PH5875

ADVANCED GENERAL RELATIVITY

January–May 2023

Lecture schedule and meeting hours

- The course will consist of about 42 lectures, including about 10 tutorial sessions. However, note that there will be no separate tutorial sessions, and they will be integrated with the lectures.
- The duration of each lecture will be 50 minutes. We will be meeting in HSB 210.
- The first lecture will be on Wednesday, January 18, and the last one will be on Thursday, April 27.
- We will meet thrice a week. We shall meet during the following hours: 11:00–11:50 AM on Wednesdays, 9:00–9:50 AM on Thursdays, and 8:00–8:50 AM on Fridays.
- We may also meet during 5:00–5:50 PM on Tuesdays for either the quizzes or to make up for lectures that I may have to miss due to other unavoidable commitments. Changes in schedule, if any, will be notified sufficiently in advance.
- If you would like to discuss with me about the course outside the lecture hours, you are welcome to meet me at my office (HSB 202) during 9:00–10:00 AM on Fridays. In case you are unable to find me in my office on more than occasion, please send me an e-mail at `sriram@physics.iitm.ac.in` with the subject line containing the name of the course, i.e. PH5875: Advanced General Relativity.

Information about the course

- All the information regarding the course such as the schedule of the lectures, the structure and the syllabus of the course, suitable textbooks and additional references will be available on the course's page on Moodle at the following URL:

  `https://courses.iitm.ac.in/`

- The exercise sheets and other additional material will also be made available on Moodle.
- A PDF file containing these information as well as completed quizzes will also be available at the link on this course at the following URL:

  `http://physics.iitm.ac.in/~sriram/professional/teaching/teaching.html`

  I will keep updating this file and the course's page on Moodle as we make progress.

Quizzes, end-of-semester exam and grading

- The grading will be based on three scheduled quizzes and an end-of-semester exam.
- I will consider the best two quizzes for grading, and the two will carry 25% weight each.
- The three quizzes will be held on February 14, March 14 and April 18. All these three dates are Tuesdays, and the quizzes will be held during 5:00–6:30 PM.
- The end-of-semester exam will be held during 9:00 AM–12:00 Noon on Friday, May 12, and the exam will carry 50% weight.

Syllabus and structure

1. **Describing curved spacetimes** [$\sim$ 4 lectures]
   (a) Manifolds and coordinates – The metric
   (b) Covariant differentiation and the affine connection – Geodesics
   (c) Isometries – The Killing equation and conserved quantities
   (d) The Riemann tensor – The equation of geodesic deviation
   (e) The curvature and the Weyl tensors

Exercise sheets 1, 2, and 3

2. **Field equations of general relativity** [$\sim$ 6 lectures]
   (a) The equivalence principle – The principle of general covariance – The principle of minimal gravitational coupling
   (b) The vacuum Einstein's equations
   (c) Derivation of vacuum Einstein's equations from the action – The Bianchi identities
   (d) The stress-energy tensor – The cases of perfect fluid, scalar and electromagnetic fields
   (e) Non-canonical scalar fields – Relation to relativistic fluids
   (f) The structure of the Einstein's equations

Exercise sheets 4 and 5

Quiz I

Additional exercises I

3. **Schwarzschild geometry and tests of general relativity** [$\sim$ 6 lectures]
   (a) The general static isotropic metric – The Schwarzschild solution
   (b) Motion of particles and photons in the Schwarzschild metric
   (c) Precession of the perihelion of Mercury – Bending of light – Gravitational redshift

Exercise sheets 6 and 7

4. **Static and stationary black holes** [$\sim$ 6 lectures]
   (a) Schwarzschild black hole – Event horizon – Singularities
   (b) The Kruskal extension – Penrose diagrams
   (c) The Reissner-Nordstrom solution
   (d) The Kerr solution – Frame dragging — Ergosphere – Penrose process
   (e) Black hole thermodynamics – Hawking radiation

Exercise sheets 8 and 9

Quiz II

5. **Cosmology** [$\sim$ 12 lectures]
   (a) Homogeneity and isotropy – The Friedmann-Lemaître-Robertson-Walker line-element
   (b) Geodesics – Cosmological red-shift – Luminosity and angular diameter distances
   (c) Friedmann equations – Solutions with different types of matter

(d) Cosmological parameters – Age of the universe – Supernovae and late time acceleration

(e) Cosmic microwave background radiation – Thermal history – Big bang nucleosynthesis

(f) Horizon problem – Inflationary scenario – Generation of perturbations in the early universe

(g) Evolution of perturbations – Anisotropies in the cosmic microwave background – Recent constraints

Exercise sheets 10, 11, 12 and 13

Additional exercises II

Quiz III

6. **Gravitational waves** [$\sim$ 8 lectures]

(a) Linearized Einstein's equations – Transverse-traceless gauge – Solutions to the wave equation

(b) Polarization of gravitational waves – Effects of gravitational waves on a ring of masses

(c) Generation of gravitational waves – The quadrupole formula for the energy loss – Hulse-Taylor binary pulsar

(d) Gravitational waves from binary systems – Interferometric detectors – Detection of gravitational waves

(e) Observed events – Gravitational wave astronomy – Implications for astrophysics of compact sources and theories of gravity

(f) The stochastic gravitational wave background – Current constraints – Implications for the physics of the early universe

Exercise sheets 14 and 15

End-of-semester exam

Advanced problems

Note: The topics highlighted in red could not be covered for want of time.

Basic textbooks

1. L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Course of Theoretical Physics, Volume 2), Fourth Edition (Pergamon Press, New York, 1975).
2. B. F. Schutz, *A First Course in General Relativity* (Cambridge University Press, Cambridge, 1990).
3. E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley, Redwood City, California, 1990).
4. R. d'Inverno, *Introducing Einstein's Relativity* (Oxford University Press, Oxford, 1992).
5. J. B. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (Pearson Education, Delhi, 2003).
6. S. Carroll, *Spacetime and Geometry* (Addison Wesley, New York, 2004).
7. M. P. Hobson, G. P. Efstathiou and A. N. Lasenby, *General Relativity: An Introduction for Physicists* (Cambridge University Press, Cambridge, 2006).
8. S. Weinberg, *Cosmology* (Oxford University Press, Oxford, England, 2008).

Additional references

1. S. Weinberg, *Gravitation and Cosmology* (John Wiley, New York, 1972).
2. A. P. Lightman, W. H. Press, R. H. Price and S. A. Teukolsky, *Problem Book in Relativity and Gravitation* (Princeton University Press, New Jersey, 1975).
3. S. Dodelson, *Modern Cosmology* (Academic Press, SanDiego, U.S.A., 2003).
4. V. F. Mukhanov, *Physical Foundations of Cosmology* (Cambridge University Press, Cambridge, England, 2005).
5. M. Maggiore, *Gravitational Waves: Volume 1: Theory and Experiments* (Oxford University Press, Oxford, England, 2007).
6. T. Padmanabhan, *Gravitation: Foundation and Frontiers* (Cambridge University Press, Cambridge, 2010).
7. A. Zee, *Einstein Gravity in a Nutshell* (Princeton University Press, Princeton, New Jersey, 2013).

Advanced texts

1. S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge University Press, Cambridge, 1973).
2. C. W. Misner, K. S. Thorne and J. W. Wheeler, *Gravitation* (W. H. Freeman and Company, San Francisco, 1973).
3. R. M. Wald, *General Relativity* (The University of Chicago Press, Chicago, 1984).
4. E. Poisson, *A Relativist's Toolkit* (Cambridge University Press, Cambridge, 2004).
5. M. Maggiore, *Gravitational Waves: Volume 2: Astrophysics and Cosmology* (Oxford University Press, Oxford, England, 2018).

Exercise sheet 1

Covariant differentiation, Christoffel symbols, geodesics and Killing vectors

1. *Geodesics on a plane:* Working in the polar coordinates, arrive at the geodesic equations on the plane. Solve the equations to show that the geodesics are straight lines.

   Note: You are expected to solve the geodesic equation involving the Christoffel symbols, and not the more familiar variation of the problem!

2. *Geodesics on a cone:* Consider a cone with a semi-vertical angle α.

   (a) Determine the line element on the cone.

   (b) Obtain the equations governing the geodesics on the cone.

   (c) Solve the equations to arrive at the geodesics.

3. *Geodesics in a Poincaré half plane:* Consider the so-called Poincaré half plane described by the line-element
$$\mathrm{d}l^2 = \frac{a^2}{y^2}\,\left(\mathrm{d}x^2 + \mathrm{d}y^2\right),$$
   where $-\infty < x < \infty$, while $0 < y < \infty$. Determine the trajectory $y(x)$ of geodesics in this geometry.

4. *Energy of photons in a Friedmann universe:* The spatially flat Friedmann universe is described by the line element
$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\,\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right),$$
   where $a(t)$ is known as the scale factor that characterizes the expansion of the universe.

   (a) Obtain the Christoffel symbols corresponding to this line element.

   (b) Explicitly write down the time and the spatial components of the geodesic equation governing a photon in the spatially flat Friedmann universe.

   (c) Solve the geodesic equation and show that the energy, say, E, of the photon behaves as $E \propto 1/a$.

   Note: Recall that the time component of the four momentum of a particle represents its energy. The above result implies that the energy of photons constantly decreases as the universe expands, a phenomenon that is known as cosmological redshift.

5. *Killing vectors of a plane in polar coordinates:* Consider the two dimensional Euclidean plane described in terms of the polar coordinates.

   (a) What is the line element of the Euclidean plane in terms of the polar coordinates?

   (b) Evaluate all the Christoffel symbols associated with the line element.

   (c) Write down the equations describing the Killing vectors in the polar coordinates.

   (d) Obtain all the Killing vectors by solving the equations and interpret the solutions.

Exercise sheet 2

Fields in curved spacetime

1. *Klein-Gordon equation in a curved spacetime:* Consider the following action that describes a scalar field, say, ϕ, in a generic curved spacetime:

$$S[\phi] = \frac{1}{c} \int \mathrm{d}^4\tilde{x}\, \sqrt{-g}\, \left(\frac{1}{2}\, g_{\mu\nu}\, \partial^\mu \phi\, \partial^\nu \phi - \frac{1}{2}\, \sigma^2\, \phi^2\right),$$

where $g_{\mu\nu}$ is the metric tensor that describes the curved spacetime, while the quantity σ is related to the mass of the field. Also, the quantity g denotes the determinant of the metric tensor $g_{\mu\nu}$.

   (a) Vary the above action to arrive at the equation of motion for the scalar field.

   (b) Show that equation of motion of the scalar field can be written as

$$\nabla_\mu \nabla^\mu \phi + \sigma^2 \phi \equiv \phi^{;\mu}{}_{;\mu} + \sigma^2\, \phi = 0.$$

2. *Tachyons:* Consider a scalar field T that is described by the action

$$S[T] = -\frac{1}{c} \int \mathrm{d}^4\tilde{x}\, \sqrt{-g}\, V(T)\, \sqrt{1 - \alpha^2\, \partial_\mu T\, \partial^\mu T},$$

where α is a constant of suitable dimensions.

Note: The field T is often referred to as the tachyon.

   (a) Vary the action with respect to the scalar field T to arrive at the equation of motion governing the field in a curved spacetime.

   (b) Construct the stress-energy tensor associated with the field T.

   (c) Show that the conservation of the stress-energy tensor leads to the equation of motion governing the field T.

3. *Generic scalar fields:* Consider a generic scalar field ϕ that is described by the action

$$S[\phi] = \frac{1}{c} \int \mathrm{d}^4\tilde{x}\, \sqrt{-g}\, \mathcal{L}(X, \phi),$$

where X denotes the kinetic energy of the scalar field and is given by

$$X = \frac{1}{2}\, g^{\mu\nu}\, \partial_\mu \phi\, \partial_\nu \phi.$$

   (a) Let the Lagrangian density $\mathcal{L}$ be an arbitrary function of the kinetic term X and the field ϕ. Vary the above action with respect to the metric tensor and obtain the corresponding stress-energy tensor.

   Note: Such scalar fields are often referred to as k-essence.

   (b) Assuming $\mathcal{L} = X - V(\phi)$, where $V(\phi)$ is the potential describing the scalar field, determine the corresponding stress-energy tensor. From the conservation of the stress-energy tensor, arrive at the equation of motion governing the scalar field for the case wherein $V(\phi) = \sigma^2\, \phi^2/2$.

4. *Maxwell's equations in curved spacetime:* Typically, the equations governing fields in a curved spacetime can be arrived at by replacing the partial derivatives encountered in the Minkowski spacetime by the corresponding covariant derivatives.

   (a) Show that

$$F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu} = A_{\nu,\mu} - A_{\mu,\nu}.$$

(b) Establish that the first pair of Maxwell's equations in a curved spacetime, viz.

$$F_{\mu\nu;\lambda} + F_{\lambda\mu;\nu} + F_{\nu\lambda;\mu} = 0,$$

actually reduce to

$$F_{\mu\nu,\lambda} + F_{\lambda\mu,\nu} + F_{\nu\lambda,\mu} = 0.$$

(c) Show that the second pair of Maxwell's equations in a curved spacetime, viz.

$$F^{\mu\nu}{}_{;\nu} = \frac{4\,\pi}{c}\, j^{\mu},$$

can be written as

$$\frac{1}{\sqrt{-g}}\,\partial_{\nu}\left(\sqrt{-g}\,F^{\mu\nu}\right) = \frac{4\,\pi}{c}\, j^{\mu}.$$

5. *Conformal invariance of the electromagnetic action:* Recall that, in a curved spacetime, the dynamics of the source free electromagnetic field is governed by the action

$$S[A^{\mu}] = -\frac{1}{16\,\pi\,c}\int \mathrm{d}^4\tilde{x}\,\sqrt{-g}\,F_{\mu\nu}\,F^{\mu\nu},$$

where

$$F_{\mu\nu} = A_{\mu;\nu} - A_{\nu;\mu} = A_{\mu,\nu} - A_{\nu,\mu},$$

while the commas and semi-colons, as usual, represent partial and covariant differentiation, respectively. Show that this action is invariant under the following conformal transformation:

$$x^{\mu} \to x^{\mu}, \quad A_{\mu} \to A_{\mu} \quad \text{and} \quad g_{\mu\nu} \to \Omega^2\, g_{\mu\nu}.$$

Exercise sheet 3

Riemann, Ricci tensors and scalar curvature

1. *Algebraic identity involving the Riemann tensor:* Recall that, the Riemann tensor is defined as

$$R^{a}{}_{bcd} = \Gamma^{a}_{bd,c} - \Gamma^{a}_{bc,d} + \Gamma^{a}_{ec}\,\Gamma^{e}_{bd} - \Gamma^{a}_{ed}\,\Gamma^{e}_{bc}.$$

   Using this expression, establish that

$$R^{a}{}_{bcd} + R^{a}{}_{dbc} + R^{a}{}_{cdb} = 0.$$

2. *The number of independent components of the Riemann tensor:* Show that, on a n-dimensional manifold, the number of independent components of the Riemann tensor are $\left(n^2/12\right)\,\left(n^2-1\right)$.

