^2}{\Delta}\,\mathrm{d}r^2 - \rho^2\,\mathrm{d}\theta^2,$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta,\;\; \Delta = r^2 - 2\,\mu\,r + a^2,\;\; \Sigma^2 = (r^2+a^2)^2 - a^2\,\Delta\,\sin^2\theta,\;\; \omega = \frac{2\,\mu\,c\,r\,a}{\Sigma^2},\;\; a = \frac{J}{M\,c}.$$

The quantities M, Q and J are constants that denote the mass, the electric charge and the angular momentum associated with the black holes, respectively. Note that, since the trace of the stress-energy tensor vanishes, we expect the Ricci tensor $R_{\alpha\beta}$ and the Ricci scalar R to vanish in these cases. As in the case of the Schwarzschild black hole, calculate the Kretschmann scalar $R_{\alpha\beta\gamma\delta}\,R^{\alpha\beta\gamma\delta}$ and determine the locations of the singularities in the spacetimes of these two black holes.

Exercise sheet 10

Kinematics of the FLRW universe

1. <u>*Spaces of constant curvature:*</u> Consider spaces of constant curvature that are described by the metric tensor g_{ab}.

   (a) Argue that, the Riemann tensor associated with such a space can be expressed in terms of the metric g_{ab} as follows:
   $$R_{abcd} = \kappa\ \left(g_{ac}\, g_{bd} - g_{ad}\, g_{bc}\right),$$
   where κ is a constant.

   (b) Show that the Ricci tensor corresponding to the above Riemann tensor is given by
   $$R_{ab} = 2\,\kappa\, g_{ab}.$$

   Note: Examples of spacetimes with a constant scalar curvature are the Einstein static universe, the de Sitter and the anti de Sitter spacetimes.

2. <u>*Visualizing the Friedmann metric:*</u> The Friedmann universe is described by the line-element
   $$\mathrm{d}s^2 = c^2\, \mathrm{d}t^2 - a^2(t)\, \mathrm{d}\ell^2,$$
   where
   $$\mathrm{d}\ell^2 = \frac{\mathrm{d}r^2}{(1-\kappa\, r^2)} + r^2\ \left(\mathrm{d}\theta^2 + \sin^2\!\theta\, \mathrm{d}\phi^2\right)$$
   and $\kappa = 0, \pm 1$.

   (a) Let us define a new coordinate χ as follows:
   $$\chi = \int \frac{dr}{\sqrt{1-\kappa\, r^2}}.$$
   Show that in terms of the coordinate χ the spatial line element $\mathrm{d}\ell^2$ reduces to
   $$\mathrm{d}\ell^2 = \mathrm{d}\chi^2 + S_\kappa^2(\chi)\ \left(\mathrm{d}\theta^2 + \sin^2\!\theta\, \mathrm{d}\phi^2\right),$$
   where
   $$S_\kappa(\chi) = \begin{cases} \sin\chi & \text{for} \quad \kappa = 1, \\ \chi & \text{for} \quad \kappa = 0, \\ \sinh\chi & \text{for} \quad \kappa = -1. \end{cases}$$

   (b) Show that, for $\kappa = 1$, the spatial line-element $\mathrm{d}\ell^2$ can be described as the spherical surface
   $$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$$
   embedded in an Euclidean space described by the line-element
   $$\mathrm{d}\ell^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \mathrm{d}x_4^2.$$

   (c) Show that, for $\kappa = -1$, the spatial line-element $\mathrm{d}\ell^2$ can be described as the hyperbolic surface
   $$x_1^2 + x_2^2 + x_3^2 - x_4^2 = -1$$
   embedded in a Lorentzian space described by the line-element
   $$\mathrm{d}\ell^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 - \mathrm{d}x_4^2.$$

3. *Geodesic equations in a FLRW universe:* Obtain the following non-zero components of the Christoffel symbols for the FLRW line element:

$$\Gamma^t_{ij} = \frac{a\,\dot{a}}{c}\,\sigma_{ij},$$

where σ_{ij} denotes the spatial metric defined through the relation $\mathrm{d}\ell^2 = \sigma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$. Use these Christoffel symbols to arrive at the geodesic equations corresponding to the t coordinate for massive as well as massless particles in a FLRW universe.

4. *Weyl tensor and conformal invariance:* In $(3+1)$-spacetime dimensions, the Weyl tensor $C_{\alpha\beta\gamma\delta}$ is defined as follows:

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + \frac{1}{2}\,(g_{\alpha\delta}\,R_{\beta\gamma} + g_{\beta\gamma}\,R_{\alpha\delta} - g_{\alpha\gamma}\,R_{\beta\delta} - g_{\beta\delta}\,R_{\alpha\gamma}) + \frac{1}{6}\,(g_{\alpha\gamma}\,g_{\delta\beta} - g_{\alpha\delta}\,g_{\gamma\beta})\,R.$$

   (a) Show that the Weyl tensor vanishes for the FLRW metric.

