PH5870

GENERAL RELATIVITY AND COSMOLOGY

January–May 2012

Lecture schedule

- The course will consist of about 45 lectures, including about 10–12 tutorial sessions. However, note that there will be no separate tutorial sessions, and they will be integrated with the lectures.
- The duration of each lecture will be 50 minutes.
- The first lecture will be on January 2, 2012 and the last lecture on April 20, 2012.
- We will meet for three hours every week. The lectures are scheduled for 11:10 AM–12:50 PM on Mondays and 10:00–10:50 AM on Thursdays. We may also meet during 12:00–12:50 PM on Thursdays for either the quizzes or to make up for any lecture that I may have to miss due to travel.
- We will be meeting in HSB 210.
- Changes in schedule, if any, will be notified sufficiently in advance.

Quizzes, end-of-semester exam and grading

- The grading will be based on three scheduled quizzes and an end-of-semester exam.
- I will consider the best of the two quizzes for grading, and the best two will carry 25% weight each.
- The three quizzes will be on January 30, March 8 and April 5. The first of these three dates is a Monday, and the quiz will be held during 4:45–5:35 PM on the day. The remaining two dates are Thursdays, and the quizzes will be held during 12:15–1:45 PM on these dates.
- The end-of-semester exam will be during 2:00–5:00 PM on May 2, and the exam will carry 50% weight.

Syllabus and structure

1. **Introduction** [$\sim$ 3 lectures]
   (a) The scope of the general theory of relativity
   (b) Geometry and physics
   (c) Space, time and gravity in Newtonian physics

2. **Spacetime and relativity** [$\sim$ 8 lectures]
   (a) The Michelson-Morley interferometric experiment – Postulates of special relativity
   (b) Lorentz transformations – The relativity of simultaneity – Length contraction and time dilation
   (c) Transformation of velocities and acceleration – Uniform acceleration – Doppler effect
   (d) Four vectors – Action for the relativistic free particle – Charges in an electromagnetic field and the Lorentz force law
   (e) Conservation of relativistic energy and momentum

Exercise sheets 1, 2 and 3

Quiz I

3. **Tensor algebra and tensor calculus** [$\sim$ 16 lectures]
   (a) Manifolds and coordinates – Curves and surfaces
   (b) Transformation of coordinates – Contravariant, covariant and mixed tensors – Elementary operations with tensors
   (c) The partial derivative of a tensor – Covariant differentiation and the affine connection
   (d) The metric – Geodesics
   (e) Isometries – The Killing equation and conserved quantities
   (f) The Riemann tensor – The equation of geodesic deviation
   (g) The curvature and the Weyl tensors

Additional exercises I

Exercise sheets 4, 5, 6 and 7

4. **The principles of general relativity** [$\sim$ 2 lectures]
   (a) The equivalence principle – The principle of general covariance – The principle of minimal gravitational coupling

5. **The field equations of general relativity** [$\sim$ 4 lectures]
   (a) The vacuum Einstein equations
   (b) Derivation of vacuum Einstein equations from the action – The Bianchi identities
   (c) The stress-energy tensor – The cases of perfect fluid, scalar and electromagnetic fields
   (d) The structure of the Einstein equations

Exercise sheet 8

Quiz II

6. **The Schwarzschild solution, and black holes** [$\sim$ 6 Lectures]

   (a) The Schwarzschild solution – Properties of the metric – Symmetries and conserved quantities

   (b) Motion of particles in the Schwarzschild metric – Precession of the perihelion – Bending of light

   (c) Black holes – Event horizon, its properties and significance – Singularities

   (d) The Kruskal extension – Penrose diagrams

Exercise sheet 9

Quiz III

7. **The Friedmann-Lemaitre-Robertson-Walker cosmology** [$\sim$ 6 lectures]

   (a) Homogeneity and isotropy – The Friedmann line-element

   (b) Friedmann equations – Solutions with different types of matter

   (c) Red-shift – Luminosity and angular diameter distances

   (d) The horizon problem – The inflationary scenario

Additional exercises II

Exercise sheets 10 and 11

8. **Gravitational waves** [$\sim$ 3 lectures]

   (a) The linearized Einstein equations – Solutions to the wave equation – Production of weak gravitational waves

   (b) Gravitational radiation from binary stars – The quadrupole formula for the energy loss

Exercise sheet 12

End-of-semester exam

Note: The topics in red could not be covered for want of time.