3. *The flatness of the cylinder:* Calculate the Riemann tensor of a cylinder of constant radius, say, R, in three dimensional Euclidean space. What does the result you find imply?

   Note: The surface of the cylinder is actually two-dimensional.

4. *The curvature of the two-sphere:* Calculate all the components of the Riemann and the Ricci tensors, and also the corresponding scalar curvature associated with the two sphere.

   Note: Given the Riemann tensor $R^{a}{}_{bcd}$, the Ricci tensor R_{ab} and the Ricci scalar R are defined as

$$R_{ab} = R^{c}{}_{acb} \qquad \text{and} \qquad R = g^{ab}\,R_{ab}.$$

5. *Identities involving the covariant derivative and the Riemann tensor:* Establish the following relations:

   (a) $\nabla_c\,\nabla_b\,A_a - \nabla_b\,\nabla_c\,A_a = R^{d}{}_{abc}\,A_d$,

   (b) $\nabla_d\,\nabla_c\,A_{ab} - \nabla_c\,\nabla_d\,A_{ab} = R^{e}{}_{bcd}\,A_{ae} + R^{e}{}_{acd}\,A_{eb}$.

Exercise sheet 4

Einstein's equations

1. *The Palatini relation:* Show that

$$\delta R^{a}{}_{bcd} = \nabla_c \left(\delta\Gamma^a_{bd}\right) - \nabla_d \left(\delta\Gamma^a_{bc}\right)$$

and, hence,

$$\delta R_{ab} = \nabla_c \left(\delta\Gamma^c_{ab}\right) - \nabla_b \left(\delta\Gamma^c_{ac}\right).$$

2. *Einstein tensor:* Recall that we can write

$$\sqrt{-g}\, R = \sqrt{-g}\, \left[G + \frac{1}{\sqrt{-g}}\, \partial_\alpha \left(\sqrt{-g}\, w^\alpha\right)\right],$$

where

$$G = g^{ab} \left(\Gamma^c_{ad}\, \Gamma^d_{bc} - \Gamma^c_{cd}\, \Gamma^d_{ab}\right) \quad \text{and} \quad w^a = g^{bc}\, \Gamma^a_{bc} - g^{ab}\, \Gamma^c_{bc}.$$

Show that the Einstein tensor can be written as

$$G_{ab} = R_{ab} - \frac{1}{2}\, g_{ab}\, R = \frac{1}{\sqrt{-g}}\, \left\{\frac{\partial(\sqrt{-g}\, G)}{\partial g^{ab}} - \partial_c \left[\frac{\partial(\sqrt{-g}\, G)}{\partial\left(\partial_c g^{ab}\right)}\right]\right\}.$$

3. *Newtonian limit and Poisson equation:* Recall that, in the non-relativistic limit, the metric corresponding to the Newtonian potential ϕ is given by

$$\mathrm{d}s^2 = c^2\, \left[1 + \frac{2\, \phi(\boldsymbol{x})}{c^2}\right]\, \mathrm{d}t^2 - \mathrm{d}\boldsymbol{x}^2.$$

Let the energy density of the matter field that is giving rise to the Newtonian potential ϕ be $\rho\, c^2$. Show that, in such a case, the time-time component of the Einstein's equations reduces to the conventional Poisson equation in the limit of large c.

Note: It is this Newtonian limit that determines the overall constant in the Einstein-Hilbert action.

4. *The Bianchi identity:* Recall that, the Riemann tensor is defined as

$$R_{abcd} = g_{ae}\, R^{e}{}_{bcd} = g_{ae}\, \left(\Gamma^e_{bd,c} - \Gamma^e_{bc,d} + \Gamma^e_{fc}\, \Gamma^f_{bd} - \Gamma^e_{fd}\, \Gamma^f_{bc}\right).$$

Also, note that, given the Riemann tensor $R^{a}{}_{bcd}$, the Ricci tensor R_{ab} and the Ricci scalar R are defined as

$$R_{ab} = R^{c}{}_{acb} \qquad \text{and} \qquad R = g^{ab}\, R_{ab}.$$

Further, the Einstein tensor is given by

$$G_{ab} = R_{ab} - \frac{1}{2}\, R\, g_{ab}.$$

   (a) Using the expression for the Riemann tensor, establish the following Bianchi identity:

$$\nabla_e\, R_{abcd} + \nabla_d\, R_{abec} + \nabla_c\, R_{abde} = 0.$$

   Note: It will be convenient to work in the so-called local coordinates, where the Christoffel symbols vanish, but their derivatives do not.

   (b) Using the above identity, show that

$$\nabla_b\, G^b_a = 0.$$

(c) Using the Bianchi identity, show that the covariant derivative of the Ricci tensor R^a_b can be expressed in terms of the partial derivative of the Ricci scalar R as follows:

$$\nabla_a R^a_b = \frac{1}{2}\,\partial_b R.$$

5. *Structure of the Einstein's equations:* Establish the following properties of the Einstein's equations.

(a) Show that the G^0_0 and G^0_i (i.e. the time-time and time-space) components of the Einstein tensor do not depend on $\dot{g}_{00}$ and $\dot{g}_{0i}$, where the overdots denote differentiation with respect to time. Also, show that these components depend only on $\dot{g}_{ij}$.

(b) Moreover, show that it is only the G^i_j (i.e. the space-space) components of the Einstein tensor that depend on $\ddot{g}_{ij}$.

Note: These imply that the time-time and time-space components of the Einstein's equations are constraints and it is the spatial components of the equations that govern the dynamics of the metric components g_{ij}.

Exercise sheet 5

Stress-energy tensor

1. *Conservation of the stress-energy tensor:* Let the action describing a matter field be given by

$$S = \frac{1}{c}\int \mathrm{d}^4\tilde{x}\,\sqrt{-g}\,\mathcal{L},$$

where $\mathcal{L}$ is the Lagrangian density describing the matter field. Consider the following variation of the spacetimes coordinates: $x^\mu \to x'^\mu = x^\mu + \xi^\mu$, where ξ^μ is an infinitesimal quantity.

   (a) Show that, under such a transformation, the contravariant and covariant metric tensors transform as follows:

$$g^{\mu\nu} \to g'^{\mu\nu} = g^{\mu\nu} + \delta g^{\mu\nu} \quad \text{and} \quad g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu},$$

   where $\delta g^{\mu\nu}$ and $\delta g_{\mu\nu}$ are given by

$$\delta g^{\mu\nu} = \xi^{\mu;\nu} + \xi^{\nu;\mu} \quad \text{and} \quad \delta g_{\mu\nu} = -\xi_{\mu;\nu} - \xi_{\nu;\mu}.$$

   (b) Show that, under such a variation, the corresponding variation in the action can be written as

$$\delta S = -\frac{1}{c}\int \mathrm{d}^4\tilde{x}\,\sqrt{-g}\,\left(\nabla_\mu T^\mu_\nu\right)\,\xi^\nu,$$

   where T^μ_ν denotes the stress-energy tensor of the matter field defined through the relation

$$\frac{1}{2}\sqrt{-g}\,T_{\mu\nu} = \frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial g^{\mu\nu}} - \partial_\lambda\left[\frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial\left(\partial_\lambda g^{\mu\nu}\right)}\right].$$

   (c) Argue that, invariance of the action under the transformation requires that

$$\nabla_\mu T^\mu_\nu = 0,$$

   i.e. the stress-energy tensor of the matter field is covariantly conserved.

2. *The stress-energy tensor of an ideal fluid:* Consider an ideal fluid described by the energy density $\rho\, c^2$ (with ρ being the mass density) and pressure p. Further, assume that the fluid does not possess any anisotropic stress.

   (a) Argue that, in the comoving frame, the stress energy tensor of the fluid is given by

$$T^\mu_\nu = \mathrm{diag.}\,\left(\rho\, c^2, -p, -p, -p\right).$$

   (b) Further, show that, in a general frame, the stress energy tensor of the fluid can be written as

$$T^\mu_\nu = (\rho\, c^2 + p)\, u^\mu\, u_\nu - p\,\delta^\mu_\nu,$$

   where u^μ is the four velocity of the fluid.

   (c) Using the law governing the conservation of the stress energy tensor, arrive at the equations of motion that describe an ideal fluid in Minkowski spacetime.

3. *The stress-energy tensor of a scalar field:* Recall that, given an action that describes a matter field, the stress-energy tensor associated with the matter field is given by the variation of the action with respect to the metric tensor as follows:

$$\delta S = \frac{1}{2\,c}\int \mathrm{d}^4\tilde{x}\,\sqrt{-g}\,T_{\mu\nu}\,\delta g^{\mu\nu} = -\frac{1}{2\,c}\int \mathrm{d}^4\tilde{x}\,\sqrt{-g}\,T^{\mu\nu}\,\delta g_{\mu\nu}.$$

Consider a scalar field ϕ that is governed by the following action:

$$S[\phi] = \frac{1}{c}\int \mathrm{d}^4x\,\sqrt{-g}\,\left[\frac{1}{2}\,g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right],$$

where $V(\phi)$ is the potential describing the scalar field.

(a) Upon varying this action with respect to the metric tensor, arrive at the stress energy tensor of the scalar field.

(b) Show that the conservation of the stress-energy tensor leads to the equation of motion of the scalar field.

4. *The stress-energy tensor of the electromagnetic field:* In a curved spacetime, the action describing the electromagnetic field is given by

$$S[A^\mu] = -\frac{1}{16\,\pi\,c} \int \mathrm{d}^4\tilde{x}\,\sqrt{-g}\,F_{\mu\nu}\,F^{\mu\nu},$$

where

$$F_{\mu\nu} = A_{\mu;\nu} - A_{\nu;\mu} = A_{\mu,\nu} - A_{\nu,\mu}.$$

(a) Construct the stress-energy tensor associated with the electromagnetic field.

(b) What are the time-time and the time-space components of the stress energy tensor of the electromagnetic field in flat spacetime?

5. *Energy conditions:* There are five so-called energy conditions — viz. weak energy condition (WEC), null energy condition (NEC), dominant energy condition (DEC), dominant null energy condition (DNEC) and the strong energy condition (SEC) — which are often used to characterize the stress-energy tensor of matter fields. Let t^μ and l^μ denote timelike and lightlike vectors, i.e. $t_\mu\,t^\mu > 0$ and $l_\mu\,l^\mu = 0$. The energy conditions are defined in terms of the vectors t^μ and l^μ as follows:

- WEC: $T_{\mu\nu}\,t^\mu\,t^\nu \geq 0$,
- NEC: $T_{\mu\nu}\,l^\mu\,l^\nu \geq 0$,
- DNEC: $T_{\mu\nu}\,t^\mu\,t^\nu \geq 0$ and $T_{\mu\nu}\,t^\mu$ is a non-spacelike vector, i.e. $T_{\mu\nu}\,T^\nu_\lambda\,t^\mu\,t^\lambda \geq 0$,
- NDEC: $T_{\mu\nu}\,l^\mu\,l^\nu \geq 0$ and $T_{\mu\nu}\,l^\mu$ is a non-spacelike vector, i.e. $T_{\mu\nu}\,T^\nu_\lambda\,l^\mu\,l^\lambda \geq 0$,
- SEC: $T_{\mu\nu}\,t^\mu\,t^\nu \geq \frac{1}{2}\,T^\mu_\mu\,t^\nu\,t_\nu$.

Recall that the stress-energy tensor of an ideal fluid is given by

$$T^\mu_\nu = (\rho\,c^2 + p)\,u^\mu\,u_\nu - p\,\delta^\mu_\nu,$$

where ρ, p and u^μ denote the mass density, pressure and four velocity of the fluid. Determine the conditions on ρ and p of an ideal fluid that correspond to the different energy conditions.

Quiz I

From describing curved spacetimes to Einstein's equations

1. *Transformation properties of Christoffel symbols:* Recall that, the Christoffel symbols, say, Γ^a_{bc}, are introduced through the following definition of the covariant derivative:

$$\nabla_b A^a = \partial_b A^a + \Gamma^a_{cb}\, A^c, \quad \nabla_b A_a = \partial_b A_a - \Gamma^c_{ab}\, A_c.$$

   (a) Using the transformation properties of the covariant derivative, obtain the transformation rule for the Christoffel symbols Γ^a_{bc}. 8 marks

   (b) Show that the following difference of the Christoffel symbols:

$$T^a_{bc} = \Gamma^a_{bc} - \Gamma^a_{cb}$$

   transforms as a tensor. 2 marks

   Note: We are not yet assuming that the Christoffel symbols can be expressed in terms of the metric tensor. The tensor T^a_{bc} is referred to as torsion.

2. *Parallel transporting a vector on a sphere:* Given a vector, say, A^a, on a sphere at $(\theta = \theta_0, , \phi = 0,$ parallel transport the vector along the curve $\theta = \theta_0$ and determine the vector at arbitrary ϕ such that $0 < \phi < 2\,\pi$. 10 marks

3. *Lie derivative:* Consider the following transformation from a point P to Q generated by the vector X^a:

$$x^{a\prime} = x^a + \delta\lambda\, X^a,$$

   where $\delta\lambda$ is an infinitesimal quantity.

   (a) Consider a tensor T^{ab}. The above transformation can be thought of dragging the tensor T^{ab} from the point P to the point Q. Express $T_{ab}(x^{c\prime})$ in terms of $T_{ab}(x^c)$. 3 marks

   (b) The Lie derivative of the tensor T^{ab} with respect to the vector X^a is defined as follows:

$$L_X T^{ab} = \lim_{\delta\lambda\to 0}\left[\frac{T^{ab}(x^{c\prime}) - T^{ab\prime}(x^{c\prime})}{\delta\lambda}\right].$$

   Show that the above Lie derivative can be expressed as 4 marks

$$L_X T^{ab} = X^c\, \partial_c T^{ab} - T^{ac}\, \partial_c X^b - T^{cb}\, \partial_c X^a.$$

   (c) Let ξ^a denote a Killing vector. Show that the Lie derivative of the metric tensor g_{ab} along the Killing vector ξ^a vanishes, i.e. $L_\xi g_{ab} = 0$. 3 marks

4. *Is the spacetime curved?* The Friedmann-Lemaître-Robertson-Walker (FLRW) universe is described by the line-element

$$\mathrm{d}s^2 = c^2\, \mathrm{d}t^2 - a^2(t)\, \left[\frac{\mathrm{d}r^2}{1-\kappa\, r^2} + r^2\, \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right)\right],$$

   where the function $a(t)$ is referred to as the scale factor and $\kappa = 0, \pm 1$.