   (b) The vanishing Weyl tensor implies that there exists a coordinate system in which the FLRW metric (for all κ) is conformal to the Minkowski metric. It is straightforward to check that the metric of the $\kappa = 0$ (i.e. the spatially flat) FLRW universe can be expressed in the following form:

$$g_{\mu\nu} = a^2(\eta)\,\eta_{\mu\nu},$$

   where η is the conformal time coordinate defined by the relation

$$\eta = \int \frac{\mathrm{d}t}{a(t)},$$

   and $\eta_{\mu\nu}$ denotes the flat spacetime metric. Construct the coordinate systems in which the metrics corresponding to the $\kappa = \pm 1$ FLRW universes can be expressed in a form wherein they are conformally related to flat spacetime.

5. *Consequences of conformal invariance:* As we have seen, the action of the electromagnetic field in a curved spacetime is invariant under the conformal transformation.

   (a) Utilizing the conformal invariance of the electromagnetic action, show that the electromagnetic waves in the spatially flat FLRW universe can be written in terms of the conformal time coordinate η as follows:

$$A_\mu \propto \exp -(i\,k\,\eta) = \exp -\left[i\,k\,\int dt/a(t)\right].$$

   (b) Since the time derivative of the phase defines the instantaneous frequency $\omega(t)$ of the wave, conclude that $\omega(t) \propto a^{-1}(t)$.

Exercise sheet 11

Dynamics of the FLRW universe

1. *The Friedmann equations:* Recall that the FLRW universe is described by the line element

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-\kappa\,r^2} + r^2\,\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right],$$

where $\kappa = 0, \pm 1$.

(a) Arrive at the following expressions for the Ricci tensor R^μ_ν, the scalar curvature R, and the Einstein tensor G^μ_ν for the above Friedmann metric:

$$\begin{aligned} R^t_t &= -\frac{3\,\ddot{a}}{c^2\,a}, \\ R^i_j &= -\left[\frac{\ddot{a}}{c^2\,a} + 2\left(\frac{\dot{a}}{c\,a}\right)^2 + \frac{2\,\kappa}{a^2}\right]\delta^i_j, \\ R &= -6\left[\frac{\ddot{a}}{c^2\,a} + \left(\frac{\dot{a}}{c\,a}\right)^2 + \frac{\kappa}{a^2}\right], \\ G^t_t &= 3\left[\left(\frac{\dot{a}}{c\,a}\right)^2 + \frac{\kappa}{a^2}\right], \\ G^i_j &= \left[\frac{2\,\ddot{a}}{c^2\,a} + \left(\frac{\dot{a}}{c\,a}\right)^2 + \frac{\kappa}{a^2}\right]\delta^i_j, \end{aligned}$$

where the overdots denote differentiation with respect to the cosmic time t.

(b) Consider a fluid described by the stress energy tensor

$$T^\mu_\nu = \mathrm{diag.}\left(\rho\,c^2, -p, -p, -p\right),$$

where ρ and p denote the mass density and the pressure associated with the fluid. In a smooth Friedmann universe, the quantities ρ and p depend only on time. Using the above Einstein tensor, obtain the following Friedmann equations for such a source:

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa\,c^2}{a^2} &= \frac{8\,\pi\,G}{3}\,\rho, \\ \frac{2\,\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa\,c^2}{a^2} &= -\,\frac{8\,\pi\,G}{c^2}\,p. \end{aligned}$$

(c) Show that these two Friedmann equations lead to the equation

$$\frac{\ddot{a}}{a} = -\,\frac{4\,\pi\,G}{3}\,\left(\rho + \frac{3\,p}{c^2}\right).$$

Note: This relation implies that $\ddot{a} > 0$, i.e. the universe will undergo accelerated expansion, only when $\left(\rho\,c^2 + 3\,p\right) < 0$.

2. *Conservation of the stress energy tensor in a FLRW universe:* Recall that the conservation of the stress energy tensor is described by the equation $T^\mu_{\nu;\mu} = 0$.

(a) Show that the time component of the stress energy tensor conservation law leads to the following equation in a Friedmann universe:

$$\dot{\rho} + 3\,H\,\left(\rho + \frac{p}{c^2}\right) = 0,$$

where $H = \dot{a}/a$, a quantity that is known as the Hubble parameter.

(b) Also arrive at this equation from the two Friedmann equations obtained above.

(c) Show that the above equation can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\, a^3\right) = -\frac{p}{c^2}\left(\frac{\mathrm{d}a^3}{\mathrm{d}t}\right).$$

3. *Evolution of energy density in a FLRW universe:* The different types of matter that are present in the universe are often described by an equation of state, i.e. the relation between the density and the pressure associated with the matter. Consider the following equation of state $p = w\,\rho\,c^2$, where w is a constant.