Basic textbooks

1. L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Course of Theoretical Physics, Volume 2), Fourth Edition (Pergamon Press, New York, 1975).
2. B. F. Schutz, *A First Course in General Relativity* (Cambridge University Press, Cambridge, 1990).
3. R. d'Inverno, *Introducing Einstein's Relativity* (Oxford University Press, Oxford, 1992).
4. J. B. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (Pearson Education, Delhi, 2003).

Additional references

1. S. Weinberg, *Gravitation and Cosmology* (John Wiley, New York, 1972).
2. A. P. Lightman, W. H. Press, R. H. Price and S. A. Teukolsky, *Problem Book in Relativity and Gravitation* (Princeton University Press, New Jersey, 1975).
3. S. Carroll, *Spacetime and Geometry* (Addison Wesley, New York, 2004).
4. M. P. Hobson, G. P. Efstathiou and A. N. Lasenby, *General Relativity: An Introduction for Physicists* (Cambridge University Press, Cambridge, 2006).
5. W. Rindler, *Relativity: Special, General and Cosmological* (Oxford University Press, Oxford, 2006).
6. T. Padmanabhan, *Gravitation: Foundation and Frontiers* (Cambridge University Press, Cambridge, 2010).

Advanced texts

1. S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge University Press, Cambridge, 1973).
2. C. W. Misner, K. S. Thorne and J. W. Wheeler, *Gravitation* (W. H. Freeman and Company, San Francisco, 1973).
3. R. M. Wald, *General Relativity* (The University of Chicago Press, Chicago, 1984).
4. E. Poisson, *A Relativist's Toolkit* (Cambridge University Press, Cambridge, 2004).

Exercise sheet 1

Special relativity: Lorentz transformations and some consequences

1. *Superluminal motion:* Consider a blob of plasma that is moving at a speed v along a direction that makes an angle θ with respect to the line of sight. Show that the *apparent* transverse speed of the source, projected on the sky, will be related to the actual speed v by the relation

$$v_{\rm app} = \left(\frac{v \sin\theta}{1 - (v/c)\cos\theta}\right).$$

   From this expression conclude that the apparent speed $v_{\rm app}$ can exceed the speed of light.

2. *Aberration of light:* Consider two inertial frames S and S', with the frame S' moving along the x-axis with a velocity v with respect to the frame S. Let the velocity of a particle in the frames S and S' be $\mathbf{u}$ and $\mathbf{u}'$, and let θ and θ' be the angles subtended by the velocity vectors with respect to the common x-axis, respectively.

   (a) Show that

$$\tan\theta = \left(\frac{u' \sin\theta'}{\gamma\,[u'\cos\theta' + v]}\right),$$

   where $\gamma = \left[1 - (v/c)^2\right]^{-1/2}$.

   (b) For $u = u' = c$, show that

$$\cos\theta = \left(\frac{\cos\theta' + (v/c)}{1 + (v/c)\cos\theta'}\right)$$

   and

$$\sin\theta = \left(\frac{\sin\theta'}{\gamma\,[1 + (v/c)\cos\theta']}\right).$$

   (c) For $(v/c) \ll 1$, show that

$$\Delta\theta = (v/c)\,\sin\theta',$$

   where $\Delta\theta = (\theta' - \theta)$.

3. *Decaying muons:* Muons are unstable and decay according to the radioactive decay law $N = N_0\,\exp -(0.693\,t/t_{1/2})$, where N_0 and N are the number of muons at times $t = 0$ and t, respectively, while $t_{1/2}$ is the half life. The half life of the muons in their own rest frame is 1.52×10^{-6} s. Consider a detector on top of a $2,000$ m mountain which counts the number of muons traveling at the speed of $v = 0.98\,c$. Over a given period of time, the detector counts 10^3 muons. When the relativistic effects are taken into account, how many muons can be expected to reach the sea level?

4. *Binding energy:* As you may know, the deuteron which is the nucleus of deuterium, an isotope of hydrogen, consists of one proton and one neutron. Given that the mass of a proton and a neutron are $m_{\rm p} = 1.673\times10^{-27}$ kg and $m_{\rm n} = 1.675\times10^{-27}$ kg, while the mass of the deuteron is $m_{\rm d} = 3.344\times10^{-27}$ kg, show that the binding energy of the deuteron in about 2.225 MeV.

