PH7080

FOUNDATIONS IN THEORETICAL PHYSICS
ESSENTIAL CLASSICAL ELECTRODYNAMICS

January–May 2025

Lecture schedule

- This part of the course will consist of about 16 lectures including 4 tutorial sessions. However, note that there will be no separate tutorial sessions, and they will be integrated with the lectures.
- The duration of each lecture will be 50 minutes. We will be meeting in HSB 217.
- We will meet twice a week. The lectures are scheduled for 9:00–11:00 AM on Mondays and for 8:00–10:00 AM on Tuesdays.
- Specifically, we will be meeting on February 10, 11, 24, 25 and March 3, 4, 10, 11.

Syllabus and structure

1. **Special relativity** [$\sim$ 2 Lectures]
   (a) The Michelson-Morley interferometric experiment – Postulates of special relativity
   (b) Lorentz transformations – The relativity of simultaneity – Length contraction and time dilation
   (c) Composition law for velocities – Doppler effect

Exercise sheet 1

2. **Four vectors, tensors, and transformations** [$\sim$ 2 Lectures]
   (a) Transformation of coordinates – Four vectors
   (b) Contravariant, covariant and mixed tensors – Elementary operations with tensors
   (c) The Minkowski line-element – Action for the relativistic free particle
   (d) Conservation of relativistic energy and momentum – Elastic collisions of particles

Exercise sheet 2

3. **Charges in electromagnetic fields** [$\sim$ 3 Lectures]
   (a) Four vector potential – Equation of motion of a charge in an electromagnetic field
   (b) Gauge invariance
   (c) Motion in constant, uniform electric and magnetic fields
   (d) The electromagnetic field tensor – Lorentz transformation of the field – Invariants of the field

Exercise sheet 3

4. **The electromagnetic field equations** [$\sim$ 4 Lectures]
   (a) The first pair of Maxwell's equations
   (b) The action for the electromagnetic field
   (c) The four dimensional current vector and the equation of continuity
   (d) The second pair of Maxwell's equations
   (e) Energy density and energy flux – Energy momentum tensor of the electromagnetic field

Exercise sheet 4

5. **Electromagnetic waves and the propagation of light** [$\sim$ 1 Lecture]
   (a) The wave equation – Plane waves – Monochromatic plane waves
   (b) Spectral resolution – Partially polarized light
   (c) The Fourier resolution of the electrostatic field

Exercise sheet 5

6. **Radiation of electromagnetic waves** [$\sim$ 4 Lectures]
   (a) The retarded potentials – The Lienard-Wiechart potentials
   (b) The field of a system of charges at large distances
   (c) Dipole radiation
   (d) Radiation from a rapidly moving charge

Exercise sheet 6

Essential texts

1. L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Course of Theoretical Physics, Volume 2), Fourth Edition (Pergamon Press, New York, 1975).

2. L. D. Landau, E. M. Lifshitz and L. P. Pitaevskii, *Electrodynamics of Continuous Media* (Course of Theoretical Physics, Volume 8), Second Edition (Pergamon Press, New York, 1984).

3. D. J. Griffiths, *Introduction to Electrodynamics*, Third Edition (Prentice Hall of India, New Delhi, 1999).

4. J. D. Jackson, *Classical Electrodynamics*, Third Edition (John Wiley and Sons, Singapore, 1999).

Additional references

1. R. d'Inverno, *Introducing Einstein's Relativity* (Oxford University Press, Oxford, 1992).

2. W. Greiner, *Classical Electrodynamics* (Springer-Verlag, New York, 1998).

3. J. B. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (Pearson Education, Delhi, 2003).

4. M. Longair, *Theoretical Concepts in Physics*, Second Edition (Cambridge University Press, Cambridge, England, 2003).

Exercise sheet 1

Special relativity

1. *Decaying muons:* Muons are unstable and decay according to the radioactive decay law $N = N_0 \exp -(0.693\, t/t_{1/2})$, where N_0 and N are the number of muons at times $t = 0$ and t, respectively, while $t_{1/2}$ is the half life. The half life of the muons in their own rest frame is 1.52×10^{-6} s. Consider a detector on top of a $2,000$ m mountain which counts the number of muons traveling at the speed of $v = 0.98\, c$. Over a given period of time, the detector counts 10^3 muons. When the relativistic effects are taken into account, how many muons can be expected to reach the sea level?

2. (a) *Radial Doppler effect:* Consider a source of photons that is moving radially away with a velocity v from an observer who is at rest. Let ω_{E} and ω_{O} denote the frequency of the photons in the frames of the source and the observer, respectively. Obtain the relation between ω_{E} and ω_{O} in terms of the velocity v.

