EP2102

CLASSICAL DYNAMICS

July–November 2024

Lecture schedule and meeting hours

- The course will consist of about 54 lectures, including about 8–10 tutorial sessions. However, note that there will be no separate tutorial sessions, and they will be integrated with the lectures.
- The duration of each lecture will be 50 minutes. We will be meeting in HSB 210.
- The first lecture will be on Monday, July 29, and the last lecture will be on Friday, May 3.
- We will meet four times a week. The lectures are scheduled for 11:00–11:50 AM on Mondays, 10:00–10:50 AM on Tuesdays, 9:00–9:50 AM on Wednesdays. and 1:00–1:50 PM on Thursdays.
- We will be meeting four times week so that I can make up for lectures that I may have to miss due to, say, travel. Changes in schedule, if any, will be notified sufficiently in advance.
- If you would like to discuss with me about the course outside the lecture hours, you are welcome to meet me at my office (in HSB 202) during 5:00–6:00 PM on Mondays. In case you are unable to find me in my office on more than occasion, please send me an e-mail at `sriram@physics.iitm.ac.in`.

Information about the course

- Information such as schedule of lectures, structure and syllabus of the course, suitable textbooks and additional references will be available on the course's page on Moodle at the following URL:

  `https://courses.iitm.ac.in/`
- The exercise sheets and other additional material will also be made available on Moodle.
- A PDF file containing these information as well as completed quizzes will also be available at the link on this course at the following URL:

  `http://www.physics.iitm.ac.in/~sriram/professional/teaching/teaching.html`

  I will keep updating this file and the course's page on Moodle as we make progress.

Quizzes, end-of-semester exam and grading

- The grading will be based on three scheduled quizzes and an end-of-semester exam.
- I will consider the best two quizzes for grading, and the two will carry 25% weight each.
- The three quizzes will be held on August 22, September 26 and October 24. The three dates are Thursdays, and the quizzes will be held during 5:00–6:30 PM on these days.
- The end-of-semester exam will be held during 9:00 AM–12:00 NOON on Wednesday, November 20, and the exam will carry 50% weight.

Syllabus and structure

1. **Overview of mechanics** [$\sim$ 1 lecture]
2. **Calculus of variations** [$\sim$ 4 lectures]
   (a) Advantages of the variational principle
   (b) Concept of variation – Euler equation for one-dependent and one-independent variable
   (c) Applications of the Euler equation
   (d) Generalization of the Euler equation to several dependent variables
   (e) Lagrange multipliers – Variation subject to constraints

Exercise sheets 1 and 2

3. **Lagrangian formulation of mechanics** [$\sim$ 7 lectures]
   (a) Degrees of freedom – Generalized coordinates and velocities
   (b) Principle of least action – Lagrange equations of motion – Additivity of Lagrangian for non-interacting systems
   (c) Inertial frames of reference – Newton's first law – Galileo's relativity principle –Galilean transformations
   (d) The Lagrangian for a free particle and a system of particles – Equations of motion in an external field – Newton's second law
   (e) Examples of constrained systems – Holonomic and non-holonomic constraints – Difficulties faced with constraints
   (f) Dealing with holonomic constraints – Extension of the Lagrangian formulation to non-holonomic systems

Exercise sheets 3 and 4

Additional exercises I

4. **Symmetries and conservation laws** [$\sim$ 4 lectures]
   (a) Homogeneity of space and time – Conservation of energy and momentum – Newton's third law – Center of mass
   (b) Isotropy of space – Conservation of angular momentum
   (c) Mechanical similarity – Virial theorem

Exercise sheet 5

Quiz I

5. **Integration of the Lagrange equations of motion** [$\sim$ 8 lectures]
   (a) Motion in one dimension – Determination of the potential energy from the period of oscillation
   (b) Two body problem – Reduced mass and the equivalent one-dimensional problem – Motion in a central field – Kepler's second law
   (c) Classification of orbits – Integrable power law potentials – Conditions for closed orbits
   (d) Kepler problem – Kepler's first and third laws – Motion in a repulsive field – Laplace-Runge-Lenz vector
   (e) Three body problem – The restricted case – Hill's curve and Lagrange points

Exercise sheets 6 and 7

Additional exercises II

6. **Collisions between particles** [$\sim$ 5 lectures]
   (a) Disintegration of particles – Elastic collisions
   (b) Scattering in a central field – Rutherford's formula – Total cross section – Small angle scattering
   (c) Rainbow scattering – Orbiting – Glory scattering

Exercise sheet 8

Quiz II

7. **Small oscillations** [$\sim$ 5 lectures]
   (a) Free, damped and forced oscillations in one dimension – Resonance – Parametric resonance
   (b) Oscillations of systems with more than one degree of freedom – Vibrations of molecules
   (c) Anharmonic oscillations – Resonance in non-linear oscillations
   (d) Motion in a rapidly oscillating field

Exercise sheets 9 and 10

8. **Motion of a rigid body** [$\sim$ 5 lectures]
   (a) Angular velocity – The inertia tensor – Angular momentum of a rigid body
   (b) The equations of motion of a rigid body
   (c) Eulerian angles – Euler's equations
   (d) The asymmetrical top – Heavy symmetrical top with one point fixed
   (e) Motion in a rotating frame – Coriolis force

Exercise sheet 11

Quiz III

9. **Hamiltonian formulation** [$\sim$ 5 lectures]
   (a) Conjugate momentum – Legendre's transformation – Hamiltonian and Hamilton's equations
   (b) Poisson brackets – Poisson's theorem
   (c) Canonical transformations – Invariance of Poisson brackets under canonical transformations
   (d) Phase space – Dynamics in the phase space – Phase portraits – Liouville's theorem
   (e) Hamilton-Jacobi equation – Separation of variables and solutions
   (f) Adiabatic invariants – Accuracy of conservation of the adiabatic invariant
   (g) Action-angle variables – Conditionally periodic motion
   (h) Integrable Hamiltonians – Properties of integrable systems – Examples – Motion on the tori

Exercise sheets 12 and 13

Additional exercises III

10. **Special relativity** [$\sim$ 6 Lectures]
    (a) The Michelson-Morley interferometric experiment – Postulates of special relativity
    (b) Lorentz transformations – The relativity of simultaneity – Length contraction and time dilation
    (c) Composition law for velocities – Doppler effect
    (d) Four vectors

Exercise sheet 14

End-of-semester exam

Advanced problems

Note: The parts in red will be covered if time permits.

Basic textbooks

Classical mechanics

1. L. D. Landau and E. M. Lifshitz, *Mechanics*, Course of Theoretical Physics, Volume 1, Third Edition (Pergamon Press, New York, 1976).
2. D. Kleppner and R. J. Kolenkow, *An Introduction to Mechanics* (Tata McGraw-Hill, New Delhi, 1999).
3. H. Goldstein, C. Poole and J. Safko, *Classical Mechanics*, Third Edition (Pearson Education, Singapore, 2002).
4. S. T. Thornton and J. B. Marion, *Classical Dynamics of Particles and Systems* (Cengage Learning, Singapore, 2004).
5. T. W. B. Kibble, *Classical Mechanics*, Fifth Edition (Imperial College Press, London, 2004).
6. D. Morin, *Introduction to Classical Mechanics* (Cambridge University Press, Cambridge, England, 2008).

Special relativity

1. A. P. French, *Special Relativity* (W. W. Norton, New York, 1968).
2. R. Resnick, *Introduction to Special Relativity* (Wiley Eastern, New Delhi, 1985).
3. E. F. Taylor and J. A. Wheeler, *Spacetime Physics* (W. H. Freeman, San Francisco, 1992).

Additional references

Mathematical methods

1. J. Mathews and R. L. Walker, *Mathematical Methods of Physics*, Second Edition (Addison Wesley, New York, 1970).
2. G. B. Arfken and H. J. Weber, *Mathematical Methods for Physicists*, Sixth Edition (Academic Press, New York, 2005).

Classical mechanics

1. D. T. Greenwood, *Principles of Dynamics*, Second Edition (Prentice-Hall of India, New Delhi, 1988).
2. L. N. Hand and J. D. Finch, *Analytical Mechanics* (Cambridge University Press, Cambridge, 1998).
3. W. Greiner, *Classical Mechanics: Systems of Particles and Hamiltonian Dynamics* (Springer-Verlag, New York, 2003).

Special relativity

1. W. Rindler, *Introduction to Special Relativity* (Oxford University Press, Oxford, 2004).

Advanced textbooks

1. I. Percival and D. Richards, *Introduction to Dynamics* (Cambridge University Press, Cambridge, England, 1982).

2. M. Tabor, *Chaos and Integrability in Nonlinear Dynamics—An Introduction* (John Wiley, New York, 1989).

3. W. Dittrich and M. Reuter, *Classical and Quantum Dynamics* (Springer-Verlag, Berlin, 1994).

4. L. N. Hand and J. D. Finch, *Analytical Mechanics* (Cambridge University Press, Cambridge, England, 1998).

5. J. V. Jose and E. J. Saletan, *Classical Dynamics: A Contemporary Approach* (Cambridge University Press, Cambridge, England, 1998).

