PH7080

FOUNDATIONS IN THEORETICAL PHYSICS

January–May 2026

Lecture schedule

- The course will consist of about 56 lectures including about 12 tutorial sessions. However, note that there will be no separate tutorial sessions, and they will be integrated with the lectures.
- We will meet thrice a week. The lectures are scheduled for 10:00–11:50 AM on Mondays, 9:00–10:50 AM on Tuesdays, and 8:00–9:50 AM on Wednesdays. We will be meeting in HSB 263.
- The first lecture will be on Monday, February 2, 2026, and the last lecture will be on or before Wednesday, April 29, 2026.
- Changes in schedule, if any, will be notified sufficiently in advance.
- If you would like to discuss with me about the course outside the lecture hours, you are welcome to meet me at my office (HSB 202A) during 5:00–6:00 PM on Fridays. In case you are unable to find me in my office on more than occasion, please send me an e-mail at `sriram@physics.iitm.ac.in`.

Information about the course

- Information, such as the schedule of the lectures, the structure and the syllabus of the course, suitable textbooks and additional references, as well as exercise sheets, will be available on Moodle at the following URL:
  `https://courses.iitm.ac.in/.`
- A PDF file containing these information as well as completed quizzes will also be available at the link on this course at the following URL:
  `http://www.physics.iitm.ac.in/~sriram/professional/teaching/teaching.html.`
  I will keep updating the file as we make progress.

Quizzes, end-of-semester exam and grading

- The grading will be based on two scheduled quizzes and an end-of-semester exam.
- The quizzes will be held on Fridays, March 6, 2026, and April 10, 2026. They will be for a duration of 1.5 hours and will carry 25% weight each.
- The end-of-semester exam will be on Friday, May 8, 2026. The exam will be for a duration of 3 hours and will carry 50% weight.

Syllabus and structure

Classical Mechanics

1. **Calculus of variations and Lagrangian formulation of mechanics** [$\sim$ 3 lectures]
   (a) Concept of variation – Euler equation
   (b) Degrees of freedom – Generalized coordinates and velocities
   (c) Principle of least action – Lagrange equations of motion
   (d) Inertial frames of reference – Newton's first law – Galileo's relativity principle – Galilean transformations
   (e) Lagrangian for a free particle and a system of particles – Equations of motion in an external field – Newton's second law

Exercise sheet 1

2. **Symmetries and conservation laws** [$\sim$ 2 lectures]
   (a) Homogeneity of space and time – Conservation of energy and momentum – Newton's third law – Center of mass
   (b) Isotropy of space – Conservation of angular momentum
   (c) Mechanical similarity – Virial theorem

Exercise sheet 2

3. **Integration of the Lagrange equations of motion** [$\sim$ 5 lectures]
   (a) Motion in one dimension – Determination of the potential energy from the period of oscillation
   (b) Two body problem – Reduced mass and the equivalent one-dimensional problem – Motion in a central field – Kepler's second law
   (c) Kepler problem – Kepler's first and third laws – Motion in a repulsive field – Laplace-Runge-Lenz vector
   (d) Scattering in a central field – Rutherford's formula – Total cross section

Exercise sheets 3 and 4

4. **Small oscillations** [$\sim$ 2 lectures]
   (a) Free, damped and forced oscillations in one dimension – Resonance – Parametric resonance
   (b) Oscillations of systems with more than one degree of freedom – Vibrations of molecules

Exercise sheet 5

5. **Hamiltonian formulation** [$\sim$ 4 lectures]
   (a) Conjugate momentum – Legendre's transformation – Hamiltonian and Hamilton's equations
   (b) Poisson brackets – Poisson's theorem
   (c) Canonical transformations – Invariance of Poisson brackets under canonical transformations
   (d) Phase space – Dynamics in the phase space – Phase portraits – Liouville's theorem

Exercise sheet 6

Classical Electrodynamics

6. **Special relativity** [$\sim$ 2 Lectures]
   (a) The Michelson-Morley interferometric experiment – Postulates of special relativity
   (b) Lorentz transformations – The relativity of simultaneity – Length contraction and time dilation
   (c) Composition law for velocities – Doppler effect

Exercise sheet 7

7. **Four vectors, tensors, and transformations** [$\sim$ 2 Lectures]
   (a) Transformation of coordinates – Four vectors
   (b) Contravariant, covariant and mixed tensors – Elementary operations with tensors
   (c) The Minkowski line-element – Action for the relativistic free particle
   (d) Conservation of relativistic energy and momentum – Elastic collisions of particles

Exercise sheet 8
Quiz I

8. **Charges in electromagnetic fields** [$\sim$ 3 Lectures]
   (a) Four vector potential – Equation of motion of a charge in an electromagnetic field
   (b) Gauge invariance
   (c) Motion in constant, uniform electric and magnetic fields
   (d) The electromagnetic field tensor – Lorentz transformation of the field – Invariants of the field

Exercise sheet 9

9. **The electromagnetic field equations** [$\sim$ 4 Lectures]
   (a) The first pair of Maxwell's equations
   (b) The action for the electromagnetic field
   (c) The four dimensional current vector and the equation of continuity
   (d) The second pair of Maxwell's equations
   (e) Energy density and energy flux – Energy momentum tensor of the electromagnetic field

Exercise sheet 10

10. **Electromagnetic waves and the propagation of light** [$\sim$ 1 Lecture]
    (a) The wave equation – Plane waves – Monochromatic plane waves
    (b) Spectral resolution – Partially polarized light
    (c) The Fourier resolution of the electrostatic field

Exercise sheet 11

11. **Radiation of electromagnetic waves** [$\sim$ 4 Lectures]
    (a) The retarded potentials – The Lienard-Wiechart potentials
    (b) The field of a system of charges at large distances
    (c) Dipole radiation
    (d) Radiation from a rapidly moving charge

Exercise sheet 12

Quantum Mechanics

12. **Postulates of quantum mechanics and the Schrodinger equation** [$\sim$ 2 lectures]
    (a) Observables and operators
    (b) Expectation values and fluctuations
    (c) Measurement and the collapse of the wave function
    (d) Time-dependent Schrodinger equation

Exercise sheet 13

13. **Time-independent Schrodinger equation in one dimension** [$\sim$ 4 lectures]
    (a) The time-independent Schrodinger equation – Stationary states
    (b) The infinite square well
    (c) Reflection and transmission in potential barriers
    (d) The delta function potential
    (e) The free particle
    (f) Linear harmonic oscillator

Exercise sheet 14

14. **Essential mathematical formalism** [$\sim$ 4 lectures]
    (a) Hilbert space
    (b) Observables – Hermitian operators – Eigen functions and eigen values of hermitian operators
    (c) Orthonormal basis – Expansion in terms of a complete set of states
    (d) Position and momentum representations
    (e) Generalized statistical interpretation – The generalized uncertainty principle
    (f) Studying the simple harmonic oscillator using the operator method
    (g) Unitary evolution

