PH5460

CLASSICAL FIELD THEORY

July–November 2013

Lecture schedule and meeting hours

- The course will consist of about 43 lectures, including about 8–10 tutorial sessions. However, note that there will be no separate tutorial sessions, and they will be integrated with the lectures.
- The duration of each lecture will be 50 minutes. We will be meeting in HSB 210.
- The first and the last lectures will be on Wednesdays, July 31 and November 13, respectively.
- We will meet thrice a week. The lectures are scheduled for 11:00–11:50 AM on Wednesdays, 9:00–9:50 AM on Thursdays, and 8:00–8:50 AM on Fridays.
- We may also meet during 4:45–5:35 PM on Tuesdays for either the quizzes or to make up for any lecture that I may have to miss due to, say, travel. Changes in schedule, if any, will be notified sufficiently in advance.
- If you would like to discuss with me about the course outside the lecture hours, you are welcome to meet me at my office (HSB 202A) during 12:00–13:00 PM on Wednesdays. In case you are unable to find me in my office, please send me an e-mail at `sriram@physics.iitm.ac.in`.

Information about the course

- I will be distributing hard copies containing information such as the schedule of the lectures, the structure and the syllabus of the course, suitable textbooks and additional references, as well as exercise sheets.
- A PDF file containing these information as well as completed quizzes will also made be available at the link on this course at the following URL:
  `http://www.physics.iitm.ac.in/~sriram/professional/teaching/teaching.html`
  I will keep updating the file as we make progress.

Quizzes, end-of-semester exam and grading

- The grading will be based on three scheduled quizzes and an end-of-semester exam.
- I will consider the best two quizzes for grading, and the two will carry 25% weight each.
- The three quizzes will be on August 27, September 24 and October 29. All these three dates are Tuesdays, and the quizzes will be held during 4:45–5:35 PM.
- The end-of-semester exam will be held during 9:00 AM – 12:00 NOON on Tuesday, November 26, and the exam will carry 50% weight.

Syllabus and structure

Classical Field Theory

1. **Essential special relativity** [$\sim$ 6 lectures]

   (a) Lorentz transformations – Length contraction and time dilation
   (b) Metric tensor – The light cone
   (c) Contravariant and covariant vectors – Tensors and transformations
   (d) Infinitesimal generators of translations, rotations and boosts
   (e) Algebra of the generators – The Lorenz and the Poincare groups

Exercise sheets 1, 2 and 3

2. **Theory of a real scalar field** [$\sim$ 5 lectures]

   (a) An illustrative example – Action formulation for a string
   (b) Action describing a real, canonical, scalar field – The Euler-Lagrange field equation
   (c) The conjugate momentum – Hamiltonian density
   (d) The stress-energy tensor – Physical interpretation
   (e) Non-canonical scalar fields – Relation to relativistic fluids

Exercise sheet 4

Quiz I

3. **More on real scalar fields** [$\sim$ 4 lectures]

   (a) The angular momentum tensor
   (b) Invariance under rotations and Lorentz transformations – Conservation of angular momentum
   (c) Symmetrization of the stress-energy tensor

Exercise sheet 5

4. **The case of the complex scalar field** [$\sim$ 5 lectures]

   (a) Action governing the complex scalar field – Equations of motion
   (b) Global gauge invariance
   (c) Local gauge invariance and the need for the electromagnetic field

Exercise sheet 6

Quiz II

Additional exercises I

5. **Symmetries and conservation laws** [$\sim$ 7 lectures]

   (a) Noether's theorem
   (b) External symmetries – Symmetry under translations, rotations and Lorentz transformations – Conserved quantities
   (c) Dilatations – The conformal stress-energy tensor
   (d) Internal symmetries and gauge transformations – The Abelian case of the complex scalar field interacting with the electromagnetic field

(e) The iso-vector, Lorentz-scalar field and the Yang-Mills field as a non-Abelian example

Exercise sheet 7

6. **The theory of the electromagnetic field** [$\sim$ 6 lectures]

   (a) The electromagnetic field tensor – The first pair of Maxwell's equations
   (b) The four current vector – The continuity equation – Charge conservation
   (c) Action governing the free electromagnetic field – Interaction of the electromagnetic field with charges and currents – The second pair of Maxwell's equations
   (d) Gauge invariance of the electromagnetic field – The Lorentz and the Coulomb gauges
   (e) Equations governing the free field – Electromagnetic waves – Polarization
   (f) Energy density – Poynting vector – The stress-energy tensor of the electromagnetic field
   (g) Lorentz transformation properties of the electric and the magnetic fields
   (h) The retarded Green's function – The Lienard-Wiechart potentials – Radiation from moving charges
   (i) The case of the massive vector field – The Proca equations

Exercise sheets 8 and 9

Quiz III

7. **Spontaneous symmetry breaking and formation of topological defects** [$\sim$ 8 lectures]

   (a) The concept of spontaneous symmetry breaking – Simple illustrative examples
   (b) Spontaneous breaking of global symmetries – The Goldstone theorem
   (c) Spontaneous breaking of local symmetries – The Higgs mechanism in Abelian and non-Abelian models
   (d) Formation of topological defects – Domain walls – Cosmic strings – The t'Hooft-Polyakov monopole
   (e) Time-dependent solutions – Solitons

Exercise sheet 10

Additional exercises II

8. **Basic aspects of general relativity** [$\sim$ 2 lectures]

   (a) Principle of general covariance
   (b) The concept of a metric tensor – Geodesics
   (c) Curvature of spacetime – The Einstein-Hilbert action

End-of-semester exam

Basic textbooks

1. L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Course of Theoretical Physics, Volume 2), Fourth Revised English Edition (Pergamon Press, New York, 1975).

2. D. E. Soper, *Classical Field Theory* (Dover, New York, 1976).

3. A. O. Barut, *Electrodynamics and Classical Theory of Fields and Particles* (Dover, New York, 1980).

4. F. Scheck, *Classical Field Theory* (Springer, Heidelberg, 2012).

Additional references

1. S. Coleman, *Aspects of Symmetry* (Cambridge University Press, Cambridge, England, 1988).

2. R. Rajaraman, *Solitons and Instantons* (North-Holland, Amsterdam, 1989).

3. L. H. Ryder, *Quantum Field Theory*, Second Edition (Cambridge University Press, Cambridge, England, 1996).

4. B. Felsager, *Geometry, Particles and Fields* (Springer, Heidelberg, 1998).

5. E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley, New York, 1990).

Exercise sheet 1

Special relativity: Lorentz transformations and some consequences

1. *Superluminal motion:* Consider a blob of plasma that is moving at a speed v along a direction that makes an angle θ with respect to the line of sight. Show that the *apparent* transverse speed of the source, projected on the sky, will be related to the actual speed v by the relation

$$v_{\rm app} = \frac{v\,\sin\theta}{1-(v/c)\cos\theta}.$$

From this expression conclude that the apparent speed $v_{\rm app}$ can exceed the speed of light.

2. *Aberration of light:* Consider two inertial frames S and S', with the frame S' moving along the x-axis with a velocity v with respect to the frame S. Let the velocity of a particle in the frames S and S' be $\mathbf{u}$ and $\mathbf{u}'$, and let θ and θ' be the angles subtended by the velocity vectors with respect to the common x-axis, respectively.

   (a) Show that

$$\tan\theta = \frac{u'\sin\theta'}{\gamma\,[u'\cos\theta'+v]},$$

   where $\gamma = \left[1-(v/c)^2\right]^{-1/2}$.

   (b) For $u = u' = c$, show that

$$\cos\theta = \frac{\cos\theta'+(v/c)}{1+(v/c)\cos\theta'}$$

   and

$$\sin\theta = \frac{\sin\theta'}{\gamma\,[1+(v/c)\cos\theta']}.$$

   (c) For $(v/c) \ll 1$, show that

$$\Delta\theta = (v/c)\,\sin\theta',$$

   where $\Delta\theta = (\theta'-\theta)$.

3. *Decaying muons:* Muons are unstable and decay according to the radioactive decay law $N = N_0\,\exp -(0.693\,t/t_{1/2})$, where N_0 and N are the number of muons at times $t=0$ and t, respectively, while $t_{1/2}$ is the half life. The half life of the muons in their own rest frame is 1.52×10^{-6} s. Consider a detector on top of a $2,000$ m mountain which counts the number of muons traveling at the speed of $v = 0.98\,c$. Over a given period of time, the detector counts 10^3 muons. When the relativistic effects are taken into account, how many muons can be expected to reach the sea level?

4. *Binding energy:* As you may know, the deuteron which is the nucleus of deuterium, an isotope of hydrogen, consists of one proton and one neutron. Given that the mass of a proton and a neutron are $m_{\rm p} = 1.673\times10^{-27}$ kg and $m_{\rm n} = 1.675\times10^{-27}$ kg, while the mass of the deuteron is $m_{\rm d} = 3.344\times10^{-27}$ kg, show that the binding energy of the deuteron in about 2.225 MeV.

   Note: MeV refers to Million electron Volts, and an electron Volt is 1.602×10^{-19} J.

5. *Form invariance of the Minkowski line element:* Show that the following Minkowski line element is invariant under the Lorentz transformations:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}\boldsymbol{x}^2.$$

Exercise sheet 2

Special relativity: Working in terms of four vectors

1. *Compton effect using four vectors:* Consider the scattering between a photon of frequency ω and a relativistic electron with velocity $\mathbf{v}$ leading to a photon of frequency ω' and electron with velocity $\mathbf{v}'$. Such a scattering is known as Compton scattering. Let α be the angle between the incident and the scattered photon. Also, let θ and θ' be the angles subtended by the directions of propagation of the incident and the scattered photon with the velocity vector of the electron before the collision.

   (a) Using the conservation of four momentum, show that

   $$\frac{\omega'}{\omega} = \frac{1-(v/c)\cos\theta}{1-(v/c)\cos\theta' + (\hbar\,\omega/\gamma\, m_{\rm e}\, c^2)\,(1-\cos\alpha)},$$

   where $\gamma = \left[1-(v/c)^2\right]^{-1/2}$ and $m_{\rm e}$ is the mass of the electron.

   (b) When $\hbar\,\omega \ll \gamma\, m_{\rm e}\, c^2$, show that the frequency shift of the photon can be written as

   $$\frac{\Delta\omega}{\omega} = \frac{(v/c)\;(\cos\theta - \cos\theta')}{1-(v/c)\cos\theta'},$$

   where $\Delta\omega = (\omega' - \omega)$.

2. *Creation of electron-positron pairs:* A purely relativistic process corresponds to the production of electron-positron pairs in a collision of two high energy gamma ray photons. If the energies of the photons are ϵ_1 and ϵ_2 and the relative angle between their directions of propagation is θ, then, by using the conservation of energy and momentum, show that the process can occur only if

   $$\epsilon_1\,\epsilon_2 > \frac{2\,m_{\rm e}^2\,c^4}{1-\cos\theta},$$

   where $m_{\rm e}$ is the mass of the electron.

3. *Transforming four vectors and invariance under Lorentz transformations:* Consider two inertial frames K and K', with K' moving with respect to K, say, along the common x-axis with a certain velocity.

   (a) Given a four vector A^μ in the K frame, construct the corresponding contravariant and covariant four vectors, say, $A^{\mu\prime}$ and A'_μ, in the K' frame.

   (b) Explicitly illustrate that the scalar product $A_\mu\, A^\mu$ is a Lorentz invariant quantity, i.e. show that $A_\mu\, A^\mu = A'_\mu\, A^{\mu\prime}$.

4. *Lorentz invariance of the wave equation:* Show that the following wave equation:

   $$\frac{1}{c^2}\,\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi = 0$$

   satisfied by, say, light, is invariant under the Lorentz transformations.

5. *Mirrors in motion:* A mirror moves with the velocity v in a direction perpendicular its plane. A ray of light of frequency ν_1 is incident on the mirror at an angle of incidence θ, and is reflected at an angle of reflection ϕ and frequency ν_2.