6. J. B. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (Pearson Education, Singapore, 2003).

Exercise sheet 1

Calculus of variations

1. <u>*Snell's law:*</u> Two homogeneous media of refractive indices n_1 and n_2 are placed adjacent to each other. A ray of light propagates from a point in the first medium to a point in the second medium. According to the Fermat's principle, the light ray will follow a path that minimizes the transit time between the two points. Use Fermat's principle to derive the Snell's law of refraction, viz. that

$$n_1 \sin\theta_1 = n_2 \sin\theta_2,$$

where θ_1 and θ_2 are the angles of incidence and refraction at the interface.

Note: As the complete path is not differentiable at the interface, the problem is not an Euler equation problem.

2. <u>*Need for dependence on the derivative:*</u> Show that the condition that the integral

$$J = \int dx\, f(y,x)$$

has a stationary value (a) leads to $f(y,x)$ being independent of y, and (b) yields no information about any x-dependence.

Note: In other words, the above variational problem does not lead to a continuous and differentiable solution. This clearly suggests that, for a variational problem to be meaningful, the integrand must depend on derivatives of y. However, as we shall see later, the situation will change when constraints are introduced.

3. <u>*Equivalent forms of the Euler equation:*</u> Show that the two forms of the Euler's equation:

$$\left(\frac{\partial f}{\partial y}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y_x}\right) = 0$$

where $y_x = \mathrm{d}y/\mathrm{d}x$, and

$$\left(\frac{\partial f}{\partial x}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left[f - y_x\left(\frac{\partial f}{\partial y_x}\right)\right] = 0$$

are equivalent.

4. <u>*Brachistochrone problem:*</u> Consider a particle that is moving in a constant force field starting at rest from some point to a lower point. Determine the path that allows the particle to accomplish the transit in the least possible time.

Note: The resulting curve is referred to as the brachistochrone, i.e. the curve of the fastest descent.

5. <u>*Geodesics in Euclidean space:*</u> Geodesics are curves that connect any two points in a given space along the shortest distance.

   (a) Prove that, in any dimension, the shortest distance between two points in Euclidean space is a straight line.

   (b) Working in the polar coordinates, obtain the result that the shortest distance between two points on a plane is a straight line.

Exercise sheet 2

Calculus of variations subject to constraints

1. *Minimizing the surface area of a cylinder:* Find the ratio of the radius to the height that will minimize the total surface area of a right-circular cylinder of fixed volume.

2. *Inscribing a rectangle inside an ellipse:* Consider the ellipse
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$
Find the inscribed rectangle of maximum area. Show that the ratio of the area of the maximum area rectangle to the area of the ellipse is $2/\pi$.

3. *The gradient:* Find the maximum value of the directional derivative of $\varphi(x,y,z)$
$$\frac{\mathrm{d}\varphi}{\mathrm{d}s}=\frac{\partial\varphi}{\partial x}\cos\alpha+\frac{\partial\varphi}{\partial y}\cos\beta+\frac{\partial\varphi}{\partial z}\cos\gamma$$
subject to the constraint
$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1.$$

4. *Maxwell-Boltzmann, Fermi-Dirac and Bose Einstein distributions:* Let g_i denote the number of distinct quantum states between energies E_i and $E_i+\mathrm{d}E_i$ of a system. Let n_i be the number of particles distributed among these states subject to the constraints that the total number of particles N and the total energy E are fixed, i.e.
$$\sum_i n_i=N,\quad \sum_i n_i\,E_i=E.$$
   (a) For distinguishable particles, the probability of a given arrangement is
$$W_{\mathrm{MB}}=N!\prod_{i=1}^{\infty}\frac{g_i^{n_i}}{n_i!}.$$
   Show that maximizing W_{MB} subject to a fixed total number of particles and fixed total energy leads to the following distribution:
$$n_i=g_i\,\mathrm{e}^{-\lambda_1-\lambda_2\,E_i}.$$
   (b) For identical particles obeying the Pauli exclusion principle, the probability of a given arrangement is
$$W_{\mathrm{FD}}=\prod_{i=1}^{\infty}\frac{g_i!}{n_i!\,(g_i-n_i)!}.$$
   Show that maximizing W_{FD} subject to a fixed total number of particles and fixed total energy leads to the following distribution:
$$n_i=\frac{g_i}{\mathrm{e}^{\lambda_1+\lambda_2\,E_i}+1}.$$
   (c) For identical particles with no restriction on the number of particles in a given state, the probability of a given arrangement is
$$W_{\mathrm{BE}}=\prod_{i=1}^{\infty}\frac{(n_i+g_i-1)!}{n_i!\,(g_i-1)!}.$$

Show that maximizing W_{BE} subject to a fixed total number of particles and fixed total energy leads to the following distribution:

$$n_i = \frac{g_i}{\mathrm{e}^{\lambda_1+\lambda_2\, E_i} - 1}.$$

Hint: Use the so-called Stirling's approximation, according to which

$$\ln(n!) \simeq n \ln n - n.$$

Note: For $\lambda_1 = -\mu/(k_{\mathrm{B}}\, T)$ and $\lambda_2 = 1/(k_{\mathrm{B}}\, T)$, these correspond to the Maxwell-Boltzmann, Fermi-Dirac and Bose-Einstein distributions.

5. *Extremizing area and perimeter of a closed curve:* Show that: (a) for a perimeter of fixed length, the curve with maximum area is a circle, and (b) for a fixed area, the curve with minimum perimeter is a circle.

Exercise sheet 3

Lagrangian formulation of mechanics I

1. *Lagrangian of a free particle in different coordinate systems:* Write down the Lagrangian and the equations of motion for a free particle in (a) Cartesian, (b) cylindrical and (c) spherical polar coordinates.

2. *Behavior of Euler-Lagrange equation under point transformations:* Let $(q_1, q_2, \ldots, q_n)$ be a set of independent generalized coordinates for a system with n degrees of freedom. Let the system be described by the Lagrangian $L(q_i, \dot{q}_i, t)$ and let us transform to a new set of independent coordinates $(s_1, s_2, \ldots, s_n)$ by means of the transformation equations

$$q_i = q_i\left(s_1, s_2, \ldots, s_n, t\right), \text{ for } i = 1, 2, \ldots, n.$$

Such a transformation is known as a point transformation. Show that, if the Lagrangian $L(q_i, \dot{q}_i, t)$ is expressed as a function of s_j, $\dot{s}_j$ and t through a point transformation, then the system satisfies the Lagrange's equations with respect to the new coordinates, viz.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{s}_j}\right) - \left(\frac{\partial L}{\partial s_j}\right) = 0, \text{ for } j = 1, 2, \ldots, n.$$

Note: This implies that the form of the Lagrange's equations remain invariant under a point transformation.

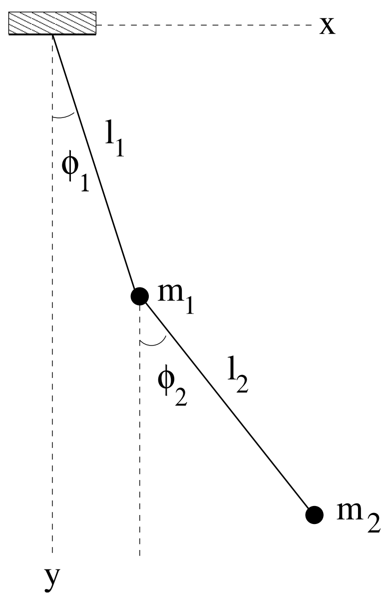
3. *Lagrangian for a constrained system:* A bead slides without friction down a wire that has the shape $y = f(x)$ in a uniform gravitational field (corresponding to an acceleration g). Write down the Lagrangian for the system and obtain the equation of motion. In particular, consider the case wherein the shape of the wire is a cycloid, i.e. when the shape $y = f(x)$ can be parametrically written as

$$x = a\ (\theta - \sin\theta)\,,\ y = a\ (1 + \cos\theta)\,.$$

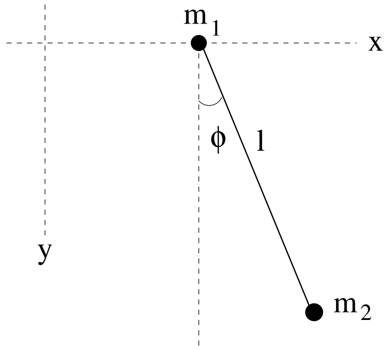
What is the equation of motion for $u = \cos(\theta/2)$?