   Note: MeV refers to Million electron Volts, and an electron Volt is 1.602×10^{-19} J.

5. *Form invariance of the Minkowski line element:* Show that the following Minkowski line element is invariant under the Lorentz transformations:

$${\rm d}s^2 = c^2\,{\rm d}t^2 - {\rm d}\mathbf{x}^2.$$

Exercise sheet 2

Special relativity: Working in terms of four vectors

1. *Compton effect using four vectors:* Consider the scattering between a photon of frequency ω and a relativistic electron with velocity $\mathbf{v}$ leading to a photon of frequency ω' and electron with velocity $\mathbf{v}'$. Such a scattering is known as Compton scattering. Let α be the angle between the incident and the scattered photon. Also, let θ and θ' be the angles subtended by the directions of propagation of the incident and the scattered photon with the velocity vector of the electron before the collision.

   (a) Using the conservation of four momentum, show that

   $$\left(\frac{\omega'}{\omega}\right) = \left(\frac{1-(v/c)\cos\theta}{1-(v/c)\cos\theta' + (\hbar\,\omega/\gamma\, m_{\rm e}\, c^2)\,(1-\cos\alpha)}\right),$$

   where $\gamma = \left[1-(v/c)^2\right]^{-1/2}$ and $m_{\rm e}$ is the mass of the electron.

   (b) When $(\hbar\,\omega) \ll \left(\gamma\, m_{\rm e}\, c^2\right)$, show that the frequency shift of the photon can be written as

   $$\left(\frac{\Delta\omega}{\omega}\right) = \left[\frac{(v/c)\ (\cos\theta - \cos\theta')}{1-(v/c)\cos\theta'}\right],$$

   where $\Delta\omega = (\omega' - \omega)$.

2. *Creation of electron-positron pairs:* A purely relativistic process corresponds to the production of electron-positron pairs in a collision of two high energy gamma ray photons. If the energies of the photons are ϵ_1 and ϵ_2 and the relative angle between their directions of propagation is θ, then, by using the conservation of energy and momentum, show that the process can occur only if

   $$(\epsilon_1\,\epsilon_2) > \left(\frac{2\, m_{\rm e}^2\, c^4}{1-\cos\theta}\right),$$

   where $m_{\rm e}$ is the mass of the electron.

3. *Transforming four vectors and invariance under Lorentz transformations:* Consider two inertial frames K and K', with K' moving with respect to K, say, along the common x-axis with a certain velocity.

   (a) Given a four vector A^μ in the K frame, construct the corresponding contravariant and covariant four vectors, say, $A^{\mu\prime}$ and A'_μ, in the K' frame.

   (b) Explicitly illustrate that the scalar product $(A_\mu\, A^\mu)$ is a Lorentz invariant quantity, i.e. show that $(A_\mu\, A^\mu) = \left(A'_\mu\, A^{\mu\prime}\right)$.

4. *Lorentz invariance of the wave equation:* Show that the following wave equation:

   $$\left(\frac{1}{c^2}\right)\left(\frac{\partial^2\phi}{\partial t^2}\right) - \nabla^2\phi = 0$$

   satisfied by, say, light, is invariant under the Lorentz transformations.

5. *Mirrors in motion:* A mirror moves with the velocity v in a direction perpendicular its plane. A ray of light of frequency ν_1 is incident on the mirror at an angle of incidence θ, and is reflected at an angle of reflection ϕ and frequency ν_2.

   (a) Show that

   $$\left(\frac{\tan(\theta/2)}{\tan(\phi/2)}\right) = \left(\frac{c+v}{c-v}\right) \qquad \text{and} \qquad \left(\frac{\nu_2}{\nu_1}\right) = \left(\frac{c+v\cos\theta}{c-v\cos\phi}\right).$$

   (b) What happens if the mirror was moving parallel to its plane?

Exercise sheet 3

Electromagnetism in tensorial notation

1. *The Lorentz force:* In Minkowski spacetime, the action for a relativistic particle that is interacting with the electromagnetic field is given by

$$S[x^\mu] = -m\,c \int \mathrm{d}s - \frac{e}{c} \int \mathrm{d}x_\mu\, A^\mu,$$

where m is the mass of the particle, while e is its electronic charge. The quantity $A^\mu = (\phi, \mathbf{A})$ is the four vector potential that describes the electromagnetic field, with, evidently, ϕ and $\mathbf{A}$ being the conventional scalar and three-vector potentials.

