PH3500

CLASSICAL PHYSICS

August–November 2011

Lecture schedule

- The course will consist of about 40 lectures, including about 8–10 tutorial sessions. However, note that there will be no separate tutorial sessions, and they will be integrated with the lectures.
- The duration of each lecture will be 50 minutes.
- The first lecture will be on August 1, 2011 and the last lecture on November 18, 2011.
- We will meet thrice a week. The lectures are scheduled for 1:00–1:50 PM on Mondays, 10:00–10:50 AM on Thursdays, and 9:00–9:50 AM on Fridays. We may also meet during 4:45–5:35 PM on Mondays for either the quizzes or to make up for any lecture that I may have to miss due to travel.
- We will be meeting in HSB 210.
- Changes in schedule, if any, will be notified sufficiently in advance.

Quizzes, end-of-semester exam and grading

- The grading will be based on three scheduled quizzes and an end-of-semester exam.
- I will consider the best of the two quizzes for grading, and the best two will carry 25% weight each.
- The three quizzes will be on August 29, September 26 and October 31. All these three dates are Mondays, and the quizzes will be held during 4:45–5:35 PM.
- The end-of-semester exam will be on November 29, and the exam will carry 50% weight.

Syllabus and structure

Classical mechanics

1. **Calculus of variations** [$\sim$ 4 Lectures]

   (a) The concept of variation – Euler equation for one-dependent and one-independent variable
   (b) Applications of the Euler equation
   (c) Generalization to several dependent and independent variables
   (d) Lagrangian multipliers – Variation subject to constraints

Exercise sheet 1

2. **The action principle and the Lagrangian formulation** [$\sim$ 5 Lectures]

   (a) Degrees of freedom – Generalized coordinates and velocities
   (b) The principle of least action – The Euler-Lagrange equations of motion
   (c) Inertial frames of reference – Newton's first law – Galileo's principle of relativity – Galilean transformations
   (d) The Lagrangian for a free particle and a system of particles – Equations of motion in an external field – Newton's second law
   (e) Examples

Exercise sheet 2

Quiz I

Exercise sheet 3

3. **Symmetries, conserved quantities and the integration of the equations of motion** [$\sim$ 8 Lectures]

   (a) Homogeneity of time – Conservation of energy – Motion in one dimension – Period as a function of energy
   (b) Homogeneity of space – Conservation of momentum – Newton's third law – Centre of mass
   (c) Isotropy of space – Conservation of angular momentum
   (d) Virial theorem – Simple applications
   (e) The reduced mass – Motion in a central field – Centrifugal barrier and the effective potential
   (f) The Kepler problem – Kepler's laws – The Laplace-Runge-Lenz vector

Exercise sheet 4

4. **The Hamiltonian formulation** [$\sim$ 6 Lectures]

   (a) Conjugate momentum – Legendre's transformation – Hamiltonian and Hamilton's equations
   (b) Poisson brackets – Integrals of motion
   (c) Phase space – Dynamics in the phase space – Phase portraits
   (d) Liouville's theorem

Quiz II

Exercise sheets 5 and 6

Additional exercises

Statistical physics

5. **The need for a statistical approach** [$\sim$ 1 Lecture]

6. **Statistical description of systems of particles** [$\sim$ 5 Lectures]

   (a) Specification of the state of a system – Microstates and macrostates

   (b) Statistical ensemble – Basic postulates – Probabilistic calculations

   (c) Coarse graining of phase space – Density of states for physical systems – Examples

Exercise sheet 7

7. **Statistical thermodynamics** [$\sim$ 5 Lectures]

   (a) Systems in equilibrium – The concept of temperature – The zeroth law of thermodynamics

   (b) Interaction between macroscopic systems – Thermal and mechanical interactions – Dependence of the density of states on the external parameters – The zeroth law of thermodynamics

   (c) The approach to thermal equilibrium – Reversibility – The second law of thermodynamics

Quiz III

Exercise sheet 8

8. **The canonical ensemble** [$\sim$ 4 Lectures]

   (a) The microcanonical ensemble

   (b) Heat reservoirs – System in contact with a heat bath

   (c) Calculations of mean values and the dispersion in the canonical ensemble – Partition functions and their properties

   (d) Applications – Paramagnetism – Equipartition theorem – Quantum oscillator in a thermal bath – Specific heats of solids

Exercise sheet 9

Special relativity

9. **Spacetime and relativity** [$\sim$ 4 Lectures]

   (a) The Michelson-Morley interferometric experiment – Postulates of special relativity

   (b) Lorentz transformations – The relativity of simultaneity – Length contraction and time dilation

   (c) Composition law for velocities – Doppler effect

   (d) Four vectors

Exercise sheet 10

End-of-semester exam

Basic textbooks

Classical mechanics

1. H. Goldstein, C. Poole and J. Safko, *Classical Mechanics*, Third Edition (Pearson Education, Singapore, 2002).
2. W. Greiner, *Classical Mechanics: Systems of Particles and Hamiltonian Dynamics* (Springer-Verlag, New York, 2003).
3. S. T. Thornton and J. B. Marion, *Classical Dynamics of Particles and Systems* (Cengage Learning, Singapore, 2004).
4. T. W. B. Kibble, *Classical Mechanics*, Fifth Edition (Imperial College Press, London, 2004).

Statistical physics

1. F. Reif, *Fundamentals of Statistical and Thermal Physics* (McGraw-Hill, New York, 1965).
2. W. Greiner, L. Neise and H. Stocker, *Thermodynamics and Statistical Mechanics* (Springer-Verlag, New York, 1995).
3. F. Reif, *Statistical Physics*, Berkeley Physics Course, Volume V (Tata McGraw-Hill, New Delhi, 2008).

Special relativity

1. R. Resnick, *Introduction to Special Relativity* (Wiley Eastern, New Delhi, 1985).

Additional references

Mathematical methods

1. G. Arfken, *Mathematical Methods for Physicists*, Third Edition (Academic Press, New York, 1985).
2. J. Mathews and R. L. Walker, *Mathematical Methods of Physics*, Second Edition (Addison Wesley, New York, 1970).

Classical mechanics

1. L. D. Landau and E. M. Lifshitz, *Mechanics*, Course of Theoretical Physics, Volume 1, Third Edition (Pergamon Press, New York, 1976).
2. D. T. Greenwood, *Principles of Dynamics*, Second Edition (Prentice-Hall of India, New Delhi, 1988).
3. D. Kleppner and R. J. Kolenkow, *An Introduction to Mechanics* (Tata McGraw-Hill, New Delhi, 1999).
4. L. N. Hand and J. D. Finch, *Analytical Mechanics* (Cambridge University Press, Cambridge, 1998).