   (a) Show that

   $$\frac{\tan(\theta/2)}{\tan(\phi/2)} = \frac{c+v}{c-v} \qquad \text{and} \qquad \frac{\nu_2}{\nu_1} = \frac{c+v\cos\theta}{c-v\cos\phi}.$$

   (b) What happens if the mirror was moving parallel to its plane?

Exercise sheet 3

Tensors and transformations

1. *Four velocity and four acceleration:* The four acceleration of a relativistic particle is defined as $a^\mu = \mathrm{d}u^\mu/\mathrm{d}s$, where $u^\mu = \mathrm{d}x^\mu/\mathrm{d}s$ is the four velocity of the particle.

   (a) Express a^μ in terms of the three velocity $\mathbf{v}$ and the three acceleration $\mathbf{a} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t$ of the particle.

   (b) Evaluate $a^\mu\, u_\mu$ and $a^\mu\, a_\mu$ in terms of $\boldsymbol{v}$ and $\boldsymbol{a}$.

2. *The Lorentz force:* The action for a relativistic particle that is interacting with the electromagnetic field is given by

$$S[x^\mu(s)] = -m\,c \int \mathrm{d}s - \frac{e}{c} \int \mathrm{d}x_\mu\, A^\mu,$$

   where m is the mass of the particle, while e is its electric charge. The quantity $A^\mu = (\phi, \boldsymbol{A})$ is the four vector potential that describes the electromagnetic field, with, evidently, ϕ and $\mathbf{A}$ being the conventional scalar and three vector potentials.

   (a) Vary the above action with respect to x^μ to arrive at the following Lorentz force law:

$$m\,c\, \frac{\mathrm{d}u^\mu}{\mathrm{d}s} = \frac{e}{c}\, F^{\mu\nu}\, u_\nu,$$

   where u^μ is the four velocity of the particle and the electromagnetic field tensor $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} = \partial_\mu\, A_\nu - \partial_\nu\, A_\mu$$

   with $\partial_\mu \equiv \partial/\partial x^\mu$.

   (b) Show that the components of the field tensor $F_{\mu\nu}$ are given by

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix},$$

   where (E_x, E_y, E_z) and (B_x, B_y, B_z) are the components of the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ which are related to the components of the four vector potential by the following standard expressions:

$$\boldsymbol{E} = -\frac{1}{c}\,\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi \qquad \text{and} \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A}.$$

   (c) Express the above equation governing the motion of the charge in the more familiar three vector notation. What does the zeroth component of the equation describe?

3. *Reducing tensors to vectors:* If $X^\lambda{}_{\mu\nu}$ is a mixed tensor of rank $(1,2)$, show that the contracted quantity $Y_\mu = X^\nu{}_{\mu\nu}$ is a covariant vector.

4. *Transformation of electric and magnetic fields:* Consider two inertial frames, say, K and K', with the frame K' moving with a velocity v with respect to the frame K along the common x-axes.

   (a) Given the components of the electric and the magnetic fields, say, $\mathbf{E}$ and $\mathbf{B}$, in the frame K, using the transformation properties of the electromagnetic field tensor $F_{\mu\nu}$, construct the corresponding components in the frame K'.

   (b) Show that $|\boldsymbol{E}|^2 - |\boldsymbol{B}|^2$ is invariant under the Lorentz transformations.

   (c) Express the quantity $|\boldsymbol{E}|^2 - |\boldsymbol{B}|^2$ explicitly as a scalar in terms of the field tensor $F_{\mu\nu}$.

5. *The Lorentz invariant four volume:* Show that the differential spacetime volume $\mathrm{d}^4x = c\, \mathrm{d}t\, \mathrm{d}^3\boldsymbol{x}$ is a Lorentz invariant quantity.

Exercise sheet 4

Theories of real scalar fields

1. *An unconventional scalar field:* Consider a scalar field, say, ϕ, that is governed by the following unconventional action:
$$S[\phi(\tilde{x})] = \frac{1}{c}\int \mathrm{d}^4\tilde{x}\, \left[\left(\frac{X}{\alpha^2}\right)^n - V(\phi)\right],$$
where $\mathrm{d}^4\tilde{x} = c\, \mathrm{d}t\, \mathrm{d}^3\boldsymbol{x}$, the quantity X is given by
$$X = \frac{1}{2}\, \partial_\mu\phi\, \partial^\mu\phi,$$
while $V(\phi)$ denotes the potential describing the scalar field and α is a constant of suitable dimensions.
   (a) Determine the dimension of ϕ/α.
   (b) What is the equation of motion governing the scalar field?
   (c) What is the corresponding stress-energy tensor?

2. *Tachyons:* Consider a scalar field, say, T, that is described by the following action:
$$S[T(\tilde{x})] = -\frac{1}{c}\int \mathrm{d}^4\tilde{x}\, V(T)\, \sqrt{1 - \partial_\mu T\, \partial^\mu T}.$$
   (a) What is the dimension of the scalar field T?
   (b) Vary the action to arrive at the equation of motion for T.
   (c) Construct the corresponding stress-energy tensor.

   Note: Under conditions when the potential $V(T)$ contains a maxima and no minima, the field T is referred to as the tachyon.

3. *Relativistic ideal fluids:* The stress-energy tensor of a relativistic, ideal fluid with energy density ε and pressure p is given by
$$T^\mu_\nu = (\varepsilon + p)\, u^\mu\, u_\nu - p\, \delta^\mu_\nu,$$
where u^μ is the four velocity of the fluid.
   (a) Show that, in a frame that is comoving with the fluid, the stress-energy tensor has the following simple form: $T^\mu_\nu = \mathrm{diag.}\ (\varepsilon, -p, -p, -p)$.
   (b) Using the conservation of the stress-energy tensor, arrive at the equations of motions governing the dynamics of the fluid in a generic frame.
   (c) Express the time and the spatial components of the equations of motion in terms of the three vector $\boldsymbol{v}$ of the fluid.
   (d) Using these equations of motion, arrive at the Euler equation that governs the fluid in the non-relativistic limit.

   Note: The energy density ε is related to the mass density ρ as follows: $\varepsilon = \rho\, c^2$.

4. *A generic non-canonical scalar field:* The action for a generic scalar field can be written as
$$S[\phi(\tilde{x})] = \frac{1}{c}\int \mathrm{d}^4\tilde{x}\, \mathcal{L}(X, \phi),$$
where the Lagrangian density $\mathcal{L}$ is an arbitrary function of the kinetic term X and the field ϕ.

(a) Assume that the scalar field ϕ is homogeneous, i.e. it is not dependent on the spatial coordinates. Show that, in such a case, the stress-energy tensor associated with the scalar field reduces to the following form: $T^{\mu}_{\nu} = \text{diag.}\,(\varepsilon, -p, -p, -p)$, with the energy density ε and the pressure p being given by

$$\varepsilon = 2\,X\,\frac{\partial \mathcal{L}}{\partial X} - \mathcal{L} \qquad \text{and} \quad p = \mathcal{L}.$$

(b) Show that a homogeneous scalar field that is described by the above action satisfies the following equation of motion:

$$\left(\frac{\partial \mathcal{L}}{\partial X} + 2\,X\,\frac{\partial^2 \mathcal{L}}{\partial X^2}\right)\frac{\partial^2 \phi}{\partial x_0^2} + 2\,X\,\frac{\partial^2 \mathcal{L}}{\partial X\,\partial \phi} - \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$

5. *Barotropic scalar fields:* Consider a scalar field which is described by a Lagrangian density that is not directly dependent on the field, but is an arbitrary function of the kinetic term X.

(a) Evaluate the energy density and pressure of the field.

(b) Argue that, in such a situation, the scalar field behaves as a barotropic fluid wherein the pressure p can be expressed as a function of the energy density ε.

(c) What is the four velocity of the fluid in such a case?

Quiz I

Special relativity and real scalar fields

1. (a) *Colliding particles I:* A particle of mass m_1 and velocity $\boldsymbol{v}_1$ collides with a particle at rest of mass m_2, and is absorbed by it. Determine the mass as well as the velocity of the compound system. 4 marks

   (b) *Colliding particles II:* A particle of mass m and kinetic energy T_i collides with a stationary particle of the same mass. Determine the kinetic energy of the incident particle after the collision, if it is scattered by an angle θ. 6 marks

   Note: The kinetic energy associated with a particle of mass m and energy E is $(E - m\,c^2)$.

2. *Motion in a constant and uniform electric field:* Consider a particle that is moving in a constant and uniform electric field that is directed, say, along the positive x-axis. Let the relativistic three momentum $\boldsymbol{p}$ of the particle at the time, say, $t = 0$, be zero.

   (a) Solve the equation of motion to arrive at $x(t)$. 7 marks

   Hint: It is useful to note that we can write $\boldsymbol{v} = \mathrm{d}\boldsymbol{x}/\mathrm{d}t = \boldsymbol{p}\,c^2/\mathcal{E}$, where $\mathcal{E}/c = \sqrt{\boldsymbol{p}^2 + m^2\,c^2}$.

   (b) Plot the trajectory of the particle in the $c\,t$-x plane. 3 marks

3. (a) *Symmetry under transformations:* Show that the electromagnetic field tensor $F_{\mu\nu}$ remains anti-symmetric under an arbitrary coordinate transformation. 5 marks

   (b) *A second Lorentz scalar involving the electromagnetic field:* Show that $\boldsymbol{E}\cdot\boldsymbol{B}$ is a Lorentz invariant quantity. 5 marks

4. *The first pair of Maxwell's equations:* Recall that the electromagnetic field tensor $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} = \partial_\mu\,A_\nu - \partial_\nu\,A_\mu.$$

   (a) Show that 3 marks

$$\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0.$$

   (b) Express these equations in terms of the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$. 7 marks

5. *Equation of motion of a scalar field:* Consider the following action that describes a scalar field, say, ϕ, in Minkowski spacetime:

$$S[\phi(\tilde{x})] = \frac{1}{c}\int c\,\mathrm{d}t\,\mathrm{d}^3\boldsymbol{x}\,\left(\frac{1}{2}\,\eta_{\mu\nu}\,\partial^\mu\phi\,\partial^\nu\phi - \frac{1}{2}\,\sigma^2\,\phi^2\right),$$

   where $\eta_{\mu\nu}$ is the metric tensor in flat spacetime, while σ is a constant.

   (a) What are the dimensions of ϕ and σ? 3 marks

   (b) Explicitly *vary* the above action to obtain the equation of motion for the scalar field. 7 marks

   Note: The resulting equation of motion is called the Klein-Gordon equation.

Exercise sheet 5

More on real scalar fields

1. *Solutions to the Klein-Gordon equation:* Consider a scalar field ϕ obeying the following Klein-Gordon equation:

$$\left(\Box + \sigma^2\right)\,\phi = 0.$$

   (a) Write the solution to the scalar field as

$$\phi(\tilde{x}) = q_{\boldsymbol{k}}(t)\,\exp\left(i\,\boldsymbol{k}\cdot\boldsymbol{x}\right)$$

   and show that $q_{\boldsymbol{k}}(t)$ satisfies the equation of motion of a simple harmonic oscillator.
   Note: For convenience in notation, here and hereafter, we shall use $\tilde{x}$ to denote $x^\mu = (c\,t, \boldsymbol{x})$.

   (b) Determine the relation between the frequency ω of the oscillator, the wave vector $\boldsymbol{k}$ and the quantity σ.

   (c) Since ϕ is a scalar, the solution has to be a Lorentz invariant quantity. Express the solution in an explicitly Lorentz invariant form.

   (d) As the Klein-Gordon equation is a linear equation, a superposition of the individual solutions will also be a solution. Write down the most general solution possible, and express it in an explicitly Lorentz invariant manner.