Exercise sheets 15 and 16

15. **Schrodinger equation in three dimensions and particle in a central potential** [$\sim$ 3 lectures]
    (a) The Schrodinger equation in three dimensions
    (b) Particle in a three-dimensional box – The harmonic oscillator in three-dimensions
    (c) Motion in a central potential – Orbital angular momentum
    (d) Hydrogen atom – Energy levels
    (e) Degeneracy

Exercise sheet 17

16. **Angular momentum and spin** [$\sim$ 3 lectures]
    (a) Angular momentum – Eigen values and eigen functions
    (b) Electron spin – Pauli matrices
    (c) Application to magnetic resonance

Exercise sheet 18

Essential texts

Classical mechanics

1. L. D. Landau and E. M. Lifshitz, *Mechanics*, Course of Theoretical Physics, Volume 1, Third Edition (Pergamon Press, New York, 1976).
2. D. Kleppner and R. J. Kolenkow, *An Introduction to Mechanics* (Tata McGraw-Hill, New Delhi, 1999).
3. H. Goldstein, C. Poole and J. Safko, *Classical Mechanics*, Third Edition (Pearson Education, Singapore, 2002).
4. S. T. Thornton and J. B. Marion, *Classical Dynamics of Particles and Systems* (Cengage Learning, Singapore, 2004).
5. T. W. B. Kibble, *Classical Mechanics*, Fifth Edition (Imperial College Press, London, 2004).
6. D. Morin, *Introduction to Classical Mechanics* (Cambridge University Press, Cambridge, England, 2008).

Classical Electrodynamics

1. L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Course of Theoretical Physics, Volume 2), Fourth Edition (Pergamon Press, New York, 1975).
2. L. D. Landau, E. M. Lifshitz and L. P. Pitaevskii, *Electrodynamics of Continuous Media* (Course of Theoretical Physics, Volume 8), Second Edition (Pergamon Press, New York, 1984).
3. D. J. Griffiths, *Introduction to Electrodynamics*, Third Edition (Prentice Hall of India, New Delhi, 1999).
4. J. D. Jackson, *Classical Electrodynamics*, Third Edition (John Wiley and Sons, Singapore, 1999).

Quantum Mechanics

1. P. A. M. Dirac, *The Principles of Quantum Mechanics*, Fourth Edition (Oxford University Press, Oxford, 1958).
2. S. Gasiorowicz, *Quantum Physics*, Third Edition (John Wiley and Sons, New York, 2003).
3. R. L. Liboff, *Introductory Quantum Mechanics*, Fourth Edition (Pearson Education, Delhi, 2003).
4. W. Greiner, *Quantum Mechanics*, Fourth Edition (Springer, Delhi, 2004).
5. D. J. Griffiths, *Introduction to Quantum Mechanics*, Second Edition (Pearson Education, Delhi, 2005).
6. R. W. Robinett, *Quantum Mechanics*, Second Edition (Oxford University Press, Oxford, 2006).
7. R. Shankar, *Principles of Quantum Mechanics*, Second Edition (Springer, Delhi, 2008).

Additional references

Classical Mechanics

1. D. T. Greenwood, *Principles of Dynamics*, Second Edition (Prentice-Hall of India, New Delhi, 1988).
2. L. N. Hand and J. D. Finch, *Analytical Mechanics* (Cambridge University Press, Cambridge, 1998).
3. W. Greiner, *Classical Mechanics: Systems of Particles and Hamiltonian Dynamics* (Springer-Verlag, New York, 2003).

Classical Electrodynamics

1. R. d'Inverno, *Introducing Einstein's Relativity* (Oxford University Press, Oxford, 1992).
2. W. Greiner, *Classical Electrodynamics* (Springer-Verlag, New York, 1998).
3. J. B. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (Pearson Education, Delhi, 2003).
4. M. Longair, *Theoretical Concepts in Physics*, Second Edition (Cambridge University Press, Cambridge, England, 2003).

Quantum Mechanics

1. L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Course of Theoretical Physics, Volume 3), Third Edition (Pergamon Press, New York, 1977).
2. J. J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley, Singapore, 1994).

Exercise sheet 1

Calculus of variations and Lagrangian formulation of mechanics

1. *Snell's law:* Two homogeneous media of refractive indices n_1 and n_2 are placed adjacent to each other. A ray of light propagates from a point in the first medium to a point in the second medium. According to the Fermat's principle, the light ray will follow a path that minimizes the transit time between the two points. Use Fermat's principle to derive the Snell's law of refraction, viz. that

$$n_1 \sin\theta_1 = n_2 \sin\theta_2,$$

where θ_1 and θ_2 are the angles of incidence and refraction at the interface.

Note: As the complete path is not differentiable at the interface, the problem is not an Euler equation problem.

2. *Brachistochrone problem:* Consider a particle that is moving in a constant force field starting at rest from some point to a lower point. Determine the path that allows the particle to accomplish the transit in the least possible time.

Note: The resulting curve is referred to as the brachistochrone, i.e. the curve of the fastest descent.

3. *Lagrangian of a free particle in different coordinate systems:* Write down the Lagrangian and the equations of motion for a free particle in (a) Cartesian, (b) cylindrical and (c) spherical polar coordinates.

4. *Behavior of Euler-Lagrange equation under point transformations:* Let $(q_1, q_2, \ldots, q_n)$ be a set of independent generalized coordinates for a system with n degrees of freedom. Let the system be described by the Lagrangian $L(q_i, \dot{q}_i, t)$ and let us transform to a new set of independent coordinates $(s_1, s_2, \ldots, s_n)$ by means of the transformation equations

$$q_i = q_i\left(s_1, s_2, \ldots, s_n, t\right), \text{ for } i = 1, 2, \ldots, n.$$

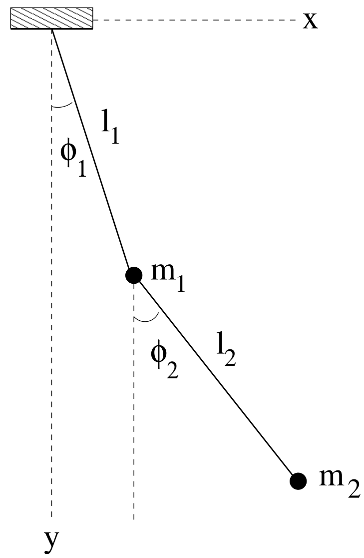
Such a transformation is known as a point transformation. Show that, if the Lagrangian $L(q_i, \dot{q}_i, t)$ is expressed as a function of s_j, $\dot{s}_j$ and t through a point transformation, then the system satisfies the Lagrange's equations with respect to the new coordinates, viz.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{s}_j}\right) - \left(\frac{\partial L}{\partial s_j}\right) = 0, \text{ for } j = 1, 2, \ldots, n.$$

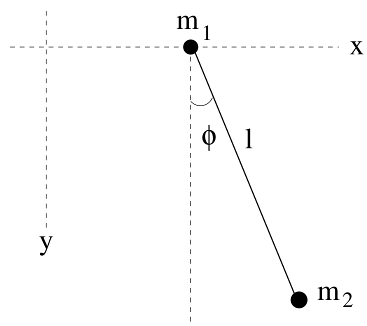
Note: This implies that the form of the Lagrange's equations remain invariant under a point transformation.