Statistical physics

1. C. Kittel, *Elementary Statistical Physics* (Wiley, New York, 1966).
2. F. Mandl, *Statistical Physics*, Second Edition (Wiley, New York, 1988).
3. A. J. Walton, *Three Phases of Matter*, Second Edition (Oxford University Press, Oxford, 1992).
4. J. M. Yeomans, *Statistical Mechanics of Phase Transitions* (Clarendon Press, Oxford, 1992).

Special relativity

1. A. P. French, *Special Relativity* (W. W. Norton, New York, 1968).
2. E. F. Taylor and J. A. Wheeler, *Spacetime Physics* (W. H. Freeman, San Francisco, 1992).
3. J. B. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (Pearson Education, Singapore, 2003).
4. W. Rindler, *Introduction to Special Relativity* (Oxford University Press, Oxford, 2004).

Exercise sheet 1

Calculus of variations

1. *Geodesics on a sphere:* A geodesic is a curve that represents the shortest path between two points in any space. Find the geodesics on the surface of a sphere.

2. *The brachistochrone problem:* Consider a particle that is moving in a constant force field starting at rest from some point to a lower point. Determine the path that allows the particle to accomplish the transit in the least possible time.

   Note: The resulting curve is referred to as the brachistochrone, i.e. the curve of the fastest descent.

3. *Shortest path in Euclidean space of arbitrary dimension:* Prove that the shortest distance between two points in Euclidean space is a straight line in any dimension.

4. *Snell's law of refraction:* Two homogeneous media of refractive indices n_1 and n_2 are placed adjacent to each other. A ray of light propagates from a point in the first medium to a point in the second medium. According to the Fermat's principle, the light ray will follow a path that minimizes the transit time between the two points. Use Fermat's principle to derive the Snell's law of refraction, viz. that

   $$n_1 \sin\theta_1 = n_2 \sin\theta_2,$$

   where θ_1 and θ_2 are the angles of incidence and refraction at the interface.

   Note: As the complete path is not differentiable at the interface, actually, the problem is not an Euler equation problem.

5. *Variation involving higher derivatives:* Show that the Euler equation corresponding to the integral

   $$J[y(x)] = \int_{x_1}^{x_2} \mathrm{d}x\, f(y, y_x, y_{xx}, , x),$$

   where $y_{xx} \equiv \left(\mathrm{d}^2 y/\mathrm{d}x^2\right)$, is given by

   $$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial f}{\partial y_{xx}}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y_x}\right) + \left(\frac{\partial f}{\partial y}\right) = 0.$$

   Note: In order to obtain this equation, the variation as well its first derivative need to be set to zero at the end points.

Exercise sheet 2

The Lagrangian formulation of mechanics

1. *Form invariance of the Lagrange's equations under a point transformation:* Let $(q_1, q_2, \ldots, q_n)$ be a set of independent generalized coordinates for a system with n degrees of freedom. Let the system be described by the Lagrangian $L(q_i, \dot{q}_i, t)$ and let us transform to a new set of independent coordinates $(s_1, s_2, \ldots, s_n)$ by means of the transformation equations

$$q_i = q_i\,(s_1, s_2, \ldots, s_n, t)\,, \qquad i = 1, 2, \ldots, n.$$

Such a transformation is known as a point transformation.

Show that if the Lagrangian $L(q_i, \dot{q}_i, t)$ is expressed as a function of s_j, $\dot{s}_j$ and t through the above transformation equations, then the system satisfies the Lagrange's equations with respect to the new coordinates, viz.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}_j}\right) - \left(\frac{\partial L}{\partial s_j}\right) = 0, \qquad j = 1, 2, \ldots, n.$$

2. *Coupled oscillators:* A Lagrangian for a particular system can be written as

$$L = \left(\frac{m}{2}\right)\left(a\,\dot{x}^2 + 2b\,\dot{x}\dot{y} + c\,\dot{y}^2\right) - \left(\frac{K}{2}\right)\left(a\,x^2 + 2b\,xy + c\,y^2\right),$$

where a, b and c are arbitrary constants, but subject to the condition that $(b^2 - ac) \neq 0$. Obtain the equations of motion corresponding to the above Lagrangian and examine the two cases $a = c = 0$ and $b = 0,\ c = -a$. Can you identify the physical system described by the Lagrangian? What is the significance of the condition on the value of $(b^2 - ac)$?

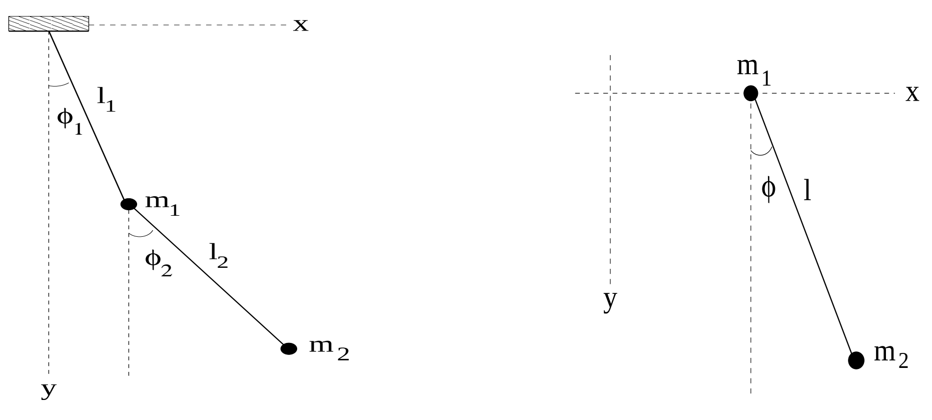
3. *A bead on a wire:* A bead slides without friction down a wire that has the shape $y = f(x)$ in a uniform gravitational field (corresponding to an acceleration g). Write down the Lagrangian for the system and obtain the equation of motion. In particular, consider the case wherein the shape of the wire is a cycloid, i.e. when the shape $y = f(x)$ can be parametrically written as

$$x = a\,(\theta - \sin\theta)\,, \quad y = a\,(1 + \cos\theta)\,.$$

What is the equation of motion for $u = \cos(\theta/2)$?