(a) Vary the above action with respect to x^μ to arrive at the following Lorentz force law:

$$m\,c \left(\frac{\mathrm{d}u^\mu}{\mathrm{d}s}\right) = \left(\frac{e}{c}\right) F^{\mu\nu}\, u_\nu,$$

where $u^\mu = (\mathrm{d}x^\mu/\mathrm{d}s)$ is the four velocity of the particle and the electromagnetic field tensor $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} = (\partial_\mu\, A_\nu - \partial_\nu\, A_\mu)$$

with $\partial_\mu \equiv (\partial/\partial x^\mu)$.

(b) Show that the components of the field tensor $F_{\mu\nu}$ are given by

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix},$$

where (E_x, E_y, E_z) and (B_x, B_y, B_z) are the components of the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ which are related to the components of the four vector potential by the following standard expressions:

$$\mathbf{E} = -\left(\frac{1}{c}\right)\left(\frac{\partial \mathbf{A}}{\partial t}\right) - \nabla\phi \qquad \text{and} \qquad \mathbf{B} = \nabla \times \mathbf{A}.$$

(c) Establish that the spatial components of the above equation of motion for the charge can be written as

$$\left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\right) = e\,\mathbf{E} + \left(\frac{e}{c}\right)(\mathbf{v} \times \mathbf{B})$$

with $\mathbf{p} = (\gamma\, m\, \mathbf{v})$ being the relativistic three momentum of the particle.

Note: This equation reduces to the familiar equation of motion for a charge driven by the Lorentz force in the non-relativistic limit [i.e. when terms of order $(v/c)^2$ can be ignored] wherein $\mathbf{p} \simeq (m\,\mathbf{v})$.

(d) Show that the time component of the the above equation of motion for the charge reduces to

$$\left(\frac{\mathrm{d}E_{\mathrm{KE}}}{\mathrm{d}t}\right) = e\,(\mathbf{v}\cdot\mathbf{E}),$$

where $E_{\mathrm{KE}} = \left(\gamma\, m\, c^2\right)$ is the kinetic energy of the particle.

2. *The first pair of Maxwell's equations:* Show that the above definition of $F_{\mu\nu}$ leads to the following Maxwell's equations in flat spacetime:

$$\left(\frac{\partial F_{\mu\nu}}{\partial x^{\lambda}}\right) + \left(\frac{\partial F_{\nu\lambda}}{\partial x^{\mu}}\right) + \left(\frac{\partial F_{\lambda\mu}}{\partial x^{\nu}}\right) = 0.$$

   Also, show that these equations correspond to the following two source free Maxwell's equations:

$$(\nabla \times \mathbf{E}) = -\left(\frac{1}{c}\right)\left(\frac{\partial \mathbf{B}}{\partial t}\right) \qquad \text{and} \qquad (\nabla \cdot \mathbf{B}) = 0.$$

3. *The Lorentz invariant four volume:* Show that the spacetime volume $\mathrm{d}^4x = (c\,\mathrm{d}t\,\mathrm{d}^3\mathbf{x})$ is a Lorentz invariant quantity.

4. *The second pair of Maxwell's equations:* Let the four current $j^{\mu} = (\rho\, c, \mathbf{j})$ represent the charge density ρ and the three-current $\mathbf{j}$ that source the electric and magnetic fields. In flat spacetime, the action describing the electromagnetic field that is sourced by the four current j^{μ} is given by

$$S[A^{\mu}] = -\left(\frac{1}{c^2}\right)\int \mathrm{d}^4x\ (A_{\mu}\, j^{\mu}) - \left(\frac{1}{16\,\pi\, c}\right)\int \mathrm{d}^4x\ (F^{\mu\nu}\, F_{\mu\nu}).$$