5. *Lagrangian for different systems:* Construct the Lagrangian and obtain the equations of motion for the following systems when they are placed in a uniform gravitational field (corresponding to an acceleration g)

   (a) a coplanar double pendulum as shown in the figure below,

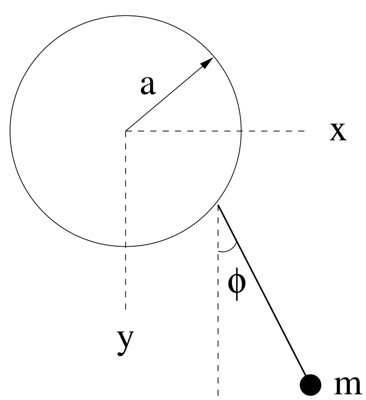


(b) a simple pendulum of mass m_2, with a mass m_1 at the point of support which can move on a horizontal line lying in the plane in which m_2 moves as shown in the figure below,



(c) a simple pendulum of mass m whose point of support

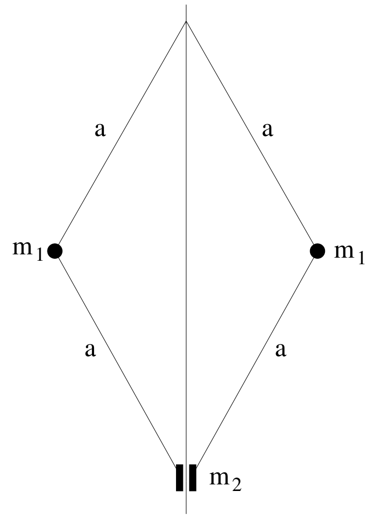
i. moves uniformly on a vertical circle with constant frequency γ as shown in the figure below,



ii. oscillates horizontally in the plane of motion of the pendulum according to the law $x = a\cos(\gamma\, t)$,

iii. oscillates vertically according to the law $y = a\cos(\gamma\, t)$,

(d) In the system shown in the figure below, the mass m_2 moves on a vertical axis and the whole system rotates about this axis with a constant angular velocity Ω.


a
a
m1
m1
a
a
m2

Exercise sheet 2

Symmetries and conservation laws

1. *Utilizing conservation of energy and momentum:* Consider a region of space divided by a plane. The potential energy of a particle on one half of the plane is U_1 and, on the other half, it is U_2, where U_1 and U_1 are constants. A particle of mass m which has a speed'v_1 moves from the first half to the second. If θ_1 and θ_2 are the angles subtended by the trajectory of particle with respect to the normal on either side of the plane, show that

$$\frac{\sin\theta_1}{\sin\theta_2} = \left(1 + \frac{U_1 - U_2}{m\,v_1^2/2}\right)^{1/2}.$$

   What is the optical analog of the problem?

2. *Components and magnitude of angular momentum:* Obtain the expressions for the Cartesian components and the magnitude of the angular momentum of a particle in the cylindrical and the spherical polar coordinates.

3. *Virial theorem:* Given a function $f(t)$, the average value of the function is defined as

$$\langle f(t)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{d}t\, f(t).$$

   Consider a system in which the forces acting on the particles consist of conservative forces $\boldsymbol{F}_i$, determined by the potential $U(\boldsymbol{r}_i)$. Show that, for such a system, the so-called virial theorem holds in the form

$$2\,\langle T\rangle = -\left\langle \sum_i \boldsymbol{r}_i \cdot \boldsymbol{F}_i \right\rangle = \left\langle \sum_i \boldsymbol{r}_i \cdot \frac{\partial U}{\partial \boldsymbol{r}_i} \right\rangle.$$

4. *Perfect gas law:* A perfect gas is defined as one in which the forces of interaction between the molecules of the gas are negligible. This occurs, for example, when the gas is so dilute that the collisions between the molecules are rare when compared to the collisions with the walls of the container. Using these information and the virial theorem, obtain the perfect gas law, viz.

$$P\,V = N\,k_{\mathrm{B}}\,T,$$

   where P, V, T and N denote the pressure, the volume, the temperature and the number of molecules of the gas, respectively, and k_{B} is the Boltzmann constant.

5. *An application of the virial theorem:* Using virial theorem, show that the total mass M of a spherical cluster of stars (or galaxies) of uniform density and radius R is given by

$$M = \frac{5\,R\,\langle v^2\rangle}{3\,G},$$

   where $\langle v^2\rangle$ is the mean-squared velocity of the individual stars and G is, of course, the gravitational constant.

   Note: The above relation allows us to obtain an estimate of the mass of a cluster of stars or galaxies if we can measure the mean-squared velocity, say, from the Doppler spread of the spectral lines and the radius of the cluster, say, from its known distance and angular size.

Exercise sheet 3

Integration of the Lagrange equations of motion I

1. *Period of oscillation:* Determine the period of oscillation as a function of the energy, say, E, when a particle of mass m is moving in one dimension (along the x-axis) under the influence of the following potentials: (a) $U = A\,|x|^{2\,n}$ (with $A > 0$ and $n > 0$), (b) $U(x) = -U_0/\cosh^2(\alpha\,x)$ (for $-U_0 < E < 0$), and (c) $U(x) = U_0 \tan^2(\alpha\,x)$.

2. *Explicitly establishing the virial theorem:* Evaluate the time averages (i.e. the average over one complete period) of the kinetic and potential energies, say, T and U, for a particle that is moving along an elliptic orbit in the Keplerian central potential, and establish the virial theorem for the case, viz. that $2\,\langle T\rangle = -\langle U\rangle$, where the angular brackets denote the averages.

3. *Time evolution in the Keplerian potential:* Consider a particle of reduced mass m that is moving on a hyperbolic trajectory in the central potential $U(r) = -\alpha/r$, where $\alpha > 0$. Show that the time evolution of the trajectory can be parametrically expressed as follows:

$$r = a\,(e\cosh\xi - 1)\,,\ t = \sqrt{m\,a^3/\alpha}\,(e\sinh\xi - \xi)$$

or, equivalently, as

$$x = a\,(e - \cosh\xi)\,,\ y = a\,\sqrt{e^2-1}\,\sinh\xi,$$

where the quantity a and the eccentricity of the orbit e are given in terms of the energy $E > 0$ and the angular momentum ℓ by the relations

$$a = \frac{\alpha}{2\,E},\ e = \sqrt{1 + \frac{2\,E\,\ell^2}{m\,\alpha^2}},$$

while $-\infty < \xi < \infty$.