   (a) Consider the case wherein $a(t) = c\,t$ and $\kappa = -1$. What does the metric describe? Is it a curved spacetime? 4 marks

Note: For the FLRW line element, the non-zero components of the Ricci tensor and scalar curvature are given by

$$\begin{aligned} R^0_0 &= -\frac{3\,\ddot{a}}{c^2\,a}, \\ R^i_j &= -\left[\frac{\ddot{a}}{c^2\,a} + 2\left(\frac{\dot{a}}{c\,a}\right)^2 + \frac{2\,\kappa}{a^2}\right]\delta^i_j, \\ R &= -6\left[\frac{\ddot{a}}{c^2\,a} + \left(\frac{\dot{a}}{c\,a}\right)^2 + \frac{\kappa}{a^2}\right]. \end{aligned}$$

(b) Can you construct a coordinate transformation that reduces the FLRW line-element with $a(t) = c\,t$ and $\kappa = -1$ to the Minkowskian form? 6 marks

5. <u>*Stress-energy tensor of the Kalb-Ramond field:*</u> The Kalb-Ramond field $H_{\alpha\beta\gamma}$ is defined in terms of the anti-symmetric tensor $B_{\alpha\beta}$ through the following relation in a curved spacetime:

$$H_{\alpha\beta\gamma} = \nabla_\alpha B_{\beta\gamma} + \nabla_\gamma B_{\alpha\beta} + \nabla_\beta B_{\gamma\alpha}.$$

(a) Show that 2 marks

$$H_{\alpha\beta\gamma} = \nabla_\alpha B_{\beta\gamma} + \nabla_\gamma B_{\alpha\beta} + \nabla_\beta B_{\gamma\alpha} = \partial_\alpha B_{\beta\gamma} + \partial_\gamma B_{\alpha\beta} + \partial_\beta B_{\gamma\alpha}.$$

(b) Show that the Kalb-Ramond field is invariant under the gauge transformation

$$B_{\alpha\beta} \to B_{\alpha\beta} + \nabla_\alpha \epsilon_\beta - \nabla_\beta \epsilon_\alpha,$$

where ϵ_α is an arbitrary four vector. 4 marks

(c) The free Kalb-Ramond field is described by the action

$$S[B_{\alpha\beta}(\tilde{x})] = -\frac{1}{6\,\kappa^2\,c}\int \mathrm{d}^4\tilde{x}\,\sqrt{-g}\; H_{\alpha\beta\gamma}\,H^{\alpha\beta\gamma}.$$

Vary the action with respect to the metric tensor to arrive at the stress energy tensor associated with the Kalb-Ramond field $H^{\alpha\beta\gamma}$. 4 marks

Additional exercises I

Tensor calculus and general relativity

1. <u>*Rewriting the Riemann tensor:*</u> Recall that, the Riemann tensor is defined as

$$R_{abcd} = g_{ae}\, R^{e}{}_{bcd} = g_{ae}\, \left(\Gamma^{e}_{bd,c} - \Gamma^{e}_{bc,d} + \Gamma^{e}_{fc}\,\Gamma^{f}_{bd} - \Gamma^{e}_{fd}\,\Gamma^{f}_{bc}\right).$$

Show that this can be rewritten as

$$R_{abcd} = \frac{1}{2}\, \left(g_{ad,bc} + g_{bc,ad} - g_{ac,bd} - g_{bd,ac}\right) + g_{ef}\, \left(\Gamma^{e}_{bc}\,\Gamma^{f}_{ad} - \Gamma^{e}_{bd}\,\Gamma^{f}_{ac}\right),$$

an expression which reflects the symmetries of the Riemann tensor more readily.

2. <u>*Properties of Killing vectors:*</u> If ξ^a is a Killing vector, show that

   (a) $\xi_{a;b;c} = R_{dcba}\,\xi^d$,

   (b) $\xi_{a;b}{}^{;b} + R_{ac}\,\xi^c = 0$.

3. <u>*Geodesic deviation:*</u> Consider two nearby geodesics, say, $x^a(\lambda)$ and $\bar{x}^a(\lambda)$, where λ is an affine parameter. Let $\xi^a(\lambda)$ denote a 'small vector' that connects these two geodesics. Working in the locally geodesic coordinates, show that ξ^a satisfies the differential equation

$$\frac{\mathrm{D}^2\xi^a}{\mathrm{D}\lambda^2} + R^{a}_{bcd}\,\dot{x}^b\,\xi^c\,\dot{x}^d = 0,$$

where

$$\frac{\mathrm{D}^2\xi^a}{\mathrm{D}\lambda^2} \equiv \left(\dot{\xi}^a + \Gamma^{a}_{bc}\,\xi^b\,\dot{x}^c\right)^{\boldsymbol{\cdot}},$$

while the overdots denote differentiation with respect to λ.

Note: This implies that a non-zero Riemann tensor R^{a}_{bcd} will lead to a situation where geodesics, in general, will not remain parallel as, for instance, on the surface of the two sphere $\mathbb{S}^2$.

4. <u>*Scalar curvature in two dimensions:*</u> Consider the following $(1+1)$-dimensional line element:

$$\mathrm{d}s^2 = f^2(\eta,\xi)\, \left(\mathrm{d}\eta^2 - d\xi^2\right),$$

where $f(\eta,\xi)$ is an arbitrary function of the coordinates η and ξ. Show that the scalar curvature associated with this line-element can be expressed as

$$R = -\nabla_\mu\nabla^\mu\,\ln f^2 = -\Box\,\ln f^2.$$

Note: In $(1+1)$-dimensions, any metric can be reduced to the above, so-called conformally flat form.

5. <u>*Conformal transformations:*</u> Show that, under the conformal transformation,

$$g_{ab}(x^c) \to \Omega^2(x^c)\; g_{ab}(x^c),$$

the Christoffel symbols Γ^{a}_{bc}, the Ricci tensor R^{a}_{b}, and the scalar curvature R of a n-dimensional manifold are modified as follows:

$$\begin{aligned}
\Gamma^{a}_{bc} &\;\to\; \Gamma^{a}_{bc} + \Omega^{-1}\, \left(\delta^{a}_{b}\,\Omega_{;c} + \delta^{a}_{c}\,\Omega_{;b} - g_{bc}\,g^{ad}\,\Omega_{;d}\right),\\
R^{a}_{b} &\;\to\; \Omega^{-2}\,R^{a}_{b} - (n-2)\,\Omega^{-1}\,g^{ac}\,\left(\Omega^{-1}\right)_{;bc} + \frac{1}{n-2}\,\Omega^{-n}\,\delta^{a}_{b}\,g^{cd}\,\left[\Omega^{(n-2)}\right]_{;cd},\\
R &\;\to\; \Omega^{-2}\,R + 2\,(n-1)\,\Omega^{-3}\,g^{ab}\,\Omega_{;ab} + (n-1)\,(n-4)\,\Omega^{-4}\,g^{ab}\,\Omega_{;a}\,\Omega_{;b}.
\end{aligned}$$

6. *Weyl tensor:* In $(3+1)$-spacetime dimensions, the Weyl tensor $C_{\alpha\beta\gamma\delta}$ is defined as follows:

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + \frac{1}{2}\,(g_{\alpha\delta}\,R_{\beta\gamma} + g_{\beta\gamma}\,R_{\alpha\delta} - g_{\alpha\gamma}\,R_{\beta\delta} - g_{\beta\delta}\,R_{\alpha\gamma}) + \frac{1}{6}\,(g_{\alpha\gamma}\,g_{\delta\beta} - g_{\alpha\delta}\,g_{\gamma\beta})\,R.$$

Show that the Weyl tensor is invariant under conformal transformations.

Note: This implies that a spacetime is conformally flat if the Weyl tensor vanishes.

7. *Useful identities:* Establish the following identities.

   (a) Show that

$$\frac{\partial g^{cd}}{\partial g_{ab}} = -\frac{1}{2}\,\left(g^{ac}\,g^{bd} + g^{ad}\,g^{bc}\right).$$

   (b) If $\bar{g}_{ab} = \sqrt{-g}\,g_{ab}$, show that

$$\bar{g}^{ab}, c = \bar{g}^{ab}\,\Gamma^{d}_{dc} - \bar{g}^{db}\,\Gamma^{a}_{dc} - \bar{g}^{ad}\,\Gamma^{b}_{dc}.$$

   (c) Show that

$$\frac{\partial\,(\sqrt{-g}\,R)}{\partial g_{ab,cd}} = \sqrt{-g}\,\left[\frac{1}{2}\,\left(g^{ac}\,g^{bd} + g^{ad}\,g^{bc}\right) - g^{ab}\,g^{cd}\right].$$

8. *Actions involving higher derivatives:* There is another approach (apart from one I had discussed in the lectures) to arrive at the Einstein's equations. Recall that, in the introductory course, we had considered a scalar field ϕ that is described by the action

$$S[\phi(\tilde{x})] = \frac{1}{c}\int \mathrm{d}^4\tilde{x}\,\mathcal{L}(\phi, \partial_\mu\phi, \partial_\mu\partial_\nu\phi).$$

Note that, in addition to the usual dependence on the scalar field ϕ and its first derivative $\partial_\mu\phi$, the Lagrangian density $\mathcal{L}$ depends on the second derivatives $\partial_\mu\partial_\nu\phi$.

   (a) Derive the Euler-Lagrange equation for the system.

   Note: In order to obtain the Euler-Lagrange equation, the variation of the field *as well as* its first derivative need to be set to zero at the initial and final hypersurfaces.

   (b) Consider an action of the following form that describes a theory of gravitation:

$$S[g_{\mu\nu}(\tilde{x})] = \int \mathrm{d}^4\tilde{x}\,\sqrt{-g}\,\mathcal{L}_{\mathrm{G}}(g_{\mu\nu}, \partial_\alpha g_{\mu\nu}, \partial_\alpha\partial_\beta g_{\mu\nu}).$$

   Note that, apart from the metric tensor $g_{\mu\nu}$ and its first derivative $g_{\mu\nu,\alpha}$, the above action depends on the second derivative of the metric tensor, viz. $g_{\mu\nu,\alpha\beta}$, as well. In a manner similar to that of the scalar field that we considered above, vary the above action with respect to the metric tensor to arrive at the following equation of motion:

$$\partial_\alpha\partial_\beta\left(\frac{\partial\mathcal{L}_{\mathrm{G}}}{\partial g_{\mu\nu,\alpha\beta}}\right) - \partial_\alpha\left(\frac{\partial\mathcal{L}_{\mathrm{G}}}{\partial g_{\mu\nu,\alpha}}\right) + \frac{\partial\mathcal{L}_{\mathrm{G}}}{\partial g_{\mu\nu}} = 0.$$

   (c) Assuming that gravitation is described by the standard Einstein-Hilbert action, i.e. with the Lagrangian density being given by

$$\mathcal{L}_{\mathrm{G}} = -\frac{c^3}{16\,\pi\,G}\,R,$$

   use the above Euler-Lagrange equation governing the metric tensor to arrive at the Einstein's equations in vacuum.

9. *Palatini formalism:* According to the Palatini approach, the metric, specifically $\bar{g}_{ab} = \sqrt{-g}\, g_{ab}$, and the metric connection Γ^a_{bc} that it symmetric in its covariant indices are to be treated as separate dynamical variables.

   (a) Vary the Einstein-Hilbert action with respect to $\bar{g}_{ab}$ and show that, in vacuum, the equation of motion governing the gravitational field is given by $R_{ab} = 0$.

   (b) Vary the Einstein-Hilbert action with respect to Γ^a_{bc} and show that it leads to the condition that the covariant derivative of the metric tensor vanishes identically.

   Note: As we have discussed earlier, it is this condition that helps us express the Christoffel symbols in terms of the derivatives of the metric tensor.

10. *The nature of a worm hole:* The spacetime of a worm hole is described by the line-element

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}r^2 - \left(b^2 + r^2\right)\,\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right),$$

where b is a constant with the dimensions of length that reflects the size of the 'traversable' region. Show that the energy density of matter has to be negative to sustain such a spacetime.

Exercise sheet 6

Schwarzschild spacetime

1. *Spherically symmetric spacetimes:* Consider the following line element that describes spherically symmetric spacetimes in $(3+1)$-dimensions:

$$\mathrm{d}s^2 = c^2\, e^{\Phi(t,r)}\, \mathrm{d}t^2 - e^{\Psi(t,r)}\, \mathrm{d}r^2 - r^2\, \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right),$$

where $\Phi(t,r)$ and $\Psi(t,r)$ are arbitrary functions of the coordinates t and r.

   (a) Find $g_{\mu\nu}$ and $g^{\mu\nu}$ corresponding to this line element.
   (b) Evaluate the resulting $\Gamma^{\alpha}_{\mu\nu}$.
   (c) Also, calculate the corresponding $R_{\mu\nu}$ and R.

2. *Utilizing the Bianchi identities:* Compute the Einstein tensor corresponding to the above line element and show that its non-zero components are given by

$$\begin{aligned}
G^t_t &= \left(\frac{\Psi'}{r} - \frac{1}{r^2}\right) \mathrm{e}^{-\Psi} + \frac{1}{r^2},\\
G^r_t &= -\frac{\dot{\Psi}}{c\, r\, \mathrm{e}^{\Psi}} = -G^t_r\, \mathrm{e}^{(\Psi-\Phi)},\\
G^r_r &= -\left(\frac{\Phi'}{r} + \frac{1}{r^2}\right) \mathrm{e}^{-\Psi} + \frac{1}{r^2},\\
G^\theta_\theta &= G^\phi_\phi = \frac{1}{2}\left(\frac{\Phi'\,\Psi'}{2} + \frac{\Psi'}{r} - \frac{\Phi'}{r} - \frac{\Phi'^2}{2} - \Phi''\right) \mathrm{e}^{-\Psi} + \frac{1}{2\,c^2}\left(\ddot{\Psi} + \frac{\dot{\Psi}^2}{2} - \frac{\dot{\Phi}\,\dot{\Psi}}{2}\right) \mathrm{e}^{-\Phi},
\end{aligned}$$

where the overdots and the overprimes denote differentiation with respect to t and r, respectively. Show that the contracted Bianchi identities, viz. $\nabla_\mu\, G^\mu_\nu = 0$, imply that the last of the above equations vanishes, if the remaining three equations vanish.

3. *Spherically symmetric vacuum solution of the Einstein's equations:* In the absence of any sources, the above components of the Einstein tensor should vanish. Integrate the equations to arrive at the following Schwarzschild line element:

$$\mathrm{d}s^2 = c^2\left(1 - \frac{2\,G\,M}{c^2\,r}\right)\mathrm{d}t^2 - \left(1 - \frac{2\,G\,M}{c^2\,r}\right)^{-1} \mathrm{d}r^2 - r^2\, \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right),$$

where M is a constant of integration that denotes the mass of the central object that is responsible for the gravitational field.