4. *Lagrangian for different systems:* Construct the Lagrangian and obtain the equations of motion for the following systems when they are placed in a uniform gravitational field (corresponding to an acceleration g)

   (a) a coplanar double pendulum as shown in the figure below,

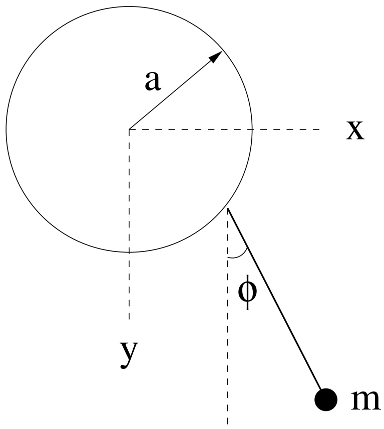


(b) a simple pendulum of mass m_2, with a mass m_1 at the point of support which can move on a horizontal line lying in the plane in which m_2 moves as shown in the figure below,



(c) a simple pendulum of mass m whose point of support

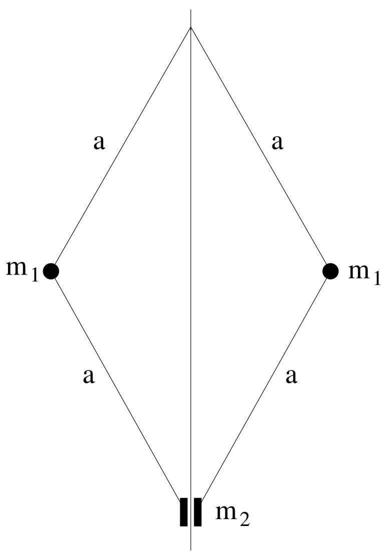
i. moves uniformly on a vertical circle with constant frequency γ as shown in the figure below,



ii. oscillates horizontally in the plane of motion of the pendulum according to the law $x = a\cos(\gamma\, t)$,

iii. oscillates vertically according to the law $y = a\cos(\gamma\, t)$,

(d) In the system shown in the figure below, the mass m_2 moves on a vertical axis and the whole system rotates about this axis with a constant angular velocity Ω.



5. _Lagrangian describing a coupled system:_ A Lagrangian for a particular system can be written as

$$L = \frac{m}{2}\left(a\,\dot{x}^2 + 2\,b\,\dot{x}\,\dot{y} + c\,\dot{y}^2\right) - \frac{k}{2}\left(a\,x^2 + 2b\,x\,y + c\,y^2\right),$$

where a, b and c are arbitrary constants, but subject to the condition that $(b^2 - a\,c) \neq 0$.

(a) Obtain the equations of motion corresponding to the above Lagrangian and examine the two cases $a = c = 0$ and $b = 0,\ c = -a$.

(b) Can you identify the physical system described by the Lagrangian?

(c) What is the significance of the condition on the value of $(b^2 - a\,c)$?

Exercise sheet 4

Lagrangian formulation of mechanics II

1. *Particle sliding off a surface:* A particle slides off a cylindrical surface. Using the method of Lagrange multipliers, obtain the critical angle at which the particle flies off from the surface.

2. *A suspended cable:* A flexible cable is suspended from two fixed points. The length of the cable is fixed. Using the method of Lagrange multipliers, find the curve that will minimize the total gravitational potential energy of the cable.

3. *Example of a non-holonomic constraint:* A particle moves in the x-y plane under the constraint that its velocity vector is always directed towards a point on the x-axis whose abscissa is some given function of time $f(t)$. Show that for $f(t)$ differentiable, but otherwise arbitrary, the constraint is non-holonomic.

4. *Mixture of holonomic and non-holonomic constraints:* Two wheels of radius a are mounted on the ends of a common axle of length b such that the wheels rotate independently. The whole combination rolls without slipping on a plane.

   (a) Show that there are arise two non-holonomic equations of constraint, viz.

   $$\begin{aligned} \cos\theta\,\mathrm{d}x + \sin\theta\,\mathrm{d}y &= 0, \\ \sin\theta\,\mathrm{d}x - \cos\theta\,\mathrm{d}y &= a\,\left(\mathrm{d}\phi + \mathrm{d}\phi'\right), \end{aligned}$$

   where θ, ϕ and ϕ' have meanings similar to those in the case of a single vertical disk we had considered and (x, y) are the coordinates of a point on the axle midway between the two wheels.

   (b) Also, show that there arises one holonomic equation of constraint, viz.

   $$\theta = C - \frac{a}{b}\,\left(\phi - \phi'\right),$$

   where C is a constant.

5. *Motion of a particle on a surface of revolution:* A particle of mass m moves under the influence of gravity on the inner surface of the paraboloid of revolution $x^2 + y^2 = a\,z$, which is assumed to be frictionless.

   (a) Obtain the equations of motion using the method of Lagrange multipliers.

   (b) Prove that the particle will describe a horizontal circle in the plane $z = h$ provided that it is given an angular velocity whose magnitude is $\omega = \sqrt{2\,g/a}$.

Additional exercises I

Calculus of variations and Lagrangian formulation of mechanics

1. *Back to the brachistochrone problem:* Recall that, the solution to the brachistochrone problem consisted of extremizing integrals of the type

$$I = \int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{2\,g\,y}}\,\left(1+y_x^2\right)^{1/2} = \int_{y_1}^{y_2} \frac{\mathrm{d}y}{\sqrt{2\,g\,y}}\,\left(1+x_y^2\right)^{1/2},$$

where in the first integral y and x assume their usual roles, i.e. y is the dependent variable and x the independent variable, whereas their roles are reversed in the second integral (with $x_y = \mathrm{d}x/\mathrm{d}y$.) Also, we had obtained the solution to the brachistochrone problem by solving the Euler's equation corresponding to the second integral. Using the first integral and the second form of the Euler's equation in the last problem, obtain the solution to the brachistochrone problem.

2. *Geodesics on a sphere:* Geodesics on a given surface are curves that connect any two points on the surface along the shortest distance. Show that the geodesics of a spherical surface are great circles, i.e. circles whose centers lie at the center of the sphere.

3. *Variation involving higher derivatives:*Show that the Euler equation corresponding to the integral

$$J[y(x)] = \int_{x_1}^{x_2} \mathrm{d}x\, f(y, y_x, y_{xx},, x),$$

where $y_{xx} = \mathrm{d}^2y/\mathrm{d}x^2$, is given by

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial f}{\partial y_{xx}}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y_x}\right) + \frac{\partial f}{\partial y} = 0.$$

Note: In order to obtain this equation, the variation as well its first derivative need to be set to zero at the end points.

4. *Fermat's principle in inhomogeneous media:* Consider a light ray propagating between two points, say, (x_1, y_1) and (x_2, y_2), in a two-dimensional inhomogeneous medium whose refractive index is described by the function $n(y, x)$. The velocity of light at any given point in the medium is $c/n(y, x)$ and, hence, according to the Fermat's principle, the light ray will follow a path between the two points such that the integral

$$I = \int_{(x_1,y_1)}^{(x_2,y_2)} \mathrm{d}l\, n(y, x)$$

is a minimum. (Note that $\mathrm{d}l$ denotes the infinitesimal arc length.) For $x_1 = -x_2 = -1$ and $y_1 = y_2 = 1$, find the path $y(x)$ of the light ray in media for which $n = e^y$ and $n = a\,(y - y_0)$, where $y_0 > 0$.

5. *Parallelepiped inside an ellipsoid:* A rectangular parallelepiped is inscribed in an ellipsoid of semi-axes a, b and c. Using the method of Lagrange multipliers, maximize the volume of the inscribed rectangular parallelepiped. Show that the ratio of the maximum volume to the volume of the ellipsoid is $2/(\sqrt{3}\,\pi)$.

6. *A different Lagrangian:* A particle of mass m moves in one dimension such that it has the Lagrangian

$$L = \left(\frac{m^2}{12}\right)\dot{x}^4 + m\,\dot{x}^2\,V(x) - V^2(x),$$

where V is a differentiable function of x. Obtain the equation of motion for $x(t)$ and describe the physical nature of the system on the basis of this equation.

7. *Equations of motion in non-inertial frames:* Obtain the Lagrangian and the equations of motion of a free particle in: (a) a frame that is rotating at a constant angular velocity, say, Ω, with respect to the z-axis, and (b) a frame that is accelerating with a uniform acceleration, say, g, in the direction of positive x-axis.

8. *Actions for simple systems:* Evaluate the action for the following one-dimensional systems: (a) a free particle (of mass m), (b) a particle (of mass m) that is subjected to a constant force (say, of magnitude α), and (c) a simple harmonic oscillator (of mass m and frequency ω). Assume that these systems are at the points q_1 and q_2 at times t_1 and t_2, respectively.

9. *Bead on a wire:* A bead of mass m slides without friction in a uniform gravitational field on a vertical hoop of radius R. The hoop is constrained to rotate at a fixed angular frequency Ω about its vertical diameter. Let θ denote the position of the bead on the hoop as measured from its lowest point. Write down the Lagrangian for the system and find how the equilibrium values of θ depend on Ω.

10. *A coupled system:* Consider two particles of masses m_1 and m_2. Let m_1 be confined to move on a circle of radius a in the $z = 0$ plane, centered at $x = y = 0$. Let m_2 be confined to move on a circle of radius b in the $z = c$ plane, centered at $x = y = 0$. A light and massless spring of spring constant k is attached between the two particles.