   (b) *Transverse Doppler effect:* What is the relation between ω_{E} and ω_{O} if the source is moving transversely to the direction of the photon, as it occurs, say, when the source is on a circular trajectory about the observer?

3. *Superluminal motion:* Consider a blob of plasma that is moving at a speed v along a direction that makes an angle θ with respect to the line of sight. Show that the *apparent* transverse speed of the source, projected on the sky, will be related to the actual speed v by the relation

$$v_{\mathrm{app}} = \frac{v \sin\theta}{1 - (v/c)\cos\theta}.$$

   From this expression conclude that the apparent speed v_{app} can exceed the speed of light.

4. *Aberration of light:* Consider two inertial frames S and S', with the frame S' moving along the x-axis with a velocity v with respect to the frame S. Let the velocity of a particle in the frames S and S' be $\mathbf{u}$ and $\mathbf{u}'$, and let θ and θ' be the angles subtended by the velocity vectors with respect to the common x-axis, respectively.

   (a) Show that

$$\tan\theta = \frac{u' \sin\theta'}{\gamma\,[u'\cos\theta' + v]},$$

   where $\gamma = \left[1 - (v/c)^2\right]^{-1/2}$.

   (b) For $u = u' = c$, show that

$$\cos\theta = \frac{\cos\theta' + (v/c)}{1 + (v/c)\cos\theta'}$$

   and

$$\sin\theta = \frac{\sin\theta'}{\gamma\,[1 + (v/c)\cos\theta']}.$$

   (c) For $(v/c) \ll 1$, show that

$$\Delta\theta = (v/c)\sin\theta',$$

   where $\Delta\theta = (\theta' - \theta)$.

5. *Form invariance of the Minkowski line element:* Show that the following Minkowski line element is invariant under the Lorentz transformations:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}\boldsymbol{x}^2.$$

Exercise sheet 2

Four vectors, tensors, and transformations

1. *Transforming four vectors and invariance under Lorentz transformations:* Consider two inertial frames K and K', with K' moving with respect to K, say, along the common x-axis with a certain velocity.
   (a) Given a four vector A^μ in the K frame, construct the corresponding contravariant and covariant four vectors, say, $A^{\mu\prime}$ and A'_μ, in the K' frame.
   (b) Explicitly illustrate that the scalar product $A_\mu\, A^\mu$ is a Lorentz invariant quantity, i.e. show that $A_\mu\, A^\mu = A'_\mu\, A^{\mu\prime}$.

2. *Four velocity and four acceleration:* The four acceleration of a relativistic particle is defined as $a^\mu = \mathrm{d}u^\mu/\mathrm{d}s$, where $u^\mu = \mathrm{d}x^\mu/\mathrm{d}s$ is the four velocity of the particle.
   (a) Express a^μ in terms of the three velocity $\boldsymbol{v}$ and the three acceleration $\boldsymbol{a} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t$ of the particle.
   (b) Evaluate $a^\mu\, u_\mu$ and $a^\mu\, a_\mu$ in terms of $\boldsymbol{v}$ and $\boldsymbol{a}$.

3. *Compton effect using four vectors:* Consider the scattering between a photon of frequency ω and a relativistic electron with velocity $\mathbf{v}$ leading to a photon of frequency ω' and electron with velocity $\mathbf{v}'$. Such a scattering is known as Compton scattering. Let α be the angle between the incident and the scattered photon. Also, let θ and θ' be the angles subtended by the directions of propagation of the incident and the scattered photon with the velocity vector of the electron before the collision.
   (a) Using the conservation of four momentum, show that
   $$\frac{\omega'}{\omega} = \frac{1-(v/c)\cos\theta}{1-(v/c)\cos\theta' + (\hbar\,\omega/\gamma\, m_{\mathrm{e}}\, c^2)\,(1-\cos\alpha)},$$
   where $\gamma = \left[1-(v/c)^2\right]^{-1/2}$ and m_{e} is the mass of the electron.
   (b) When $\hbar\,\omega \ll \gamma\, m_{\mathrm{e}}\, c^2$, show that the frequency shift of the photon can be written as
   $$\frac{\Delta\omega}{\omega} = \frac{(v/c)\ (\cos\theta - \cos\theta')}{1-(v/c)\cos\theta'},$$
   where $\Delta\omega = (\omega' - \omega)$.