2. *Normalization and completeness of the modes:* Consider the set of modes

$$u_{\boldsymbol{k}}(\tilde{x}) = \frac{1}{\sqrt{(2\,\pi)^3\,(2\,\omega)}}\,\exp-\left(i\,k_\mu\,x^\mu\right) = \frac{1}{\sqrt{(2\,\pi)^3\,(2\,\omega)}}\,\exp-\left[i\,\left(\omega\,t - \boldsymbol{k}\cdot\boldsymbol{x}\right)\right],$$

where, evidently, the four vector k^μ denotes $k^\mu = (\omega/c, \boldsymbol{k})$. The canonical, real scalar field can be decomposed in terms of the modes $u_{\boldsymbol{k}}(\tilde{x})$ as follows:

$$\phi(\tilde{x}) = \int \mathrm{d}^3\boldsymbol{k}\,\left[a_{\boldsymbol{k}}\,u_{\boldsymbol{k}}(\tilde{x}) + a_{\boldsymbol{k}}^*\,u_{\boldsymbol{k}}^*(\tilde{x})\right],$$

where the $a_{\boldsymbol{k}}$'s are $\boldsymbol{k}$-dependent constants. The scalar product of the modes is defined as

$$(u_{\boldsymbol{k}}, u_{\boldsymbol{k}'}) = -i\int_\Sigma \mathrm{d}\Sigma^\mu\,\left(u_{\boldsymbol{k}}\,\overset{\leftrightarrow}{\partial}_\mu\,u_{\boldsymbol{k}'}^*\right),$$

where

$$u_{\boldsymbol{k}}\,\overset{\leftrightarrow}{\partial}_\mu\,u_{\boldsymbol{k}'}^* \equiv u_{\boldsymbol{k}}\,\partial_\mu u_{\boldsymbol{k}'}^* - u_{\boldsymbol{k}'}^*\partial_\mu u_{\boldsymbol{k}}$$

and $\mathrm{d}\Sigma^\mu = \mathrm{d}\Sigma\;\hat{n}^\mu$, with $\hat{n}^\mu$ being a future-directed unit vector orthogonal to the spacelike hypersurface Σ and $\mathrm{d}\Sigma$ is the volume element in Σ.

   (a) Choosing $\mathrm{d}\Sigma^\mu$ to be a constant time hypersurface, show that the scalar product defined above is independent of time.

   (b) Also, show that

$$\begin{aligned}
(u_{\boldsymbol{k}}, u_{\boldsymbol{k}'}) &= \delta^{(3)}\left(\boldsymbol{k} - \boldsymbol{k}'\right),\\
(u_{\boldsymbol{k}}, u_{\boldsymbol{k}'}^*) &= 0,\\
(u_{\boldsymbol{k}}^*, u_{\boldsymbol{k}'}^*) &= -\delta^{(3)}\left(\boldsymbol{k} - \boldsymbol{k}'\right).
\end{aligned}$$

(c) Further establish that, on a constant time hypersurface, i.e. when $t = t'$,

$$-i \int \mathrm{d}^3\boldsymbol{k}\, \left[u_{\boldsymbol{k}}(\tilde{x})\, \partial_{t'} u^*_{\boldsymbol{k}}(\tilde{x}') - u^*_{\boldsymbol{k}}(\tilde{x}')\, \partial_t u_{\boldsymbol{k}}(\tilde{x})\right]_{t=t'} = \delta^{(3)}\left(\boldsymbol{x} - \boldsymbol{x}'\right).$$

Note: The above two sets of conditions imply that the modes $u_{\boldsymbol{k}}$ form an orthonormal and complete set.

3. *Green's functions:* Consider a real scalar field ϕ that is sourced by a charge density ρ. Such a scalar field would be governed by the following equation of motion:

$$\left(\Box + \sigma^2\right)\, \phi = \alpha\, \rho,$$

where α is a quantity of suitable dimensions. This inhomogeneous partial differential equation can be solved using the method of Green's functions as follows.

(a) Show that the inhomogeneous solution to the above equation can be expressed as

$$\phi(\tilde{x}) = \alpha \int \mathrm{d}^4\tilde{x}\; G(\tilde{x}, \tilde{x}')\; \rho(\tilde{x}),$$

where the Green's function $G(\tilde{x}, \tilde{x}')$ satisfies the differential equation

$$\left(\Box_{\tilde{x}} + \sigma^2\right)\, G(\tilde{x}, \tilde{x}') = \delta^{(4)}\left(\tilde{x} - \tilde{x}'\right).$$

(b) Express the Green's function as a Fourier transform as

$$G(\tilde{x}, \tilde{x}') = \int \frac{\mathrm{d}^4\tilde{k}}{(2\,\pi)^4}\; G(\tilde{k})\;\, \exp\left[i\, k^\mu \left(x_\mu - x'_\mu\right)\right],$$

and substitute it into the above equation to determine the form of $G(\tilde{k})$.

4. *The retarded Green's function for a massless field:* Using the form of $G(\tilde{k})$, evaluate the above integral to determine the Green's function $G(\tilde{x}, \tilde{x}')$ for a field with $\sigma = 0$.

   Note: As we have discussed, in units wherein $c = \hbar = 1$, σ has dimensions of mass.

5. *Conservation of the stress-energy tensor and the equation of motion:* Show that demanding the conservation of the stress-energy tensor of a scalar field leads to its equation of motion.

Exercise sheet 6

The case of the complex scalar field

1. *Two real scalar fields:* Consider the following action that describes two real scalar fields, say, ϕ_1 and ϕ_2:
$$S[\phi_1(\tilde{x}), \phi_2(\tilde{x})] = \frac{1}{c} \sum_{s=1,2} \int \mathrm{d}^4\tilde{x} \, \left(\frac{1}{2}\, \eta_{\mu\nu}\, \partial^\mu \phi_s\, \partial^\nu \phi_s - \frac{1}{2}\, \sigma^2\, \phi_s^2\right).$$
   (a) Vary the action with respect to ϕ_1 and ϕ_2 to arrive at the equations of motion.
   (b) Evaluate the stress-energy tensor of the complete system.

2. *Variation of the action governing a complex scalar field:* As we have discussed, the above system involving two real scalar fields can be described in terms of a single *complex* scalar field, say, ϕ, that is defined as
$$\phi = (\phi_1 + i\, \phi_2)\,.$$
   (a) Express the above action for the fields ϕ_1 and ϕ_2 in terms of the complex field ϕ.
   (b) Vary the action to arrive at the equations of motions describing ϕ and ϕ^*.

3. *From the conservation of the stress-energy tensor to the equations of motion:* Consider a complex scalar field that is described by the canonical kinetic term, but is governed by an arbitrary potential energy density of the following form: $V(|\phi|^2)$, where $|\phi|^2 = \phi\, \phi^*$.
   (a) Obtain the equations of motion for the field ϕ and its complex conjugate in such a case.
   (b) Construct the stress-energy tensor associated with the complex scalar field and show that the conservation of the stress-energy tensor also leads to these equations of motion.

4. *Non-linear Schrodinger equation, the Madelung transformation and superfluids:* Consider a complex wavefunction $\psi(t, \boldsymbol{x})$ that describes a non-relativistic, quantum mechanical system that is governed by a non-linear Schrodinger equation of the following form:
$$i\, \hbar\, \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2}\, \nabla^2 \psi + \frac{\partial V\left(|\psi|^2\right)}{\partial \psi^*},$$
where the quantity $V\left(|\psi|^2\right)$ describes self-interactions whose nature can be very similar to those in the case of the complex scalar field discussed in the previous exercise.
   (a) Express the wavefunction $\psi(t, \boldsymbol{x})$ as
$$\psi(t, \boldsymbol{x}) = \sqrt{\rho(t, \boldsymbol{x})}\, \exp\left[i\, \chi(t, \boldsymbol{x})/\hbar\right]$$
   and show that the imaginary part of the above non-linear Schrodinger equation reduces to the following continuity equation that describes a non-relativistic fluid with density ρ and velocity $\boldsymbol{v}$:
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\, \boldsymbol{v}) = 0,$$
   where $\boldsymbol{v} = \nabla \chi$.
   Note: The functions $\rho(t, \boldsymbol{x})$ and $\chi(t, \boldsymbol{x})$ are real quantities.
   (b) From the real part of the equation, also arrive at the following Euler equation governing the fluid:
$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{v} + \nabla f(\rho) = \frac{\hbar^2}{2}\, \nabla \left(\frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}}\right),$$
   where $f(\rho) = \mathrm{d}V(\rho)/\mathrm{d}\rho$.

Note: The transformation above which helps in reducing the non-linear Schrodinger equation to the hydrodynamical form is referred to as the Madelung transformation. It is essentially due to this reason that systems such as superfluids can be described by the above non-linear Schrodinger equation, which, for suitable forms of $V\left(|\psi|^2\right)$, is known as the Gross-Pitaevskii equation.

5. *A complex scalar field in an external electric field:* Consider a complex scalar field, say, ϕ, that is propagating in a constant and uniform electric field background. Let the strength of the electric field be E, and let it be pointed towards the positive x-direction. Such an electric field can be described by either the vector potential $A_1^\mu = (-E\,x, 0, 0, 0)$ or by the potential $A_2^\mu = (0, -E\,c\,t, 0, 0)$.

   (a) Obtain the equation of motion that governs the scalar field in the two gauges.

   (b) Using the method of separation of variables to solve a partial differential equation, arrive at the differential equations governing the modes along the time coordinate and the three spatial directions.

   (c) Since the gauge A_1^μ is independent of the t, y and the z coordinates, the modes along these directions can be easily arrived at. Express these modes in terms of simple functions. Can you identify the nature of the modes along the x-direction?

   (d) Carry out the corresponding exercise in the gauge A_2^μ.

Quiz II

Real and complex scalar fields

1. *Components of the stress-energy tensor:* Consider a canonical, real scalar field, say, ϕ, of mass σ (in suitable units). Recall that, such a scalar field can be decomposed in terms of the normal modes $u_{\boldsymbol{k}}(\tilde{x})$ as follows:

$$\phi(\tilde{x}) = \int \mathrm{d}^3\boldsymbol{k}\ \left[a_{\boldsymbol{k}}\, u_{\boldsymbol{k}}(\tilde{x}) + a_{\boldsymbol{k}}^*\, u_{\boldsymbol{k}}^*(\tilde{x})\right],$$

where the $a_{\boldsymbol{k}}$'s are, in general, $\boldsymbol{k}$-dependent constants and the modes $u_{\boldsymbol{k}}(\tilde{x})$ are given by

$$u_{\boldsymbol{k}}(\tilde{x}) = \frac{1}{\sqrt{(2\,\pi)^3\ (2\,\omega)}}\ \exp-(i\,k_\mu\,x^\mu) = \frac{1}{\sqrt{(2\,\pi)^3\ (2\,\omega)}}\ \exp-[i\ (\omega\,t - \boldsymbol{k}\cdot\boldsymbol{x})]$$

with, evidently, $k^\mu = (\omega/c, \boldsymbol{k})$. Moreover, note that, in such a case, $\omega/c = \left(|\boldsymbol{k}|^2 + \sigma^2\right)^{1/2}$, with ω being assumed to be a positive definite quantity.

Let $a_{\boldsymbol{k}} = \alpha\,\delta^{(3)}\,(\boldsymbol{k} - \boldsymbol{p})$, where α is real constant, while $\boldsymbol{p}$ is a constant vector.

   (a) Express the stress-energy tensor of the scalar field associated with the above $a_{\boldsymbol{k}}$ completely in terms of the four vector p^μ. **[4 marks]**

   (b) Explicitly write down the various components of the stress-energy tensor in terms of the components of the four vector p^μ. **[6 marks]**

2. *Equation of motion of a non-canonical scalar field:* Consider a non-canonical and real scalar field ϕ that is governed by action

$$S[\phi(\tilde{x})] = \frac{1}{c}\int \mathrm{d}^4\tilde{x}\ \left[f(X) - V(\phi)\right],$$

where X, as usual, denotes the following kinetic term:

$$X = \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi,$$

while $V(\phi)$ denotes the potential describing the scalar field. Explicitly vary the action to arrive at the equation of motion for the scalar field. **[10 marks]**

3. *Chapygin gas as a scalar field:* Consider a real scalar field ϕ that is described by the Lagrangian density

$$\mathcal{L}(X) = -V_0\,\sqrt{1 - 2\,X},$$

where X denotes the standard kinetic term, while V_0 is a constant. As we have discussed, scalar fields which depend only on the kinetic term X behave as barotropic fluids wherein the pressure can be expressed as a function of the energy density.