    (a) Write down the Lagrangian for the system.

    (b) Obtain the equations of motion of the system using the method of Lagrange multipliers and give a physical interpretation for each multiplier.

Quiz I

Calculus of variations and Lagrangian formulation of mechanics

1. *Geodesics on a cone:* Consider a cone with a semi-vertical angle α.

   (a) Determine the line element on the cone. $\boxed{\text{2 marks}}$

   (b) Obtain the equation(s) governing the geodesics on the cone. $\boxed{\text{4 marks}}$

   (c) Solve the equation(s) to arrive at the geodesics. $\boxed{\text{4 marks}}$

   Note:

$$\int \frac{\mathrm{d}x}{x\,\sqrt{x^2-a^2}} = \frac{1}{a}\,\tan^{-1}\left(\frac{\sqrt{x^2-a^2}}{a}\right).$$

2. *Shape of water:* A fixed volume of water in a cylinder is rotating with constant angular velocity Ω. Let the radius of the cylinder be R and let the height of the water in the cylinder before it starts rotating be H. Using the method of Lagrange multipliers, find the curve of the water surface that will minimize the total potential energy of the water in the combined gravitational-centrifugal force field. $\boxed{\text{10 marks}}$

3. *Atwood machine:* Consider the so-called Atwood machine, i.e. two masses m_1 and m_2 which are connected by a string (of, say, length l) running through a pulley. Let the whole system be located in a uniform gravitational field with acceleration due to gravity being g.

   (a) How many degrees of freedom does the system have? $\boxed{\text{1 marks}}$

   (b) Write down the Lagrangian for the system? $\boxed{\text{6 marks}}$

   (c) Obtain the equation(s) of motion. $\boxed{\text{3 marks}}$

4. *Bead on a wire:* A bead of mass m slides without friction in a uniform gravitational field (with acceleration due to gravity being g) on a vertical hoop of radius R. The hoop is constrained to rotate at a fixed angular frequency Ω about its vertical diameter. Let θ denote the position of the bead on the hoop as measured from its lowest point.

   (a) Write down the Lagrangian for the system. $\boxed{\text{4 marks}}$

   (b) Obtain the equation of motion governing the system. $\boxed{\text{3 marks}}$

   (c) Determine the stable and unstable equilibrium points of the system. How do they depend on Ω, g and R? $\boxed{\text{3 marks}}$

   Note: The points of equilibrium correspond to locations where the first derivative of complete potential describing the system vanishes. The stable and unstable points of equilibrium correspond to locations where the second derivative of the potential are positive and negative, respectively.

5. *Non-relativistic particle in an electromagnetic field:* A non-relativistic particle that is moving in an electromagnetic field described by the scalar potential ϕ and the vector potential $\boldsymbol{A}$ is governed by the Lagrangian $\boxed{\text{10 marks}}$

$$L = \frac{m\,\boldsymbol{v}^2}{2} + q\,\left(\frac{\boldsymbol{v}}{c}\cdot\boldsymbol{A}\right) - q\,\phi,$$

where m and q are the mass and the charge of the particle, while c denotes the velocity of light. Show that the equation of motion of the particle is given by

$$m\,\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = q\,\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\right),$$

where $\boldsymbol{E}$ and the $\boldsymbol{B}$ are the electric and the magnetic fields given by

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\,\frac{\partial \boldsymbol{A}}{\partial t} \qquad \text{and} \qquad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}.$$

Note: The scalar and the vector potentials, viz. ϕ and $\boldsymbol{A}$, are dependent on time *as well as* space. Further, given two vectors, say, $\boldsymbol{C}$ and $\boldsymbol{D}$, one can write,

$$\boldsymbol{\nabla}\left(\boldsymbol{C}\cdot\boldsymbol{D}\right) = \left(\boldsymbol{D}\cdot\boldsymbol{\nabla}\right)\boldsymbol{C} + \left(\boldsymbol{C}\cdot\boldsymbol{\nabla}\right)\boldsymbol{D} + \boldsymbol{D}\times\left(\boldsymbol{\nabla}\times\boldsymbol{C}\right) + \boldsymbol{C}\times\left(\boldsymbol{\nabla}\times\boldsymbol{D}\right).$$

Also, since $\boldsymbol{A}$ depends on time as well as space, we have

$$\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} = \frac{\partial \boldsymbol{A}}{\partial t} + \left(\boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\boldsymbol{A}.$$

Exercise sheet 5

Symmetries and conservation laws

1. *Utilizing conservation of energy and momentum:* Consider a region of space divided by a plane. The potential energy of a particle on one half of the plane is U_1 and, on the other half, it is U_2, where U_1 and U_1 are constants. A particle of mass m which has a speed'v_1 moves from the first half to the second. If θ_1 and θ_2 are the angles subtended by the trajectory of particle with respect to the normal on either side of the plane, show that

$$\frac{\sin\theta_1}{\sin\theta_2} = \left(1 + \frac{U_1 - U_2}{m\, v_1^2/2}\right)^{1/2}.$$

What is the optical analog of the problem?

2. *Components and magnitude of angular momentum:* Obtain the expressions for the Cartesian components and the magnitude of the angular momentum of a particle in the cylindrical and the spherical polar coordinates.

3. *Virial theorem:* Given a function $f(t)$, the average value of the function is defined as

$$\langle f(t)\rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{d}t\, f(t).$$

Consider a system in which the forces acting on the particles consist of conservative forces $\boldsymbol{F}_i$, determined by the potential $U(\boldsymbol{r}_i)$. Show that, for such a system, the so-called virial theorem holds in the form

$$2\,\langle T\rangle = -\left\langle \sum_i \boldsymbol{r}_i \cdot \boldsymbol{F}_i \right\rangle = \left\langle \sum_i \boldsymbol{r}_i \cdot \frac{\partial U}{\partial \boldsymbol{r}_i} \right\rangle.$$

4. *Perfect gas law:* A perfect gas is defined as one in which the forces of interaction between the molecules of the gas are negligible. This occurs, for example, when the gas is so dilute that the collisions between the molecules are rare when compared to the collisions with the walls of the container. Using these information and the virial theorem, obtain the perfect gas law, viz.

$$P\,V = N\,k_{\mathrm{B}}\,T,$$

where P, V, T and N denote the pressure, the volume, the temperature and the number of molecules of the gas, respectively, and k_{B} is the Boltzmann constant.

5. *An application of the virial theorem:* Using virial theorem, show that the total mass M of a spherical cluster of stars (or galaxies) of uniform density and radius R is given by

$$M = \frac{5\,R\,\langle v^2\rangle}{3\,G},$$

where $\langle v^2\rangle$ is the mean-squared velocity of the individual stars and G is, of course, the gravitational constant.

Note: The above relation allows us to obtain an estimate of the mass of a cluster of stars or galaxies if we can measure the mean-squared velocity, say, from the Doppler spread of the spectral lines and the radius of the cluster, say, from its known distance and angular size.

Exercise sheet 6

Integration of the Lagrange equations of motion I

1. *Period of oscillation:* Determine the period of oscillation as a function of the energy, say, E, when a particle of mass m is moving in one dimension (along the x-axis) under the influence of the following potentials: (a) $U = A\,|x|^{2\,n}$ (with $A > 0$ and $n > 0$), (b) $U(x) = -U_0/\cosh^2(\alpha\,x)$ (for $-U_0 < E < 0$), and (c) $U(x) = U_0 \tan^2(\alpha\,x)$.

2. *Explicitly establishing the virial theorem:* Evaluate the time averages (i.e. the average over one complete period) of the kinetic and potential energies, say, T and U, for a particle that is moving along an elliptic orbit in the Keplerian central potential, and establish the virial theorem for the case, viz. that $2\,\langle T\rangle = -\langle U\rangle$, where the angular brackets denote the averages.

3. *Time evolution in the Keplerian potential:* Consider a particle of reduced mass m that is moving on a hyperbolic trajectory in the central potential $U(r) = -\alpha/r$, where $\alpha > 0$. Show that the time evolution of the trajectory can be parametrically expressed as follows:

$$r = a\,\left(e\cosh\xi - 1\right),\; t = \sqrt{m\,a^3/\alpha}\,\left(e\sinh\xi - \xi\right)$$

   or, equivalently, as

$$x = a\,\left(e - \cosh\xi\right),\; y = a\,\sqrt{e^2-1}\,\sinh\xi,$$

   where the quantity a and the eccentricity of the orbit e are given in terms of the energy $E > 0$ and the angular momentum ℓ by the relations

$$a = \frac{\alpha}{2\,E},\; e = \sqrt{1 + \frac{2\,E\,\ell^2}{m\,\alpha^2}},$$

   while $-\infty < \xi < \infty$.