4. *Isotropic coordinates:* Consider a transformation such that the coordinates (t, θ, ϕ) of the Schwarzschild line element remain unchanged, while the radial coordinate is transformed to a new coordinate, i.e. $r \to \rho = \rho(r)$.

   (a) If the above Schwarzschild line element can be expressed as

$$\mathrm{d}s^2 = c^2\left(1 - \frac{2\,G\,M}{c^2\,r}\right)\mathrm{d}t^2 - \lambda^2(\rho)\, \left[\mathrm{d}\rho^2 + \rho^2\, \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right)\right],$$

   determine the function $\lambda(\rho)$.

   (b) Also, express the complete line element in terms of the new coordinates (t, ρ, θ, ϕ).

Note: The new set of coordinates (t, ρ, θ, ϕ) are known as the isotropic coordinates.

5. *The Schwarzschild singularity:* Evaluate the so-called Kretschmann scalar, viz. $R_{\alpha\beta\gamma\delta}\, R^{\alpha\beta\gamma\delta}$, for the case of the Schwarzschild metric. Show that, whereas the quantity is finite at the Schwarzschild radius $r_{\rm S} = 2\, G\, M/c^2$, it diverges at the origin.

   Note: This implies that, while the Schwarzschild radius is a coordinate singularity (which can be avoided with a better choice of coordinates to describe the spacetime), the singularity at the origin is an unavoidable, physical one.

Exercise sheet 7

Classical tests of general relativity

1. *Gravitational redshift:* Consider two observers located at radii r_1 and r_2 in the static and spherically symmetric gravitational field around an object of mass M. Let the infinitesimal proper times in the frames of the two observers at r_1 and r_2 be $\mathrm{d}\tau_1$ and $\mathrm{d}\tau_2$, respectively.

   (a) Assuming that the static spacetime around the central object of mass M is described by the Schwarzschild line element, arrive at the relation between the proper times $\mathrm{d}\tau_1$ and $\mathrm{d}\tau_2$.

   (b) If the proper times denote the characteristic frequencies of atomic clocks located at the two radii, determine the relation between the corresponding angular frequencies, say, ω_1 and ω_2, or, equivalently, the associated energies E_1 and E_2.

   (c) If the energies E_1 and E_2 are associated with photons that climb out of the potential well, show that photons lose energy or, equivalently, their wavelengths grow longer.

   (d) Consider a photon that is emitted at the surface of the Sun, which reaches the Earth. If λ_{E} and λ_{O} denote the emitted and observed wavelengths, determine the extent to which the wavelength of the photon shifts as it reaches the Earth, i.e. calculate $(\lambda_{\mathrm{E}} - \lambda_{\mathrm{O}})/\lambda_{\mathrm{E}}$.

   Note: The radius of the Sun is $6.96 \times 10^8\,\mathrm{m}$, while the radius of Earth's orbit is $1.5 \times 10^{11}\,\mathrm{m}$.

   Note: This phenomenon where the wavelengths of the photons exhibit a shift towards the red end of the electromagnetic spectrum, as they climb out of a potential well, is known as gravitational redshift.

2. *Effective potential governing the motion of massive particles in Schwarzschild spacetime:* Consider a particle of mass m that is moving in the Schwarzschild spacetime.

   (a) Using the relation $p^\mu\, p_\mu = m^2\, c^2$ and the conserved energy, say, E, and the conserved angular momentum, say, L, show that the motion of the particle is described by the equation

   $$\left(\frac{\mathrm{d}r}{c\,\mathrm{d}\tau}\right)^2 = \tilde{E}^2 - \tilde{V}_{\mathrm{eff}}^2(r),$$

   where the effective potential $\tilde{V}_{\mathrm{eff}}^2(r)$ is given by

   $$\tilde{V}_{\mathrm{eff}}^2(r) = \left(1 - \frac{2\,G\,M}{c^2\,r}\right)\,\left(1 + \frac{\tilde{L}^2}{2\,r^2}\right),$$

   while $\tilde{E} = E/(m\,c^2)$ and $\tilde{L} = L/(m\,c)$.

   (b) Plot the effective potential $\tilde{V}_{\mathrm{eff}}^2(r)$ as a function of r/r_{S}, where $r_{\mathrm{S}} = 2\,G\,M/c^2$ is the Schwarzschild radius.

   (c) What are the radii at which circular orbits arise? Do these correspond to stable or unstable circular orbits?

   (d) Determine the value of $\tilde{L}$ below which no circular orbits are possible.

3. *Effective potential governing the motion of photons in Schwarzschild spacetime:* Consider a photon that is moving in the Schwarzschild spacetime. Let λ be the affine parameter that describes the trajectory of the photon.

   (a) Using the relation $p^\mu\, p_\mu = 0$ and the conserved energy E and angular momentum L, show that the motion of the photon is described by the equation

   $$\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 = \frac{E^2}{c^2} - B_{\mathrm{eff}}^2(r),$$

where the 'effective potential' $B^2_{\mathrm{eff}}(r)$ is given by

$$B^2_{\mathrm{eff}}(r) = \left(1 - \frac{2\,G\,M}{c^2\,r}\right)\,\frac{L^2}{r^2}.$$

(b) Plot the 'effective potential' $B^2_{\mathrm{eff}}(r)$ as a function of r/r_{S}, and determine the location of the circular orbit. Is the circular orbit stable or unstable?

(c) Are stable orbits possible for any value of L?

4. *Precession of the perihelion of Mercury:* Consider a particle of mass m moving in the Schwarzschild spacetime.

   (a) From the equation of motion obtained above, upon setting $u = 1/r$, arrive at the following differential equation describing the orbital motion of massive particles:

   $$\frac{\mathrm{d}^2\,u}{\mathrm{d}\phi^2} + u = \frac{G\,M}{\tilde{L}^2} + \frac{3\,G\,M}{c^2}\,u^2.$$

   (b) The second term on the right hand side of the above equation would have been absent in the case of the conventional, non-relativistic, Kepler problem. Treating the term as a small perturbation, show that the orbits are no more closed, and the perihelion precesses by the angle

   $$\Delta\phi \simeq \frac{6\,\pi\,(G\,M)^2}{\tilde{L}^2\,c^2} = \frac{6\,\pi\,G\,M}{a\,(1-e^2)\,c^2}\ \ \text{radians/revolution},$$

   where e and a are the eccentricity and the semi-major axis of the original closed, Keplerian elliptical orbit.

   (c) For the case of the planet Mercury, $a = 5.8 \times 10^{10}$ m, while $e = 0.2$. Also, the period of the Mercury's orbit around the Sun is 88 days. Further, the mass of the Sun is $M_{\odot} = 2 \times 10^{30}$ kg. Use these information to determine the angle by which the perihelion of Mercury would have shifted in a century.

   Note: The measured precession of the perihelion of the planet Mercury proves to be $5599''.7 \pm 0''.4$ per century, but a large part of it is caused due to the influences of the other planets. When the other contributions have been subtracted, the precession of the perihelion of the planet Mercury due to the purely relativistic effects amounts to 43.1 ± 0.5 seconds of arc per century.

5. *Gravitational bending of light:* Consider the propagation of photons in the Schwarzschild spacetime.

   (a) From the equation of motion obtained above, upon setting $u = 1/r$, arrive at the following differential equation describing the orbital motion of massless particles:

   $$\frac{\mathrm{d}^2\,u}{\mathrm{d}\phi^2} + u = \frac{3\,G\,M}{c^2}\,u^2,$$

   (b) Establish that, in the absence of the term on the right hand side, the photons will travel in straight lines.

   (c) As in the previous exercise, treating the term on the right hand side as a small perturbation, show that it leads to a deflection of a photon's trajectory by the angle

   $$\Delta\phi \simeq \frac{4\,G\,M}{c^2\,b},$$

   where $b = E/(c\,L)$ is the impact parameter of the photon (i.e. the distance of the closest approach of the photon to the central mass).

(d) Given that the radius of the Sun is 6.96×10^8 m, determine the deflection angle $\Delta\phi$ for a ray of light that grazes the Sun.

Note: The famous 1919 eclipse expedition led by Eddington led to two sets of results, viz.

$$\Delta\phi = 1''.98 \pm 0''.16 \quad \text{and} \quad \Delta\phi = 1''.61 \pm 0''.4,$$

both of which happen to be consistent with the theory.

Exercise sheet 8

Static black holes

1. *Coordinate time versus proper time:* Consider a particle of mass m that is falling radially into a Schwarzschild black hole. Assume that the particle starts at the radius r_0 at the proper time τ_0 and the coordinate time t_0.

   (a) Determine the radial position of the particle as a function of the proper time τ. What is the proper time when the particle reaches the event horizon of the black hole? When does it reach the singularity at $r = 0$?

   (b) Determine the radial position of the particle as a function of the coordinate time t. When does the particle reach the horizon of the black hole in terms of the coordinate time? Does it cross the event horizon of the black hole?

   Note: The Schwarzschild radius $r_{\mathrm{S}} = 2\,G\,M/c^2$ is referred to as the event horizon of the black hole. It is called so since it represents the boundary of all events that can be observed by observers located outside the radius.

2. *Spacetime diagram in the Schwarzschild coordinates:* Consider radially ingoing and outgoing null rays propagating in the Schwarzschild line element. Solve for the trajectories of these rays and plot them for different initial values of the radius r.

3. *Kruskal-Szekeres coordinates and the maximal extension:* Let us focus on the time and radial parts of the Schwarzschild metric.

   (a) For $r > r_{\mathrm{S}}$, determine the new radial coordinate, say, r^*, in terms of which the Schwarzschild line-element can be expressed in the following form:

   $$\mathrm{d}s^2 = \left(1 - \frac{2\,G\,M}{c^2\,r}\right)\,\left(c^2\,\mathrm{d}t^2 - \mathrm{d}r^{*2}\right).$$

   Note: The radial coordinate r^* is often referred to as the tortoise coordinate.

   (b) In terms of the null coordinates $u = c\,t - r^*$ and $v = c\,t + r^*$, show that the above line element reduces to

   $$\mathrm{d}s^2 = \left(1 - \frac{2\,G\,M}{c^2\,r}\right)\,\mathrm{d}u\,\mathrm{d}v.$$

   (c) Let us introduce two new coordinates, say, u' and v', through the relations

   $$u' = -\frac{4\,G\,M}{c^2}\,\exp\left(\frac{-c^2\,u}{4\,G\,M}\right), \quad v' = \frac{4\,G\,M}{c^2}\,\exp\left(\frac{c^2\,v}{4\,G\,M}\right).$$

   Express the Schwarzschild metric in terms of the coordinates u' and v'.

   (d) Defining new coordinates t' and $r^{*\prime}$ as

   $$c\,t' = \frac{1}{2}\,(v' + u'), \quad r^{*\prime} = \frac{1}{2}\,(v' - u'),$$

   express the line element in terms of the t' and $r^{*\prime}$.

   (e) Arrive at the relation between the coordinates $(c\,t, r)$ and $(c\,t', r^{*\prime})$.

   (f) In the $c\,t'$-$r^{*\prime}$-plane, draw the curves corresponding to constant t and r.

   (g) Repeat the above exercises for $r < r_{\mathrm{S}}$.

   (h) Extend the above arguments for the domain $r^{*\prime} < 0$.

4. *Penrose diagram of a Schwarzschild black hole:* Construct suitable conformal transformations to arrive at the Penrose diagram of the Schwarzschild black hole in the maximally extended Kruskal-Szekeres coordinates.

5. *Reissner-Nordstrom black holes:* Consider a spherically symmetric black hole of mass M that also carries an electric charge Q. In such a case, to determine the line element describing the spacetime around the black hole, apart from solving the vacuum Einstein's equations, we also need to solve the following Maxwell's equations in the source free region:

$$\nabla_\mu F^{\mu\nu} = 0, \quad \partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0.$$

As in the Schwarzschild case, one can assume that the line element, in general, can be expressed in terms of the functions $\Phi(t,r)$ and $\Psi(t,r)$ as follows:

$$\mathrm{d}s^2 = c^2\,\mathrm{e}^{\Phi(t,r)}\,\mathrm{d}t^2 - \mathrm{e}^{\Psi(t,r)}\,\mathrm{d}r^2 - r^2\,\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right).$$

(a) Assuming the spacetime to be static, show that the electromagnetic field tensor can be expressed as follows:

$$F_{\mu\nu} = E(r)\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(b) Establish that, under these assumptions, the Maxwell's equations lead to

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[r^2\,\mathrm{e}^{-(\Phi+\Psi)/2}\,E\right] = 0.$$

(c) Integrate the equation to determine the electric field $E(r)$ in terms of the charge Q carried by the black hole.

(d) Using the Einstein's and Maxwell's equations, arrive at the following equation governing Φ and Ψ:

$$\frac{\mathrm{d}}{\mathrm{d}r}\,(\Phi + \Psi) = 0.$$

(e) Solve the Einstein's equations to arrive at

$$\mathrm{e}^{\Phi} = 1 - \frac{2\,G\,M}{c^2\,r} + \frac{G\,Q^2}{4\,\pi\,\epsilon_0\,c^4\,r^2}.$$

Note: The resulting line element is of the form

$$\mathrm{d}s^2 = c^2\,\left(1 - \frac{2\,G\,M}{c^2\,r} + \frac{G\,Q^2}{4\,\pi\,\epsilon_0\,c^4\,r^2}\right)\mathrm{d}t^2 - \left(1 - \frac{2\,G\,M}{c^2\,r} + \frac{G\,Q^2}{4\,\pi\,\epsilon_0\,c^4\,r^2}\right)^{-1}\mathrm{d}r^2 - r^2\,\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right),$$

which, evidently, reduces to the Schwarzschild metric when $Q = 0$.

Note: Black holes with mass and electric charge are referred to as Reissner-Nordstrom black holes.

Quiz II

From classical tests of general relativity to static black holes

1. *Gravitational time dilation near the surface of the Earth:* Consider two clocks, one located on the surface of the Earth and another located at a height h above the Earth's surface. Let $\Delta\tau_1$ and $\Delta\tau_2$ denote the lapse in the proper time of the two clocks corresponding to the change Δt in the coordinate time of the Schwarzschild metric.