   (a) Evaluate the energy density and pressure associated with the above scalar field. **[6 marks]**

   (b) Obtain the relation between the pressure and the energy density. **[4 marks]**

Note: The equation of state (viz. the relation between the pressure and the energy density) that you will arrive at describes the so-called Chaplygin gas.

4. *Mutually interacting scalar fields:* Two real scalar fields, say, ϕ and χ, are governed by the following action:

$$S[\phi(\tilde{x}), \chi(\tilde{x})] = \frac{1}{c}\int \mathrm{d}^4\tilde{x}\ \left(\frac{1}{2}\,\partial^\mu\phi\,\partial_\mu\phi + \frac{1}{2}\,\partial^\mu\chi\,\partial_\mu\chi - \frac{\sigma_\phi^2}{2}\,\phi^2 - \frac{\sigma_\chi^2}{2}\,\chi^2 - \frac{\lambda}{4!}\,\phi^2\,\chi^2\right),$$

where σ_ϕ and σ_χ denote the masses of the fields ϕ and χ in suitable units, while λ is a constant.

(a) Obtain the equations of motion for the fields ϕ and χ. $\boxed{\text{3 marks}}$

(b) Construct the stress-energy tensor of the system. $\boxed{\text{3 marks}}$

(c) From the equation governing the conservation of the stress-energy tensor, arrive at the equations of motion describing the two fields. $\boxed{\text{4 marks}}$

5. *A complex scalar field in an external magnetic field:* Consider a complex scalar field, say, ϕ, that is propagating in a given constant and uniform magnetic field. Let the magnetic field be of strength B that is pointed towards the positive z-direction. Such a magnetic field can be described by the vector potential $A^\mu = (0, 0, B\,x, 0)$.

   (a) Write down the equation of motion governing the scalar field in the given magnetic field. $\boxed{\text{3 marks}}$

   (b) Using the method of separation of variables to solve a partial differential equation, arrive at the differential equations governing the modes along the time coordinate and the three spatial directions. $\boxed{\text{3 marks}}$

   (c) Since the system exhibits translational invariance along the t, y and the z directions, you can easily identify the normal modes of the field along these directions. Express them in terms of simple functions. $\boxed{\text{2 marks}}$

   (d) Can you identify the nature of the modes along the x-direction? $\boxed{\text{2 marks}}$

Additional exercises I

From special relativity to the case of the complex scalar field

1. Algebra of the infinitesimal generators of the Lorentz group: Recall that the infinitesimal generators of rotation and the Lorentz transformations, viz. (J_x, J_y, J_z) and (K_x, K_y, K_z), respectively, can be written in a matrix form as follows:

$$J_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$K_x = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_y = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_z = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}.$$

Utilizing these representations, establish the following commutation relations:

$$[J_i, J_j] = i\,\epsilon_{ijk}\,J_k, \quad [K_i, K_j] = -i\,\epsilon_{ijk}\,J_k \quad \text{and} \quad [J_i, K_j] = i\,\epsilon_{ijk}\,K_k,$$

where ϵ_{ijk} represents the completely anti-symmetric tensor and, as is our convention, the Latin indices i, j, k, take values $(1, 2, 3)$.

2. The generators as differential operators: As we had discussed, the generators (J_x, J_y, J_z) and (K_x, K_y, K_z) can also be represented in the differential form as follows:

$$J_x = -i\left(y\,\frac{\partial}{\partial z} - z\,\frac{\partial}{\partial y}\right), \quad J_y = -i\left(z\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial z}\right), \quad J_z = -i\left(x\,\frac{\partial}{\partial y} - y\,\frac{\partial}{\partial x}\right),$$

and

$$K_x = i\left(c\,t\,\frac{\partial}{\partial x} + \frac{x}{c}\,\frac{\partial}{\partial t}\right), \quad K_y = i\left(c\,t\,\frac{\partial}{\partial y} + \frac{y}{c}\,\frac{\partial}{\partial t}\right), \quad K_z = i\left(c\,t\,\frac{\partial}{\partial z} + \frac{z}{c}\,\frac{\partial}{\partial t}\right).$$

Use these representations to establish all the commutation relations listed in the previous exercise.

3. The Lorentz group in terms of Pauli matrices: Construct a representation of the Lorentz group, consisting of the six infinitesimal generators (J_x, J_y, J_z) and (K_x, K_y, K_z), in terms of the following Pauli matrices and the unit matrix:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note: This is known as the complex uni-modular matrix representation of the Lorentz group.

4. Thomas rotation: Show that two successive, arbitrary, Lorentz boosts, say, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, is equivalent to a pure boost, say, $\boldsymbol{v}_3$, followed by a pure rotation, say, $\theta\,\hat{n}$, where $\hat{n}$ is the unit vector along the axis of rotation. Determine the angle θ in terms of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, and also establish that $\hat{n}\cdot\boldsymbol{v}_3 = 0$.

   Hint: The easiest way to solve this problem would be to make use of the uni-modular representation of the Lorentz group discussed in the previous exercise.

   Note: The rotation involved is known as Thomas (or, occasionally, Wigner) rotation.

5. *The Poincare group:* As we have discussed, the Poincare group consists of the operators (J_x, J_y, J_z) and (K_x, K_y, K_z), which generate rotations and Lorentz transformations, as well as the generators of translations, which can be represented as $P_\mu \equiv i\,\partial_\mu$. Let us represent the six elements of the Lorentz group as the components of an anti-symmetric tensor $J_{\mu\nu}$ as follows:

$$J_{\mu\nu} = \begin{cases} J_{0i} &= -J_{i0} = K_i, \\ J_{ij} &= -J_{ji} = i\,\epsilon_{ijk}\,J_k. \end{cases}$$

Using the representations of $J_{\mu\nu}$ and P_μ in terms of differential operators, establish the following commutation relations:

$$\begin{aligned} [J_{\mu\nu}, J_{\rho\sigma}] &= i\,(\eta_{\nu\rho}\,J_{\mu\sigma} - \eta_{\mu\rho}\,J_{\nu\sigma} + \eta_{\mu\sigma}\,J_{\nu\rho} - \eta_{\nu\sigma}\,J_{\mu\rho})\,, \\ [P_\mu, J_{\rho\sigma}] &= i\,(\eta_{\mu\rho}\,P_\sigma - \eta_{\mu\sigma}\,P_\rho)\,. \end{aligned}$$

6. *The retarded and the advanced Green's functions:* Recall that, the Green's function associated with a massless field can be expressed as

$$G(\tilde{x}, \tilde{x}') = \int \frac{\mathrm{d}^4\tilde{k}}{(2\,\pi)^4}\, G(\tilde{k})\, \exp - \left[i\,k^\mu\,(x_\mu - x'_\mu)\right],$$

with $G(\tilde{k})$ being given by

$$G(\tilde{k}) = -\frac{1}{k^\mu\,k_\mu}.$$

It is important to notice that the phase factor in the exponential now contains an overall minus sign in contrast to the expression which we had worked with before [see Exercise sheet 5, Exercise 3 (b)]. Both the representations are indeed correct, and they are simply a matter of convention.

(a) Show that, in such a case, when $t > t'$, the poles on the real k^0 axis (at $k = \pm|\boldsymbol{k}|$) now need to be provided a small and *negative* imaginary part (i.e. they have to be pushed downwards instead of upwards as we had done earlier) and the contour has to be closed in the lower half of the the complex k^0-plane to arrive at the retarded Green's function, say, $D_{\mathrm{ret}}(\tilde{x}, \tilde{x}')$.

Note: As we had discussed, the spacetime coordinates $\tilde{x}$ denote the point of observation, while the $\tilde{x}'$ represent the position of the source. Hence, the condition $t > t'$ essentially implies causality. Also, it should pointed out that the Green's functions associated with massless fields are often denoted as $D(\tilde{x}, \tilde{x}')$.

(b) Assuming $t < t'$ and pushing the poles upward by providing them with a small and *positive* imaginary part, arrive at the so-called advanced Green's function $D_{\mathrm{adv}}(\tilde{x}, \tilde{x}')$ by closing the contour in the upper half of the complex k^0-plane.

Note: You will find that the Green's function $D_{\mathrm{adv}}(\tilde{x}, \tilde{x}')$ is non-zero only along the *past* light cone.

(c) Establish that the difference between the retarded and the advanced Green's functions satisfies the homogeneous wave equation.

7. *Retarded Green's function in the presence of a boundary:* Consider a free, real and massless scalar field, say, ϕ. Let the field vanish on a boundary that is located on the $x = 0$ plane. i.e. $\phi(c\,t, x = 0, y, z) = 0$ for all t, y and z. Determine the retarded Green's function associated with the field in such a case.

8. *The retarded Green's function in terms of spherical polar coordinates:* Consider a situation wherein you are required to work in terms of the spherical polar coordinates instead of the more conventional and, not to mention, convenient, Cartesian coordinates. Express the Green's function as a suitable integral and sum over the modes associated with the d'Alembertian in the spherical polar coordinates, and carry out the sums and integrals involved to arrive at the standard result for the retarded Green's function of a massless field.

9. *Complex scalar fields and gauge transformations:* We had earlier considered a complex scalar field propagating in a constant and uniform electric field background described by either the vector potential $A_1^\mu = (-E\,x, 0, 0, 0)$ or by the potential $A_2^\mu = (0, -E\,c\,t, 0, 0)$.

   (a) Construct the gauge transformation that takes one from the gauge A_1^μ to the gauge A_2^μ.

   (b) Determine how the scalar field transforms as one moves from one gauge to the other.

   (c) Explicitly check that the transformed solution indeed satisfies the equation of motion in the new gauge.

10. *Multiple scalar fields:* Consider a system involving N scalar fields that is described by the canonical kinetic term and a potential. Write down the action of such a system assuming it is invariant under the global transformations of the $SO(N)$ group.

Exercise sheet 7

Symmetries and conservation laws

1. *Conserved current in the presence of the electromagnetic field:* We had earlier derived the conserved current associated with the symmetry of a complex scalar field, say, ϕ, under global gauge transformations. Recall that, the action governing a massive, complex scalar field that is interacting with the electromagnetic field described by the vector potential A^μ can be written as

$$S[\phi(\tilde{x}), A^\mu(\tilde{x})] = \frac{1}{c} \int \mathrm{d}^4\tilde{x} \left[(D_\mu\phi)\,(D^\mu\phi)^* - \sigma^2\,\phi\,\phi^*\right],$$

where the quantity $D_\mu\phi$ is given by

$$D_\mu\phi = \partial_\mu\phi + i\,e\,A_\mu\,\phi,$$

with e denoting the coupling constant.

   (a) Show that the above action is invariant under local gauge transformations of the form

$$\phi(\tilde{x}) \to \exp - [i\,e\,\Lambda(\tilde{x})]\;\phi(\tilde{x}) \qquad \text{and} \qquad A_\mu(\tilde{x}) \to A_\mu(\tilde{x}) + \partial_\mu\Lambda(\tilde{x}).$$

   (b) Determine the conserved current associated with this local symmetry.

2. *Scale invariance:* Consider a real scalar field ϕ that is governed by the action

$$S[\phi(\tilde{x})] = \frac{1}{c} \int \mathrm{d}^4\tilde{x} \left(\frac{1}{2}\,\partial^\mu\phi\,\partial_\mu\phi - \lambda\,\phi^4\right).$$

   (a) Show that this action is invariant under the following scale transformations:

$$x^\mu \to b\,x^\mu \quad \text{and} \quad \phi \to \phi/b,$$

   where b is a constant.