4. *Laplace-Runge-Lenz vector:* Recall that, for a particle with s degrees of freedom, we require $2\,s-1$ constants of motion in order to arrive at a unique trajectory for the particle. According to this argument, for the Kepler problem, we would then need five integrals of motion to obtain the solution. We had expressed the solution in terms of the energy E of the system and the amplitude of the angular momentum vector $\boldsymbol{L}$, both of which were conserved. However, these quantities, viz. the energy E and the three components of the angular momentum vector $\boldsymbol{L}$, only add up to four constants of motion. Evidently, it will be interesting to examine if we can identify the fifth integral of motion associated with the system.

   (a) Show that, for a particle moving in the Keplerian central potential, i.e. $U(r) = -\alpha/r$ with $\alpha > 0$, the following vector is an integral of motion:

$$\boldsymbol{A} = m\,\boldsymbol{v}\times\boldsymbol{L} - \frac{m\,\alpha\,\boldsymbol{r}}{r}.$$

   Note: The conserved vector $\boldsymbol{A}$ is known as the Laplace-Runge-Lenz vector.

   (b) Show that the vector $\boldsymbol{A}$ lies in the plane of the orbit.

   (c) Indicate the amplitude and the direction of $\boldsymbol{A}$ associated with a planet as it moves in an elliptical orbit around the Sun.

   Hint: Determine the amplitude and the direction of $\boldsymbol{A}$ at, say, the perihelion and the aphelion.

   (d) If E, $\boldsymbol{L}$ and $\boldsymbol{A}$ are all constants, then, we seem to have seven integrals of motion instead of the required five to arrive at a unique solution! How does seven reduce to five?

   Hint: Examine if there exist any relations between $\boldsymbol{A}$ and $\boldsymbol{L}$ and/or E.

5. *Passing through the centre:* Consider a particle that describes a circular orbit under the influence of an attractive central force directed towards a point on the circle.

(a) Show that the force varies as the inverse-fifth power of the distance.

(b) Also, show that for the orbit described the total energy of the particle is zero.

(c) Moreover, obtain the period of the motion.

Exercise sheet 7

Integration of the Lagrange equations of motion II

1. *Falling onto each other:* Consider two particles which are moving under the influence of their mutual gravitational force. Let the particles follow circular orbits about one another with a time period T. Show that, if the particles are suddenly stopped in their orbits and allowed to gravitate towards each other, they will collide after a time $T/(4\sqrt{2})$.

2. *Precessing orbits:* Show that the motion of a particle in the central potential

$$U(r) = -\frac{\alpha}{r} + \frac{h}{r^2},$$

with $\alpha > 0$, is the same as that of motion under the Kepler potential alone when expressed in terms of a coordinate system rotating or precessing around the center of the force.

3. *Precession of the perihelion of Mercury:* When a small correction $\delta U(r)$ is added to the potential energy $U = -\alpha/r$, with $\alpha > 0$, the paths of finite motion are no longer closed and in each revolution the perihelion is displaced through a small angle, say, $\delta\phi$.

   (a) Find $\delta\phi$ when: (i) $\delta U = \beta/r^2$ and (ii) $\delta U = \gamma/r^3$. Compare the result for the first case with the result from the previous exercise.

   (b) For the case of the planet Mercury, the value of the semi-major axis of its orbit is $a = 5.79 \times 10^{10}$ m and its eccentricity is $e = 0.206$. Also, the period of the Mercury's orbit around the Sun is 88 days. Further, the value of the gravitational constant is $G = 6.673 \times 10^{-11}\,\mathrm{kg^{-1}\,m^3\,s^{-2}}$, the speed of light is $c = 2.998 \times 10^8$ m/s, and the mass of the Sun is $M_\odot = 1.989 \times 10^{30}$ kg. Estimate the extent of precession of Mercury's orbit if $\delta U = -\alpha\,\ell^2/(m^2\,c^2\,r^3)$, where $\alpha = G\,M\,m$, ℓ is the angular momentum, and c is the velocity of light.

   Note: The δU mentioned above arises due to general relativistic effects. The measured precession of the perihelion of the planet Mercury turns out to be $5599''.7 \pm 0''.4$ per century, but a large part of it is caused due to the influences of the other planets. When the other contributions have been subtracted, the precession of the perihelion of the planet Mercury due to the purely general relativistic effects amounts to 43.1 ± 0.5 seconds of arc per century.

4. *Motion in the Yukawa potential:* A particle moves in the Yukawa potential described by

$$V(r) = -\frac{k}{r}\,\exp -(r/a),$$

where k and a are positive quantities.

   (a) Write down the equations of motion and reduce them to the equivalent one-dimensional problem.

   (b) Use the effective potential to discuss the qualitative nature of the orbits for different values of the energy and the angular momentum.

   (c) Also, show that if the orbit is nearly circular, the apsides will advance approximately by $\pi\,\rho/a$ per revolution, where ρ is the radius of the circular orbit.

   Note: The radial distances to the turning points are known as apsidal distances.

5. *Eliminating the centre of mass:* A system consists of one particle of mass M and n particles with equal masses m. Eliminate the motion of the centre of mass and thereby reduce the problem to one involving n particles.

Additional exercises II

Symmetries and conservation laws and integration of the equations of motion

1. *Motion in one dimension:* Consider a particle that is in motion in the following one-dimensional potential:
$$U(x) = \alpha\,\left(\mathrm{e}^{-2\,\beta\,x} - 2\,\mathrm{e}^{-\beta\,x}\right), \text{ where } (\alpha, \beta) > 0.$$
Let E be the energy of the particle.
   (a) Obtain the solutions describing the time evolution of the particle for the cases $E < 0$, $E = 0$ and $E > 0$.
   (b) Evaluate the period of oscillation of the particle when $E < 0$.

2. *Time period of the Duffing's oscillator:* A Duffing oscillator is a non-linear oscillator that is governed by the potential
$$U(x) = \frac{1}{2}\,m\,\omega^2\,x^2 + \frac{1}{4}\,\epsilon\,\alpha\,x^4,$$
where ϵ is a dimensionless, infinitesimal and positive quantity, while $\alpha > 0$ has suitable dimensions. Express the time period of the oscillator in terms of its amplitude, say, A, at the order ϵ.

3. *Zero potential:* Solve the orbital equation when the potential is zero. What is the shape of the trajectory?

4. *Two massive bodies in a constant field:* Two bodies move in a constant external gravitational field. Show that their motion can be reduced to an equivalent one body problem, as in the case of the Kepler problem.

5. *Intersecting orbits:* Two masses, M and $2\,M$, orbit around their centre of mass. If the orbits are circular, they do not intersect. But if they are very elliptical, they do. What is the smallest value of the eccentricity for which they intersect?

6. *Potentials permitting closed orbits:* Obtain the condition on attractive power law central potentials if the orbits in the potentials are to remain closed.

   Note: The result is known as Bertrand's theorem.

7. *The isotropic oscillator:* Consider the isotropic oscillator described by the potential
$$U(r) = \frac{1}{2}\,m\,\omega^2\,r^2.$$
   (a) Solve the corresponding orbital equation.
   (b) What is the shape of the general trajectory?
   (c) Also, solve for the time dependence of the radial and the angular coordinates.

8. *Ratio of the mass of the Sun to the Earth:* Evaluate, approximately, the ratio of the mass of the Sun to that of the Earth, using only the lengths of the year and of the lunar month (which is 27.3 days) and the mean radii of the orbits of the Earth and the Moon, which are 1.49×10^8 km and 3.8×10^5 km, respectively.

9. *Return of comets Hyakutake and Hale-Bopp:* The comets Hyakutake and Hale-Bopp appeared in March–May 1996 and March–May 1997. If their eccentricities and perihelion are $e = 0.999846$ and 0.995075 and 0.230123 AU and 0.913959 AU, respectively, determine when the comets will be seen again.

10. *Galaxy rotation curves and dark matter:* Many spiral galaxies are thin disks (with possibly a bulge and/or a bar in the middle), which rotate about an axis through their centre. Let the visible matter in the galaxy extend until radius R.

   (a) Assuming a constant mass density until a radius R, determine the rotational velocity of the galaxy as a function of the radius r for $r < R$ and $r > R$.

   (b) The so-called galaxy rotation curves of spiral galaxies indicate that the velocity is a constant for $r > R$. If so, assuming a simple power law for the density as a function of the radius, determine the radial dependence of the density of mass in the galaxy. What does the result imply?

   Note: This suggests the existence of dark matter, i.e. matter that does not interact with light.

   (c) If the rotation curves remain a constant until the radius of, say, $r = 8\,R$, estimate the fraction of mass of the galaxy that is in the form of dark matter?

Exercise sheet 8

Collisions between particles

1. *Angles in spontaneous disintegration:* In the laboratory system, find: (a) the relation between the angles θ_1 and θ_2 after a spontaneous disintegration of a particle into two particles, and (b) the angular distribution of the resulting particles.

2. *Velocities in an elastic collision:* Express the velocity of each particle after an elastic collision between a moving particle (say, m_1) and another at rest (say, m_2) in terms of their directions of motion in the laboratory system.