4. *A variety of systems:* Construct the Lagrangian and obtain the equations of motion for the following systems when they are placed in a uniform gravitational field (corresponding to an acceleration g):

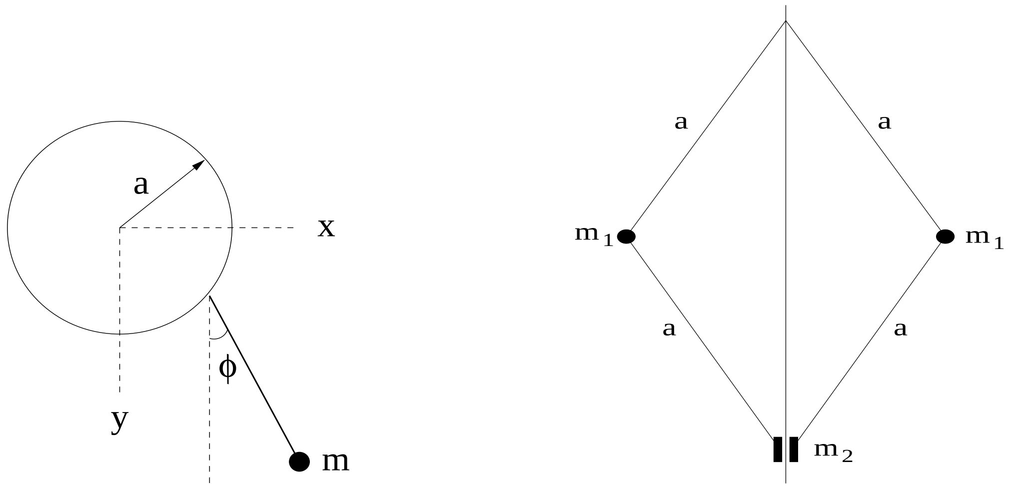
   (a) a coplanar double pendulum as shown on the left in the figure below,



   (b) a simple pendulum of mass m_2, with a mass m_1 at the point of support which can move on a horizontal line lying in the plane in which m_2 moves as shown on the right in the figure above,

(c) a simple pendulum of mass m whose point of support

i. moves uniformly on a vertical circle with constant frequency γ as shown on the left in the figure below,



ii. oscillates horizontally in the plane of motion of the pendulum according to the law $x = a\cos(\gamma t)$,

iii. oscillates vertically according to the law $y = a\cos(\gamma t)$,

(d) In the system shown on the right in the figure above, the mass m_2 moves on a vertical axis and the whole system rotates about this axis with a constant angular velocity Ω.

Quiz I

Calculus of variations and the Lagrangian formulation of mechanics

1. *Straight line in polar coordinates:* Recall that, working in the Cartesian coordinates, we had shown that the shortest distance between two points on an Euclidean plane is a straight line. Establish the result in the polar coordinates.

2. *Geodesic on a cylinder:* Determine the geodesic on a right circular cylinder of a fixed radius, say, R.

3. *Spherical pendulum:* Recall that the ordinary pendulum consists of a mass m, attached to a rod of fixed length, say, l, which is free to move along the polar angle in a uniform gravitational field. In contrast, the spherical pendulum is free to move along the polar *as well as* the azimuthal angles. Write down the Lagrangian for the spherical pendulum, and obtain the equations of motion.

4. *Non-relativistic particle in an electromagnetic field:* A non-relativistic particle that is moving in an electromagnetic field described by the scalar potential ϕ and the vector potential $\mathbf{A}$ is governed by the Lagrangian
$$L = \frac{m\,\mathbf{v}^2}{2} + q\,(\mathbf{v}\cdot\mathbf{A}) - q\,\phi,$$
where m and q are the mass and the charge of the particle. Show that the equation of motion of the particle is given by
$$m\,\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q\,(\mathbf{E} + \mathbf{v}\times\mathbf{B})\,,$$
where $\mathbf{E}$ and the $\mathbf{B}$ are the electric and the magnetic fields given by
$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} \qquad \text{and} \qquad \mathbf{B} = \nabla\times\mathbf{A}.$$
Note: The scalar and the vector potentials, viz. ϕ and $\mathbf{A}$, are dependent on time *as well as* space. Further, given two vectors, say, $\mathbf{C}$ and $\mathbf{D}$, one can write,
$$\nabla\,(\mathbf{C}\cdot\mathbf{D}) = (\mathbf{D}\cdot\nabla)\;\mathbf{C} + (\mathbf{C}\cdot\nabla)\;\mathbf{D} + \mathbf{D}\times(\nabla\times\mathbf{C}) + \mathbf{C}\times(\nabla\times\mathbf{D})\,.$$
Also, since $\mathbf{A}$ depends on time as well as space, we have,
$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \frac{\partial\mathbf{A}}{\partial t} + (\mathbf{v}\cdot\nabla)\;\mathbf{A}.$$

5. *Equation of motion for a relativistic particle:* The Lagrangian for a relativistic particle moving in a potential $U(\mathbf{r})$ is given by
$$L = -m\,c^2\,\sqrt{1 - \frac{\mathbf{v}^2}{c^2}} - U(\mathbf{r}),$$
where m is the mass of the particle and c is a constant that denotes the speed of light. Obtain the equation of motion of the relativistic particle. What happens to the equation of motion when $|\mathbf{v}| \ll c$?

Exercise sheet 3

Constraints and Lagrangian multipliers

1. *Particle sliding off a surface:* A particle slides off a cylindrical surface. Using the method of Lagrange multipliers, obtain the critical angle at which the particle flies off from the surface.

2. *Curve circumscribing the largest area:* Consider a curve $y(x)$ of a given length, say, l, that runs from $(-a, 0)$ to $(a, 0)$ in the x-y plane. Determine the function $y(x)$ that encloses the largest area, when bounded by x-axis.

3. *Suspended cable:* A flexible cable of a given length is suspended from two fixed points. Using the method of Lagrange multipliers, find the curve that will minimize the total gravitational potential energy of the cable.

4. *A rotating bucket of water:* A fixed volume of water is rotating in a cylindrical bucket with a constant angular velocity ω. Find the curve of the water surface that will minimize the potential energy of the water in the combined gravitational-centrifugal force field.

5. *Particle moving on a paraboloid:* A particle of mass m moves under the influence of gravity on the inner surface of the paraboloid of revolution $(x^2 + y^2) = (a\, z)$ which is assumed to be frictionless. Obtain the equations of motion using the method of Lagrange multipliers. Prove that the particle will describe a horizontal circle in the plane $z = h$ provided that it is given an angular velocity whose magnitude is $\omega = \sqrt{2\, g/a}$.

Exercise sheet 4

Conserved quantities and the integration of the equations of motion

1. *Period associated with bounded, one-dimensional motion:* Determine the period of oscillation as a function of the energy, say, E, when a particle of mass m moves in fields for which the potential energy is given by (a) $U = -\left[U_0/\cosh^2(\alpha\, x)\right]$, for E such that $-U_0 < E < 0$, and (b) $U = \left[U_0 \tan^2(\alpha\, x)\right]$.