(a) Using the above equation which governs the evolution of ρ in a FLRW universe, show that, in such a case,

$$\rho \propto a^{-3\,(1+w)}.$$

(b) While the quantity w vanishes for pressure free non-relativistic matter (such as baryons and cold dark matter), $w = 1/3$ for relativistic particles (such as photons and the nearly massless neutrinos). Note that the energy density does not change with time when $w = -1$ or, equivalently, when $p = -\rho\,c^2$. Such a type of matter is known as the cosmological constant. Utilizing the above result, express the total density of a universe filled with non-relativistic (NR) and relativistic (R) matter as well as the cosmological constant (Λ) as follows:

$$\rho(a) = \rho_{\mathrm{NR}}^0 \left(\frac{a_0}{a}\right)^3 + \rho_{\mathrm{R}}^0 \left(\frac{a_0}{a}\right)^4 + \rho_{\Lambda},$$

where ρ_{NR}^0 and ρ_{R}^0 denote the density of non-relativistic and relativistic matter today (i.e. at, say, $t = t_0$, corresponding to the scale factor $a = a_0$).

(c) Also, further rewrite the above expression as

$$\rho(a) = \rho_{\mathrm{C}}\left[\Omega_{\mathrm{NR}}\left(\frac{a_0}{a}\right)^3 + \Omega_{\mathrm{R}}\left(\frac{a_0}{a}\right)^4 + \Omega_{\Lambda}\right] = \rho_{\mathrm{C}}\left[\Omega_{\mathrm{NR}}\,(1+z)^3 + \Omega_{\mathrm{R}}\,(1+z)^4 + \Omega_{\Lambda}\right],$$

where $\Omega_{\mathrm{NR}} = \rho_{\mathrm{NR}}^0/\rho_{\mathrm{C}}$, $\Omega_{\mathrm{R}} = \rho_{\mathrm{R}}^0/\rho_{\mathrm{C}}$ and $\Omega_{\Lambda} = \rho_{\Lambda}/\rho_{\mathrm{C}}$, while ρ_{C} is the so-called critical density defined as

$$\rho_{\mathrm{C}} = \frac{3\,H_0^2}{8\,\pi\,G},$$

with the quantity H_0 being the Hubble parameter (referred to as the Hubble constant) today. Note: The quantities H_0, Ω_{NR}, Ω_{R} and Ω_{Λ} are cosmological parameters that are to be determined by observations.

(d) Observations suggest that $H_0 \simeq 72$ km s^{-1} Mpc^{-1}. Evaluate the corresponding numerical value of the critical density ρ_{c}.

Note: A parsec (pc) corresponds to 3.26 light years, and a Mega parsec (Mpc) amounts to 10^6 parsecs.

4. *The Cosmic Microwave Background:* It is found that we are immersed in a perfectly thermal and nearly isotropic distribution of radiation, which is referred to as Cosmic Microwave Background (CMB), as it energy density peaks in the microwave region of the electromagnetic spectrum. The CMB is a relic of an earlier epoch when the universe was radiation dominated, and it provides the dominant contribution to the relativistic energy density in the universe.

(a) Given that the temperature of the CMB today is $T \simeq 2.73$ K, show that one can write

$$\Omega_{\mathrm{R}}\,h^2 \simeq 2.56 \times 10^{-5},$$

where h is related to the Hubble constant H_0 as follows:

$$H_0 \simeq 100\,h\ \mathrm{km\ s^{-1}\ Mpc^{-1}}.$$

(b) Show that the redshift $z_{\rm eq}$ at which the energy density of matter and radiation were equal is given by

$$1 + z_{\rm eq} = \frac{\Omega_{\rm NR}}{\Omega_{\rm R}} \simeq 3.9 \times 10^4 \left(\Omega_{\rm NR}\, h^2\right).$$

(c) Also, show that the temperature of the radiation at this epoch is given by

$$T_{\rm eq} \simeq 9.24 \left(\Omega_{\rm NR}\, h^2\right) \,{\rm eV}.$$

5. *Solutions to the Friedmann equations:* We had discussed the solutions to Friedmann equations in the presence of a single component when the universe is spatially flat (i.e. when $\kappa = 0$). It proves to be difficult to obtain analytical solutions for the scale factor when all the three components of matter (viz. non-relativistic and relativistic matter as well as the cosmological constant) are simultaneously present. However, the solutions can be obtained for the cases wherein two of the components are present.

(a) Integrate the first Friedmann equation for a $\kappa = 0$ universe with matter and radiation to obtain that

$$a(\eta) = \sqrt{\Omega_{\rm R}\, a_0^4}\; (H_0\, \eta) + \frac{\Omega_{\rm NR}\, a_0^3}{4}\; (H_0\, \eta)^2,$$

where η is the conformal time coordinate. Show that, at early (i.e. for small η) and late times (i.e. for large η), this solution reduces to the behavior in the radiation and matter dominated epochs, respectively, as required.

Note: In obtaining the above result, it has been assumed that $a = 0$ at $\eta = 0$.

(b) Integrate the Friedmann equation for a $\kappa = 0$ universe with matter and cosmological constant to obtain that

$$\frac{a(t)}{a_0} = \left(\frac{\Omega_{\rm NR}}{\Omega_\Lambda}\right)^{1/3} \sinh^{2/3}\left(3\,\sqrt{\Omega_\Lambda}\, H_0\, t/2\right).$$

Also, show that, at early times, this solution simplifies to $a \propto t^{2/3}$, while at late times, it behaves as $a \propto \exp\,(\Omega_\Lambda^{3/2}\, H_0\, t/\Omega_{\rm NR})$, as expected.