3. *Scattering by a rigid sphere:* Consider scattering by a perfectly rigid sphere of radius a, i.e. when the interaction is such that $U(r) = \infty$ for $r < a$ and $U(r) = 0$ for $r > a$. Determine the effective cross-section for the scattering of particles by the rigid sphere. What is the total cross-section?

4. *Rutherford scattering in the centre-of-mass frame:* Consider scattering by the repulsive potential $U(r) = \alpha/r$ with $\alpha > 0$.

   (a) Find the effective cross-section for scattering.

   (b) What is the total cross-section? What does the result imply?

   Note: This corresponds to the famous experiment by Rutherford wherein α-particles were scattered by atoms in a gold foil.

5. *Scattering cross-section in the laboratory frame:* Determine the cross-section in the laboratory frame in the cases of scattering by a hard sphere and Rutherford scattering.

Quiz II

From symmetries and conservation laws to integration of the equations of motion

1. *Bounded motion in one-dimension:* Consider a particle moving in the following one-dimensional potential:

$$U(x) = \frac{a}{x^2} + b\, x^2, \text{ where } (a, b) > 0.$$

   (a) Obtain the solution describing the time evolution of the particle. $\boxed{\text{7 marks}}$

   (b) Also, evaluate the time period of the particle as a function of its energy. $\boxed{\text{3 marks}}$

2. *Unbounded orbit in a gravitational field:* Find the time dependence of the coordinates of a particle with energy $E = 0$ moving along a parabola in the central potential $U = -\alpha/r$ with $\alpha > 0$. $\boxed{\text{10 marks}}$

3. *Motion in a repulsive central potential:* Consider a particle that is moving in the repulsive central potential $U(r) = \alpha/r$, where $\alpha > 0$.

   (a) Obtain the second order differential equation satisfied by the quantity $u = 1/r$ in terms of the angular coordinate ϕ. $\boxed{\text{3 marks}}$

   (b) Solve the second order differential equation to obtain the orbital trajectory, and express the constants of integration in terms of the energy and the angular momentum of the particle. $\boxed{\text{3 marks}}$

   (c) Assuming that the trajectory of the particle to be in the x-y plane, show that the trajectory is a hyperbola with the following asymptotes: $\boxed{\text{2 marks}}$

$$y = \pm\sqrt{e^2 - 1}\;\; x - \frac{e\, p}{\sqrt{e^2 - 1}},$$

   where $p = \ell^2/(m\,\beta)$, while e denotes the eccentricity which is given by

$$e = \sqrt{1 + \frac{2\,E\,\ell^2}{m\,\alpha^2}}.$$

   (d) Also, plot the hyperbolic trajectory for the corresponding attractive central potential (i.e. when $\alpha < 0$). $\boxed{\text{2 marks}}$

4. (a) *Period of a binary system:* Two double stars of the same mass as the Sun rotate about their common center of mass. Their separation is 4 light years. Estimate their time period of revolution. $\boxed{\text{5 marks}}$

   (b) *Perihelion and aphelion of the Halley's comet:* Halley's comet, which passed around the Sun early in 1986, moves in a highly elliptical orbit with an eccentricity of 0.967 and a period of 76 years. Calculate the perihelion and aphelion of the comet. $\boxed{\text{5 marks}}$

   Note: The mass of the Sun is about 2×10^{30} kg, while the value of the gravitational constant is $G = 6.673 \times 10^{-11}\, \mathrm{m^3\, kg^{-1}\, s^{-2}}$.

5. *From the orbit to the force:* Consider a particle that is moving in a central field.

   (a) Determine the central force, given that the particle is known to move on the following spiral orbit: $\boxed{\text{4 marks}}$

$$r(\phi) = \beta\, \mathrm{e}^{\gamma\,\phi}.$$

   (b) Obtain the corresponding time evolution of the radial and the angular coordinates r and ϕ. $\boxed{\text{4 marks}}$

   (c) What is the energy associated with the above orbit? $\boxed{\text{2 marks}}$

Exercise sheet 9

Small oscillations I

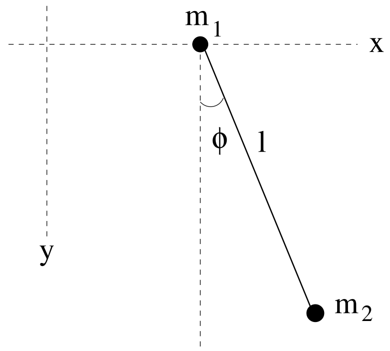
1. *Frequency of oscillations I:* Find the frequency of oscillations of a particle of mass m which is attached to a spring whose other end is fixed at a distance l and that is

   (a) Free to move along a line,

   (b) Moving on a circle of radius r.

   Assume that a force F is required to extend the spring to the length l.-

2. *Frequency of oscillations II:* Find the frequency of small oscillations of a pendulum whose point of support carries a mass m_1 and is free to move horizontally as shown in the figure on the left below.



3. *Frequency independent of amplitude:* Determine the form of a curve such that the frequency of oscillations of a particle on it under the force of gravity is independent of the amplitude.

4. *Forced oscillations:* Assuming that, at time $t = 0$, a system is at rest in equilibrium (i.e. $x(0) = \dot{x}(0) = 0$), determine the forced oscillations of the system under a force $F(t)$ of the following forms: (a) $F = F_0$, (b) $F = a\,t$, (c) $F = F_0\,\exp-(\alpha\,t)$, and (d) $F = F_0\,\exp-(\alpha\,t)\,\cos(\beta\,t)$, with F_0, a, α and β being constants.

5. *Final amplitude:* Assuming that a system is at rest in equilibrium up to time $t = 0$, determine the final amplitude for the oscillations of the system under: (a) a force which is zero for $t < 0$, $F_0\,t/T$ for $0 < t < T$, and F_0 for $t > T$, (b) a constant force F_0 that acts for a finite time T, and (c) forces (i) $F_0\,t/T$ and (ii) $F_0\,\sin(\omega t)$, which act between $t = 0$ and $t = T$.

Exercise sheet 10

Small oscillations II

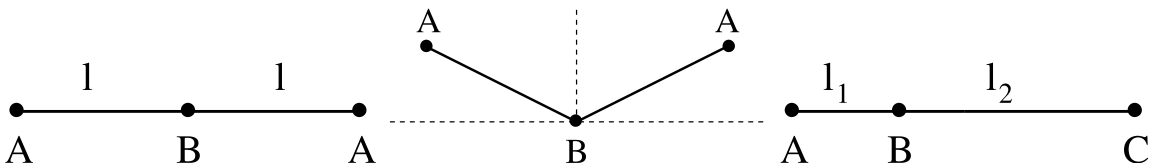
1. *Small oscillations along a curve:* A particle of mass m moves in a constant vertical gravitational field along the curve defined by $y = a\,x^4$, where y is the vertical direction and $a > 0$. Find the equation of motion for small oscillations about the position of equilibrium.

2. *System with two degrees of freedom:* Determine the oscillations of a system with two degrees of freedom (say, x and y) whose Lagrangian is given by

$$L = \frac{1}{2}\,(\dot{x}^2 + \dot{y}^2) - \frac{{\omega_0}^2}{2}\,(x^2 + y^2) + \alpha\,x\,y.$$

3. *Space oscillator:* Consider a particle moving in the central potential $U(r) = k\,r^2/2$. Determine the trajectory of the particle in its plane of motion.

4. *Vibrations of molecules:* Determine the frequencies of vibrations of (a) a symmetrical linear tri-atomic molecule ABA shown on the left in the figure below, (b) a triangular molecule ABA shown in the middle, and (c) an asymmetrical linear molecule ABC shown on the right in the figure below.



Note: Assume that the potential energy of the molecule depends only on the distances between the atoms and the angle subtended by the two bonds.

5. *Forced oscillations in the presence of friction:* Determine the forced oscillations due to the external force $F = F_0\,\exp{(\alpha\,t)}\,\cos{(\gamma\,t)}$ in the presence of friction.

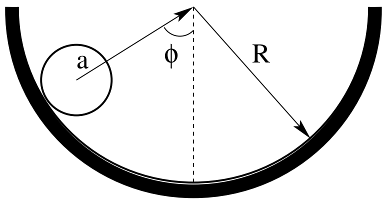
Exercise sheet 11

Motion of a rigid body

1. *Moments of inertia of rigid bodies:* Determine the principal moments of inertia for the following homogeneous bodies: (a) a thin rod of length l, (b) a sphere of radius R, (c) a circular cylinder of radius R and height h, and (d) a rectangular parallelepiped of sides a, b and c.

2. *Small oscillations of a compound pendulum:* Determine the frequency of small oscillations of a compound pendulum, i.e. a rigid body swinging about a fixed horizontal axis in a gravitational field.

3. *Kinetic energy of a rolling cylinder:* Obtain the kinetic energy of a homogeneous cylinder of radius a that is rolling inside a cylindrical surface of radius R as shown in the figure below



4. *Deflection of a freely falling body:* Find the deflection of a freely falling body from the vertical caused by the Earth's rotation, assuming the angular velocity of this rotation to be small.