2. *Motion in one dimension:* Obtain the solutions describing the time evolution of a particle moving in the one-dimensional potential

$$U(x) = \alpha\left(e^{-2\beta x} - 2\, e^{-\beta x}\right), \quad \text{where} \quad \alpha, \beta > 0,$$

for the cases $E < 0$, $E = 0$ and $E > 0$ (E being the energy of the particle). Also, evaluate the period of oscillation of the particle when $E < 0$.

3. *The damped oscillator:* Obtain the equations of motion for the Lagrangian

$$L = e^{\gamma\, t}\left(\frac{m\,\dot{x}^2}{2} - \frac{m\,\omega^2\, x^2}{2}\right).$$

Suppose we make a point transformation of the form

$$q = x\; e^{\gamma\, t/2}.$$

What are the Lagrangian and the equation of motion in terms of q? How would you describe the system? Are there any constants of motion?

4. *Application of the virial theorem:* Using virial theorem, show that the total mass M of a spherical cluster of stars (or galaxies) of uniform density and radius R is given by

$$M = \left(\frac{5\, R\, \langle v^2\rangle}{3\, G}\right),$$

where $\langle v^2\rangle$ is the mean-squared velocity of the individual stars and G is, of course, the gravitational constant.

Note: The above relation allows us to obtain an estimate of the mass of a cluster if we can measure the mean-squared velocity, say, from the Doppler spread of the spectral lines and the radius of the cluster, say, from its known distance and angular size.

5. *Integrating the equations of motion:* Using the conservation of energy and angular momentum, integrate the equations of motion of (a) the spherical pendulum and (b) the particle that is constrained to move on the inner surface of a cone, when the systems are in a uniform gravitational field.

Exercise sheet 5

Motion in a central potential

1. *Time evolution in the Keplerian potential:* Consider a particle of reduced mass m that is moving on a hyperbolic trajectory in the central potential $U(r) = -\alpha/r$, where $\alpha > 0$. Show that the time evolution of the trajectory can be parametrically expressed as follows:

$$r = a\ (e \cosh\xi - 1) \quad \text{and} \quad t = \sqrt{m\, a^3/\alpha}\ \ (e \sinh\xi - \xi)$$

   or, equivalently, as

$$x = a\ (e - \cosh\xi) \quad \text{and} \quad y = a\,\sqrt{e^2 - 1}\ \sinh\xi,$$

   where the quantity a and the eccentricity of the orbit e are given in terms of the energy E and the angular momentum M by the relations

$$a = \frac{\alpha}{2\,E} \quad \text{and} \quad e = \sqrt{1 + \frac{2\,E\,M^2}{m\,\alpha^2}},$$

   while $-\infty < \xi < \infty$.

2. *Explicitly establishing the virial theorem:* Evaluate the time averages (i.e. the average over one complete period) of the kinetic and potential energies, say, T and U, for a particle that is moving along an elliptic orbit in the Keplerian central potential, and establish the virial theorem for the case, viz. that $2\,\langle T\rangle = -\langle U\rangle$, where the angular brackets denote the averages.

3. *From the orbit to the force:* Consider a particle that is moving in a central field.

   (a) Determine the central force, given that the particle is known to move on the following, logarithmic spiral, orbit:
$$r(\phi) = \beta\, e^{\gamma\,\phi}.$$

   (b) Obtain the corresponding time evolution of the radial and the angular coordinates r and ϕ.

   (c) What is the energy associated with the above orbit?

4. *Passing through the centre:* A particle moves under the influence of a central force given by $F(r) = -\delta/r^n$, where $\delta > 0$. Show that $n = 5$ if the orbit of the particle is circular and it passes through the centre of the force.

5. *Falling onto each other:* Consider two particles which are moving under the influence of their mutual gravitational force. Let the particles follow circular orbits about one another with a time period T. Show that, if the particles are suddenly stopped in their orbits and allowed to gravitate towards each other, they will collide after a time $T/4\,\sqrt{2}$.

Exercise sheet 6

The Hamiltonian formulation and the structure in phase space

1. *Bead on a helical wire:* A bead is moving on a helical wire under the influence of a uniform gravitational field. Let the helical wire be described by the relations $z = \alpha\,\theta$ and $\rho = constant$. Construct the Hamiltonian of the system and obtain the Hamilton's equations of motion.

2. *Particle in the Keplerian central potential:* Working in the plane polar coordinates, write down the Hamiltonian and the Hamilton's equations for a particle of mass m that is moving in the Keplerian central potential $U(r) = -\alpha/r$, where $\alpha > 0$.

3. *Poisson brackets:* Establish the following relations: $[q_i, q_j] = 0$, $[p_i, p_j] = 0$, $[q_i, p_j] = \delta_{ij}$, $\dot{q}_i = [q_i, H]$ and $\dot{p}_i = [p_i, H]$, where q_i, p_i and H represent the generalized coordinates, the corresponding conjugate momenta and the Hamiltonian, respectively, while the square brackets denote the Poisson brackets.

   Note: The quantity δ_{ij} is called the Kronecker symbol and it takes on the following values:

$$\delta_{ij} = \begin{cases} 1 & \text{when } i = j, \\ 0 & \text{when } i \neq j. \end{cases}$$

4. *Phase portraits:* Draw the phase portraits of a particle moving in the following one dimensional potentials: (a) $U(x) = a\,|x|^n$ and (b) $U(\theta) = -a\cos\theta$, where $a > 0$ and $n > 2$.

5. *An illustration of the Liouville's theorem:* Consider, a collection of non-interacting particles, each of mass m, in a uniform gravitational field. At an initial time, let the particles have momenta and energies in the range $p_1 < p < p_2$ and $E_1 < E < E_2$.

   (a) What is the area occupied by the particles in the phase space at the initial time?

   (b) Evaluate the area of the phase space occupied at a later time and show that it is the same as the area occupied by the particles initially.

Additional exercises

From the calculus of variations to the Hamiltonian formalism

1. *Walking around a volcano:* Consider a conical surface defined by the relation $z = 1 - \sqrt{x^2 + y^2}$.

   (a) What is the geodesic on the surface?

   Note: This geodesic describes the shortest mountain path around a volcano.

   (b) Determine the distance between the points $(x, y, z) = (0, -1, 0)$ and $(0, 1, 0)$ along the geodesic on the surface.

2. *The action for simple systems:* Evaluate the action for the following one-dimensional systems: (a) a free particle (of mass m), (b) a particle (of mass m) that is subjected to a constant force (say, of magnitude α), and (c) a simple harmonic oscillator (of mass m and frequency ω). Assume that these systems are at the points q_1 and q_2 at times t_1 and t_2, respectively.