4. *Creation of electron-positron pairs:* A purely relativistic process corresponds to the production of electron-positron pairs in a collision of two high energy gamma ray photons. If the energies of the photons are ϵ_1 and ϵ_2 and the relative angle between their directions of propagation is θ, then, by using the conservation of energy and momentum, show that the process can occur only if
   $$\epsilon_1\,\epsilon_2 > \frac{2\, m_{\mathrm{e}}^2\, c^4}{1-\cos\theta},$$
   where m_{e} is the mass of the electron.

5. *Mirrors in motion:* A mirror moves with the velocity v in a direction perpendicular its plane. A ray of light of frequency ν_1 is incident on the mirror at an angle of incidence θ, and is reflected at an angle of reflection ϕ and frequency ν_2.
   (a) Show that
   $$\frac{\tan(\theta/2)}{\tan(\phi/2)} = \frac{c+v}{c-v} \qquad \text{and} \qquad \frac{\nu_2}{\nu_1} = \frac{c+v\cos\theta}{c-v\cos\phi}.$$
   (b) What happens if the mirror was moving parallel to its plane?

Exercise sheet 3

Charges in electromagnetic fields

1. <u>*The Lorentz force:*</u> The action for a relativistic particle that is interacting with the electromagnetic field is given by

$$S[x^\mu(s)] = -m\,c \int \mathrm{d}s - \frac{e}{c} \int \mathrm{d}x_\mu\, A^\mu,$$

where m is the mass of the particle, while e is its electric charge. The quantity $A^\mu = (\phi, \boldsymbol{A})$ is the four vector potential that describes the electromagnetic field, with, evidently, ϕ and $\mathbf{A}$ being the conventional scalar and three vector potentials.

(a) Vary the above action with respect to x^μ to arrive at the following Lorentz force law:

$$m\,c\,\frac{\mathrm{d}u^\mu}{\mathrm{d}s} = \frac{e}{c}\, F^{\mu\nu}\, u_\nu,$$

where u^μ is the four velocity of the particle and the electromagnetic field tensor $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} = \partial_\mu\, A_\nu - \partial_\nu\, A_\mu$$

with $\partial_\mu \equiv \partial/\partial x^\mu$.

(b) Show that the components of the field tensor $F_{\mu\nu}$ are given by

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix},$$

where (E_x, E_y, E_z) and (B_x, B_y, B_z) are the components of the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ which are related to the components of the four vector potential by the following standard expressions:

$$\boldsymbol{E} = -\frac{1}{c}\,\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi \qquad \text{and} \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A}.$$

(c) Express the above equation governing the motion of the charge in the more familiar three vector notation. What does the zeroth component of the equation describe?

2. <u>*Electron in an electric field:*</u> An electron moving relativistically enters a region of constant electric field that is pointed along the positive y-axis. Let the relativistic three-momentum of the electron as it enters the region of the electric field at time, say, $t = 0$, be $\boldsymbol{p} = (p_x^0, p_y^0, 0)$.

(a) Integrate the equation of motion describing the electron to determine p_x and p_y as function of time. Express the energy of the electron in terms of $p_x(t)$ and $p_y(t)$.

(b) From the equation governing the conservation of energy of the electron and the above expression for energy, arrive at the expression for v_y in terms of time. Using the above results, also arrive at the expression for v_x in terms of time. Determine the asymptotic (i.e. the large time) behavior of v_x and v_y.

Hint: It is useful to note that we can write $\boldsymbol{v} = \mathrm{d}\boldsymbol{x}/\mathrm{d}t = \boldsymbol{p}\,c^2/\mathcal{E}$, where $\mathcal{E}/c = \sqrt{\boldsymbol{p}^2 + m^2\,c^2}$.

(c) Arrive at $y(t)$. Assuming that $p_x^0 = p_y^0 = 0$, plot the trajectory of the particle in the $c\,t$-y plane.

3. <u>*Motion in a constant and uniform magnetic field:*</u> Consider a particle of mass m and charge e that is moving in a magnetic field of strength B that is directed, say, along the positive z-axis.

(a) Show that the energy $\mathcal{E} = \gamma\, m\, c^2$ of the particle is a constant.

(b) Determine the trajectory $\boldsymbol{x}(t)$ of the particle and show that, in the absence of any initial momentum along the z-direction, the particle describes a circular trajectory in the x-y plane with the angular frequency $\omega = e\,c\,B/\mathcal{E}$.

4. *Transformation of electric and magnetic fields:* Consider two inertial frames, say, K and K', with the frame K' moving with a velocity v with respect to the frame K along the common x-axes.