   (a) Determine $\Delta = (\Delta\tau_1 - \Delta\tau_2)/\Delta\tau_1$. $\boxed{\text{7 marks}}$

   (b) Estimate Δ for $h = 10^3\,\mathrm{m}$. $\boxed{\text{3 marks}}$

2. *First order solutions to the orbital equations:* When we had considered the orbits of massive particles and photons in the Schwarzschild spacetime, we had solved the orbital equations perturbatively. Recall that, we had encountered equations of the following form at the first order in the perturbations for massive particles and photons:

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2} + u_1 = f(\phi).$$

   Using the method of Green's function, obtain the solutions for $u_1(\phi)$ when $f(\phi)$ is (a) A, (b) $A\,\cos\phi$, and (c) $A\,\cos^2\phi$, where A is a constant. $\boxed{\text{3+4+3 marks}}$

3. *Behavior of photons around the Schwarzschild black hole:* Recall that photons propagating in the Schwarzschild metric are described by the equation

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 = \frac{E^2}{c^2} - V_{\mathrm{eff}}^2(r),$$

   where the 'effective potential' $V_{\mathrm{eff}}^2(r)$ is given by

$$V_{\mathrm{eff}}^2(r) = \left(1 - \frac{2\,\mu}{r}\right)\frac{L^2}{r^2},$$

   with $\mu = G\,M/c^2$. Note that E and L denote the energy and angular momentum of the photon, while λ is an affine parameter that helps us track the trajectory of the photon.

   (a) Determine the radius of the circular orbit for the photon. Establish that the circular orbit is unstable. $\boxed{\text{2 marks}}$

   (b) What is the time period of the circular orbit in terms of the proper time of an observer located at the same radius? $\boxed{\text{3 marks}}$

   (c) Determine the threshold impact parameter $b = L\,c/E$ below which the photon will be captured by the Schwarzschild black hole. $\boxed{\text{2 marks}}$

   (d) For an unbounded orbit, determine the relation between the impact the parameter b and the point of closest approach, say, r_0. $\boxed{\text{3 marks}}$

4. *Scalar field in a Schwarzschild spacetime:* Consider the following action that describes a scalar field, say, ψ, in a generic curved spacetime:

$$S[\psi] = \frac{1}{c}\int \mathrm{d}^4\tilde{x}\,\sqrt{-g}\,\frac{1}{2}\,g_{\mu\nu}\,\partial^\mu\psi\,\partial^\nu\psi,$$

   where $g_{\mu\nu}$ is the metric tensor that describes the curved spacetime.

   (a) Arrive at the equation governing the scalar field ψ in the Schwarzschild metric. $\boxed{\text{3 marks}}$

(b) Assuming the solution to be of the form

$$\psi(t, r, \theta, \phi) = \mathrm{e}^{-i\,\omega\,t}\,\frac{u(r)}{r}\,Y_{lm}(\theta, \phi),$$

obtain the equation describing the function $u(r)$. $\boxed{\text{3 marks}}$

Note: The quantities $Y_{lm}(\theta, \phi)$ denote spherical harmonics which satisfy the differential equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial Y_{lm}}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\,\frac{\partial^2 Y_{lm}}{\partial\phi^2} + l\,(l+1)\,Y_{lm} = 0.$$

(c) Show that, in terms of the tortoise coordinate r^* defined through the relation,

$$\frac{\mathrm{d}r^*}{\mathrm{d}r} = \frac{1}{1-(2\,\mu/r)},$$

the differential equation governing $u(r)$ reduces to the form

$$\frac{\mathrm{d}^2 u}{\mathrm{d}r^{*2}} + \left[\frac{\omega^2}{c^2} - V_{\mathrm{eff}}(r)\right]\,u = 0$$

and arrive at the form of $V_{\mathrm{eff}}(r)$. $\boxed{\text{4 marks}}$

5. *Quasi normal modes:* Consider a field ψ under the influence of the potential $V(x)$ and satisfying the following equation:

$$\frac{1}{c^2}\,\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial x^2} + V(x)\,\psi = 0. \tag{1}$$

(a) Show that the solution to the field ψ can be written as

$$\psi(t, x) = \mathrm{e}^{-i\,\omega\,t}\,u(x),$$

where the function $u(x)$ satisfies the equation $\boxed{\text{2 marks}}$

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \left[\omega^2 - V(x)\right]\,u = 0.$$

(b) Let the potential be given by

$$V(x) = \alpha\,\delta^{(1)}(x).$$

In such a potential, determine the value of ω assuming that the solutions to u are outgoing waves at the left and right extremes, i.e. $u(x) \propto \exp\,(-i\,\omega\,x/c)$ as $x \to -\infty$ and $u(x) \propto \exp\,(i\,\omega\,x/c)$ as $x \to \infty$. $\boxed{\text{8 marks}}$

Note: The result for ω that you obtain implies that the field ψ decays with time. Such decaying perturbations (arising, say, when a wine glass is gently tapped with a knife) are known as quasi normal modes (in contrast to the normal modes which ring forever). In principle, you can solve the equation for the scalar field around the Schwarzschild black hole (that you obtained in the previous problem), along with suitable boundary conditions, to arrive at the quasi normal modes of the black hole.

Exercise sheet 9

Stationary black holes

1. *Frame dragging:* Consider a particle that is dropped radially (i.e. with zero angular momentum) into a Kerr black hole from infinity.

   (a) Working, say, in the Boyer-Lindquist coordinates, show that the particle picks up a non-zero angular velocity.

   (b) Express the angular velocity in terms of the components of the contravariant metric tensor.

   Note: This effect is known as dragging of inertial frames. In a sense, it is similar to the effect of a spinning charge generating a magnetic field in electromagnetism. For this reason, such effects are referred to as gravitomagnetism.

2. *Event horizons of the Kerr black hole:* As we had discussed, an event horizon is the boundary that separates the spacetime into events that can communicate with distant observers and events that cannot. In the case of the Schwarzschild black hole, it is clear that the event horizon is located at the Schwarzschild radius. This is because of the fact that, once inside the Schwarzschild radius, the null rays (even outgoing ones) are inevitably dragged towards the singularity at the centre of the black hole.

   (a) Mathematically, the location of the event horizon can be said to be the hypersurface of constant radius where the normal to the hypersurface becomes null. Determine the locations of the two horizons in the case of the Kerr black hole.

   (b) Also, evaluate the area of the outer event horizon.

3. *Infinite redshift surfaces:* What is the condition that determines the radius at which photons exhibit infinite redshift with respect to an observer at infinity? Use the condition to determine the infinite redshift surfaces of the Kerr black hole.

   Note: In the case of the Schwarzschild black hole, the location of the event horizon and the infinite redshift surface coincide. This is not so in the case of the Kerr black hole.

4. *Ergosphere of the Kerr black hole:* Consider photons that are moving along the $\pm\phi$-directions in the equatorial plane (i.e. wherein $\theta = \pi/2$) at a fixed radius r around the Kerr black hole.

   (a) Assuming that $\mathrm{d}r = \mathrm{d}\theta = 0$ and using the fact that $\mathrm{d}s^2 = 0$ for photons, express the two solutions for the angular velocity of the photon, viz. $\mathrm{d}\phi/\mathrm{d}t$, in terms of the components of the metric tensor.

   (b) Determine the angular velocities when the metric component g_{00} vanishes.

   Note: The region between the outer event horizon and location wherein the metric component g_{00} vanishes is known as the ergosphere.

   (c) Determine the location of the ergosphere of the Kerr black hole.

   (d) Show that, once inside the ergosphere, photons cannot move in the direction opposite to the rotation of the black hole. Also, argue that, inside the ergosphere, the photons have to co-rotate with the hole, independent of their angular momentum.

   (e) Understand the structure of the ergosphere in relation to the horizon of the Kerr black hole.

5. *Singularities in the Reissner-Nordstrom and Kerr black holes:* Use the given `Mathematica` code to evaluate the Christoffel symbols, the Riemann, the Ricci, and the Einstein tensors as well as the Ricci scalar around the charged Reissner-Nordstrom and the Kerr black holes that are described by the following line elements:

$$\mathrm{d}s^2 = c^2 \left(1 - \frac{2\,\mu}{r} + \frac{q^2}{r^2}\right) \mathrm{d}t^2 - \left(1 - \frac{2\,\mu}{r} + \frac{q^2}{r^2}\right)^{-1} \mathrm{d}r^2 - r^2 \left(\mathrm{d}\theta^2 + \sin^2\!\theta\, \mathrm{d}\phi^2\right),$$

where

$$\mu = \frac{G\,M}{c^2} \quad \text{and} \quad q^2 = \frac{G\,Q^2}{4\,\pi\,\epsilon_0\,c^4},$$

and

$$\mathrm{d}s^2 = c^2\,\frac{\rho^2\,\Delta}{\Sigma^2}\,\mathrm{d}t^2 - \frac{\Sigma^2\sin^2\theta}{\rho^2}\,\left(\mathrm{d}\phi - \omega\,\mathrm{d}t\right)^2 - \frac{\rho^2}{\Delta}\,\mathrm{d}r^2 - \rho^2\,\mathrm{d}\theta^2,$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta,\;\; \Delta = r^2 - 2\,\mu\,r + a^2,\;\; \Sigma^2 = (r^2+a^2)^2 - a^2\,\Delta\,\sin^2\theta,\;\; \omega = \frac{2\,\mu\,c\,r\,a}{\Sigma^2},\;\; a = \frac{J}{M\,c}.$$

The quantities M, Q and J are constants that denote the mass, the electric charge and the angular momentum associated with the black holes, respectively. Note that, since the trace of the stress-energy tensor vanishes, we expect the Ricci tensor $R_{\alpha\beta}$ and the Ricci scalar R to vanish in these cases. As in the case of the Schwarzschild black hole, calculate the Kretschmann scalar $R_{\alpha\beta\gamma\delta}\,R^{\alpha\beta\gamma\delta}$ and determine the locations of the singularities in the spacetimes of these two black holes.

Exercise sheet 10

Kinematics of the FLRW universe

1. *Spaces of constant curvature:* Consider spaces of constant curvature that are described by the metric tensor g_{ab}.

   (a) Argue that, the Riemann tensor associated with such a space can be expressed in terms of the metric g_{ab} as follows:
   $$R_{abcd} = \kappa\ (g_{ac}\, g_{bd} - g_{ad}\, g_{bc}),$$
   where κ is a constant.

   (b) Show that the Ricci tensor corresponding to the above Riemann tensor is given by
   $$R_{ab} = 2\,\kappa\, g_{ab}.$$

   Note: Examples of spacetimes with a constant scalar curvature are the Einstein static universe, the de Sitter and the anti de Sitter spacetimes.

2. *Visualizing the Friedmann metric:* The Friedmann universe is described by the line-element
   $$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\ \mathrm{d}\ell^2,$$
   where
   $$\mathrm{d}\ell^2 = \frac{\mathrm{d}r^2}{(1-\kappa\, r^2)} + r^2\ \left(\mathrm{d}\theta^2 + \sin^2\!\theta\, \mathrm{d}\phi^2\right)$$
   and $\kappa = 0, \pm 1$.

   (a) Let us define a new coordinate χ as follows:
   $$\chi = \int \frac{dr}{\sqrt{1-\kappa\, r^2}}.$$
   Show that in terms of the coordinate χ the spatial line element $\mathrm{d}\ell^2$ reduces to
   $$\mathrm{d}\ell^2 = \mathrm{d}\chi^2 + S_\kappa^2(\chi)\ \left(\mathrm{d}\theta^2 + \sin^2\!\theta\, \mathrm{d}\phi^2\right),$$
   where
   $$S_\kappa(\chi) = \begin{cases} \sin\chi & \text{for} \quad \kappa = 1, \\ \chi & \text{for} \quad \kappa = 0, \\ \sinh\chi & \text{for} \quad \kappa = -1. \end{cases}$$

   (b) Show that, for $\kappa = 1$, the spatial line-element $\mathrm{d}\ell^2$ can be described as the spherical surface
   $$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$$
   embedded in an Euclidean space described by the line-element
   $$\mathrm{d}\ell^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \mathrm{d}x_4^2.$$

   (c) Show that, for $\kappa = -1$, the spatial line-element $\mathrm{d}\ell^2$ can be described as the hyperbolic surface
   $$x_1^2 + x_2^2 + x_3^2 - x_4^2 = -1$$
   embedded in a Lorentzian space described by the line-element
   $$\mathrm{d}\ell^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 - \mathrm{d}x_4^2.$$

3. *Geodesic equations in a FLRW universe:* Obtain the following non-zero components of the Christoffel symbols for the FLRW line element:

$$\Gamma^t_{ij} = \frac{a\,\dot{a}}{c}\,\sigma_{ij},$$

where σ_{ij} denotes the spatial metric defined through the relation $\mathrm{d}\ell^2 = \sigma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$. Use these Christoffel symbols to arrive at the geodesic equations corresponding to the t coordinate for massive as well as massless particles in a FLRW universe.

4. *Weyl tensor and conformal invariance:* In $(3+1)$-spacetime dimensions, the Weyl tensor $C_{\alpha\beta\gamma\delta}$ is defined as follows:

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + \frac{1}{2}\,(g_{\alpha\delta}\,R_{\beta\gamma} + g_{\beta\gamma}\,R_{\alpha\delta} - g_{\alpha\gamma}\,R_{\beta\delta} - g_{\beta\delta}\,R_{\alpha\gamma}) + \frac{1}{6}\,(g_{\alpha\gamma}\,g_{\delta\beta} - g_{\alpha\delta}\,g_{\gamma\beta})\,R.$$

   (a) Show that the Weyl tensor vanishes for the FLRW metric.

   (b) The vanishing Weyl tensor implies that there exists a coordinate system in which the FLRW metric (for all κ) is conformal to the Minkowski metric. It is straightforward to check that the metric of the $\kappa = 0$ (i.e. the spatially flat) FLRW universe can be expressed in the following form:

$$g_{\mu\nu} = a^2(\eta)\,\eta_{\mu\nu},$$

   where η is the conformal time coordinate defined by the relation

$$\eta = \int \frac{\mathrm{d}t}{a(t)},$$

   and $\eta_{\mu\nu}$ denotes the flat spacetime metric. Construct the coordinate systems in which the metrics corresponding to the $\kappa = \pm 1$ FLRW universes can be expressed in a form wherein they are conformally related to flat spacetime.

5. *Consequences of conformal invariance:* As we have seen, the action of the electromagnetic field in a curved spacetime is invariant under the conformal transformation.

   (a) Utilizing the conformal invariance of the electromagnetic action, show that the electromagnetic waves in the spatially flat FLRW universe can be written in terms of the conformal time coordinate η as follows:

$$A_\mu \propto \exp -(i\,k\,\eta) = \exp -\left[i\,k\int dt/a(t)\right].$$

   (b) Since the time derivative of the phase defines the instantaneous frequency $\omega(t)$ of the wave, conclude that $\omega(t) \propto a^{-1}(t)$.

Exercise sheet 11

Dynamics of the FLRW universe

1. *The Friedmann equations:* Recall that the FLRW universe is described by the line element

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-\kappa\,r^2} + r^2\,\left(\mathrm{d}\theta^2 + \sin^2\!\theta\,\mathrm{d}\phi^2\right)\right],$$

where $\kappa = 0, \pm 1$.