3. *A messy Lagrangian for a well-known system:* Find the equation of motion corresponding to the Lagrangian

$$L = \mathrm{e}^{-\left(x^2+\dot{x}^2\right)} + 2\,\dot{x}\,\mathrm{e}^{-x^2} \int_0^{\dot{x}} \mathrm{d}\alpha\, \mathrm{e}^{-\alpha^2}.$$

   Obtain the energy integral for the system and also construct a simpler Lagrangian that will lead to the same equation of motion.

4. *Energy integral for a Lagrangian involving a higher derivative:* Construct the energy integral for a system whose Lagrangian depends on $\ddot{q}$, in addition to q and $\dot{q}$.

5. *An application of the conservation of linear momentum:* Consider a region of space divided by a plane. The potential energy of a particle on one half of the plane is U_1 and, on the other half, it is U_2, where U_1 and U_1 are constants. A particle of mass m which has a speed v_1 moves from the first half to the second. If θ_1 and θ_2 are the angles subtended by the trajectory of particle with respect to the normal on either side of the plane, show that

$$\frac{\sin\theta_1}{\sin\theta_2} = \left(1 + \frac{U_1 - U_2}{m\, v_1^2/2}\right)^{1/2}.$$

   What is the optical analog of the problem?

6. *Period of a binary system:* Two double stars of the same mass as the Sun rotate about their common center of mass. Their separation is 4 light years. What is their period of revolution?

   Note: The mass of the Sun is about 2×10^{30} kg, while the value of the gravitational constant is $G = 6.673 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$.

7. *Perihelion and aphelion of the Halley's comet:* Halley's comet, which passed around the Sun early in 1986, moves in a highly elliptical orbit with an eccentricity of 0.967 and a period of 76 years. Calculate the perihelion and aphelion of the Halley's comet.

   Note: It would be worthwhile to compare these distances with the radius of the orbits of Venus and Neptune.

8. *Motion in a repulsive central potential:* Consider a particle that is moving in the repulsive central potential $U(r) = \alpha/r$, where $\alpha > 0$.

   (a) Determine the orbital trajectory of particle.

   (b) How does the trajectory compare with the corresponding unbounded trajectory in the Keplerian, attractive central potential?

9. *More Poisson brackets:* Establish the following relations:

$$\begin{aligned}[f\,g, h] &= f\,[g, h] + g\,[f, h],\\ [f, [g, h]] + [h,\ [f, g]] + [g,\ [h, f]] &= 0,\end{aligned}$$

where f, g and h and are arbitrary functions of the generalized coordinates and the corresponding conjugate momenta, while the square brackets, as before, denote the Poisson brackets.

Note: The second relation, with its characteristic cyclic structure, is known as the Jacobi's identity.

10. *A messy Lagrangian and the corresponding Hamiltonian:* If a system has the Lagrangian

$$L = \frac{1}{2}\,G(q,t)\;\dot{q}^2 + F(q,t)\;\dot{q} - V(q,t),$$

show that the corresponding Hamiltonian is given by

$$H = \frac{[p - F(q,t)]^2}{2\,G(q,t)} + V(q,t),$$

where

$$p = G(q,t)\,\dot{q} + F(q,t).$$

Quiz II

From conserved quantities to the phase space structure

1. *Bounded motion in one-dimension:* Consider a particle moving in the following one-dimensional potential:

$$U(x) = \frac{a}{x^2} + b\,x^2, \quad \text{where} \quad a,\ b > 0.$$

   (a) Obtain the solution describing the time evolution of the particle. $\boxed{\text{7 marks}}$

   (b) Also, evaluate the time period of the particle as a function of its energy. $\boxed{\text{3 marks}}$

2. *Solving the Kepler problem in a different fashion:* Consider a particle of mass m that is moving in the Keplerian central potential $U(r) = -\alpha/r$, where $\alpha > 0$.

   (a) Show that the equation of motion can be written as

$$\left(\frac{\mathrm{d}^2 v}{\mathrm{d}\phi^2}\right) + v = \left(\frac{m\,\alpha}{M^2}\right),$$

   where $v = (1/r)$ and M is the angular momentum of the particle. $\boxed{\text{5 marks}}$

   (b) Solve the above equation to arrive at the corresponding orbit equation. $\boxed{\text{5 marks}}$

3. *The isotropic oscillator:* Consider a particle moving in the central potential $U(r) = \alpha\,r^2$, where $\alpha > 0$.

   (a) Obtain the orbit equation. $\boxed{\text{5 marks}}$

   (b) Solve for the radial motion in time. $\boxed{\text{5 marks}}$

4. *A bead on a wire:* Consider a bead that is sliding without friction down a wire in a uniform gravitational field. Let the shape of the wire be $y = f(x)$.

   (a) Obtain the Hamiltonian for the system. $\boxed{\text{6 marks}}$

   (b) Also, write down the corresponding Hamilton's equations of motion. $\boxed{\text{4 marks}}$

5. *Phase portraits:* Draw the phase portrait for the following cases:

   (a) A particle moving vertically in a uniform gravitational field. $\boxed{\text{4 marks}}$

   (b) A particle moving in the one-dimensional potential $U(x) = \alpha\,x^2\,e^{-\beta\,x^2}$, where α and β are positive definite quantities. $\boxed{\text{6 marks}}$

Exercise sheet 7

Basic concepts in probability

1. <u>*The binomial distribution:*</u> Consider an ideal system of N spin $\frac{1}{2}$ particles, each having an associated magnetic moment μ_0. Suppose that the system is located in an external magnetic field. Let us assume that the system is in equilibrium so that a statistical ensemble consisting of $\mathcal{N}$ such systems is time-independent. Let p and q denote the probability that the magnetic moment of one of the spins points up or down, respectively.

   (a) Show that the probability $P(n)$ that n of the N moments point up is given by
   $$P(n) = \frac{N!}{n!\,(N-n)!}\ p^n\, q^{N-n}.$$

   (b) Argue that $P(n)$ is the coefficient of the $p^n\, q^{N-n}$ term in the binomial expansion $(p+q)^N$.

   (c) Evaluate
   $$\sum_{n=0}^{N} P(n).$$

   (d) Plot $P(n)$ for a given N.