5. *Effects of Earth's rotation on a pendulum:* Determine the effect of the Earth's rotation on small oscillations of a pendulum.

   Note: This effect is best illustrated by the so-called Foucault's pendulum.

Quiz III

From collisions between particles to small oscillations

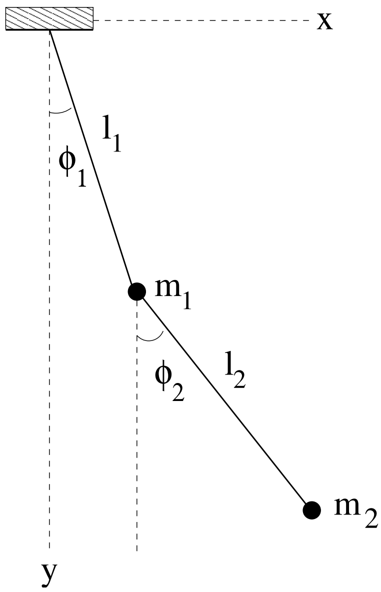
1. *Cross-section in a repulsive potential:* Consider a particle being scattered by the repulsive central potential $U = \alpha/r^2$, where $\alpha > 0$.

   (a) Determine $r_{\min}$, i.e. the radius corresponding to the closest approach of the particle to the centre of the potential. 3 marks

   (b) Obtain the expression for the deflection angle θ in terms of the impact parameter s. 4 marks

   (c) Using the expression for the deflection angle θ, obtain the differential cross-section $\mathrm{d}\sigma$ for scattering. 3 marks

2. *Frequency of small oscillations:* Consider the coplanar double pendulum shown in the figure below.



   (a) Obtain the Lagrangian for the system. 3 marks

   (b) Arrive at the Lagrangian for small oscillations of the double pendulum. 2 marks

   (c) What are the normal frequencies of the system? 3 marks

   (d) What are the frequencies in the limit $m_1 \gg m_2$? 2 marks

3. *Diagonalizing a symmetric matrix:* Consider the following real, symmetric matrix:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

   (a) What are the eigen values of the matrix? 2 marks

   (b) Determine the eigen vectors, say, X, corresponding to the eigen values and normalize them, i.e. ensure that $X^{\mathrm{T}} X = 1$, where X is a column vector and X^{T} is its transpose. 3 marks

   Note: Given an eigen value λ, the eigen vector, say, X, of a matrix A is described by the equation $A\,X = \lambda\,X$.

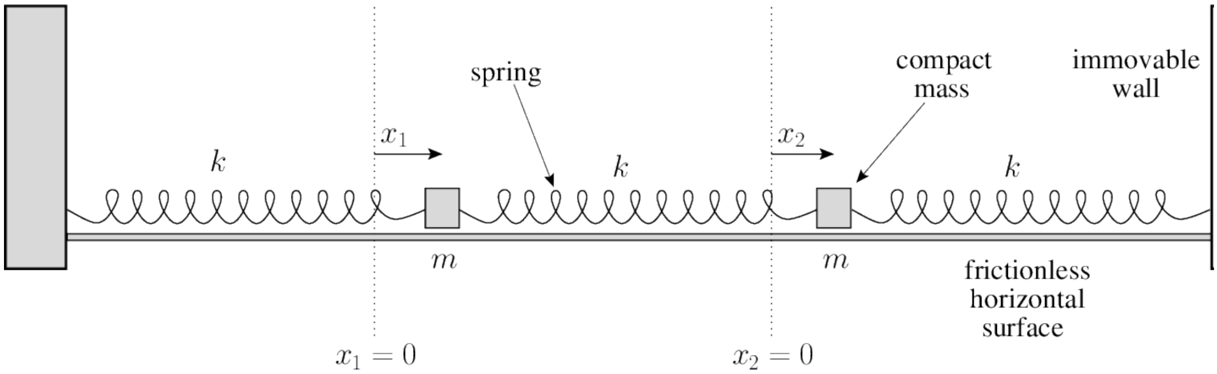
   (c) Construct an orthogonal matrix, say, S, using the eigen vectors. 3 marks

   Note: A matrix that is the same as its transpose is known as an orthogonal matrix.

   (d) Show that the matrix $S^{\mathrm{T}} A\, S$ is diagonal. 2 marks

   Note: Observe what the diagonal components are!

4. *Two masses connected by springs:* Two identical blocks of mass m are connected to each other and two immovable walls by springs with spring constant k, as shown in the figure below.



   (a) Write down Lagrangian of the system of two masses. 3 marks

   (b) What are the equations of motion for the two masses? 2 marks

   (c) What are the characteristic frequencies of oscillation of the system? 2 marks

   (d) By combining the equations of motion for the two degrees of freedom, construct the normal coordinates of the system associated with the two frequencies. 3 marks

5. *Under-damped, forced oscillator:* Consider an under-damped (or weakly damped) oscillator with natural frequency ω_0 that is subject to the following periodic force:

$$f(t) = f_0 \cos(\gamma t).$$

   (a) What is the complete solution for the system? What is the solution describing the behavior of the oscillator at very late times? 4 marks

   (b) Find the amplitude, say, A, and phase, say, δ, of the solution at late times. 3 marks

   (c) What are the values of δ when $\gamma \gg \omega_0$, $\gamma = \omega_0$ and $\gamma \ll \omega_0$? 2 marks

   (d) What do these values of δ imply? 1 mark

Exercise sheet 12

Hamiltonian formulation I

1. *Bead on a wire:* A bead slides without friction down a wire that has the shape $y = f(x)$ in a uniform gravitational field (corresponding to an acceleration g).

   (a) Obtain the Hamiltonian of the system.

   (b) Also, write down the Hamilton's equation of motion.

2. *A complicated Lagrangian:* Consider a system described by the Lagrangian

$$L = a\,\dot{x}^2 + b\,\frac{\dot{y}}{x} + c\,\dot{x}\,\dot{y} + f\,y^2\,\dot{x}\,\dot{z} + g\,\dot{y} - k\sqrt{x^2 + y^2},$$

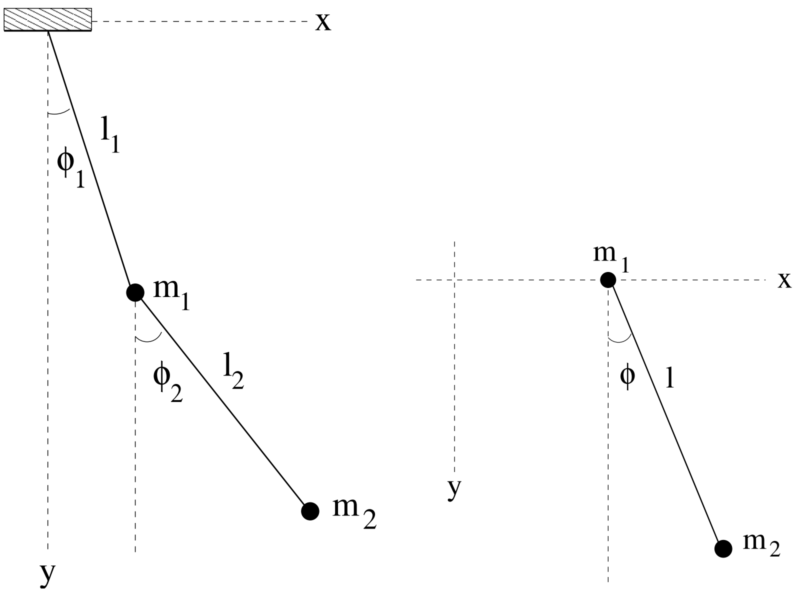
   where (a, b, c, g, k) are constants.

   (a) What is the Hamiltonian of the system?

   (b) What are the quantities that are conserved?

3. *Hamiltonian of various systems:* Construct the Hamiltonian and obtain the Hamilton's equations of motion for the following systems when they are placed in a uniform gravitational field (corresponding to an acceleration g)

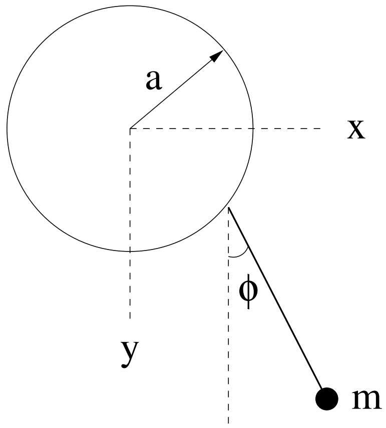
   (a) a coplanar double pendulum as shown on the left in the figure below,

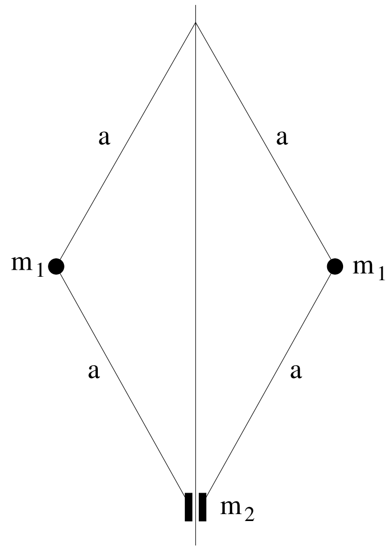


   (b) a simple pendulum of mass m_2, with a mass m_1 at the point of support which can move on a horizontal line lying in the plane in which m_2 moves as shown on the right in the figure above,

   (c) a simple pendulum of mass m whose point of support

      i. moves uniformly on a vertical circle with constant frequency γ as shown on the left in the figure below,





ii. oscillates horizontally in the plane of motion of the pendulum according to the law $x = a\cos(\gamma\, t)$,

iii. oscillates vertically according to the law $y = a\cos(\gamma\, t)$,

(d) In the system shown on the right in the figure above, the mass m_2 moves on a vertical axis and the whole system rotates about this axis with a constant angular velocity Ω.