4. *Laplace-Runge-Lenz vector:* Recall that, for a particle with s degrees of freedom, we require $2\,s-1$ constants of motion in order to arrive at a unique trajectory for the particle. According to this argument, for the Kepler problem, we would then need five integrals of motion to obtain the solution. We had expressed the solution in terms of the energy E of the system and the amplitude of the angular momentum vector $\boldsymbol{L}$, both of which were conserved. However, these quantities, viz. the energy E and the three components of the angular momentum vector $\boldsymbol{L}$, only add up to four constants of motion. Evidently, it will be interesting to examine if we can identify the fifth integral of motion associated with the system.

   (a) Show that, for a particle moving in the Keplerian central potential, i.e. $U(r) = -\alpha/r$ with $\alpha > 0$, the following vector is an integral of motion:

   $$\boldsymbol{A} = m\,\boldsymbol{v}\times\boldsymbol{L} - \frac{m\,\alpha\,\boldsymbol{r}}{r}.$$

   Note: The conserved vector $\boldsymbol{A}$ is known as the Laplace-Runge-Lenz vector.

   (b) Show that the vector $\boldsymbol{A}$ lies in the plane of the orbit.

   (c) Indicate the amplitude and the direction of $\boldsymbol{A}$ associated with a planet as it moves in an elliptical orbit around the Sun.

   Hint: Determine the amplitude and the direction of $\boldsymbol{A}$ at, say, the perihelion and the aphelion.

   (d) If E, $\boldsymbol{L}$ and $\boldsymbol{A}$ are all constants, then, we seem to have seven integrals of motion instead of the required five to arrive at a unique solution! How does seven reduce to five?

   Hint: Examine if there exist any relations between $\boldsymbol{A}$ and $\boldsymbol{L}$ and/or E.

5. *Passing through the centre:* Consider a particle that describes a circular orbit under the influence of an attractive central force directed towards a point on the circle.

(a) Show that the force varies as the inverse-fifth power of the distance.

(b) Also, show that for the orbit described the total energy of the particle is zero.

(c) Moreover, obtain the period of the motion.

Exercise sheet 4

Integration of the Lagrange equations of motion II

1. *Falling onto each other:* Consider two particles which are moving under the influence of their mutual gravitational force. Let the particles follow circular orbits about one another with a time period T. Show that, if the particles are suddenly stopped in their orbits and allowed to gravitate towards each other, they will collide after a time $T/(4\sqrt{2})$.

2. *Precessing orbits:* Show that the motion of a particle in the central potential

$$U(r) = -\frac{\alpha}{r} + \frac{h}{r^2},$$

with $\alpha > 0$, is the same as that of motion under the Kepler potential alone when expressed in terms of a coordinate system rotating or precessing around the center of the force.

3. *Precession of the perihelion of Mercury:* When a small correction $\delta U(r)$ is added to the potential energy $U = -\alpha/r$, with $\alpha > 0$, the paths of finite motion are no longer closed and in each revolution the perihelion is displaced through a small angle, say, $\delta\phi$.

   (a) Find $\delta\phi$ when: (i) $\delta U = \beta/r^2$ and (ii) $\delta U = \gamma/r^3$. Compare the result for the first case with the result from the previous exercise.

   (b) For the case of the planet Mercury, the value of the semi-major axis of its orbit is $a = 5.79 \times 10^{10}$ m and its eccentricity is $e = 0.206$. Also, the period of the Mercury's orbit around the Sun is 88 days. Further, the value of the gravitational constant is $G = 6.673 \times 10^{-11}\,\mathrm{kg^{-1}\,m^3\,s^{-2}}$, the speed of light is $c = 2.998 \times 10^8$ m/s, and the mass of the Sun is $M_\odot = 1.989 \times 10^{30}$ kg. Estimate the extent of precession of Mercury's orbit if $\delta U = -\alpha\,\ell^2/(m^2\,c^2\,r^3)$, where $\alpha = G\,M\,m$, ℓ is the angular momentum, and c is the velocity of light.

   Note: The δU mentioned above arises due to general relativistic effects. The measured precession of the perihelion of the planet Mercury turns out to be $5599''.7 \pm 0''.4$ per century, but a large part of it is caused due to the influences of the other planets. When the other contributions have been subtracted, the precession of the perihelion of the planet Mercury due to the purely general relativistic effects amounts to 43.1 ± 0.5 seconds of arc per century.

4. *Motion in the Yukawa potential:* A particle moves in the Yukawa potential described by

$$V(r) = -\frac{k}{r}\,\exp -(r/a),$$

where k and a are positive quantities.

   (a) Write down the equations of motion and reduce them to the equivalent one-dimensional problem.

   (b) Use the effective potential to discuss the qualitative nature of the orbits for different values of the energy and the angular momentum.

   (c) Also, show that if the orbit is nearly circular, the apsides will advance approximately by $\pi\,\rho/a$ per revolution, where ρ is the radius of the circular orbit.

   Note: The radial distances to the turning points are known as apsidal distances.

5. (a) *Scattering by a rigid sphere:* Consider scattering by a perfectly rigid sphere of radius a, i.e. when the interaction is such that $U(r) = \infty$ for $r < a$ and $U(r) = 0$ for $r > a$. Determine the effective cross-section for the scattering of particles by the rigid sphere. What is the total cross-section?

   (b) *Rutherford scattering in the centre-of-mass frame:* Consider scattering by the repulsive potential $U(r) = \alpha/r$ with $\alpha > 0$.

i. Find the effective cross-section for scattering.

ii. What is the total cross-section? What does the result imply?

Note: This corresponds to the famous experiment by Rutherford wherein α-particles were scattered by atoms in a gold foil.

Exercise sheet 5

Small oscillations

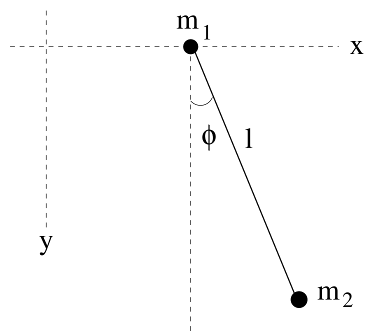
1. (a) *Frequency of oscillations I:* Find the frequency of oscillations of a particle of mass m which is attached to a spring whose other end is fixed at a distance l and that is

   i. Free to move along a line,

   ii. Moving on a circle of radius r.

   Assume that a force F is required to extend the spring to the length l.-

   (b) *Frequency of oscillations II:* Find the frequency of small oscillations of a pendulum whose point of support carries a mass m_1 and is free to move horizontally as shown in the figure on the left below.