(a) Arrive at the following expressions for the Ricci tensor R^μ_ν, the scalar curvature R, and the Einstein tensor G^μ_ν for the above Friedmann metric:

$$\begin{aligned}
R^t_t &= -\frac{3\,\ddot{a}}{c^2\,a},\\
R^i_j &= -\left[\frac{\ddot{a}}{c^2\,a} + 2\left(\frac{\dot{a}}{c\,a}\right)^2 + \frac{2\,\kappa}{a^2}\right]\delta^i_j,\\
R &= -6\left[\frac{\ddot{a}}{c^2\,a} + \left(\frac{\dot{a}}{c\,a}\right)^2 + \frac{\kappa}{a^2}\right],\\
G^t_t &= 3\left[\left(\frac{\dot{a}}{c\,a}\right)^2 + \frac{\kappa}{a^2}\right],\\
G^i_j &= \left[\frac{2\,\ddot{a}}{c^2\,a} + \left(\frac{\dot{a}}{c\,a}\right)^2 + \frac{\kappa}{a^2}\right]\delta^i_j,
\end{aligned}$$

where the overdots denote differentiation with respect to the cosmic time t.

(b) Consider a fluid described by the stress energy tensor

$$T^\mu_\nu = \mathrm{diag.}\left(\rho\,c^2, -p, -p, -p\right),$$

where ρ and p denote the mass density and the pressure associated with the fluid. In a smooth Friedmann universe, the quantities ρ and p depend only on time. Using the above Einstein tensor, obtain the following Friedmann equations for such a source:

$$\begin{aligned}
\left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa\,c^2}{a^2} &= \frac{8\,\pi\,G}{3}\,\rho,\\
\frac{2\,\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa\,c^2}{a^2} &= -\frac{8\,\pi\,G}{c^2}\,p.
\end{aligned}$$

(c) Show that these two Friedmann equations lead to the equation

$$\frac{\ddot{a}}{a} = -\frac{4\,\pi\,G}{3}\left(\rho + \frac{3\,p}{c^2}\right).$$

Note: This relation implies that $\ddot{a} > 0$, i.e. the universe will undergo accelerated expansion, only when $\left(\rho\,c^2 + 3\,p\right) < 0$.

2. *Conservation of the stress energy tensor in a FLRW universe:* Recall that the conservation of the stress energy tensor is described by the equation $T^\mu_{\nu;\mu} = 0$.

(a) Show that the time component of the stress energy tensor conservation law leads to the following equation in a Friedmann universe:

$$\dot{\rho} + 3\,H\left(\rho + \frac{p}{c^2}\right) = 0,$$

where $H = \dot{a}/a$, a quantity that is known as the Hubble parameter.

(b) Also arrive at this equation from the two Friedmann equations obtained above.

(c) Show that the above equation can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\, a^3\right) = -\frac{p}{c^2}\left(\frac{\mathrm{d}a^3}{\mathrm{d}t}\right).$$

3. *Evolution of energy density in a FLRW universe:* The different types of matter that are present in the universe are often described by an equation of state, i.e. the relation between the density and the pressure associated with the matter. Consider the following equation of state $p = w\,\rho\, c^2$, where w is a constant.

   (a) Using the above equation which governs the evolution of ρ in a FLRW universe, show that, in such a case,
   $$\rho \propto a^{-3\,(1+w)}.$$

   (b) While the quantity w vanishes for pressure free non-relativistic matter (such as baryons and cold dark matter), $w = 1/3$ for relativistic particles (such as photons and the nearly massless neutrinos). Note that the energy density does not change with time when $w = -1$ or, equivalently, when $p = -\rho\, c^2$. Such a type of matter is known as the cosmological constant. Utilizing the above result, express the total density of a universe filled with non-relativistic (NR) and relativistic (R) matter as well as the cosmological constant (Λ) as follows:
   $$\rho(a) = \rho_{_{\mathrm{NR}}}^0\left(\frac{a_0}{a}\right)^3 + \rho_{_{\mathrm{R}}}^0\left(\frac{a_0}{a}\right)^4 + \rho_{_\Lambda},$$
   where $\rho_{_{\mathrm{NR}}}^0$ and $\rho_{_{\mathrm{R}}}^0$ denote the density of non-relativistic and relativistic matter today (i.e. at, say, $t = t_0$, corresponding to the scale factor $a = a_0$).

   (c) Also, further rewrite the above expression as
   $$\rho(a) = \rho_{_{\mathrm{C}}}\left[\Omega_{_{\mathrm{NR}}}\left(\frac{a_0}{a}\right)^3 + \Omega_{_{\mathrm{R}}}\left(\frac{a_0}{a}\right)^4 + \Omega_{_\Lambda}\right] = \rho_{_{\mathrm{C}}}\left[\Omega_{_{\mathrm{NR}}}\,(1+z)^3 + \Omega_{_{\mathrm{R}}}\,(1+z)^4 + \Omega_{_\Lambda}\right],$$
   where $\Omega_{_{\mathrm{NR}}} = \rho_{_{\mathrm{NR}}}^0/\rho_{_{\mathrm{C}}}$, $\Omega_{_{\mathrm{R}}} = \rho_{_{\mathrm{R}}}^0/\rho_{_{\mathrm{C}}}$ and $\Omega_{_\Lambda} = \rho_{_\Lambda}/\rho_{_{\mathrm{C}}}$, while $\rho_{_{\mathrm{C}}}$ is the so-called critical density defined as
   $$\rho_{_{\mathrm{C}}} = \frac{3\,H_0^2}{8\,\pi\,G},$$
   with the quantity H_0 being the Hubble parameter (referred to as the Hubble constant) today. Note: The quantities H_0, $\Omega_{_{\mathrm{NR}}}$, $\Omega_{_{\mathrm{R}}}$ and $\Omega_{_\Lambda}$ are cosmological parameters that are to be determined by observations.

   (d) Observations suggest that $H_0 \simeq 72$ km s^{-1} Mpc^{-1}. Evaluate the corresponding numerical value of the critical density ρ_{c}.
   Note: A parsec (pc) corresponds to 3.26 light years, and a Mega parsec (Mpc) amounts to 10^6 parsecs.

4. *The Cosmic Microwave Background:* It is found that we are immersed in a perfectly thermal and nearly isotropic distribution of radiation, which is referred to as Cosmic Microwave Background (CMB), as it energy density peaks in the microwave region of the electromagnetic spectrum. The CMB is a relic of an earlier epoch when the universe was radiation dominated, and it provides the dominant contribution to the relativistic energy density in the universe.

   (a) Given that the temperature of the CMB today is $T \simeq 2.73$ K, show that one can write
   $$\Omega_{_{\mathrm{R}}}\, h^2 \simeq 2.56 \times 10^{-5},$$
   where h is related to the Hubble constant H_0 as follows:
   $$H_0 \simeq 100\, h \text{ km s}^{-1}\text{ Mpc}^{-1}.$$

(b) Show that the redshift $z_{\rm eq}$ at which the energy density of matter and radiation were equal is given by

$$1+z_{\rm eq} = \frac{\Omega_{\rm NR}}{\Omega_{\rm R}} \simeq 3.9\times 10^4\,\left(\Omega_{\rm NR}\,h^2\right).$$

(c) Also, show that the temperature of the radiation at this epoch is given by

$$T_{\rm eq} \simeq 9.24\,\left(\Omega_{\rm NR}\,h^2\right)\,{\rm eV}.$$

5. <u>*Solutions to the Friedmann equations:*</u> We had discussed the solutions to Friedmann equations in the presence of a single component when the universe is spatially flat (i.e. when $\kappa = 0$). It proves to be difficult to obtain analytical solutions for the scale factor when all the three components of matter (viz. non-relativistic and relativistic matter as well as the cosmological constant) are simultaneously present. However, the solutions can be obtained for the cases wherein two of the components are present.

(a) Integrate the first Friedmann equation for a $\kappa = 0$ universe with matter and radiation to obtain that

$$a(\eta) = \sqrt{\Omega_{\rm R}\,a_0^4}\,\left(H_0\,\eta\right) + \frac{\Omega_{\rm NR}\,a_0^3}{4}\,\left(H_0\,\eta\right)^2,$$

where η is the conformal time coordinate. Show that, at early (i.e. for small η) and late times (i.e. for large η), this solution reduces to the behavior in the radiation and matter dominated epochs, respectively, as required.

Note: In obtaining the above result, it has been assumed that $a = 0$ at $\eta = 0$.

(b) Integrate the Friedmann equation for a $\kappa = 0$ universe with matter and cosmological constant to obtain that

$$\frac{a(t)}{a_0} = \left(\frac{\Omega_{\rm NR}}{\Omega_\Lambda}\right)^{1/3}\,\sinh^{2/3}\left(3\,\sqrt{\Omega_\Lambda}\,H_0\,t/2\right).$$

Also, show that, at early times, this solution simplifies to $a \propto t^{2/3}$, while at late times, it behaves as $a \propto \exp\,(\Omega_\Lambda^{3/2}\,H_0\,t/\Omega_{\rm NR})$, as expected.

Exercise sheet 12

Thermal history of the universe

Note that the problems below have been described in units wherein $\hbar = c = k_{\mathrm{B}} = 1$.

1. *Evolution of the temperature of the CMB:* Let $f(\boldsymbol{x}, \boldsymbol{p}, t)$ denote the distribution function associated with bosons or fermions in the early universe.

   (a) The number density of particles within the phase space volume $\mathrm{d}^3\boldsymbol{x}\, \mathrm{d}^3\boldsymbol{p}$ is given by

   $$\mathrm{d}N = f(\boldsymbol{x}, \boldsymbol{p}, t)\, \mathrm{d}^3\boldsymbol{x}\, \mathrm{d}^3\boldsymbol{p}.$$

   In a FLRW universe, the distribution function will be independent of $\boldsymbol{x}$ due to the homogeneity of the background, and it will depend only on p (rather than on $\boldsymbol{p}$) due to the isotropy. Show that, if no particles are created or destroyed, then the distribution function remains invariant during the evolution of the universe.

   (b) Argue that, for a thermal distribution of photons in a FLRW universe, the invariance of the distribution function implies that the temperature of the radiation is inversely proportional to the scale factor.

2. *Number density, energy density and pressure of particles in thermal equilibrium:* Consider a collection of relativistic particles of mass m, momentum $\boldsymbol{k}$, and energy $E = \left(k^2 + m^2\right)^{1/2}$, where $k = |\boldsymbol{k}|$. If the distribution function of the particles is denoted as $f(\boldsymbol{k})$, then the number density n, energy density ρ and the pressure p of the collection of particles are given by

   $$n = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\,\pi)^3}\, f(\boldsymbol{k}), \quad \rho = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\,\pi)^3}\, f(\boldsymbol{k})\, E, \quad p = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\,\pi)^3}\, \frac{f(\mathbf{k})\, k^2}{3\, E}.$$

   In thermal equilibrium, an ideal Bose or Fermi gas is described by the distribution function

   $$f(\mathbf{k}) = g\, \frac{1}{\exp\left[(E - \mu)/T\right] \pm 1},$$

   where g is the spin degeneracy, T is the temperature and μ denotes the chemical potential. In the above expression for the distribution function, the upper sign (viz. $+$) corresponds to fermions and the lower sign (viz. $-$) to bosons.

   (a) Using the above expressions, show that, for bosons, when $\mu \ll T$ and $T \gg m$ (i.e. when the particles are relativistic), we have

   $$n = g\, \frac{\zeta(3)}{\pi^2}\, T^3, \quad \rho = g\, \frac{\pi^2}{30}\, T^4, \quad p = \frac{\rho}{3},$$

   while for $T \ll m$ (i.e. in the non-relativistic limit), we have

   $$n = g\, \left(\frac{m\, T}{2\, \pi}\right)^{3/2} \exp\left[-(m - \mu)/T\right], \quad \rho = n\, m, \quad p = n\, T \ll \rho.$$

   (b) Similarly, for fermions, when $\mu \ll T$, show that, for $T \gg m$ we have

   $$n = g\, \frac{3\, \zeta(3)}{4\, \pi^2}\, T^3, \quad \rho = g\, \frac{7\, \pi^2}{240}\, T^4, \quad p = \frac{\rho}{3},$$

   while, for $T \ll m$, we have

   $$n = g\, \left(\frac{m\, T}{2\, \pi}\right)^{3/2} \exp\left[-(m - \mu)/T\right], \quad \rho = n\, m, \quad p = n\, T \ll \rho.$$

(c) Also show that, when $T \gg m$, for bosons, we have

$$\frac{\rho}{n} \simeq 2.701\, T,$$

while, for fermions, we have

$$\frac{\rho}{n} \simeq 3.151\, T.$$

3. *Conservation of entropy and the first law of thermodynamics:* Consider a system of bosons and fermions in thermal equilibrium.

(a) Using the above definitions of n, ρ and p, show that, in a FLRW universe described by the scale factor $a(t)$, we can arrive at

$$\mathrm{d}S = -\frac{\mu}{T}\,\mathrm{d}\left(n\, a^3\right),$$

where S is the entropy of the system defined as

$$S = \frac{\rho + p - n\,\mu}{T}\, a^3.$$

Note: In obtaining the above relation for $\mathrm{d}S$, we have assumed that the chemical potential μ is a function of the temperature T.

(b) When $\mu/T \ll 1$, show that the entropy S of the system is conserved.

(c) Also show that, when $\mu/T \ll 1$, the above relation can be expressed as the first law of thermodynamics, viz. $T\,\mathrm{d}S = \mathrm{d}\mathcal{E} + p\,\mathrm{d}V$, with $\mathcal{E} = \rho\, a^3$ and $V = a^3$.

4. *Temperature of neutrinos:* Consider the time, say, t_{NR}, during the radiation dominated epoch, when the temperature falls below $T = 0.511\,\mathrm{MeV}$. The particles that are relativistic immediately prior to t_{NR} are the photons, the electrons and positrons, and the three types of neutrinos. Since the electrons and positrons become *non-relativistic* after t_{NR}, there arises a change in the effective number of relativistic degrees of freedom, say, g_{eff}. Using the above expression for entropy, the fact that it is conserved and the change in g_{eff}, express the temperature of the neutrinos, say, T_ν, in terms of the temperature of the photons, say, T_γ, after the time t_{NR}.

5. *Time during the radiation dominated epoch:* Show that, during the radiation dominated era, the age of the universe at the temperature T is given by

$$t \simeq \frac{2.42}{\sqrt{g_{\mathrm{eff}}}} \left(\frac{T}{1\,\mathrm{MeV}}\right)^{-2}\ \mathrm{s},$$

where g_{eff} is the effective number of degrees of freedom associated with the relativistic particles.

Exercise sheet 13

The inflationary epoch

Note that the problems below have been described in units wherein $\hbar = c = k_{\rm B} = 1$.

1. *Comoving wave length corresponding to the Hubble radius at the epoch of equality:* Determine the wave length of a perturbation *today* that is equal to the Hubble radius $d_{\rm H} = H^{-1}$ at the time of radiation-matter equality.