4. *Hamiltonian of a particle in a rotating frame:* Construct the Hamiltonian for a particle in a uniformly rotating frame of reference.

5. *Hamiltonian of a collection of particles:* Obtain the Hamiltonian for a system comprising one particle of mass M and n particles each of mass m, excluding the motion of the centre of mass.

Exercise sheet 13

Hamiltonian formulation II: Phase portraits and Poisson brackets

1. *Phase portraits:* Draw the phase portraits of a particle moving in the following one dimensional potentials: (a) $U = a\,x$, (b) $U = a\,x^2/2$, (c) $U = -a\,x^2/2$, (d) $U = -a\cos\theta$ and (e) $U = a\,|x|^n$, where $a > 0$ and $n > 2$.

2. *Poisson brackets:* Determine: (a) the Poisson brackets formed from the Cartesian components of the momentum $\boldsymbol{p}$ and the angular momentum $\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{p}$ of a particle, and (b) the Poisson brackets formed from the Cartesian components of the angular momentum $\boldsymbol{M}$.

3. *More Poisson brackets:* Show that: (a) $[\Phi, M_z] = 0$, where Φ is any function, spherically symmetric about the origin, of the coordinates and the momentum of a particle, and (b) $[\boldsymbol{f}, M_z] = \boldsymbol{f} \times \hat{\boldsymbol{z}}$, where $\boldsymbol{f}$ is a vector function of the coordinates and the momentum of a particle and $\hat{\boldsymbol{z}}$ is the unit vector along the positive z-axis.

4. *A condition on the Poisson brackets:* Show that, any three quantities (f, g, h), which are functions of position and momenta, satisfy the following relation involving the Poisson brackets between them:
$$[f, [g, h]] + [h, [f, g]] + [g, [h, f]] = 0.$$
Note: This condition is known as Jacobi's identity.

5. *Evolution of density in phase space:* Show that the density of points in phase space corresponding to the motion of a system of particles remains constant during the motion.

   Note: This result is known as Liouville's theorem.

Exercise sheet 14

Special relativity I: Lorentz transformations and some consequences

1. *Superluminal motion:* Consider a blob of plasma that is moving at a speed v along a direction that makes an angle θ with respect to the line of sight. Show that the *apparent* transverse speed of the source, projected on the sky, will be related to the actual speed v by the relation

$$v_{\mathrm{app}} = \frac{v \sin\theta}{1 - (v/c)\cos\theta}.$$

From this expression conclude that the apparent speed v_{app} can exceed the speed of light.

2. *Aberration of light:* Consider two inertial frames S and S', with the frame S' moving along the x-axis with a velocity v with respect to the frame S. Let the velocity of a particle in the frames S and S' be $\mathbf{u}$ and $\mathbf{u}'$, and let θ and θ' be the angles subtended by the velocity vectors with respect to the common x-axis, respectively.

   (a) Show that
   $$\tan\theta = \frac{u' \sin\theta'}{\gamma\,[u'\cos\theta' + v]},$$
   where $\gamma = \left[1 - (v/c)^2\right]^{-1/2}$.

   (b) For $u = u' = c$, show that
   $$\cos\theta = \frac{\cos\theta' + (v/c)}{1 + (v/c)\cos\theta'}$$
   and
   $$\sin\theta = \frac{\sin\theta'}{\gamma\,[1 + (v/c)\cos\theta']}.$$

   (c) For $(v/c) \ll 1$, show that
   $$\Delta\theta = (v/c)\sin\theta',$$
   where $\Delta\theta = (\theta' - \theta)$.

3. *Decaying muons:* Muons are unstable and decay according to the radioactive decay law $N = N_0 \exp -(0.693\, t/t_{1/2})$, where N_0 and N are the number of muons at times $t = 0$ and t, respectively, while $t_{1/2}$ is the half life. The half life of the muons in their own rest frame is 1.52×10^{-6} s. Consider a detector on top of a $2,000$ m mountain which counts the number of muons traveling at the speed of $v = 0.98\, c$. Over a given period of time, the detector counts 10^3 muons. When the relativistic effects are taken into account, how many muons can be expected to reach the sea level?

4. *Binding energy:* As you may know, the deuteron which is the nucleus of deuterium, an isotope of hydrogen, consists of one proton and one neutron. Given that the mass of a proton and a neutron are $m_{\mathrm{p}} = 1.673 \times 10^{-27}$ kg and $m_{\mathrm{n}} = 1.675 \times 10^{-27}$ kg, while the mass of the deuteron is $m_{\mathrm{d}} = 3.344 \times 10^{-27}$ kg, show that the binding energy of the deuteron in about 2.225 MeV.

   Note: MeV refers to Million electron Volts, and an electron Volt is 1.602×10^{-19} J.

5. *Form invariance of the Minkowski line element:* Show that the following Minkowski line element is invariant under the Lorentz transformations:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}\boldsymbol{x}^2.$$

Exercise sheet 15

Special relativity II: Working in terms of four vectors

1. *Compton effect using four vectors:* Consider the scattering between a photon of frequency ω and a relativistic electron with velocity $\mathbf{v}$ leading to a photon of frequency ω' and electron with velocity $\mathbf{v}'$. Such a scattering is known as Compton scattering. Let α be the angle between the incident and the scattered photon. Also, let θ and θ' be the angles subtended by the directions of propagation of the incident and the scattered photon with the velocity vector of the electron before the collision.

   (a) Using the conservation of four momentum, show that
   $$\frac{\omega'}{\omega} = \frac{1-(v/c)\cos\theta}{1-(v/c)\cos\theta' + (\hbar\,\omega/\gamma\, m_{\rm e}\, c^2)\,(1-\cos\alpha)},$$
   where $\gamma = \left[1-(v/c)^2\right]^{-1/2}$ and $m_{\rm e}$ is the mass of the electron.

   (b) When $\hbar\,\omega \ll \gamma\, m_{\rm e}\, c^2$, show that the frequency shift of the photon can be written as
   $$\frac{\Delta\omega}{\omega} = \frac{(v/c)\;(\cos\theta - \cos\theta')}{1-(v/c)\cos\theta'},$$
   where $\Delta\omega = (\omega' - \omega)$.

2. *Creation of electron-positron pairs:* A purely relativistic process corresponds to the production of electron-positron pairs in a collision of two high energy gamma ray photons. If the energies of the photons are ϵ_1 and ϵ_2 and the relative angle between their directions of propagation is θ, then, by using the conservation of energy and momentum, show that the process can occur only if
   $$\epsilon_1\,\epsilon_2 > \frac{2\, m_{\rm e}^2\, c^4}{1-\cos\theta},$$
   where $m_{\rm e}$ is the mass of the electron.

3. *Transforming four vectors and invariance under Lorentz transformations:* Consider two inertial frames K and K', with K' moving with respect to K, say, along the common x-axis with a certain velocity.

   (a) Given a four vector A^μ in the K frame, construct the corresponding contravariant and covariant four vectors, say, $A^{\mu\prime}$ and A'_μ, in the K' frame.

   (b) Explicitly illustrate that the scalar product $A_\mu\, A^\mu$ is a Lorentz invariant quantity, i.e. show that $A_\mu\, A^\mu = A'_\mu\, A^{\mu\prime}$.

4. *Lorentz invariance of the wave equation:* Show that the following wave equation:
   $$\frac{1}{c^2}\,\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi = 0$$
   satisfied by, say, light, is invariant under the Lorentz transformations.

5. *Mirrors in motion:* A mirror moves with the velocity v in a direction perpendicular its plane. A ray of light of frequency ν_1 is incident on the mirror at an angle of incidence θ, and is reflected at an angle of reflection ϕ and frequency ν_2.

   (a) Show that
   $$\frac{\tan(\theta/2)}{\tan(\phi/2)} = \frac{c+v}{c-v} \qquad \text{and} \qquad \frac{\nu_2}{\nu_1} = \frac{c+v\cos\theta}{c-v\cos\phi}.$$

   (b) What happens if the mirror was moving parallel to its plane?