2. <u>*Mean values and fluctuations:*</u> Suppose that a variable u of a system can take any of the following α possible distinct values
   $$u_1, u_2, u_3, \dots, u_\alpha$$
   with respective probabilities
   $$P_1, P_2, P_3, \dots, P_\alpha.$$
   This means that, in an ensemble of $\mathcal{N}$ systems (where $\mathcal{N} \to \infty$), the variable u assumes the particular value u_r in a number $\mathcal{N}_r = \mathcal{N}\, P_r$ of these systems. In such a case, the mean value $\langle u \rangle$ of u is given by
   $$\langle u \rangle = \sum_{r=1}^{\alpha} P_r\, u_r,$$
   and the dispersion of u is defined as
   $$\langle (\Delta u)^2 \rangle = \sum_{r=1}^{\alpha} P_r\, \left(u_r - \langle u \rangle\right)^2.$$

   (a) Show that, in case of the spin $\frac{1}{2}$ systems discussed in the previous problem, the mean value of the total magnetic moment M of the system is given by
   $$\langle M \rangle = N\,(p-q)\,\mu_0.$$

   (b) Also, establish that the dispersion in M is given by
   $$\langle (\Delta M)^2 \rangle = 4\,N\,p\,q\,\mu_0^2.$$

   Note: Each spin $\frac{1}{2}$ particle is a statistically independent system.

3. <u>*Distribution of molecules in an ideal gas:*</u> Consider an ideal gas of N molecules contained in a box of volume V_0. Determine the mean number and the dispersion in the number of molecules found in a sub-volume, say, V, of the box. How does the ratio of the standard deviation in the number of molecules to the mean number within the volume V behave as a function of N?

   Note: The square root of the dispersion is referred to as the standard deviation.

4. <u>*The Gaussian distribution:*</u> Let $\mathcal{P}(x)$ denote the probability density associated with a continuous variable x. The mean value and the dispersion of the quantity x in such a case are defined as follows:

$$\langle x\rangle = \int\limits_{-\infty}^{\infty} \mathrm{d}x\, \mathcal{P}(x)\, x \qquad \text{and} \qquad \langle(\Delta x)^2\rangle = \int\limits_{-\infty}^{\infty} \mathrm{d}x\, \mathcal{P}(x)\, \left(x - \langle x\rangle\right)^2 .$$

Consider the following Gaussian distribution for $\mathcal{P}(x)$:

$$\mathcal{P}(x) = \frac{1}{\sqrt{2\,\pi}\,\sigma}\, \exp -(x-\mu)^2/2\,\sigma^2,$$

where it is has been assumed that x can take values in the range $-\infty < x < \infty$.

(a) Show that this distribution is normalized, i.e.

$$\int\limits_{-\infty}^{\infty} \mathrm{d}x\, \mathcal{P}(x) = 1.$$

(b) Also, establish that, for the above Gaussian distribution, $\langle x\rangle = \mu$ and $\langle(\Delta x)^2\rangle = \sigma^2$.

5. <u>*The Poisson distribution:*</u> Consider a system such as the spin $\frac{1}{2}$ system we had discussed above wherein the number n of the N moments pointing up is given by the binomial distribution. Suppose the probability p is small (i.e. $p \ll 1$) and one is interested in the case wherein $n \ll N$. In such a case, show that the probability distribution $P(n)$ reduces to the following Poisson form:

$$P(n) = \frac{\lambda^n}{n!}\, \mathrm{e}^{-\lambda},$$

where $\lambda = N\,p$.

Note: To arrive at this result, one needs to make use of the Stirling's approximation, according to which, we can write

$$\frac{N!}{(N-n)!} \simeq N^n.$$

Exercise sheet 8

Statistical description of systems of particles

1. *Transforming probability distributions:* Suppose that a two-dimensional vector $\mathbf{B}$ of constant magnitude is equally likely to point in any direction specified by the angle θ. What is the probability, say, $P(B_x)\,\mathrm{d}B_x$ that the x component of the vector lies between B_x and $B_x + \mathrm{d}B_x$?

2. *A classical particle in a box:* Consider a particle of mass m that is moving in one dimension, say, along the x-direction. The particle is confined by a hard box, whose walls are located at $x = 0$ and $x = L$. Suppose the energy of particle is known to lie between E and $E + \delta E$. Draw the trajectory of the particle in the phase space and indicate the regions of the phase space that are accessible to the particle.

3. *One spin in thermal contact with a small spin system:* Consider a system A consisting of a spin $\frac{1}{2}$ particle having a magnetic moment μ_0, and another system A' consisting of three spin $\frac{1}{2}$ particles each having magnetic moment μ_0. The total system $A + A'$ is isolated, and both the systems are in the same magnetic field $\mathbf{B}$. The systems are placed in thermal contact with each other so that they can exchange energy. Suppose that, when the moment of A points up, two of the moments of A' point up and one points down.

   (a) Count the total number of states accessible to the combined system $A + A'$ when the moment of A points up, and when it points down.

   (b) Calculate the ratio of the probabilities of the moment A pointing down to the moment pointing up.

4. *A quantum particle in a box:* Consider a single, quantum mechanical, free particle of mass m that is confined to a one-dimensional box of width L. It can be shown that the energy levels of the particle are discrete, and are given by

$$E_n = \frac{\hbar^2\,\pi^2}{2\,m\,L^2}\,n^2,$$

   where n is a quantum number that is a non-zero and positive integer, while $\hbar = (h/2\,\pi)$ with h being the Planck's constant. Instead, had the box been a cube of side L, the energy levels of the particle would be given in terms of three non-zero and positive integers, say, n, l and m, as follows:

$$E_{nlm} = \frac{\hbar^2\,\pi^2}{2\,m\,L^2}\,\left(n^2 + l^2 + m^2\right).$$

   Determine the number of states that are accessible to the system if its energy lies between E and $E + \delta E$.

   Note: Assume that E is large enough so that it is associated with fairly large quantum numbers.

5. *Work done by a gas:* The mean pressure p of a thermally insulated gas varies with the volume V according to the relation

$$p\,V^\gamma = C,$$

   where γ and C are constants. Find the work done by the gas in a quasi-static process from a macrostate with pressure p_i and volume V_i to one with pressure p_f and volume V_f. Express the final result in terms of p_i, V_i, p_f and V_f.

Quiz III

From basic concepts in probability to statistical thermodynamics

1. *Probabilities in one-dimensional random walk:* A person starts from a lamppost in the middle of a street, taking steps of equal length. Suppose that the probabilities of the person taking a step to right or to the left are equal. What is the probability that the person will be back at the lamppost after taking N steps,

   (a) When N is odd? **4 marks**

   (b) When N is even? **6 marks**

2. *Displacement distribution of random oscillators:* Consider an ensemble of classical one-dimensional simple harmonic oscillators. Let ω and A describe the angular frequency and the amplitude of the oscillators, and let the displacement x of an oscillator as a function of time t be given by

$$x = A\cos(\omega\, t + \varphi),$$

   where φ is an arbitrary constant which can assume any value in the range $0 \le \varphi < 2\,\pi$ with equal probability. Determine the probability $P(x)\,\mathrm{d}x$ that, at any time t, the displacement lies between x and $x + \mathrm{d}x$. Express $P(x)$ in terms of A and x. **10 marks**

3. *Number of accessible states for a spin $\frac{1}{2}$ system:* Consider an ideal system of N spin $\frac{1}{2}$ particles, each having an associated magnetic moment μ_0. Suppose that the system is located in an external magnetic field $\mathbf{B}$. The energy of the system is then given by

$$E = -(n - n')\,\mu_0\, B,$$

   where n denotes the number of particles whose spins are up, while $n' = N - n$ represents the number of particles whose spins are down. Calculate the number of states $\Omega(E)$ which lie within the small energy interval E and $E + \delta\, E$ for the system. **10 marks**

   Note: It is assumed that $\delta E \gg \mu_0\, B$.