   (b) What is the conserved current associated with this symmetry?

   (c) Explicitly show that the four divergence of the conserved current vanishes.

   (d) Can you identify the reason for the existence of such a symmetry?

3. *The conserved charges associated with scalar fields interacting with the Yang-Mills field:* Consider a three component scalar field, say, $\boldsymbol{\phi} \equiv (\phi_1, \phi_2, \phi_3)$, that is interacting with the Yang-Mills field described by the gauge potential $\boldsymbol{W}_\mu$, and is governed by the action

$$S[\boldsymbol{\phi}(\tilde{x}), \boldsymbol{W}_\mu(\tilde{x})] = \frac{1}{c} \int \mathrm{d}^4\tilde{x} \left(\frac{1}{2}\,D_\mu\boldsymbol{\phi}\cdot D^\mu\boldsymbol{\phi} - \frac{\sigma^2}{2}\,\boldsymbol{\phi}\cdot\boldsymbol{\phi}\right),$$

where $D_\mu\boldsymbol{\phi}$ is given by

$$D_\mu\boldsymbol{\phi} = \partial_\mu\boldsymbol{\phi} + g\,\boldsymbol{W}_\mu \times \boldsymbol{\phi},$$

with g being the coupling constant. As we have discussed, the above action is invariant under local rotations (i.e. when the extent of rotation, say, $\boldsymbol{\Lambda}$, is dependent on the spacetime coordinates) in the internal field space and the following transformations of the gauge potential $\boldsymbol{W}_\mu$:

$$\boldsymbol{W}_\mu \to \boldsymbol{W}_\mu - \boldsymbol{\Lambda} \times \boldsymbol{W}_\mu + \frac{1}{g}\,\partial_\mu\boldsymbol{\Lambda}.$$

   (a) Construct the conserved current associated with the symmetry.

   (b) How many conserved charges are associated with the symmetry?

Note: Actually, $\mathbf{\Lambda} = \Lambda\,\hat{n}$, where Λ denotes the angle of rotation, while $\hat{n}$ is the unit vector along the axis of rotation.

4. *Gauge invariance and electromagnetism:* As you may know, the action describing a free electromagnetic field (which we would also formally discuss in due course) is given by

$$S[A^{\mu}(\tilde{x})] = -\frac{1}{16\,\pi\,c}\int \mathrm{d}^4\tilde{x}\; F_{\mu\nu}\,F^{\mu\nu},$$

where $F_{\mu\nu}$ is the field tensor defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

As should be evident, the electromagnetic field tensor $F_{\mu\nu}$ is invariant under gauge transformations of the form

$$A_\mu \to A_\mu + \partial_\mu \Lambda.$$

   (a) Determine the conserved current associated with this gauge symmetry.

   (b) What are the corresponding conserved charges?

5. *Traceless stress-energy tensors:* Suppose that the action describing a field ϕ is found to be invariant under spacetime translations as well as the following dilatations:

$$x^\mu \to b\,x^\mu \quad \text{and} \quad \phi \to \phi,$$

where b is a constant. Show that, in such a case, the trace of the corresponding stress-energy tensor vanishes.

Exercise sheet 8

The theory of the electromagnetic field I

1. *Equivalence of actions under gauge transformations:* Recall that the action governing the electromagnetic field described by the vector potential A_μ that is interacting with the four current j^μ is given by
$$S[A^\mu(\tilde{x})] = -\frac{1}{c^2}\int \mathrm{d}^4\tilde{x}\; j^\mu\, A_\mu - \frac{1}{16\,\pi\, c}\int \mathrm{d}^4\tilde{x}\; F_{\mu\nu}\, F^{\mu\nu},$$
where $F_{\mu\nu}$ is the field tensor defined as
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$
Since $F_{\mu\nu}$ is explicitly invariant under the gauge transformation
$$A_\mu \to A_\mu + \partial_\mu \Lambda,$$
evidently, the second term in the above action is invariant as well. Determine if the first term transforms to an equivalent action under the gauge transformation.

2. *The spatial components of the stress-energy tensor of the free electromagnetic field:* We had arrived at the forms of the time-time and the time-space components of the stress-energy tensor of the free electromagnetic field in terms of the components of the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$. Arrive at the corresponding expressions for the purely spatial components of the stress-energy tensor.

   Note: These components are usually referred to as the Maxwell stress tensor.

3. *From source free Maxwell's equations to the conservation of the stress-energy tensor:* Establish that the source free Maxwell's equations imply that the stress-energy tensor of the free electromagnetic field is conserved.

4. *Conservation of the stress-energy tensor in the presence of sources:* The above exercise had involved the electromagnetic field in the absence of charges. If the charges are also present, then it is the sum of the stress-energy tensors of the charges as well as the field that will be conserved. The stress-energy tensor of a collection of mutually non-interacting particles can be written as
$$T^{\mu\nu}_{\mathrm{P}} = \mu\, c\, u^\mu\, u^\nu\, \frac{\mathrm{d}s}{\mathrm{d}t},$$
where μ is the mass density associated with the particles, while u^μ denotes the four velocity of the particles.

   Note: The above expression for the stress-energy tensor for a collection of mutually non-interacting particles is equivalent to a pressureless relativistic fluid. Often, such a system is referred to as 'dust'.

   (a) Show that, upon using the second pair Maxwell's equations, in the presence of sources, the stress-energy of the electromagnetic field, say, $T^{\mu\nu}_{\mathrm{F}}$, satisfies the equation
$$\partial_\mu T^{\mu\nu}_{\mathrm{F}} = -\frac{1}{c}\, F^{\nu\lambda}\, j_\lambda.$$

   (b) As in the case of charges, the continuity equation corresponding to the mass flow can be expressed as follows:
$$\partial_\mu \left(\mu\, \frac{\mathrm{d}x^\mu}{\mathrm{d}t}\right) = 0.$$
   Using this equation and the following Lorentz force law:
$$\mu\, c\, \frac{\mathrm{d}u^\mu}{\mathrm{d}s} = \frac{\rho}{c}\, F^{\mu\nu}\, u_\nu,$$

where ρ denotes the charge density of the particles, show that

$$\partial_\mu T^{\mu\nu}_{\mathrm{P}} = \frac{1}{c}\, F^{\mu\lambda}\, j_\lambda$$

so that the total stress-energy tensor of the system, viz. $T^{\mu\nu} = T^{\mu\nu}_{\mathrm{P}} + T^{\mu\nu}_{\mathrm{F}}$, is conserved, as required.

5. *Traceless nature of the stress-energy tensor of the electromagnetic field:* Show that the trace of the stress-energy tensor of the electromagnetic field vanishes. Can you identify the reason behind the vanishing trace?

Exercise sheet 9

The theory of the electromagnetic field II

1. *The Coulomb gauge and the degrees of freedom of the electromagnetic field:* Recall that the Lorenz gauge was determined by the covariant condition $\partial_\mu A^\mu = 0$. However, even the Lorenz condition does not uniquely fix the gauge. Further gauge transformations of the form $A_\mu \to A_\mu + \partial_\mu \Lambda$ are possible, provided Λ satisfies the condition $\Box \Lambda = 0$. In such a situation, often, one breaks Lorentz covariance and works in the so-called Coulomb gauge wherein A^t is set to zero, so that the Lorenz condition reduces to $\nabla \cdot \mathbf{A} = 0$.

   (a) Show that this implies that the free electromagnetic field possesses two independent degrees of freedom.

   (b) What do these two degrees of freedom correspond to?

2. *The massive vector field:* Consider the following action that governs a massive vector field $\mathcal{A}_\mu$:

$$S[\mathcal{A}^\mu(\tilde{x})] = \frac{1}{16\,\pi\,c} \int \mathrm{d}^4\tilde{x}\ \left(-\mathcal{F}_{\mu\nu}\,\mathcal{F}^{\mu\nu} + 2\,\sigma^2\,\mathcal{A}^\mu\,\mathcal{A}_\mu\right),$$

   where $\mathcal{F}_{\mu\nu}$ represents the field tensor defined in the usual form, viz.

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu,$$

   and σ has dimensions of mass in suitable units.

   (a) Obtain the equation of motion governing the field $\mathcal{A}_\mu$.

   (b) Show that the Lorentz condition, viz. $\partial_\mu \mathcal{A}^\mu = 0$, *has* to be satisfied by the field apart from satisfying the equation of motion.

   (c) Is the action invariant under gauge transformations of the form $\mathcal{A}_\mu \to \mathcal{A}_\mu + \partial_\mu \Lambda$?

   (d) How many independent degrees of freedom does the massive field $\mathcal{A}^\mu$ possess?

   Note: The massive vector field $\mathcal{A}^\mu$ is known as the Proca field.

3. *The Lienard-Wiechart potentials:* Consider a point particle with charge e that is moving along the trajectory $r^\mu(\tau)$, where τ is the proper time in the frame of the charge. The four current associated with the charge is given by

$$j^\mu(\tilde{x}) = e\,c^2 \int \mathrm{d}\tau\ u^\mu\ \delta^{(4)}\left[\tilde{x} - \tilde{r}(\tau)\right],$$

   where $\tilde{r}(\tau) \equiv r^\mu(\tau)$ is the trajectory of the charge and $u^\mu = \mathrm{d}r^\mu/\mathrm{d}s$ is its four velocity, so that the corresponding charge and current densities are given by

$$\rho(\tilde{x}) = e\,c\,\delta^{(3)}[\boldsymbol{x} - \boldsymbol{r}(t)] \quad \text{and} \quad \boldsymbol{j}(\tilde{x}) = e\,\boldsymbol{v}(t)\,\delta^{(3)}[\boldsymbol{x} - \boldsymbol{r}(t)]$$

   with $\boldsymbol{v}(t) = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$, as required. In the Lorenz gauge, the electromagnetic vector potential A^μ satisfies the equation

$$\Box A^\mu = \frac{4\,\pi}{c}\,j^\mu.$$

   (a) Using the retarded Green's function for a massless field that we had obtained earlier, solve the above equation to arrive at the following expression for the vector potential A^μ:

$$A^\mu(\tilde{x}) = \frac{e\,u^\mu}{R_\mu\,u^\mu},$$

   where $R^\mu = x^\mu - r^\mu$ and $R_\mu\,R^\mu = 0$.

(b) Show that the above vector potential A^μ can be written in the three dimensional form as

$$\phi(\tilde{x}) = \frac{e}{R - (\boldsymbol{v}\cdot\boldsymbol{R})/c} \quad \text{and} \quad \boldsymbol{A}(\tilde{x}) = \frac{e\,\boldsymbol{v}/c}{R - (\boldsymbol{v}\cdot\boldsymbol{R})/c},$$

where $\boldsymbol{R} = \boldsymbol{x} - \boldsymbol{r}$ and $R = |\boldsymbol{R}|$, with the right hand sides evaluated at the so-called retarded time determined by the condition $R^\mu R_\mu = 0$.

Note: These are known as the Lienard-Wiechart potentials.

4. *The radiation field:* Using the above Lienard-Wiechart potentials, obtain the following expressions for the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ generated by a point charge that is moving along an arbitrary trajectory:

$$\begin{aligned}\boldsymbol{E} &= \frac{e}{\gamma^2\,\mu^3\,R^2}\,[\hat{\boldsymbol{n}} - (\boldsymbol{v}/c)] + \frac{e}{c^2\,\mu^3\,R}\,[\hat{\boldsymbol{n}} \times ([\hat{\boldsymbol{n}} - (\boldsymbol{v}/c)] \times \boldsymbol{a})]\,,\\ \boldsymbol{B} &= \hat{\boldsymbol{n}} \times \boldsymbol{E},\end{aligned}$$

where γ is the standard Lorentz factor, $\boldsymbol{a} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t$ is the acceleration of the charge, while the quantity μ is given by

$$\mu = \left(1 - \frac{\boldsymbol{v}\cdot\hat{\boldsymbol{n}}}{c}\right)^{-1},$$

with $\hat{\boldsymbol{n}} = \boldsymbol{R}/R$.