2. *The flatness problem:* Recall that, the observations of the anisotropies in the CMB suggest that the universe is nearly spatially flat with the dimensionless density parameter *today* constrained to be

$$\Omega_0 - 1 < 0.005.$$

Given this information, estimate $\Omega(t)$ at the following times in the early universe: (i) at the time of radiation-matter equality, and (ii) when the temperature of the universe was $T = 10^{14}\,{\rm GeV}$.

3. *Energy density and pressure of a homogeneous scalar field:* Consider a canonical scalar field ϕ that is governed by the action

$$S[\phi] = \int {\rm d}^4\tilde{x}\,\sqrt{-g}\,\left[\frac{1}{2}\,g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right],$$

where $V(\phi)$ is the potential describing the scalar field.

   (a) Assuming that the scalar field is *homogeneous*, obtain the expressions for the energy density ρ and pressure p corresponding to the scalar field.

   (b) Arrive at the condition on the energy density ρ and pressure p that leads to accelerated expansion.

4. *Power law inflation:* Consider the potential

$$V(\phi) = V_0\,\exp\left(-\sqrt{\frac{2}{q}}\,\frac{\phi}{M_{\rm Pl}}\right),$$

where V_0 and q are constants.

Note: The quantity $M_{\rm Pl}$ denotes the Planck mass, and is defined as $M_{\rm Pl} = (8\,\pi\,G)^{-1/2}$.

   (a) Show that the above potential leads to the following behavior for $a(t)$ and $\phi(t)$:

$$a(t) = a_1\,t^q, \quad \frac{\phi(t)}{M_{\rm Pl}} = \frac{\phi_{\rm i}}{M_{\rm Pl}} + \sqrt{2\,q}\,\ln\left(\frac{t}{t_{\rm i}}\right),$$

   where $\phi_{\rm i}$ is the value of the scalar field at the initial time $t_{\rm i}$.

   (b) Determine V_0 in terms of q, $\phi_{\rm i}$ and $t_{\rm i}$.

5. *Potential slow roll parameters:* Argue that the conditions for slow roll, viz. that $\dot{\phi}^2 \ll V$ and $\ddot{\phi} \ll 3\,H\,\dot{\phi}$, can be described in the terms of the following conditions on the potential and its derivatives:

$$\epsilon_V = \frac{M_{\rm Pl}^2}{2}\,\left(\frac{V_\phi}{V}\right)^2 \ll 1, \quad \eta_V = M_{\rm Pl}^2\,\left|\frac{V_{\phi\phi}}{V}\right| \ll 1,$$

where $V_\phi = {\rm d}V/{\rm d}\phi$ and $V_{\phi\phi} = {\rm d}^2V/{\rm d}\phi^2$.

Note: The quantities ϵ_V and η_V are often referred to as the potential slow roll parameters.

Additional exercises II

From classical tests of general relativity to cosmology

1. *Modified theories of gravitation:* Consider a theory of gravitation that is described by the action

$$S[g_{\mu\nu}(\tilde{x})] = A \int \mathrm{d}^4\tilde{x}\, \sqrt{-g}\, f(R),$$

where $f(R)$ is an arbitrary function of the scalar curvature R, and A is a constant.

(a) What is dimension of the constant A?

(b) Vary the action with respect to the metric tensor $g_{\mu\nu}$ to arrive at the equation of motion.

2. *Eddington's theory of gravitation:* Consider a theory of gravitation that is described by the action

$$S[g_{\mu\nu}(\tilde{x})] = A \int \mathrm{d}^4\tilde{x}\, \sqrt{-g}\, R_{\alpha\beta\gamma\delta}\, R^{\alpha\beta\gamma\delta},$$

where A is a constant.

(a) Determine the dimension of the constant A.

(b) Vary the action with respect to the metric tensor $g_{\mu\nu}$ to arrive at the equation of motion.

3. Apparent diameter of the Sun: All massive objects look larger than they really are.

(a) Consider a light ray grazing the surface of a sphere of mass M and coordinate radius $r > 3\, G\, M/c^2$. Determine the impact parameter of the light ray as it arrives at infinity.

(b) If we observe a light ray grazing the Sun, estimate the extent by which the apparent diameter of the Sun exceeds the coordinate diameter.
Note: The mass and radius of the Sun are $M_\odot = 2 \times 10^{30}\,\mathrm{kg}$ and $R_\odot = 7 \times 10^8\,\mathrm{m}$. Also, the value of Newton's gravitational constant is $G = 6.673 \times 10^{-11}\,\mathrm{N\,m^2/kg^2}$, while the velocity of light is $c = 2.998 \times 10^8\,\mathrm{m/s}$.

4. *Shapiro time delay:* A fourth classical test of general relativity is to measure the delay in the propagation of light as it travels around a central mass. Consider a situation wherein Earth, one of the inner planets, either Mercury or Venus, and the Sun are nearly aligned. Consider a light ray that travels from Earth to Mercury or Venus, passing the Sun, with D being the distance of the light ray to the Sun at the point of closest approach. Let us assume that the light ray is propagating in the $\theta = \pi/2$ plane in a Schwarzschild spacetime with the Sun as the central mass.

(a) Using the fact that $\mathrm{d}s^2 = 0$ for a light ray and, considering the situation wherein $\mathrm{d}\theta = 0$, arrive at the differential relation between $\mathrm{d}t$, $\mathrm{d}r$ and $\mathrm{d}\phi$.

(b) If we work at the order μ/r, where $\mu = G\, M/c^2$, we can assume that the light ray is traveling along a straight line so that $r = D \sin\phi$. Working under such an assumption, arrive at the following equation relating $\mathrm{d}t$ and $\mathrm{d}r$ upto the first order in μ/r:

$$c\,\mathrm{d}t \simeq \pm \frac{r\,\mathrm{d}r}{\sqrt{r^2 - D^2}} \left(1 + \frac{2\,\mu}{r} - \frac{\mu\, D^2}{r^3}\right)^{1/2}.$$

(c) Integrate this equation to arrive at the following expression for the time taken, say, T, for a light ray to travel from Earth to the planets Mercury or Venus:

$$\begin{aligned} c\,T &= \sqrt{D_{\mathrm{P}}^2 - D^2} - \sqrt{D_{\mathrm{E}}^2 - D^2} \\ &\quad + 2\,\mu \ln\left[\left(\sqrt{D_{\mathrm{P}}^2 - D^2} + D_{\mathrm{P}}\right) \left(\sqrt{D_{\mathrm{E}}^2 - D^2} + D_{\mathrm{E}}\right)/D^2\right] \\ &\quad - \mu \left[\left(\sqrt{D_{\mathrm{P}}^2 - D^2}/D_{\mathrm{P}}\right) + \left(\sqrt{D_{\mathrm{E}}^2 - D^2}/D_{\mathrm{E}}\right)\right], \end{aligned}$$

where $D_{\rm E}$ and $D_{\rm P}$ denote the average radii of the orbits of Earth and the planets Mercury or Venus.

Note: The two terms in the first line of the above expression for T is the result in flat spacetime in the absence of the central mass μ.

(d) Using the average radii of the Earth and orbits of the inner planets, estimate the time T.

(e) The delay is experimentally verified by sending pulsed radar signals to Venus and Mercury and using the echoes to determine the travel time as the positions of Earth and the planets change relative to the Sun. In the case of Venus, the delay has been measured to be about $200\,\mu$s. Compare the observed value with the theoretical estimate.

5. *Schwarzschild metric in the Eddington-Finkelstein coordinates:* In a Schwarzschild spacetime, the trajectory of radially ingoing/outgoing photons is found to be v

$$c\,t = \mp r \mp 2\,\mu\,\ln\left|\frac{r}{2\,\mu} - 1\right| + \text{constant},$$

where $\mu = G\,M/c^2$. Define a new time coordinate $t_\pm$ that is related to the coordinates t and r through the following relation for $r > 2\,\mu$:

$$c\,t_\pm = c\,t \pm 2\,\mu\,\ln\left|\frac{r}{2\,\mu} - 1\right|.$$

(a) Express the Schwarzschild line-element in terms of the coordinates $(t_\pm, r, \theta, \phi)$.

Note: These new coordinates are known as the Eddington-Finkelstein coordinates.

(b) Let $v = c\,t_+ + r$ and $u = c\,t_- - r$. Express the Schwarzschild line-element in terms of the coordinates (v, r, θ, ϕ) and (u, r, θ, ϕ).

Note: These new coordinates are known as the advanced or retarded Eddington-Finkelstein coordinates.

6. Painlevé-Gullstrand coordinates: Recall that the Schwarzschild line-element is given by

$${\rm d}s^2 = c^2\,\left(1 - \frac{2\,\mu}{r}\right)\,{\rm d}t^2 - \left(1 - \frac{2\,\mu}{r}\right)^{-1}\,{\rm d}r^2 - r^2\,\left({\rm d}\theta^2 + \sin^2\theta\,{\rm d}\phi^2\right),$$

where $2\,\mu = 2\,G\,M/c^2$ is the Schwarzschild radius. Let us define a new coordinate through the relation

$$c\,t = c\,T + h(r),$$

where

$$h(r) = \frac{\sqrt{2\,\mu/r}}{1 - (2\,\mu/r)}.$$

(a) Express the Schwarzschild line-element in terms of the coordinates (T, r, θ, ϕ).

(b) Arrive at the condition on $h(r)$ so that, in the new coordinates, the coefficient of ${\rm d}r^2$ reduces to -1.

Note: In such a case, the coordinates (T, r, θ, ϕ) are known as the Painlevé-Gullstrand coordinates.

(c) Express the Schwarzschild line-element in terms of the Painlevé-Gullstrand coordinates.

(d) Consider a particle falling radially from rest at infinity. What is the radial speed of particle, when defined with respect to the time T, at the Schwarzschild radius?

7. *Non-canonical scalar field:* Consider a scalar field, say, ϕ, that is governed by the following action:

$$S[\phi(\tilde{x})] = \frac{1}{c}\int \mathrm{d}^4\tilde{x}\,\sqrt{-g}\,P(X,\phi),$$

where the quantity X is given by

$$X = \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi,$$

while $P(X,\phi)$ is an arbitrary function of X and ϕ.

(a) Vary the above action with respect to the scalar field ϕ to obtain the equation of motion governing the field.

(b) Arrive at the stress-energy tensor $T_{\mu\nu}$ associated with the scalar field ϕ by varying the above action with respect to the metric tensor $g^{\mu\nu}$.

(c) Assuming the scalar field to be homogeneous, obtain the expressions for the energy density and pressure associated with the scalar field.

(d) Determine the condition on the scalar field that leads to accelerated expansion of the universe.

8. *Decoupling of neutrinos:* The rate for weak interactions, say, Γ_{W}, is given in terms of the Fermi constant $G_{\mathrm{F}}/(\hbar\,c)^3 = 1.17\times 10^{-5}\,\mathrm{GeV}^{-2}$ and the temperature T by the relation

$$\Gamma_{\mathrm{W}} \simeq 1.3\,G_{\mathrm{F}}^2\,(k_{\mathrm{B}}\,T)^5.$$

Show that, during the radiation dominated epoch, we can write

$$\frac{\Gamma_{\mathrm{W}}}{H} \simeq \left(\frac{k_{\mathrm{B}}\,T}{1.4\ \mathrm{MeV}}\right)^3.$$

Note: This implies the neutrinos decouple at a temperature of about $1.4\,\mathrm{MeV}$.

9. *Dipole moment of the CMB:* At each event in spacetime, the CMB has a mean rest frame and, as seen in such a rest frame, the CMB is isotropic and thermal at the temperature of $T_0 = 2.725\,\mathrm{K}$. Actually, the Earth moves relative to the rest frame of the CMB with a speed of about $600\,\mathrm{km\,s^{-1}}$ towards the Hydra-Centaurus region of the sky. Consider an observer on Earth who points his microwave receiver in a direction that makes an angle θ with the direction of the motion.

(a) Show that the intensity of the radiation received is precisely Planckian in form, but with the Doppler shifted temperature

$$T = T_0\left(\frac{\left[1-(v/c)^2\right]^{1/2}}{1+(v/c)\cos\theta}\right),$$

where v is the velocity of the Earth with respect to the rest frame of the CMB.

(b) Note that the θ dependence of the temperature leads to an anisotropy in the CMB as seen from Earth. Show that, because the Earth's velocity is small compared to the velocity of light, the anisotropy is dipolar in form.

(c) Estimate the magnitude of the variations in the CMB temperature due to such a dipolar anisotropy.

10. *Angular diameter of the horizon at decoupling:* Assuming the universe to be spatially flat and ignoring the effects due to the cosmological constant, determine the angular diameter of the horizon at the time of decoupling.

Quiz III

Cosmology

Note that the problems below have been described in units wherein $\hbar = c = 1$.

1. *Beyond Hubble's law:* Earlier, we had expanded the scale factor $a(t)$ about *today* (corresponding to cosmic time t_0) to arrive at the Hubble's law for small redshifts.

   (a) Expand $a(t)$ up to order $(t - t_0)^2$ and express it in terms of the Hubble constant H_0 and the so-called deceleration parameter *today*, i.e. $q_0 = -[1 + (\dot{H}/H^2)_{t_0}]$. $\boxed{\text{4 marks}}$

   (b) Express the luminosity and angular diameter distances $d_{\mathrm{L}}(z)$ and $d_{\mathrm{A}}(z)$ in terms of H_0 and q_0 up to order z^2. $\boxed{\text{3+3 marks}}$

   Note: To arrive at the expressions, it will be easier to expand $H(z)$ about $z = 0$.

2. *Einstein's static universe:* Assuming that the universe is *static* and contains matter, Einstein introduced the cosmological constant as an additional source.

   (a) Using the equation governing $\ddot{a}$, arrive at the relation between the energy densities of the cosmological constant and matter in such a case. $\boxed{\text{4 marks}}$

   (b) Using the first Friedmann equation governing $\dot{a}$, show that the spatial geometry of the universe has to be curved for such a possibility. $\boxed{\text{4 marks}}$

   (c) What is the scale of curvature of the universe? $\boxed{\text{2 marks}}$

3. *Angular diameter at decoupling:* Consider a spatially flat universe.

   (a) Recall that, decoupling occurs at a redshift of about $z \simeq 1100$. What is the Hubble radius at the time of decoupling? $\boxed{\text{3 marks}}$

   Note: For simplicity, assume that the universe is matter dominated over the redshift range of $0 < z < 1100$.