   (a) Given the components of the electric and the magnetic fields, say, $\boldsymbol{E}$ and $\boldsymbol{B}$, in the frame K, using the transformation properties of the electromagnetic field tensor $F_{\mu\nu}$, construct the corresponding components in the frame K'.

   (b) Show that $|\boldsymbol{E}|^2 - |\boldsymbol{B}|^2$ is invariant under the Lorentz transformations.

   (c) Express the quantity $|\boldsymbol{E}|^2 - |\boldsymbol{B}|^2$ explicitly as a scalar in terms of the field tensor $F_{\mu\nu}$.

5. *Invariance of the action under gauge transformations:* Show that the action for a relativistic particle that is interacting with the electromagnetic field, viz.

$$S[x^\mu(s)] = -m\,c \int \mathrm{d}s - \frac{e}{c} \int \mathrm{d}x_\mu\, A^\mu,$$

is invariant under gauge transformations of the form $A^\mu \to A^\mu + \partial^\mu \chi$, where χ is a scalar function.

Exercise sheet 4

Theory of the electromagnetic field

1. *Equivalence of actions under gauge transformations:* Recall that the action governing the electromagnetic field described by the vector potential A_μ that is interacting with the four current j^μ is given by

$$S[A^\mu(\tilde{x})] = -\frac{1}{c^2}\int \mathrm{d}^4\tilde{x}\, j^\mu\, A_\mu - \frac{1}{16\,\pi\,c}\int \mathrm{d}^4\tilde{x}\, F_{\mu\nu}\, F^{\mu\nu},$$

where $F_{\mu\nu}$ is the field tensor defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Since $F_{\mu\nu}$ is explicitly invariant under the gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \Lambda,$$

evidently, the second term in the above action is invariant as well. Determine if the first term transforms to an equivalent action under the gauge transformation.

2. *The spatial components of the stress-energy tensor of the free electromagnetic field:* We had arrived at the forms of the time-time and the time-space components of the stress-energy tensor of the free electromagnetic field in terms of the components of the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$. Arrive at the corresponding expressions for the purely spatial components of the stress-energy tensor.

   Note: These components are usually referred to as the Maxwell stress tensor.

3. *From source free Maxwell's equations to the conservation of the stress-energy tensor:* Establish that the source free Maxwell's equations imply that the stress-energy tensor of the free electromagnetic field is conserved.

4. *Conservation of the stress-energy tensor in the presence of sources:* The above exercise had involved the electromagnetic field in the absence of charges. If the charges are also present, then it is the sum of the stress-energy tensors of the charges as well as the field that will be conserved. The stress-energy tensor of a collection of mutually non-interacting particles can be written as

$$T^{\mu\nu}_{\mathrm{P}} = \mu\, c\, u^\mu\, u^\nu\, \frac{\mathrm{d}s}{\mathrm{d}t},$$

where μ is the mass density associated with the particles, while u^μ denotes the four velocity of the particles.

   Note: The above expression for the stress-energy tensor for a collection of mutually non-interacting particles is equivalent to a pressureless relativistic fluid. Often, such a system is referred to as 'dust'.

   (a) Show that, upon using the second pair Maxwell's equations, in the presence of sources, the stress-energy of the electromagnetic field, say, $T^{\mu\nu}_{\mathrm{F}}$, satisfies the equation

$$\partial_\mu T^{\mu\nu}_{\mathrm{F}} = -\frac{1}{c}\, F^{\nu\lambda}\, j_\lambda.$$

   (b) As in the case of charges, the continuity equation corresponding to the mass flow can be expressed as follows:

$$\partial_\mu\left(\mu\,\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\right) = 0.$$

Using this equation and the following Lorentz force law:

$$\mu\, c\, \frac{\mathrm{d}u^\mu}{\mathrm{d}s} = \frac{\rho}{c}\, F^{\mu\nu}\, u_\nu,$$

where ρ denotes the charge density of the particles, show that

$$\partial_\mu T^{\mu\nu}_{\mathrm{P}} = \frac{1}{c}\, F^{\mu\lambda}\, j_\lambda$$

so that the total stress-energy tensor of the system, viz. $T^{\mu\nu} = T^{\mu\nu}_{\mathrm{P}} + T^{\mu\nu}_{\mathrm{F}}$, is conserved, as required.

5. *Traceless nature of the stress-energy tensor of the electromagnetic field:* Show that the trace of the stress-energy tensor of the electromagnetic field vanishes. Can you identify the reason behind the vanishing trace?