2. *Forced oscillations:* Assuming that, at time $t = 0$, a system is at rest in equilibrium (i.e. $x(0) = \dot{x}(0) = 0$), determine the forced oscillations of the system under a force $F(t)$ of the following forms: (a) $F = F_0$, (b) $F = a\,t$, (c) $F = F_0\, \exp -(\alpha\, t)$, and (d) $F = F_0\, \exp -(\alpha\, t)\, \cos(\beta\, t)$, with F_0, a, α and β being constants.

3. *Final amplitude:* Assuming that a system is at rest in equilibrium up to time $t = 0$, determine the final amplitude for the oscillations of the system under: (a) a force which is zero for $t < 0$, $F_0\, t/T$ for $0 < t < T$, and F_0 for $t > T$, (b) a constant force F_0 that acts for a finite time T, and (c) forces (i) $F_0\, t/T$ and (ii) $F_0\, \sin(\omega t)$, which act between $t = 0$ and $t = T$.

4. *System with two degrees of freedom:* Determine the oscillations of a system with two degrees of freedom (say, x and y) whose Lagrangian is given by

$$L = \frac{1}{2}\,(\dot{x}^2 + \dot{y}^2) - \frac{{\omega_0}^2}{2}\,(x^2 + y^2) + \alpha\, x\, y.$$

5. *Forced oscillations in the presence of friction:* Determine the forced oscillations due to the external force $F = F_0\, \exp(\alpha\, t)\, \cos(\gamma\, t)$ in the presence of friction.

Exercise sheet 6

Hamiltonian formulation

1. *Bead on a wire:* A bead slides without friction down a wire that has the shape $y = f(x)$ in a uniform gravitational field (corresponding to an acceleration g).

   (a) Obtain the Hamiltonian of the system.

   (b) Also, write down the Hamilton's equation of motion.

2. *A complicated Lagrangian:* Consider a system described by the Lagrangian

$$L = a\,\dot{x}^2 + b\,\frac{\dot{y}}{x} + c\,\dot{x}\,\dot{y} + f\,y^2\,\dot{x}\,\dot{z} + g\,\dot{y} - k\sqrt{x^2+y^2},$$

   where (a, b, c, g, k) are constants.

   (a) What is the Hamiltonian of the system?

   (b) What are the quantities that are conserved?

3. *Phase portraits:* Draw the phase portraits of a particle moving in the following one dimensional potentials: (a) $U = a\,x$, (b) $U = a\,x^2/2$, (c) $U = -a\,x^2/2$, (d) $U = -a\cos\theta$ and (e) $U = a\,|x|^n$, where $a > 0$ and $n > 2$.

4. *Poisson brackets:* Determine: (a) the Poisson brackets formed from the Cartesian components of the momentum $\boldsymbol{p}$ and the angular momentum $\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{p}$ of a particle, and (b) the Poisson brackets formed from the Cartesian components of the angular momentum $\boldsymbol{M}$.

5. *Evolution of density in phase space:* Show that the density of points in phase space corresponding to the motion of a system of particles remains constant during the motion.

   Note: This result is known as Liouville's theorem.

Exercise sheet 7

Special relativity

1. *Decaying muons:* Muons are unstable and decay according to the radioactive decay law $N = N_0\, \exp -(0.693\, t/t_{1/2})$, where N_0 and N are the number of muons at times $t = 0$ and t, respectively, while $t_{1/2}$ is the half life. The half life of the muons in their own rest frame is 1.52×10^{-6} s. Consider a detector on top of a $2,000$ m mountain which counts the number of muons traveling at the speed of $v = 0.98\, c$. Over a given period of time, the detector counts 10^3 muons. When the relativistic effects are taken into account, how many muons can be expected to reach the sea level?

2. (a) *Radial Doppler effect:* Consider a source of photons that is moving radially away with a velocity v from an observer who is at rest. Let ω_{E} and ω_{O} denote the frequency of the photons in the frames of the source and the observer, respectively. Obtain the relation between ω_{E} and ω_{O} in terms of the velocity v.

   (b) *Transverse Doppler effect:* What is the relation between ω_{E} and ω_{O} if the source is moving transversely to the direction of the photon, as it occurs, say, when the source is on a circular trajectory about the observer?

3. *Superluminal motion:* Consider a blob of plasma that is moving at a speed v along a direction that makes an angle θ with respect to the line of sight. Show that the *apparent* transverse speed of the source, projected on the sky, will be related to the actual speed v by the relation

$$v_{\mathrm{app}} = \frac{v \sin\theta}{1 - (v/c)\cos\theta}.$$

   From this expression conclude that the apparent speed v_{app} can exceed the speed of light.

4. *Aberration of light:* Consider two inertial frames S and S', with the frame S' moving along the x-axis with a velocity v with respect to the frame S. Let the velocity of a particle in the frames S and S' be $\mathbf{u}$ and $\mathbf{u}'$, and let θ and θ' be the angles subtended by the velocity vectors with respect to the common x-axis, respectively.

   (a) Show that

$$\tan\theta = \frac{u' \sin\theta'}{\gamma\, [u' \cos\theta' + v]},$$

   where $\gamma = \left[1 - (v/c)^2\right]^{-1/2}$.

   (b) For $u = u' = c$, show that

$$\cos\theta = \frac{\cos\theta' + (v/c)}{1 + (v/c)\cos\theta'}$$

   and

$$\sin\theta = \frac{\sin\theta'}{\gamma\, [1 + (v/c)\cos\theta']}.$$

   (c) For $(v/c) \ll 1$, show that

$$\Delta\theta = (v/c)\, \sin\theta',$$

   where $\Delta\theta = (\theta' - \theta)$.

5. *Form invariance of the Minkowski line element:* Show that the following Minkowski line element is invariant under the Lorentz transformations:

$$\mathrm{d}s^2 = c^2\, \mathrm{d}t^2 - \mathrm{d}\boldsymbol{x}^2.$$

Exercise sheet 8

Four vectors, tensors, and transformations

1. *Transforming four vectors and invariance under Lorentz transformations:* Consider two inertial frames K and K', with K' moving with respect to K, say, along the common x-axis with a certain velocity.

   (a) Given a four vector A^μ in the K frame, construct the corresponding contravariant and covariant four vectors, say, $A^{\mu\prime}$ and A'_μ, in the K' frame.

   (b) Explicitly illustrate that the scalar product $A_\mu\, A^\mu$ is a Lorentz invariant quantity, i.e. show that $A_\mu\, A^\mu = A'_\mu\, A^{\mu\prime}$.

2. *Four velocity and four acceleration:* The four acceleration of a relativistic particle is defined as $a^\mu = \mathrm{d}u^\mu/\mathrm{d}s$, where $u^\mu = \mathrm{d}x^\mu/\mathrm{d}s$ is the four velocity of the particle.

   (a) Express a^μ in terms of the three velocity $\boldsymbol{v}$ and the three acceleration $\boldsymbol{a} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t$ of the particle.

   (b) Evaluate $a^\mu\, u_\mu$ and $a^\mu\, a_\mu$ in terms of $\boldsymbol{v}$ and $\boldsymbol{a}$.