Note: The contribution to the electric and the magnetic fields above which depends on the acceleration of the charge and behaves as $1/R$ with distance is known as the radiation field.

5. *Relativistic beaming:* Recall that the flux of energy being carried by electromagnetic radiation is described by the Poynting vector, viz.

$$\boldsymbol{S} = \frac{c}{4\pi}\,(\boldsymbol{E} \times \boldsymbol{B})\,.$$

When $\boldsymbol{B} = \hat{\boldsymbol{n}} \times \boldsymbol{E}$, the amount of energy, say, $\mathrm{d}\mathcal{E}$, that is propagating into a solid angle $\mathrm{d}\Omega$ in unit time is then given by

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}\Omega\,\mathrm{d}t} = |\boldsymbol{S}|\,R^2 = \frac{c\,|\boldsymbol{E}|^2\,R^2}{4\pi}.$$

(a) Upon using the above expressions for the radiative component of the electric field, show that the energy emitted by a point charge per unit time within a unit solid angle can be written as

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t\,\mathrm{d}\Omega} = \frac{e^2}{4\pi\,c^3}\,\left[2\,\mu^5\,(\hat{\boldsymbol{n}}\cdot\boldsymbol{a})\,(\boldsymbol{v}\cdot\boldsymbol{a}/c) + \mu^4\,a^2 - \mu^6\,\gamma^{-2}\,(\hat{\boldsymbol{n}}\cdot\boldsymbol{a})^2\right].$$

(b) Clearly, the intensity of the radiation is the largest along directions wherein $\mu \gg 1$. Show that, if θ is the angle between $\boldsymbol{v}$ and $\hat{\boldsymbol{n}}$, then, for $\theta \ll 1$ and $|\boldsymbol{v}| \simeq c$, we can write

$$\mu = \frac{2\,\gamma^2}{1 + \gamma^2\,\theta^2}.$$

(c) Argue that, for $\gamma \gg 1$, this expression is sharply peaked around $\theta = 0$, with a width $\Delta\theta \simeq \gamma^{-1}$.

Note: This effect, where most of the intensity is pointed along the direction of velocity of the charge, is known as relativistic beaming.

Quiz III

From symmetries and conservation laws to electromagnetic fields

1. *Conserved energy for a Lagrangian involving a second time derivative:* To begin with, consider the following conventional action describing a non-relativistic particle that is moving in one dimension:

$$S[q(t)] = \int \mathrm{d}t\, L(q, \dot{q}).$$

   (a) Since the Lagrangian is not explicitly dependent on time, the energy of the system should be conserved. Arrive at the standard form of the energy in terms of $\dot{q}$ and the Lagrangian L. $\boxed{\text{2 marks}}$

   Hint: You can arrive at the required result by following the procedure that we had adopted to arrive at the form of the stress-energy tensor. Differentiate the Lagrangian with respect to the independent variable in this case, viz. time, and assuming that the system satisfies the Euler-Lagrange equation of motion, identify the quantity whose total time derivative is zero.

   (b) Now, consider a system that is governed by the action

$$S[q(t)] = \int \mathrm{d}t\, L(q, \dot{q}, \ddot{q}).$$

   Assuming that, apart from the position q, the corresponding velocity $\dot{q}$ is also fixed at the end points, arrive at the Euler-Lagrange equation of motion governing the system. $\boxed{\text{3 marks}}$

   (c) Since the Lagrangian in the latter case is also not explicitly dependent on time, the corresponding 'energy' of the system will be conserved as well. Construct the form of the conserved energy in terms of $\dot{q}$, $\ddot{q}$ and the Lagrangian L. $\boxed{\text{5 marks}}$

2. *Scale invariance and the conformal stress-energy tensor:* We had earlier obtained the conserved current, say, j_μ, upon demanding that the action describing a canonical and real scalar field ϕ is invariant under spacetime translations as well as the scale transformations

$$x^\mu \to b\, x^\mu \quad \text{and} \quad \phi \to \phi/b,$$

   where b is a constant. Let us express the conserved current as

$$j_\mu = -x^\nu\, \widetilde{T}_{\mu\nu},$$

   where the new stress-energy tensor $\widetilde{T}_{\mu\nu}$ is a quantity that is not only conserved (i.e. $\partial_\mu \widetilde{T}^{\mu\nu} = 0$), but is also traceless (i.e. $\widetilde{T}^\mu_\mu = 0$). Writing $\widetilde{T}^{\mu\nu} = T^{\mu\nu} + \theta^{\mu\nu}$, show that $\theta^{\mu\nu}$ has the following form: $\boxed{\text{10 marks}}$

$$\theta^{\mu\nu} = \frac{1}{6}\left(\eta^{\mu\nu}\,\Box - \partial^\mu\,\partial^\nu\right)\phi^2.$$

   Note: The quantity $\widetilde{T}_{\mu\nu}$ is known as the conformal stress-energy tensor.

3. *Conservation of the stress-energy tensor of the Proca field:* Recall that the action governing the so-called Proca field (i.e. the massive vector field) $\mathcal{A}_\mu$ is given by

$$S[\mathcal{A}^\mu(\tilde{x})] = \frac{1}{16\,\pi\,c} \int \mathrm{d}^4\tilde{x}\, \left(-\mathcal{F}_{\mu\nu}\,\mathcal{F}^{\mu\nu} + 2\,\sigma^2\,\mathcal{A}^\mu\,\mathcal{A}_\mu\right),$$

   where $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$ denotes the field tensor.

   (a) Construct the stress-energy tensor associated with the field $\mathcal{A}^\mu$. $\boxed{\text{5 marks}}$

(b) Establish that the stress-energy tensor is conserved if the Proca equation is satisfied. 5 marks

4. *The dual field tensor in electromagnetism:* Consider the so-called dual field tensor $\widetilde{F}^{\mu\nu}$ which is defined in terms of the standard electromagnetic field tensor $F_{\mu\nu}$ as follows:

$$\widetilde{F}^{\mu\nu} = \frac{1}{2}\,\varepsilon^{\mu\nu\alpha\beta}\,F_{\alpha\beta},$$

with $\varepsilon^{\mu\nu\alpha\beta}$ denoting the completely anti-symmetric Levi-Civita tensor.

(a) Express the components of the quantity $\widetilde{F}^{\mu\nu}$ in terms of the components of the electric and the magnetic fields. 3 marks

(b) Evaluate the quantity $\widetilde{F}^{\mu\nu}\,F_{\mu\nu}$ in terms of the components of the electric and the magnetic fields. 3 marks

(c) Recall that the first pair of Maxwell's equations, viz. the source free equations, are given by

$$\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0.$$

Show that these equations can be written as 4 marks

$$\partial_\mu \widetilde{F}^{\mu\nu} = 0.$$

5. *Equations of motion for the iso-vector, Lorentz-scalar and the Yang-Mills fields:* Recall that the action governing the iso-vector and Lorentz scalar field, say, $\boldsymbol{\phi}$, which is interacting with the Yang-Mills field that is described by the vector potential $\boldsymbol{W}_\mu$, is given by

$$S[\boldsymbol{\phi}(\tilde{x}), \boldsymbol{W}^\mu(\tilde{x})] = \frac{1}{c}\int \mathrm{d}^4\tilde{x}\,\left(\frac{1}{2}\,D_\mu\boldsymbol{\phi}\cdot D_\mu\boldsymbol{\phi} - \frac{\sigma^2}{2}\,\boldsymbol{\phi}\cdot\boldsymbol{\phi} - \frac{1}{4}\,\boldsymbol{\mathcal{W}}_{\mu\nu}\cdot\boldsymbol{\mathcal{W}}^{\mu\nu}\right),$$

where the covariant derivative $D_\mu\boldsymbol{\phi}$ is defined as

$$D_\mu\boldsymbol{\phi} = \partial_\mu\boldsymbol{\phi} + g\,\boldsymbol{W}_\mu \times \boldsymbol{\phi},$$

with g denoting the coupling constant, while the field tensor $\boldsymbol{\mathcal{W}}_{\mu\nu}$ is given by

$$\boldsymbol{\mathcal{W}}_{\mu\nu} = \partial_\mu\boldsymbol{W}_\nu - \partial_\nu\boldsymbol{W}_\mu + g\,\boldsymbol{W}_\mu \times \boldsymbol{W}_\nu.$$

(a) Derive the equations of motion for the iso-vector, but Lorentz scalar, field $\boldsymbol{\phi}$. 4 marks

(b) Obtain the equations of motion governing the Yang-Mills field tensor $\boldsymbol{\mathcal{W}}_{\mu\nu}$. 6 marks

Exercise sheet 10

Spontaneous symmetry breaking and formation of topological defects

1. *Spontaneous symmetry breaking in non-relativistic systems:* Consider the case of the electron in the Hydrogen atom. The system is placed in an external magnetic field that is pointed in a given direction.

   (a) What is the original symmetry group of the Hydrogen atom?

   (b) Identify the symmetry that the system possesses when the magnetic field has been turned on.

   (c) What is the consequence of the broken symmetry in quantum mechanics?

2. *Surface energy density of a domain wall:* Recall that, a domain wall arises as a result of the spontaneously broken reflection symmetry of a real scalar field, say, ϕ, that is governed by the action

$$S[\phi(\tilde{x})] = \frac{1}{c}\int \mathrm{d}^4\tilde{x}\,\left[\frac{1}{2}\,\partial^\mu\phi\,\partial_\mu\phi - \frac{\lambda}{4}\,\left(\phi^2-\alpha^2\right)^2\right],$$

   where α is a real constant. The broken symmetry can lead to a non-trivial and time-independent solution of the following form:

$$\phi(z) = \alpha\tanh\,(z/\Delta)\,,$$

   where $\Delta = \sqrt{2/(\lambda\,\alpha^2)}$, which corresponds to a domain wall in the x-y plane.

   (a) Show that the stress-energy tensor associated with such a field configuration is given by

$$T^\mu_\nu = \frac{\lambda\,\alpha^4}{2}\,\mathrm{sech}^4(z/\Delta)\ \mathrm{diag.}\,(1,1,1,0)\,.$$

   (b) Further show that the surface energy density associated with the wall can be obtained to be

$$\eta = \int \mathrm{d}z\ T^t_t = \frac{2\,\sqrt{2\,\lambda}\,\alpha^3}{3}.$$

3. *Superconductivity and the Meissner effect:* As is well-known, superconductivity is a phenomenon wherein metals exhibit no resistance at very low temperatures. The phenomenon is a very good example of the Abelian Higgs model. In metals, under certain conditions, there arises an attractive force between the electrons, which leads to the formation of the so-called Cooper pairs. These Cooper pairs can be described by a complex wave function ψ that is described by the following Lagrangian density in static situations:

$$-\mathcal{L} = \frac{1}{2}\,(\boldsymbol{\nabla}\times\boldsymbol{A})^2 + |\,(\boldsymbol{\nabla} - i\,e\,\boldsymbol{A})\,\psi|^2 + m^2\,|\psi|^2 + \lambda\,|\psi|^4,$$

   where $m^2 = \alpha\,(T - T_{\mathrm{C}})$, with T_{C} denoting the critical temperature below which the transition to superconductivity occurs.

   (a) Assuming that the wavefunction ψ does not vary over the sample, show that the conserved current within the material is given by

$$\boldsymbol{j} = \frac{e\,m^2}{\lambda}\,\boldsymbol{A} = -k^2\,\boldsymbol{A},$$

   where $k^2 > 0$.

   Note: This relation that is known as the London equation.

   (b) Argue that, since $\boldsymbol{E} = -\,(\partial\boldsymbol{A}/\partial t) = 0$, according to Ohm's law, the resistance vanishes, thereby leading to superconductivity.

(c) Since $\boldsymbol{\nabla} \times \boldsymbol{B} = \boldsymbol{j}$ and $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$, show that the above expression for the current leads to the equation

$$\boldsymbol{\nabla}^2 \boldsymbol{B} = k^2\, \boldsymbol{B}.$$

(d) Establish that, this equation, in turn, implies that the magnetic field does not penetrate the superconductor beyond a characteristic depth of k^{-1}.