4. *The Maxwell distribution and the equipartition theorem:* Consider an ideal gas of non-interacting molecules at the temperature T. According to the Maxwell distribution, the probability of finding a molecule with a momentum p is given by

$$P(p) \propto \exp -p^2/2\,m\,k\,T,$$

   where m is the mass of the molecule, while k is the Boltzmann constant.

   (a) Normalize the above probability distribution. **2 marks**

   (b) Evaluate the mean momentum and the dispersion in the momentum of a molecule. **3 marks**

   (c) Calculate the mean energy of the molecule as well as the dispersion in its energy. **5 marks**

5. *Mean values and fluctuations associated with the Poisson distribution:* Consider an event that is characterized by the probability p, which occurs n times in N trials. The probability distribution $P(n)$ associated with such an event is given by the binomial distribution. As we have discussed, in the limit $p \ll 1$ and $n \ll N$, the binomial distribution reduces to the following, simpler, Poissonian form:

$$P(n) = \frac{\lambda^n}{n!}\,\mathrm{e}^{-\lambda},$$

   where $\lambda = N\,p$. Assuming that the sum over N can be extended to infinity,

   (a) Evaluate the mean value of n associated with the above Poisson distribution. **3 marks**

   (b) Also, calculate the dispersion in n. **3 marks**

   (c) Further, compute $\langle n^3\rangle$. **4 marks**

Exercise sheet 9

Statistical thermodynamics and the canonical ensemble

1. *Sharpness of the probability distribution:* Consider a system A that is interacting thermally with another system, say, A'. We had discussed as to how the probability of finding the system A with a given energy is peaked around the mean value $\langle E\rangle$. Argue that the probability distribution is sharply peaked around the mean value with a width that is given by
$$\frac{\Delta E}{\langle E\rangle} \simeq \frac{1}{\sqrt{N}},$$
where N denotes the degrees of freedom associated with the system A.

2. *Negative temperatures:* Recall that, for an ideal system of N spin $\frac{1}{2}$ particles, each having an associated magnetic moment μ_0, the energy of the system in an external magnetic field $\mathbf{B}$ is given by
$$E = -(n - n')\,\mu_0\, B,$$
where n denotes the number of particles whose spins are up, while $n' = N - n$ represents the number of particles whose spins are down. In such a case, we had shown that the number of states $\Omega(E)$ accessible to the system when it has an energy E is given by
$$\Omega(E) = \frac{N!}{[(N/2) + (E/2\,\mu_0\, B)]!\,[(N/2) - (E/2\,\mu_0\, B)]!}.$$
   (a) Assuming N to be large and using the Stirling's approximation for factorials, plot $(S/N\,k)$ as function of $(E/N\,\mu_0\,B)$, where $S = k\,\ln\,\Omega(E)$ is the entropy of the system, with k being the Boltzmann constant.
   (b) In the large N limit, evaluate the temperature of the system, and show that it turns negative for $E > 0$.

3. *Law of atmospheres:* Consider an ideal gas of molecules at a finite temperature in a uniform gravitational field, as on the surface of the Earth. Determine the probability of finding a molecule at a given height from the surface of the Earth.

4. *Mean energy of a quantum oscillator in a thermal bath:* Evaluate the mean energy of a quantum oscillator in a thermal bath. Determine the behavior of the mean energy at low and high temperatures.

5. *Specific heats of solids:* Consider any simple solid with Avogadro number $N_{\rm A}$ of atoms per mole. Examples of such solids could be copper, gold, aluminum or diamond. These atoms are free to oscillate about their equilibrium positions, oscillations which are referred to as lattice vibrations. Such vibrations are supposed to be small so that the potential energy of the atoms are quadratic in their displacements about their equilibrium positions. The energy associated with each of these atoms can be expressed as
$$E = \frac{\mathbf{p}^2}{2\,m} + \frac{m\,\omega^2}{2}\,\mathbf{q}^2,$$
where ω is the characteristic frequency of the oscillations. The specific heat capacity of the solid at constant volume defined as
$$c_{\rm V} = \left(\frac{\partial\langle E\rangle}{\partial T}\right)_{\rm V}.$$
Assuming the oscillators to be quantum, determine the behavior of $c_{\rm V}$ as a function of temperature.

Exercise sheet 10

Special relativity

1. *Aberration of light:* Consider two inertial frames S and S', with the frame S' moving along the x-axis with a velocity v with respect to the frame S. Let the velocity of a particle in the frames S and S' be $\mathbf{u}$ and $\mathbf{u}'$, and let θ and θ' be the angles subtended by the velocity vectors with respect to the common x-axis, respectively.

   (a) Show that
   $$\tan\theta = \left(\frac{u'\sin\theta'}{\gamma\,[u'\cos\theta' + v]}\right),$$
   where $\gamma = \left[1-(v/c)^2\right]^{-1/2}$.

   (b) For $u = u' = c$, show that
   $$\cos\theta = \left(\frac{\cos\theta' + (v/c)}{1+(v/c)\cos\theta'}\right)$$
   and
   $$\sin\theta = \left(\frac{\sin\theta'}{\gamma\,[1+(v/c)\cos\theta']}\right).$$

   (c) For $(v/c) \ll 1$, show that
   $$\Delta\theta = (v/c)\,\sin\theta',$$
   where $\Delta\theta = (\theta' - \theta)$.

2. *Decaying muons:* Muons are unstable and decay according to the radioactive decay law $N = N_0\,\exp-(0.693\,t/t_{1/2})$, where N_0 and N are the number of muons at times $t = 0$ and t, respectively, while $t_{1/2}$ is the half life. The half life of the muons in their own rest frame is 1.52×10^{-6} s. Consider a detector on top of a $2,000$ m mountain which counts the number of muons traveling at the speed of $v = 0.98\,c$. Over a given period of time, the detector counts 10^3 muons. When the relativistic effects are taken into account, how many muons can be expected to reach the sea level?