3. *Compton effect using four vectors:* Consider the scattering between a photon of frequency ω and a relativistic electron with velocity $\mathbf{v}$ leading to a photon of frequency ω' and electron with velocity $\mathbf{v}'$. Such a scattering is known as Compton scattering. Let α be the angle between the incident and the scattered photon. Also, let θ and θ' be the angles subtended by the directions of propagation of the incident and the scattered photon with the velocity vector of the electron before the collision.

   (a) Using the conservation of four momentum, show that

   $$\frac{\omega'}{\omega} = \frac{1-(v/c)\cos\theta}{1-(v/c)\cos\theta' + (\hbar\,\omega/\gamma\, m_{\mathrm{e}}\, c^2)\,(1-\cos\alpha)},$$

   where $\gamma = \left[1-(v/c)^2\right]^{-1/2}$ and m_{e} is the mass of the electron.

   (b) When $\hbar\,\omega \ll \gamma\, m_{\mathrm{e}}\, c^2$, show that the frequency shift of the photon can be written as

   $$\frac{\Delta\omega}{\omega} = \frac{(v/c)\ (\cos\theta - \cos\theta')}{1-(v/c)\cos\theta'},$$

   where $\Delta\omega = (\omega' - \omega)$.

4. *Creation of electron-positron pairs:* A purely relativistic process corresponds to the production of electron-positron pairs in a collision of two high energy gamma ray photons. If the energies of the photons are ϵ_1 and ϵ_2 and the relative angle between their directions of propagation is θ, then, by using the conservation of energy and momentum, show that the process can occur only if

   $$\epsilon_1\,\epsilon_2 > \frac{2\, m_{\mathrm{e}}^2\, c^4}{1-\cos\theta},$$

   where m_{e} is the mass of the electron.

5. *Mirrors in motion:* A mirror moves with the velocity v in a direction perpendicular its plane. A ray of light of frequency ν_1 is incident on the mirror at an angle of incidence θ, and is reflected at an angle of reflection ϕ and frequency ν_2.

   (a) Show that

   $$\frac{\tan(\theta/2)}{\tan(\phi/2)} = \frac{c+v}{c-v} \qquad \text{and} \qquad \frac{\nu_2}{\nu_1} = \frac{c+v\cos\theta}{c-v\cos\phi}.$$

   (b) What happens if the mirror was moving parallel to its plane?

Exercise sheet 9

Charges in electromagnetic fields

1. *The Lorentz force:* The action for a relativistic particle that is interacting with the electromagnetic field is given by

$$S[x^\mu(s)] = -m\,c \int \mathrm{d}s - \frac{e}{c} \int \mathrm{d}x_\mu\, A^\mu,$$

where m is the mass of the particle, while e is its electric charge. The quantity $A^\mu = (\phi, \boldsymbol{A})$ is the four vector potential that describes the electromagnetic field, with, evidently, ϕ and $\mathbf{A}$ being the conventional scalar and three vector potentials.

   (a) Vary the above action with respect to x^μ to arrive at the following Lorentz force law:

$$m\,c\, \frac{\mathrm{d}u^\mu}{\mathrm{d}s} = \frac{e}{c}\, F^{\mu\nu}\, u_\nu,$$

   where u^μ is the four velocity of the particle and the electromagnetic field tensor $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} = \partial_\mu\, A_\nu - \partial_\nu\, A_\mu$$

   with $\partial_\mu \equiv \partial/\partial x^\mu$.

   (b) Show that the components of the field tensor $F_{\mu\nu}$ are given by

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix},$$

   where (E_x, E_y, E_z) and (B_x, B_y, B_z) are the components of the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ which are related to the components of the four vector potential by the following standard expressions:

$$\boldsymbol{E} = -\frac{1}{c}\, \frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi \qquad \text{and} \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A}.$$

   (c) Express the above equation governing the motion of the charge in the more familiar three vector notation. What does the zeroth component of the equation describe?

2. *Electron in an electric field:* An electron moving relativistically enters a region of constant electric field that is pointed along the positive y-axis. Let the relativistic three-momentum of the electron as it enters the region of the electric field at time, say, $t = 0$, be $\boldsymbol{p} = (p_x^0, p_y^0, 0)$.

   (a) Integrate the equation of motion describing the electron to determine p_x and p_y as function of time. Express the energy of the electron in terms of $p_x(t)$ and $p_y(t)$.

   (b) From the equation governing the conservation of energy of the electron and the above expression for energy, arrive at the expression for v_y in terms of time. Using the above results, also arrive at the expression for v_x in terms of time. Determine the asymptotic (i.e. the large time) behavior of v_x and v_y.

   Hint: It is useful to note that we can write $\boldsymbol{v} = \mathrm{d}\boldsymbol{x}/\mathrm{d}t = \boldsymbol{p}\, c^2/\mathcal{E}$, where $\mathcal{E}/c = \sqrt{\boldsymbol{p}^2 + m^2\, c^2}$.

   (c) Arrive at $y(t)$. Assuming that $p_x^0 = p_y^0 = 0$, plot the trajectory of the particle in the $c\,t$-y plane.

3. *Motion in a constant and uniform magnetic field:* Consider a particle of mass m and charge e that is moving in a magnetic field of strength B that is directed, say, along the positive z-axis.

   (a) Show that the energy $\mathcal{E} = \gamma\, m\, c^2$ of the particle is a constant.

(b) Determine the trajectory $\boldsymbol{x}(t)$ of the particle and show that, in the absence of any initial momentum along the z-direction, the particle describes a circular trajectory in the x-y plane with the angular frequency $\omega = e\,c\,B/\mathcal{E}$.

4. *Transformation of electric and magnetic fields:* Consider two inertial frames, say, K and K', with the frame K' moving with a velocity v with respect to the frame K along the common x-axes.

   (a) Given the components of the electric and the magnetic fields, say, $\boldsymbol{E}$ and $\boldsymbol{B}$, in the frame K, using the transformation properties of the electromagnetic field tensor $F_{\mu\nu}$, construct the corresponding components in the frame K'.

   (b) Show that $|\boldsymbol{E}|^2 - |\boldsymbol{B}|^2$ is invariant under the Lorentz transformations.

   (c) Express the quantity $|\boldsymbol{E}|^2 - |\boldsymbol{B}|^2$ explicitly as a scalar in terms of the field tensor $F_{\mu\nu}$.

5. *Invariance of the action under gauge transformations:* Show that the action for a relativistic particle that is interacting with the electromagnetic field, viz.

$$S[x^\mu(s)] = -m\,c \int \mathrm{d}s - \frac{e}{c} \int \mathrm{d}x_\mu\, A^\mu,$$

   is invariant under gauge transformations of the form $A^\mu \to A^\mu + \partial^\mu \chi$, where χ is a scalar function.