Note: This effect wherein the superconductor 'repels' the magnetic field is referred to as the Meissner effect.

4. *A theorem on static solutions:* Consider a set of N real scalar fields, say, ϕ_A, where $A = (1, 2, 3, \ldots, N)$, that are described by the following action in $(D+1)$-spacetime dimensions:

$$S[\phi_{_A}(\tilde{x})] = \frac{1}{c} \int \mathrm{d}^{(D+1)}\tilde{x}\ \left[\frac{1}{2}\, \partial^\mu \phi_{_A}\, \partial_\mu \phi_{_A} - V(\phi_{_A})\right],$$

where the potential $V(\phi_{_A})$ is a positive definite function involving the fields and the summation over the repeated index A has been assumed. A static solution, say, $\phi_{_A}(\boldsymbol{x})$, obeys the equations of motion

$$\boldsymbol{\nabla}^2 \phi_{_A} = \frac{\partial V}{\partial \phi_{_A}}.$$

These equations are the extrema of the static energy functional, viz.

$$E(\phi_{_A}) = \int \mathrm{d}^D \boldsymbol{x}\ \left[\frac{1}{2}\, \boldsymbol{\nabla}\phi_{_A} \cdot \boldsymbol{\nabla}\phi_{_A} + V(\phi_{_A})\right] = E_1(\phi_{_A}) + E_2(\phi_{_A}).$$

(a) Establish that, for the one-parameter family of solutions

$$\phi_{_A}^\lambda(\boldsymbol{x}) = \phi_{_A}(\lambda\, \boldsymbol{x}),$$

the energy E of the system scales as

$$E(\phi_{_A}^\lambda) = \lambda^{2-D}\, E_1(\phi_{_A}) + \lambda^{-D}\, E_2(\phi_{_A}).$$

(b) Since $\phi_{_A}(\boldsymbol{x})$ is an extremum of $E(\phi_{_A})$, it must make $E(\phi_{_A}^\lambda)$ stationary with respect to variations in λ, i.e.

$$\frac{\mathrm{d}\, E(\phi_{_A}^\lambda)}{\mathrm{d}\lambda} = 0$$

at $\lambda = 1$. Show that this condition leads to

$$(2 - D)\, E_1(\phi_{_A}) = D\, E_2(\phi_{_A}).$$

(c) Argue that this, in turn, implies that, since $E_1(\phi_{_A})$ and $E_1(\phi_{_A})$ are positive definite quantities, $\phi_{_A}(\boldsymbol{x})$ *has* to be a trivial, space-independent, solution corresponding to one of the ground states of the potential.

Note: This theorem, which precludes non-trivial space-dependent solutions for $D \geq 3$, is known as Derrick's theorem. However, it should be stressed that the theorem holds only for static solutions and time-dependent solutions are indeed possible.

5. *Cosmic strings and Derrick's theorem:* We had seen that vortices with finite energy density can arise in the case of the Abelian Higgs model. Explain as to why such solutions do not violate Derrick's theorem.

Additional exercises II

From symmetries and conservation laws to spontaneous symmetry breaking and formation of topological defects

1. *$O(N)$ symmetry and conserved currents:* Consider N real scalar fields, say, ϕ_A, where $A = (1, 2, 3, \ldots, N)$, that are described by the following action:

$$S[\phi_A(\tilde{x})] = \frac{1}{c} \int \mathrm{d}^4\tilde{x} \left[\frac{1}{2}\, \partial^\mu \phi_A\, \partial_\mu \phi_A - V(\phi_A\, \phi_A)\right],$$

where, as per the standard convention, the repeated indices (external *as well as* the internal ones) are to be summed over. The above action, evidently, possesses $O(N)$ symmetry.

   (a) Argue that infinitesimal, global, $SO(N)$ transformations can be expressed as $\delta\phi_A = \theta_{AB}\, \phi_B$, where θ_{AB} is a constant and anti-symmetric tensor.

   (b) How many independent conserved currents are associated with the internal symmetry?

   (c) Arrive at the expressions for these currents and explicitly show that they are conserved.

2. *Scale invariance and the conformal stress-energy tensor in arbitrary spacetime dimensions:* Consider a canonical real scalar field in $(D+1)$ spacetime dimensions described by the potential, say, $V(\phi)$.

   (a) Assuming that the action describing the scalar field is invariant under spacetime translations as well as the following scale transformations:

$$x^\mu \to b\, x^\mu \quad \text{and} \quad \phi \to b^d\, \phi,$$

   with b, as before, being a constant, determine the form of the potential $V(\phi)$ and the value of the constant d (in terms of D).

   Note: The number d is known as the scaling dimension of the scalar field.

   (b) Expressing the conserved current associated with the symmetry as

$$j_\mu = x^\nu\, \widetilde{T}_{\mu\nu},$$

   obtain the form of the new, conserved and traceless, so-called conformal stress-energy tensor $\widetilde{T}_{\mu\nu}$.

3. *Angular-momentum tensor of scalar and electromagnetic fields:* Recall that, the angular momentum tensor, say, $J_{\mu\nu\lambda}$, is the conserved current associated with the symmetry under rotations and Lorentz transformations. The actions describing, say, a real and massive, canonical scalar field and the free electromagnetic field do indeed possess these symmetries.

   (a) Construct the conserved angular momentum tensor associated with these two fields.

   (b) You will find that both contain a term involving the corresponding stress-energy tensors. But, you will find that the electromagnetic case contains an additional term as well. Can you identify the reason for the appearance of the additional term in the case of the electromagnetic field?

4. *Action for the free electromagnetic field:* Motivated by the fact that we require a Lorentz invariant action that contains no more than the first derivative of the vector potential A_μ and is quadratic in the potential, we had considered an action of the electromagnetic field involving the quantity $F_{\mu\nu}\, F^{\mu\nu}$. Another quantity that satisfies the conditions demanded above would be $\widetilde{F}_{\mu\nu}\, F^{\mu\nu}$, where $\widetilde{F}_{\mu\nu}$ is the dual field tensor defined as $\widetilde{F}_{\mu\nu} = (1/2)\, \varepsilon_{\mu\nu\alpha\beta}\, F^{\alpha\beta}$. Identify the reason as to why the latter is not a suitable quantity to be considered in an action.

5. *Dipole radiation and the Larmor formula:* Recall that the energy radiated by a point charge per unit time within a unit solid angle can be written as

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t\,\mathrm{d}\Omega} = \frac{e^2}{4\,\pi\,c^3}\,\left[2\,\mu^5\,(\hat{\boldsymbol{n}}\cdot\boldsymbol{a})\,(\boldsymbol{v}\cdot\boldsymbol{a})\,/c + \mu^4\,a^2 - \mu^6\,\gamma^{-2}\,(\hat{\boldsymbol{n}}\cdot\boldsymbol{a})^2\right],$$

where e is the electric charge of the particle, $\boldsymbol{v}$ and $\boldsymbol{a}$ denote its velocity and acceleration, while $\mu = [1-(\boldsymbol{v}\cdot\hat{\boldsymbol{n}})/c]^{-1}$, with $\hat{\boldsymbol{n}}$ denoting the unit vector directed from the retarded location of the charge towards the point of observation.

(a) Show that, for non-relativistic motion wherein the acceleration is proportional to the velocity, the above expression simplifies to

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t\,\mathrm{d}\Omega} = \frac{e^2\,a^2}{4\,\pi\,c^3}\,\sin^2\theta,$$

where a is the magnitude of acceleration and θ is the angle subtended by the vectors $\boldsymbol{v}$ and $\hat{\boldsymbol{n}}$.

(b) Upon integrating over all angular directions, arrive at the result that the total energy radiated by the charge per unit time is given by

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = \frac{2\,e^2\,a^2}{3\,c^3},$$

a result that is known as the Larmor formula.

6. *Radiation reaction in the non-relativistic limit:* A radiating charge loses energy and, as a result, there ought to be a corresponding effect on the motion of the particle, an effect which is known as radiation reaction.

(a) Assuming that the charge is moving non-relativistically and using the result of the previous exercise, show that the radiation reaction force, say, $\boldsymbol{f}$, on the particle is given by

$$\boldsymbol{f} = \frac{2\,e^2}{3\,c^3}\,\dot{\boldsymbol{a}}.$$

(b) If the only force acting on the particle is the above radiation reaction force, establish that the particle can exhibit 'self-acceleration', i.e. its acceleration grows indefinitely as $\exp\left(3\,m\,c^3\,t/2\,e^2\right)$, where m is the mass of the charge.

7. *The Higgs mechanism in an Abelian model:* Recall that the Abelian Higgs model involving the complex scalar field ϕ and the electromagnetic vector potential A_μ is governed by the action

$$S[\phi(\tilde{x}), A^\mu(\tilde{x})] = \frac{1}{c}\int \mathrm{d}^4\tilde{x}\,\left[D_\mu\phi\,(D_\mu\phi)^* - m^2\,\phi\,\phi^* - \lambda\,(\phi\,\phi^*)^2 - \frac{1}{16\,\pi}\,F_{\mu\nu}\,F^{\mu\nu}\right],$$

where the covariant derivative $D_\mu\phi$ is defined as

$$D_\mu\phi = \partial_\mu\phi + i\,e\,A_\mu\,\phi,$$

while the field tensor $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

(a) Obtain the form of the above action in terms of the fields ϕ_1 and ϕ_2, which are related to the original field ϕ as follows:

$$\phi = a + \frac{1}{\sqrt{2}}\,(\phi_1 + \phi_2)\,,$$

where $a = \left[-m^2/(2\,\lambda)\right]^{1/2}$, with $m^2 < 0$.

(b) Let us introduce two new fields, say, $\bar{\phi}_1$ and $\bar{\phi}_2$, that are related to ϕ_1 and ϕ_2 through the relations

$$\bar{\phi}_1 = \phi_1 - \Lambda\,\phi_2 \quad \text{and} \quad \bar{\phi}_2 = \phi_2 + \Lambda\,\phi_1 + \sqrt{2}\,\Lambda\,a,$$

where Λ is a constant. Obtain the final form of the action by choosing the constant Λ such that the field $\bar{\phi}_2$ vanishes.

Note: Evidently, the final action will not contain the the field $\bar{\phi}_2$. Moreover, you will find that the field $\bar{\phi}_1$ as well as the gauge field A_μ possess mass.

8. *The Higgs mechanism in a non-Abelian case:* As you know, the non-Abelian Higgs model involving the iso-vector, but Lorentz, scalar field $\boldsymbol{\phi}$ that is interacting with the Yang-Mills field governed by the vector potential $\boldsymbol{W}_\mu$ is described by the action

$$S[\boldsymbol{\phi}(\tilde{x}), \boldsymbol{W}^\mu(\tilde{x})] = \frac{1}{c}\int \mathrm{d}^4\tilde{x}\,\left(\frac{1}{2}\,D_\mu\boldsymbol{\phi}\cdot D_\mu\boldsymbol{\phi} - \frac{m^2}{2}\,\boldsymbol{\phi}\cdot\boldsymbol{\phi} - \lambda\,(\boldsymbol{\phi}\cdot\boldsymbol{\phi})^2 - \frac{1}{4}\,\boldsymbol{\mathcal{W}}_{\mu\nu}\cdot\boldsymbol{\mathcal{W}}^{\mu\nu}\right),$$

where the covariant derivative $D_\mu\boldsymbol{\phi}$ is defined as

$$D_\mu\boldsymbol{\phi} = \partial_\mu\boldsymbol{\phi} + g\,\boldsymbol{W}_\mu\times\boldsymbol{\phi},$$

with g denoting the coupling constant, while the field tensor $\boldsymbol{\mathcal{W}}_{\mu\nu}$ is given by

$$\boldsymbol{\mathcal{W}}_{\mu\nu} = \partial_\mu\boldsymbol{W}_\nu - \partial_\nu\boldsymbol{W}_\mu + g\,\boldsymbol{W}_\mu\times\boldsymbol{W}_\nu.$$

(a) Upon choosing the ground state of the theory to be $\boldsymbol{\phi} = (0, 0, a)$, where $a = \left[-m^2/(4\,\lambda)\right]^{1/2}$ with $m^2 < 0$, and setting

$$\begin{aligned} D_\mu\phi_1 &= g\,(a+\chi)\,W_\mu^2, \\ D_\mu\phi_2 &= -g\,(a+\chi)\,W_\mu^1, \\ D_\mu\phi_3 &= \partial_\mu\chi, \end{aligned}$$

show that one arrives at an action involving a massless scalar field, and two massive and one massless vector fields.