3. *Binding energy:* As you may know, the deuteron which is the nucleus of deuterium, an isotope of hydrogen, consists of one proton and one neutron. Given that the mass of a proton and a neutron are $m_{\mathrm{p}} = 1.673\times10^{-27}$ kg and $m_{\mathrm{n}} = 1.675\times10^{-27}$ kg, while the mass of the deuteron is $m_{\mathrm{d}} = 3.344\times10^{-27}$ kg, show that the binding energy of the deuteron in about 2.225 MeV.

   Note: MeV refers to Million electron Volts, and an electron Volt is 1.602×10^{-19} J.

4. *Form invariance of the Minkowski line element:* Show that the following Minkowski line element is invariant under the Lorentz transformations:
   $$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}\mathbf{x}^2.$$

5. *Compton effect using four vectors:* Consider the scattering between a photon of frequency ω and a relativistic electron with velocity $\mathbf{v}$ leading to a photon of frequency ω' and electron with velocity $\mathbf{v}'$. Such a scattering is known as Compton scattering. Let α be the angle between the incident and the scattered photon and θ and θ' be the angles between the direction of propagation of photon and the velocity vector of the electron before and after the collision.

   (a) Using the conservation of four momentum, show that
   $$\left(\frac{\omega'}{\omega}\right) = \left(\frac{1-(v/c)\cos\theta}{1-(v/c)\cos\theta' + (\hbar\,\omega/\gamma\,m_{\mathrm{e}}\,c^2)\,(1-\cos\alpha)}\right),$$
   where $\gamma = \left[1-(v/c)^2\right]^{-1/2}$ and m_{e} is the mass of the electron.

(b) When $(\hbar\,\omega) \ll \left(\gamma\, m_{\rm e}\, c^2\right)$, show that the frequency shift of the photon can be written as

$$\left(\frac{\Delta\omega}{\omega}\right) = \left[\frac{(v/c)\ (\cos\theta - \cos\theta')}{1-(v/c)\cos\theta'}\right],$$

where $\Delta\omega = (\omega' - \omega)$.

End-of-semester exam

From calculus of variations to special relativity

1. *Geodesics on a cone:* Obtain the geodesics on the surface of a cone which has a semi-vertical angle, say, α. [10 marks]

2. *Invariance of the non-relativistic action for a free particle:* Show that the action for a non-relativistic free particle is invariant under the following Galilean transformations:
$$\mathbf{x} = \mathbf{x}' + \mathbf{V}\, t' \quad \text{and} \quad t = t',$$
where $\mathbf{V}$ is a constant. [10 marks]

3. *Precessing orbits:* Consider a particle that is moving in the following central potential:
$$U(r) = -\frac{\alpha}{r} + \frac{\beta}{r^2},$$
where α and β are positive definite quantities.
   (a) Obtain the orbital equation for the particle. [6 marks]
   (b) Show that, in the case of a bound orbit, for suitably small β, the orbit can be thought of as an ellipse with eccentricity e which is precessing slowly with the angular velocity
$$\omega = \frac{2\,\pi\,\beta}{T\,k\,a\,(1-e^2)},$$
where a denotes the semi-major axis of the ellipse, while T represents the time period along the elliptical orbit. [4 marks]

4. *Time evolution in the Keplerian potential:* Consider a particle that is moving on a parabolic trajectory corresponding to a unit eccentricity in the central potential $U(r) = -\alpha/r$, where $\alpha > 0$. Determine the time dependence of the radial coordinate of the particle. [10 marks]

5. *Phase portraits:* Draw the phase portraits for the following cases:
   (a) A particle moving in the one-dimensional potential
$$U(x) = \alpha\,\left(e^{-2\,\beta\,x} - 2\,e^{-\beta\,x}\right),$$
where $\alpha,\ \beta > 0$. [5 marks]
   (b) In the r-$\dot{r}$ plane, for a particle moving in the Keplerian central potential. [5 marks]

6. *Two Poissons make a Poisson:* Two types of events, say, a and b, are known to be described by Poisson distributions with means λ_a and λ_b, respectively. Consider a total of r events that could be all of type a, or one of type a and the rest of type b, and so on. The total probability for such an event to occur is
$$P(r) = \sum_{r_a=0}^{r} P(r_a, \lambda_a)\; P(r_b = r - r_a, \lambda_b),$$
where $P(r_a, \lambda_a)$ and $P(r_b, \lambda_b)$ are the distributions associated with the two types of events a and b. Show that the total probability distribution $P(r)$ is a Poisson too! [10 marks]

7. *Entropy and temperature of spin $\frac{1}{2}$ systems:* Consider a system of four spin $\frac{1}{2}$ particles, each with magnetic moment, say, μ_0, placed in an external magnetic field of strength B. It is known that the probability of finding the spin of a particle to be either up or down is the same.

(a) Determine the entropy of the system if its energy is found to be $-2\,\mu_0\,B$. $\boxed{\text{5 marks}}$

(b) Estimate the change in the temperature of the system if its energy changes from $-2\,\mu_0\,B$ to zero. Does the temperature of the system increase or decrease? $\boxed{\text{5 marks}}$

8. *Mean energy of a classical oscillator:* Evaluate the mean energy of a classical oscillator that is immersed in a heat bath which is maintained at the inverse temperature β. $\boxed{\text{10 marks}}$

9. *Lorentz invariance of the wave equation:* Show that the following wave equation:

$$\frac{1}{c^2}\,\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi = 0$$

satisfied by, say, light, is invariant under the Lorentz transformations. $\boxed{\text{10 marks}}$

10. *Creation of electron-positron pairs:* A purely relativistic process corresponds to the production of electron-positron pairs in a collision of two high energy gamma ray photons. If the energies of the photons are E_1 and E_2 and the relative angle between their directions of propagation is θ, then, by using the conservation of relativistic energy and momentum, show that the process can occur only if

$$E_1\,E_2 > \left(\frac{2\,m_{\mathrm{e}}^2\,c^4}{1-\cos\theta}\right),$$

where m_{e} is the mass of the electron. $\boxed{\text{10 marks}}$

Note: Recall that, the relativistic energy and three momentum of a particle of mass m are given by $E = \gamma\,m\,c^2$ and $\mathbf{p} = \gamma\,m\,\mathbf{v}$, where $\gamma = \sqrt{1-(|\mathbf{v}|/c)^2}$ and c is the velocity of light. These relations lead to $E^2 - |\mathbf{p}|^2\,c^2 = m^2\,c^4$. Further, for a photon, since its mass is zero, its three momentum can be written in terms of the energy E of the photon as $\mathbf{p} = (E/c)\,\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the direction of propagation of the photon.