Exercise sheet 10

Theory of the electromagnetic field

1. *Equivalence of actions under gauge transformations:* Recall that the action governing the electromagnetic field described by the vector potential A_μ that is interacting with the four current j^μ is given by
$$S[A^\mu(\tilde{x})] = -\frac{1}{c^2}\int \mathrm{d}^4\tilde{x}\; j^\mu\, A_\mu - \frac{1}{16\,\pi\,c}\int \mathrm{d}^4\tilde{x}\; F_{\mu\nu}\, F^{\mu\nu},$$
where $F_{\mu\nu}$ is the field tensor defined as
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$
Since $F_{\mu\nu}$ is explicitly invariant under the gauge transformation
$$A_\mu \to A_\mu + \partial_\mu \Lambda,$$
evidently, the second term in the above action is invariant as well. Determine if the first term transforms to an equivalent action under the gauge transformation.

2. *The spatial components of the stress-energy tensor of the free electromagnetic field:* We had arrived at the forms of the time-time and the time-space components of the stress-energy tensor of the free electromagnetic field in terms of the components of the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$. Arrive at the corresponding expressions for the purely spatial components of the stress-energy tensor.

   Note: These components are usually referred to as the Maxwell stress tensor.

3. *From source free Maxwell's equations to the conservation of the stress-energy tensor:* Establish that the source free Maxwell's equations imply that the stress-energy tensor of the free electromagnetic field is conserved.

4. *Conservation of the stress-energy tensor in the presence of sources:* The above exercise had involved the electromagnetic field in the absence of charges. If the charges are also present, then it is the sum of the stress-energy tensors of the charges as well as the field that will be conserved. The stress-energy tensor of a collection of mutually non-interacting particles can be written as
$$T^{\mu\nu}_{\mathrm{P}} = \mu\, c\, u^\mu\, u^\nu\, \frac{\mathrm{d}s}{\mathrm{d}t},$$
where μ is the mass density associated with the particles, while u^μ denotes the four velocity of the particles.

   Note: The above expression for the stress-energy tensor for a collection of mutually non-interacting particles is equivalent to a pressureless relativistic fluid. Often, such a system is referred to as 'dust'.

   (a) Show that, upon using the second pair Maxwell's equations, in the presence of sources, the stress-energy of the electromagnetic field, say, $T^{\mu\nu}_{\mathrm{F}}$, satisfies the equation
$$\partial_\mu T^{\mu\nu}_{\mathrm{F}} = -\frac{1}{c}\, F^{\nu\lambda}\, j_\lambda.$$

   (b) As in the case of charges, the continuity equation corresponding to the mass flow can be expressed as follows:
$$\partial_\mu\left(\mu\, \frac{\mathrm{d}x^\mu}{\mathrm{d}t}\right) = 0.$$

Using this equation and the following Lorentz force law:

$$\mu\, c\, \frac{\mathrm{d}u^\mu}{\mathrm{d}s} = \frac{\rho}{c}\, F^{\mu\nu}\, u_\nu,$$

where ρ denotes the charge density of the particles, show that

$$\partial_\mu T^{\mu\nu}_{\mathrm{P}} = \frac{1}{c}\, F^{\mu\lambda}\, j_\lambda$$

so that the total stress-energy tensor of the system, viz. $T^{\mu\nu} = T^{\mu\nu}_{\mathrm{P}} + T^{\mu\nu}_{\mathrm{F}}$, is conserved, as required.

5. *Traceless nature of the stress-energy tensor of the electromagnetic field:* Show that the trace of the stress-energy tensor of the electromagnetic field vanishes. Can you identify the reason behind the vanishing trace?

Exercise sheet 11

Electromagnetic waves

1. *Coulomb gauge and the degrees of freedom of the electromagnetic field:* Recall that the Lorenz gauge was determined by the covariant condition $\partial_\mu A^\mu = 0$. However, even the Lorenz condition does not uniquely fix the gauge. Further gauge transformations of the form $A_\mu \to A_\mu + \partial_\mu \Lambda$ are possible, provided Λ satisfies the condition $\Box \Lambda = 0$. In such a situation, often, one breaks Lorentz covariance and works in the so-called Coulomb gauge wherein A^t is set to zero, so that the Lorenz condition reduces to $\nabla \cdot \mathbf{A} = 0$.

   (a) Show that this implies that the free electromagnetic field possesses two independent degrees of freedom.

   (b) What do these two degrees of freedom correspond to?

2. *Massive vector field:* Consider the following action that governs a massive vector field $\mathcal{A}_\mu$:

$$S[\mathcal{A}^\mu(\tilde{x})] = \frac{1}{16\,\pi\,c} \int \mathrm{d}^4\tilde{x}\, \left(-\mathcal{F}_{\mu\nu}\,\mathcal{F}^{\mu\nu} + 2\,\sigma^2\,\mathcal{A}^\mu\,\mathcal{A}_\mu\right),$$

   where $\mathcal{F}_{\mu\nu}$ represents the field tensor defined in the usual form, viz.

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu,$$

   and σ has dimensions of mass in suitable units.

   (a) Obtain the equation of motion governing the field $\mathcal{A}_\mu$.

   (b) Show that the Lorentz condition, viz. $\partial_\mu \mathcal{A}^\mu = 0$, *has* to be satisfied by the field apart from satisfying the equation of motion.

   (c) Is the action invariant under gauge transformations of the form $\mathcal{A}_\mu \to \mathcal{A}_\mu + \partial_\mu \Lambda$?

   (d) How many independent degrees of freedom does the massive field $\mathcal{A}^\mu$ possess?

   Note: The massive vector field $\mathcal{A}^\mu$ is known as the Proca field.

3. *Polarization of electromagnetic waves:* Consider a monochromatic, plane electromagnetic wave propagating along the positive z-direction.

   (a) Write down solutions of the vector potential $\boldsymbol{A}$ that correspond to linearly and circularly polarized electromagnetic waves.

   (b) Determine the associated electric and magnetic fields. Show that they satisfy the Maxwell's equations in the absence of sources.

   (c) Determine the energy density, Poynting vector and the pressure exerted by the wave, when averaged over a cycle.

4. *Motion of charged particle in electromagnetic waves:* Determine the motion of a charged particle in the fields of: (i) a monochromatic, linearly polarized plane electromagnetic wave, and (ii) a circularly polarized electromagnetic wave.

5. *Laws of reflection and refraction:* Consider a monochromatic, plane electromagnetic wave that is incident on the homogeneous planar medium.

   (a) Determine the conditions on the electric and magnetic fields at the boundary.

   (b) Utilizing the boundary conditions, arrive at the laws of reflection and refraction.

Exercise sheet 12

Radiation of electromagnetic waves by moving charges

1. *Green's functions:* Consider a real scalar field ϕ that is sourced by a charge density ρ. Such a scalar field would be governed by the following equation of motion:

$$\left(\Box + \sigma^2\right)\, \phi = \alpha\, \rho,$$

where α is a quantity of suitable dimensions. This inhomogeneous partial differential equation can be solved using the method of Green's functions as follows.