   (b) Arrive at the expression describing the angular diameter distance as a function of redshift during the matter dominated epoch. $\boxed{\text{3 marks}}$

   (c) Determine the angle subtended in the sky by the Hubble radius at decoupling. $\boxed{\text{4 marks}}$

4. *Number density, energy density and pressure of particles in thermal equilibrium:* Consider a collection of relativistic particles of mass m, momentum $\boldsymbol{k}$, and energy $E = \left(k^2 + m^2\right)^{1/2}$, where $k = |\boldsymbol{k}|$. If the distribution function of the particles is denoted as $f(\boldsymbol{k})$, then the number density n, energy density ρ and the pressure p of the collection of particles are given by

$$n = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\,\pi)^3}\, f(\boldsymbol{k}), \quad \rho = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\,\pi)^3}\, f(\boldsymbol{k})\, E, \quad p = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\,\pi)^3}\, \frac{f(\mathbf{k})\, k^2}{3\,E}.$$

   In thermal equilibrium, an ideal Bose or Fermi gas is described by the distribution function

$$f(\mathbf{k}) = g\, \frac{1}{\exp\left[(E - \mu)/T\right] \pm 1},$$

   where g is the spin degeneracy, T is the temperature and μ denotes the chemical potential. In the above expression for the distribution function, the upper sign (viz. $+$) corresponds to fermions and the lower sign (viz. $-$) to bosons.

   (a) Using the above definitions of n, ρ and p, show that we can arrive at the following relation: $\boxed{\text{5 marks}}$

$$\frac{\mathrm{d}p}{\mathrm{d}T} = \frac{\rho + p}{T} + n\,T\, \frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{\mu}{T}\right).$$

Note: In obtaining the above relation for $\mathrm{d}S$, we have assumed that the chemical potential μ is a function of the temperature T.

(b) Using the above relation and the conservation of energy in a FLRW universe described by the scale factor $a(t)$, show that we can obtain the relation $\boxed{\text{3 marks}}$

$$\mathrm{d}S = -\frac{\mu}{T}\,\mathrm{d}\left(n\,a^3\right),$$

where S is the entropy of the system defined as

$$S = \frac{\rho + p - n\,\mu}{T}\,a^3.$$

(c) Also show that, when $\mu/T \ll 1$, the above relation can be expressed as the first law of thermodynamics, viz. $T\,\mathrm{d}S = \mathrm{d}\mathcal{E} + p\,\mathrm{d}V$, with $\mathcal{E} = \rho\,a^3$ and $V = a^3$. $\boxed{\text{2 marks}}$

5. *Dipole moment of the CMB:* At each event in spacetime, the CMB has a mean rest frame and, as seen in such a rest frame, the CMB is isotropic and thermal at the temperature of $T_0 = 2.725\,\mathrm{K}$. Actually, the Earth moves relative to the rest frame of the CMB with a speed of about $600\,\mathrm{km\,s^{-1}}$ towards the so-called Hydra-Centaurus region of the sky. Consider an observer on Earth who points his microwave receiver in a direction that makes an angle θ with the direction of the motion.

(a) Recall that, the number of particles, say, $\mathrm{d}N$, within a differential volume of phase space, say, $\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^3\boldsymbol{k}$, is given by

$$\mathrm{d}N = f(\boldsymbol{k})\,\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^3\boldsymbol{k},$$

where $f(\boldsymbol{k})$ is the distribution function. Show that the differential volume in phase space $\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^3\boldsymbol{k}$ is invariant under Lorentz transformations. $\boxed{\text{3 marks}}$

(b) Assuming that no photons are created or destroyed and that the function $f(\boldsymbol{k})$ is given by the Planck distribution, show that the temperature of the CMB as observed in the frame of the Earth is given by $\boxed{\text{4 marks}}$

$$T = T_0\left[\frac{\left(1 - v^2\right)^{1/2}}{1 - v\cos\theta}\right],$$

where v is the velocity of the Earth with respect to the rest frame of the CMB.

(c) Note that the θ dependence of the temperature leads to an anisotropy in the CMB as seen from Earth. Show that, because the Earth's velocity is small compared to the velocity of light, the anisotropy is dipolar in form. $\boxed{\text{2 mark}}$

(d) Estimate the magnitude of the variations in the CMB temperature due to such a dipolar anisotropy. $\boxed{\text{1 mark}}$

Exercise sheet 14

Gravitational waves in Minkowski spacetime

1. <u>*The linearized metric I:*</u> Consider a small perturbation to flat spacetime so that the standard Minkowski metric can be expressed as

$$g_{\mu\nu} \simeq \eta_{\mu\nu} + \epsilon\, h_{\mu\nu},$$

where ϵ is a small dimensionless quantity. Show that, at the same order in ϵ, the corresponding contravariant metric tensor and the Christoffel symbols are given by

$$g^{\mu\nu} \simeq \eta^{\mu\nu} + \epsilon\, h^{\mu\nu}$$

and

$$\Gamma^{\alpha}_{\beta\gamma} \simeq \frac{\epsilon}{2}\left(h^{\alpha}_{\gamma,\beta} + h^{\alpha}_{\beta,\gamma} - h_{\beta\gamma}{}^{,\alpha}\right),$$

respectively.

2. <u>*The linearized metric II:*</u> Let us now turn to the evaluation of the curvature and the Einstein tensors corresponding to the above metric.

   (a) Show that, at the linear order, the Riemann and the Ricci tensors and the scalar curvature are given by

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &\simeq \frac{\epsilon}{2}\left(h_{\alpha\delta,\beta\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta} - h_{\beta\delta,\alpha\gamma}\right), \\ R_{\beta\delta} &\simeq \frac{\epsilon}{2}\left(h^{\gamma}_{\delta,\beta\gamma} + h^{\alpha}_{\beta,\alpha\delta} - h_{,\beta\delta} - \eta^{\alpha\gamma}\, h_{\beta\delta,\alpha\gamma}\right) \end{aligned}$$

   and

$$R = \epsilon\left(h^{\alpha\beta}{}_{,\alpha\beta} - \Box\, h\right),$$

   where $\Box$ is the d'Alembertian corresponding to the Minkowski metric $\eta_{\mu\nu}$, while $h = \eta^{\mu\nu}\, h_{\mu\nu}$ denotes the trace of the perturbation $h_{\mu\nu}$.

   (b) Finally, show that the corresponding Einstein tensor can be expressed as

$$G_{\alpha\beta} = \frac{\epsilon}{2}\left(h^{\gamma}_{\beta,\alpha\gamma} + h^{\gamma}_{\alpha,\beta\gamma} - \Box\, h_{\alpha\beta} - h_{,\alpha\beta} - \eta_{\alpha\beta}\, h^{\gamma\delta}{}_{,\gamma\delta} + \eta_{\alpha\beta}\,\Box\, h\right).$$

3. <u>*Gauge transformations:*</u> Consider the following 'small' coordinate transformations:

$$x^{\mu} \to x'^{\mu} \simeq x^{\mu} + \epsilon\, \xi^{\mu},$$

which are of the same amplitude as the perturbation $h_{\mu\nu}$. Show that under such a transformation, the perturbation $h_{\mu\nu}$ transforms as follows:

$$h_{\mu\nu} \to h'_{\mu\nu} \simeq h_{\mu\nu} - \left(\xi_{\mu,\nu} + \xi_{\nu,\mu}\right).$$

Note: Such a 'small' transformation is known as a gauge transformation.

4. <u>*The de Donder gauge:*</u> Let us define a new set of variables $\psi_{\mu\nu}$, which are related to the metric perturbation $h_{\mu\nu}$ as follows:

$$\psi_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\, h.$$

   (a) Show that, in terms of $\psi_{\mu\nu}$, the above Einstein tensor is given by

$$G_{\alpha\beta} = \frac{\epsilon}{2}\left(\psi^{\gamma}_{\alpha,\beta\gamma} + \psi^{\gamma}_{\beta,\alpha\gamma} - \Box\,\psi_{\alpha\beta} - \eta_{\alpha\beta}\,\psi^{\gamma\delta}{}_{,\gamma\delta}\right).$$

(b) Show that, under the above-mentioned gauge transformations, the variables $\psi_{\mu\nu}$ transform as

$$\psi_{\mu\nu} \to \psi'_{\mu\nu} \simeq \psi_{\mu\nu} - (\xi_{\mu,\nu} + \xi_{\nu,\mu}) + \eta_{\mu\nu}\, \xi^{\lambda}_{,\lambda}.$$

(c) If we now impose the condition

$$\psi^{\alpha}_{\beta,\alpha} = 0,$$

show that, this corresponds to

$$\psi'^{\alpha}_{\beta,\alpha} = \psi^{\alpha}_{\beta,\alpha} - \Box\, \xi_{\beta}.$$

Note: These conditions correspond to four equations, which can be achieved using the gauge functions ξ_{μ}. A gauge wherein the condition is satisfied is known as the de Donder gauge.

(d) Also, show that the above condition corresponds to the following condition on $h_{\alpha\beta}$:

$$h^{\alpha}_{\beta,\alpha} - \frac{1}{2}\, h_{,\beta} = 0.$$

5. *The wave equation:* In the absence of sources, one has $G_{\alpha\beta} = 0$.

(a) Show that, in a gauge wherein $\psi^{\alpha}_{\beta,\alpha} = 0$, the vacuum Einstein's equations simplify to

$$\Box\, \psi_{\alpha\beta} = 0.$$

(b) Show that, in terms of $h_{\alpha\beta}$, this equation corresponds to the equation

$$\Box\, h_{\alpha\beta} = 0,$$

along with the additional condition

$$\Box\, h = 0.$$

Note: The solutions to these equations describe propagating gravitational waves in flat spacetime.

Exercise sheet 15

Generation of gravitational waves

1. <u>*Gauge invariance of the Riemann tensor:*</u> Recall that, under the gauge transformation

$$x^\mu \to x'^\mu \simeq x^\mu + \epsilon\, \xi^\mu,$$

the perturbation $h_{\mu\nu}$ transforms as follows:

$$h_{\mu\nu} \to h'_{\mu\nu} \simeq h_{\mu\nu} - (\xi_{\mu,\nu} + \xi_{\nu,\mu})\,.$$

   (a) Show that the Riemann tensor $R_{\alpha\beta\gamma\delta}$ remains invariant under the gauge transformation.

   (b) Show that the equation $\Box h_{\mu\nu} = 0$ implies that $\Box R_{\alpha\beta\gamma\delta} = 0$.

2. <u>*States of polarization of a plane gravitational wave:*</u> Consider a plane gravitational wave propagating along the positive z-direction.

   (a) Utilizing the gauge freedom, show that, in the so-called transverse-traceless gauge, the quantity $h_{\mu\nu}$ describing the gravitational wave can be expressed as

$$h_{\mu\nu}(u) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{xx}(u) & h_{xy}(u) & 0 \\ 0 & h_{xy}(u) & -h_{xx}(u) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

   where $u = (t - z)$.

   (b) Let us now understand the effects of an incoming plane gravitational wave on particles. Consider a single particle that is at rest. What is the effect of the above plane gravitational wave on the particle? Specifically, what is the force $F^\alpha = -\Gamma^\alpha_{\beta\gamma}\, u^\beta\, u^\gamma$ on the particle, where u^α is its four velocity?

   (c) Consider a circular ring of particles in the x-y-plane. What is the effect of the above plane gravitational wave on the particles when $(h_{xx} \neq 0, h_{xy} = 0)$ and $(h_{xx} = 0, h_{xy} \neq 0)$?

   Note: The two components h_{xx} and h_{xy} correspond to the $+$ and $\times$ polarizations of the gravitational wave.

3. <u>*Exact plane gravitational waves:*</u> Consider the following line element

$$\mathrm{d}s^2 = \mathrm{d}u\,\mathrm{d}v - f^2(u)\,\mathrm{d}y^2 - g^2(u)\,\mathrm{d}z^2,$$

where $u = (t - x)$ and $v = (t + x)$. The above line-element describes plane gravitational waves propagating along the positive x-direction, and is an exact solution to the Einstein's equations.

   (a) Show that the non-zero components of the Christoffel symbols are given by

$$\Gamma^x_{yy} = 2\,f\,f', \quad \Gamma^x_{zz} = 2\,g\,g', \quad \Gamma^y_{ty} = \frac{f'}{f}, \quad \Gamma^z_{tz} = \frac{g'}{g},$$

   where the overprimes denote differentiation with respect to u.

   (b) Also, show that the non-zero components of the Riemann tensor are

$$R_{tyty} = f\,f'', \quad R_{tztz} = g\,g''.$$

   (c) Lastly, show that the Einstein's equations in the vacuum lead to

$$\frac{f''}{f} + \frac{g''}{g} = 0.$$

4. *Imposing the Lorenz gauge condition:* Recall that, in the presence of a source described by the stress-energy tensor $T^{\mu\nu}$, the tensor $h^{\mu\nu}$ describing the gravitational waves satisfies the differential equation
$$\Box h^{\mu\nu} = -16\,\pi\,G\,T^{\mu\nu}.$$
With the aid of the retarded Green's function, the solution to this differential equation can be expressed as
$$h^{\mu\nu}(t,\boldsymbol{x}) = -4\,G \int \frac{\mathrm{d}^3\boldsymbol{x}'}{R}\,T^{\mu\nu}(t-R,\boldsymbol{x}'),$$
where $R = |\boldsymbol{x}-\boldsymbol{x}'|$. Using the conservation of the stress-energy tensor, i.e. $\partial_\mu\,T^{\mu\nu} = 0$, show that the above solution satisfies the Lorenz gauge condition $\partial_\mu\,h^{\mu\nu} = 0$.

5. *Gravitational wave luminosity of a binary system:* Consider two neutron stars of approximately the same mass that are moving on a circle about their common center of mass. The gravitational radiation from a rotating binary source is found to be twice the orbiting frequency, since after one-half of the period, the masses are interchanged, which is indistinguishable from the initial configuration. If each star has a mass M and the radius of the orbit is R, then, in Newtonian mechanics, the frequency of the orbital motion is
$$\omega = \left(\frac{G\,M}{4\,R^3}\right)^{1/2}.$$
Assume that, at time t, the two stars are at the positions $(x_1,y_1,z_1) = [R\cos(\omega\,t), R\sin(\omega\,t), 0]$ and $(x_2,y_2,z_2) = [-R\cos(\omega\,t), -R\sin(\omega\,t), 0]$.

   (a) Given that $I^{ij}(t) = M\,\left[x_1^i(t)\,x_1^j(t) + x_2^i(t)\,x_2^j(t)\right]$, calculate the moment of inertia tensor for the system.

   (b) Show that $\dddot{I}^{ij}(t)\,\dddot{I}_{ij}(t) = 128\,M^2\,R^4\,\omega^6$.

   (c) Using this result and the following quadrupole formula
$$L = \frac{\mathrm{d}E}{\mathrm{d}t} = \frac{G}{5\,c^2}\,\langle\dddot{I}^{ij}(t)\,\dddot{I}_{ij}(t)\rangle,$$
   show that the total luminosity of the source of gravitational waves is
$$L = \frac{2}{5}\left(\frac{G\,M}{c^2\,R}\right)^5\left(\frac{c^5}{G}\right),$$
   where c has been displayed explicitly.