Note: The above conditions correspond to working in a specific gauge, known as the unitary gauge.

(b) Identify the masses of the fields.

9. *Vortices in a non-Abelian Higgs model:* We had discussed the formation of vortices when the symmetry of the Abelian Higgs model is spontaneously broken. Can string-like objects also form when the symmetry of a non-Abelian Higgs model, say, involving the group $SO(3)$, is broken spontaneously? If they can, try to understand the structure of such objects.

10. *The Dirac monopole and the quantization condition:* Consider a magnetic monopole of strength g at the origin so that the magnetic field is radial and is given by

$$\boldsymbol{B} = \frac{g}{r}\,\boldsymbol{r} = -g\,\boldsymbol{\nabla}\left(\frac{1}{r}\right),$$

in the so-called Gaussian units. Since

$$\boldsymbol{\nabla}^2\left(\frac{1}{r}\right) = 4\,\pi\,g\,\delta^{(3)}(\boldsymbol{r}),$$

we have

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 4\,\pi\,g\,\delta^{(3)}(\boldsymbol{r}),$$

so that the total flux, say, Φ, through a sphere surrounding the origin is given by

$$\Phi = 4\,\pi\,r^2\,B = 4\,\pi\,g.$$

Let us now obtain an expression for the vector potential A_μ that leads to the above magnetic field. Since $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$, if $\boldsymbol{B}$ is regular, then $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$ and no magnetic charges will exist.

(a) Establish that the following vector potentials give rise to the above magnetic field:

$$A^1_r = A^1_\theta = 0,\ A^1_\phi = \frac{g}{r}\,\frac{1-\cos\theta}{\sin\theta}$$

and

$$A^2_r = A^2_\theta = 0,\ A^2_\phi = -\frac{g}{r}\,\frac{1+\cos\theta}{\sin\theta}.$$

(b) Note that while the first vector potential is singular along the line $r = -z$, the second is singular along $r = z$. These infinite lines of singularity are often referred to as the Dirac string. The alternative forms of the vector potentials and the different locations of the Dirac string imply that the singularity on the string is unphysical, and the physical singularity is located *only* at the origin. To avoid the singularities, let us divide the space around the monopole into two overlapping regions, one which excludes the North pole and the other the South. In these two regions, the vector potential $\boldsymbol{A}$ is defined differently and is given by, say, the above two expressions. It is then clear that the vector potentials are both finite in their own domains. Also, in the region of overlap, we expect them to be related by a gauge transformation. Show that

$$A^2_\phi = A^1_\phi - \frac{2\,g}{\sin\theta} = A^1_\phi - \frac{i}{e}\, S\,\boldsymbol{\nabla}_\phi\, S^{-1},$$

where

$$S = \exp\left(2\,i\,e\,g\,\phi\right).$$

(c) Finally, argue that, demanding that the gauge transformation function S be single valued as $\phi \to \phi + 2\,\pi$ leads to the following Dirac quantization condition:

$$e\,g = \frac{n}{2},$$

where n is an integer.

End-of-semester exam

From special relativity to topological defects

1. *Four acceleration:* Recall that the four velocity u^μ and the four acceleration a^μ of a relativistic particle are defined as

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \quad \text{and} \quad a^\mu = \frac{\mathrm{d}u^\mu}{\mathrm{d}s}.$$

   (a) Express the components of the four acceleration a^μ in terms of the three velocity $\boldsymbol{v} = \mathrm{d}\boldsymbol{x}/\mathrm{d}t$ and the three acceleration $\boldsymbol{a} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t$. $\boxed{\text{3 marks}}$

   (b) What is the four acceleration a^μ of the particle in a momentarily comoving inertial frame? $\boxed{\text{3 marks}}$

   (c) Evaluate the quantity $a^\mu\, a_\mu$ in terms of the three acceleration $\boldsymbol{a}$. $\boxed{\text{4 marks}}$

2. *Decaying particles:* Consider a particle A which decays into particles B and C. Let the masses of the three particles be m_{A}, m_{B} and m_{C}, respectively.

   (a) If the particle A is at rest in the lab frame, show that, in the lab frame, the particle B has the energy $\boxed{\text{4 marks}}$

$$E_{\mathrm{B}} = \frac{\left(m_{\mathrm{A}}^2 + m_{\mathrm{B}}^2 - m_{\mathrm{C}}^2\right)\, c^2}{2\, m_{\mathrm{A}}}.$$

   (b) If the particle A decays while in motion in the lab frame, find the relation between the angle at which the particle B comes off, and the energies of the particles A and B. $\boxed{\text{6 marks}}$

3. *The Dirac-Born-Infeld scalar field:* Consider a scalar field, say, ϕ, that is governed by the following so-called Dirac-Born-Infeld (DBI) action:

$$S[\phi(\tilde{x})] = -\frac{1}{c}\int \mathrm{d}^4\tilde{x}\, \left[\frac{1}{T(\phi)}\, \sqrt{1 - T(\phi)\, \partial^\mu\phi\, \partial_\mu\phi} + \frac{1}{T(\phi)} + V(\phi)\right],$$

which is encountered in certain 'brane world' scenarios. The quantity $T(\phi)$ is known as the brane tension and the function $V(\phi)$ is the potential that describes the scalar field.

   (a) Obtain the equation of motion governing the DBI scalar field ϕ. $\boxed{\text{5 marks}}$

   (b) Construct the stress-energy tensor associated with the DBI field. $\boxed{\text{5 marks}}$

4. *Products of the stress-energy tensor:* Evaluate the quantity $T^{\mu\nu}\, T_{\mu\nu}$ for the cases of a canonical, massive scalar field and the free electromagnetic field. $\boxed{\text{5+5 marks}}$

5. *Complex scalar fields and gauge transformations:* We had earlier considered a massive, complex scalar field propagating in a constant and uniform magnetic field background described by the vector potential $A^\mu = (0, 0, B\, x, 0)$. Note that this vector potential leads to the magnetic field $\boldsymbol{B} = B\, \hat{k}$, where $\hat{k}$ denotes the unit vector along the positive z-direction. Such a magnetic field can also be described by the potential $\bar{A}^\mu = (0, -B\, y, 0, 0)$.

   (a) Construct the gauge transformation that takes one from the original gauge A^μ to the new gauge $\bar{A}^\mu$. $\boxed{\text{2 marks}}$

   (b) Determine how the scalar field would correspondingly transform as one moves from the old gauge A^μ to the new one $\bar{A}^\mu$. $\boxed{\text{3 marks}}$

   (c) Explicitly check that the transformed solution indeed satisfies the equation of motion in the new gauge. $\boxed{\text{2 marks}}$

(d) You had earlier solved for the normal modes in the gauge A^μ. Can you identify the form of the normal modes in the new gauge $\bar{A}^\mu$. Are these two sets of normals modes related by the gauge transformation? If not, can you explain the origin of the difference? 3 marks

6. *Reparametrization invariance and Hamiltonians:* Recall that the action governing a relativistic free particle is given by

$$S[x^\mu(s)] = -\, m\, c \int \mathrm{d}s.$$

Instead of the quantity s, let us describe the trajectory in terms of another parameter, say, λ, as $x^\mu(\lambda)$.

(a) Show that the above action then reduces to 2 marks

$$S[x^\mu(\lambda)] = -\, m\, c \int \mathrm{d}\lambda\, \sqrt{\dot{x}^\mu\, \dot{x}_\mu},$$

where $\dot{x}^\mu \equiv \mathrm{d}x^\mu/\mathrm{d}\lambda$.

Note: The above action is invariant under reparametrizations of the form $\lambda \to f(\lambda)$.

(b) Determine the momentum p_μ that is conjugate to $\dot{x}^\mu$, i.e. $p_\mu = (\partial L/\partial \dot{x}^\mu)$. 3 marks

(c) Evaluate the corresponding Hamiltonian, viz. $H = p_\mu\, \dot{x}^\mu - L$. 5 marks

7. *$O(N)$ symmetry and conserved currents:* Consider N real scalar fields, say, $\phi_{_A}$, where $A = (1, 2, 3, \ldots, N)$, that are described by the following action:

$$S[\phi_{_A}(\tilde{x})] = \frac{1}{c} \int \mathrm{d}^4\tilde{x}\, \left[\frac{1}{2}\, \partial^\mu \phi_{_A}\, \partial_\mu \phi_{_A} - V(\phi_{_A}\, \phi_{_A})\right],$$

where, as per the standard convention, the repeated indices (external *as well as* the internal ones) are to be summed over. The above action, evidently, possesses $O(N)$ symmetry.

(a) Argue that, infinitesimal and global $SO(N)$ transformations can be expressed as $\delta\phi_{_A} = \theta_{_{AB}}\, \phi_{_B}$, where $\theta_{_{AB}}$ is a constant and anti-symmetric tensor. 3 marks

(b) How many independent conserved currents are associated with the internal symmetries? 3 marks

(c) Arrive at the expressions for these currents and explicitly show that they are indeed conserved. 4 marks

8. *Energy density of the electromagnetic field and the Poynting flux:* Let $\mathcal{E}$ be the energy density associated with the electromagnetic field. And, let $\boldsymbol{S}$ denote the Poynting flux. Show that the combination $\left(\mathcal{E} - \vert\boldsymbol{S}/c\vert^2\right)$ is a Lorentz invariant quantity. 10 marks

9. *Energy of a solitary wave:* Recall that, in $(1+1)$-spacetime dimensions, the potential $V(\phi) = (\lambda/4)\, \left(\phi^2 - \alpha^2\right)^2$ that describes a real and canonical scalar field had admitted a time-independent solution of the form

$$\phi(x) = \pm\, \alpha\, \tanh\left[(x - x_0)/\Delta\right],$$

where $\Delta = \sqrt{2/(\lambda\, \alpha^2)}$. We had also shown that the total energy, say, E, of the static solution was given by

$$E = \int_{-\infty}^{\infty} \mathrm{d}x\; T^t_t = \frac{2\,\sqrt{2\,\lambda}\, \alpha^3}{3}.$$

(a) Show that the following time-dependent function:

$$\phi(t, x) = \pm\, \alpha\, \tanh\left(\frac{x - x_0 - u\, t}{\Delta\, \sqrt{1 - (u^2/c^2)}}\right),$$

where $u < c$ is a constant, also satisfies the field equation. 4 marks

(b) Evaluate the energy corresponding to this time-dependent solution and express it in terms of the energy E associated with the earlier static solution. $\boxed{\text{6 marks}}$

10. <u>*More kinks in* $(1+1)$*-dimensions:*</u> Consider a canonical and real scalar field in $(1+1)$-spacetime dimensions, which is governed by the potential $V(\phi) = (\mu\,\phi^2/2)\,\left(\phi^2 - \alpha^2\right)^2$. Evidently, the potential contains three, degenerate, minima.

   (a) Identify the minima of the potential. $\boxed{\text{1 mark}}$

   (b) Determine the boundary conditions (as $x \to -\infty$ and $x \to \infty$) that lead to the basic types of static solutions which possess non-zero, but finite energy. $\boxed{\text{2 marks}}$

   (c) How many different solutions exist? $\boxed{\text{2 marks}}$

   (d) Obtain these solutions. $\boxed{\text{5 marks}}$