   (a) Show that the inhomogeneous solution to the above equation can be expressed as

$$\phi(\tilde{x}) = \alpha \int \mathrm{d}^4\tilde{x}\; G(\tilde{x}, \tilde{x}')\; \rho(\tilde{x}),$$

   where the Green's function $G(\tilde{x}, \tilde{x}')$ satisfies the differential equation

$$\left(\Box_{\tilde{x}} + \sigma^2\right)\, G(\tilde{x}, \tilde{x}') = \delta^{(4)}\left(\tilde{x} - \tilde{x}'\right).$$

   (b) Express the Green's function as a Fourier transform as

$$G(\tilde{x}, \tilde{x}') = \int \frac{\mathrm{d}^4\tilde{k}}{(2\,\pi)^4}\; G(\tilde{k})\; \exp\left[i\, k^\mu\left(x_\mu - x'_\mu\right)\right],$$

   and substitute it into the above equation to determine the form of $G(\tilde{k})$.

2. *The retarded Green's function for a massless field:* Using the form of $G(\tilde{k})$, evaluate the above integral to determine the Green's function $G(\tilde{x}, \tilde{x}')$ for a field with $\sigma = 0$.

3. *The Lienard-Wiechart potentials:* Consider a point particle with charge e that is moving along the trajectory $r^\mu(\tau)$, where τ is the proper time in the frame of the charge. The four current associated with the charge is given by

$$j^\mu(\tilde{x}) = e\, c^2 \int \mathrm{d}\tau\; u^\mu\; \delta^{(4)}\left[\tilde{x} - \tilde{r}(\tau)\right],$$

where $\tilde{r}(\tau) \equiv r^\mu(\tau)$ is the trajectory of the charge and $u^\mu = \mathrm{d}r^\mu/\mathrm{d}s$ is its four velocity, so that the corresponding charge and current densities are given by

$$\rho(\tilde{x}) = e\; c\; \delta^{(3)}[\boldsymbol{x} - \boldsymbol{r}(t)] \quad \text{and} \quad \boldsymbol{j}(\tilde{x}) = e\; \boldsymbol{v}(t)\; \delta^{(3)}[\boldsymbol{x} - \boldsymbol{r}(t)]$$

with $\boldsymbol{v}(t) = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$, as required. In the Lorenz gauge, the electromagnetic vector potential A^μ satisfies the equation

$$\Box A^\mu = \frac{4\,\pi}{c}\, j^\mu.$$

   (a) Using the retarded Green's function for a massless field that we had obtained earlier, solve the above equation to arrive at the following expression for the vector potential A^μ:

$$A^\mu(\tilde{x}) = \frac{e\; u^\mu}{R_\mu\, u^\mu},$$

   where $R^\mu = x^\mu - r^\mu$ and $R_\mu\, R^\mu = 0$.

   (b) Show that the above vector potential A^μ can be written in the three dimensional form as

$$\phi(\tilde{x}) = \frac{e}{R - (\boldsymbol{v}\cdot\boldsymbol{R})\,/c} \quad \text{and} \quad \boldsymbol{A}(\tilde{x}) = \frac{e\; \boldsymbol{v}/c}{R - (\boldsymbol{v}\cdot\boldsymbol{R})\,/c},$$

   where $\boldsymbol{R} = \boldsymbol{x} - \boldsymbol{r}$ and $R = |\boldsymbol{R}|$, with the right hand sides evaluated at the so-called retarded time determined by the condition $R^\mu R_\mu = 0$.

   Note: These are known as the Lienard-Wiechart potentials.

4. *The radiation field:* Using the above Lienard-Wiechart potentials, obtain the following expressions for the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ generated by a point charge that is moving along an arbitrary trajectory:

$$\begin{aligned} \boldsymbol{E} &= \frac{e}{\gamma^2\,\mu^3\,R^2}\,\left[\hat{\boldsymbol{n}} - (\boldsymbol{v}/c)\right] + \frac{e}{c^2\,\mu^3\,R}\,\left[\hat{\boldsymbol{n}} \times \left(\left[\hat{\boldsymbol{n}} - (\boldsymbol{v}/c)\right] \times \boldsymbol{a}\right)\right], \\ \boldsymbol{B} &= \hat{\boldsymbol{n}} \times \boldsymbol{E}, \end{aligned}$$

where γ is the standard Lorentz factor, $\boldsymbol{a} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t$ is the acceleration of the charge, while the quantity μ is given by

$$\mu = \left(1 - \frac{\boldsymbol{v}\cdot\hat{\boldsymbol{n}}}{c}\right)^{-1},$$

with $\hat{\boldsymbol{n}} = \boldsymbol{R}/R$.

Note: The contribution to the electric and the magnetic fields above which depends on the acceleration of the charge and behaves as $1/R$ with distance is known as the radiation field.

5. *Relativistic beaming:* Recall that the flux of energy being carried by electromagnetic radiation is described by the Poynting vector, viz.

$$\boldsymbol{S} = \frac{c}{4\,\pi}\,(\boldsymbol{E} \times \boldsymbol{B}).$$

When $\boldsymbol{B} = \hat{\boldsymbol{n}} \times \boldsymbol{E}$, the amount of energy, say, $\mathrm{d}\mathcal{E}$, that is propagating into a solid angle $\mathrm{d}\Omega$ in unit time is then given by

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}\Omega\,\mathrm{d}t} = |\boldsymbol{S}|\,R^2 = \frac{c\,|\boldsymbol{E}|^2\,R^2}{4\,\pi}.$$

(a) Upon using the above expressions for the radiative component of the electric field, show that the energy emitted by a point charge per unit time within a unit solid angle can be written as

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t\,\mathrm{d}\Omega} = \frac{e^2}{4\,\pi\,c^3}\,\left[2\,\mu^5\,(\hat{\boldsymbol{n}}\cdot\boldsymbol{a})\,(\boldsymbol{v}\cdot\boldsymbol{a}/c) + \mu^4\,a^2 - \mu^6\,\gamma^{-2}\,(\hat{\boldsymbol{n}}\cdot\boldsymbol{a})^2\right].$$

(b) Clearly, the intensity of the radiation is the largest along directions wherein $\mu \gg 1$. Show that, if θ is the angle between $\boldsymbol{v}$ and $\hat{\boldsymbol{n}}$, then, for $\theta \ll 1$ and $|\boldsymbol{v}| \simeq c$, we can write

$$\mu = \frac{2\,\gamma^2}{1 + \gamma^2\,\theta^2}.$$

(c) Argue that, for $\gamma \gg 1$, this expression is sharply peaked around $\theta = 0$, with a width $\Delta\theta \simeq \gamma^{-1}$.
Note: This effect, where most of the intensity is pointed along the direction of velocity of the charge, is known as relativistic beaming.