   (a) Vary this action with respect to the vector potential A^{μ} and arrive at the following Maxwell's equations:

$$(\partial_{\nu}\, F^{\mu\nu}) = -\left(\frac{4\,\pi}{c}\right)\, j^{\mu}.$$

   (b) Show that these equations correspond to the following two Maxwell's equations with sources:

$$(\nabla \cdot \mathbf{E}) = (4\,\pi)\,\rho \qquad \text{and} \qquad (\nabla \times \mathbf{B}) = (4\,\pi)\,\mathbf{j} + \left(\frac{1}{c}\right)\left(\frac{\partial \mathbf{E}}{\partial t}\right).$$

5. *The continuity equation:* Show that, from the second pair of Maxwell's equations above, one can arrive at the continuity equation, viz.

$$(\partial_{\mu}\, j^{\mu}) = \left(\frac{\partial \rho}{\partial t}\right) + \nabla \cdot \mathbf{j} = 0.$$

Quiz I

Special relativity

1. *Transformation of angles:* Consider two inertial frames K and K', with the frame K' moving with respect to the frame K at a given velocity along the common x-axis. A rod in the frame K' makes an angle θ' with respect to the forward direction of motion. What is the corresponding angle as seen in the frame K? [10 marks]

2. *Four velocity and four acceleration:* The four velocity and the four acceleration of a relativistic particle are defined as $u^\mu = (\mathrm{d}x^\mu/\mathrm{d}s)$ and $a^\mu = (\mathrm{d}^2x^\mu/\mathrm{d}s^2)$, respectively.

   (a) Express u^μ and a^μ in terms of the three velocity $\mathbf{v} = (\mathrm{d}\mathbf{x}/\mathrm{d}t)$ and the three acceleration $\mathbf{a} = (\mathrm{d}^2\mathbf{x}/\mathrm{d}t^2)$. [4 marks]

   (b) Evaluate $(u_\mu\, u^\mu)$, $(u_\mu\, a^\mu)$ and $(a_\mu\, a^\mu)$ in terms of $\mathbf{v}$ and $\mathbf{a}$. [6 marks]

3. (a) *Colliding particles I:* A particle of mass m_1 and velocity $\mathbf{v}_1$ collides with a particle at rest of mass m_2, and is absorbed by it. Determine the mass as well as the velocity of the compound system. [4 marks]

   (b) *Colliding particles II:* A particle of mass m and kinetic energy T_i collides with a stationary particle of the same mass. Determine the kinetic energy of the incident particle after the collision, if it is scattered by an angle θ. [6 marks]

   Note: The kinetic energy associated with a particle of mass m and energy E is $(E - m\,c^2)$.

4. *Null curves:* Consider a trajectory in Minkowski spacetime that is parametrized in terms of a quantity λ in the the following fashion:

$$c\,t = \int \mathrm{d}\lambda\; r(\lambda), \qquad x = \int \mathrm{d}\lambda\; r(\lambda)\, \sin\theta(\lambda)\, \cos\phi(\lambda),$$
$$y = \int \mathrm{d}\lambda\; r(\lambda)\, \sin\theta(\lambda)\, \sin\phi(\lambda), \qquad z = \int \mathrm{d}\lambda\; r(\lambda)\, \cos\theta(\lambda).$$

   Show that the trajectory is a null curve, i.e. the spacetime interval between any two infinitesimally separated points on the curve vanishes. [10 marks]

5. *A non-linear coordinate transformation:* Consider the following non-linear transformation from the Minkowski coordinates $(c\,t, x, y, z)$ to the coordinates $(c\,\tau, \xi, y', z')$:

$$c\,t = \xi\, \sinh\,(g\,\tau/c), \quad x = \xi\, \cosh\,(g\,\tau/c), \quad y = y' \quad \text{and} \quad z = z'.$$

   (a) Assuming ξ to be a constant, draw the trajectory associated with the coordinates $(c\,\tau, \xi)$ in the $c\,t$-x plane. [5 marks]

   (b) Express the Minkowski line element in terms of the coordinates $(c\,\tau, \xi, y', z')$. [5 marks]

   Note: The coordinates $(c\,\tau, \xi, y', z')$ describe a class of uniformly accelerated observers in flat spacetime, and these coordinates are often referred to as the Rindler coordinates.

Exercise sheet 4

Tensor algebra

1. (a) Write down the transformations from the Cartesian coordinates $x^a = (x, y, z)$ to the spherical polar coordinates $x'^a = (r, \theta, \phi)$ in $\mathbb{R}^3$.

   (b) Express the transformation matrices $[\partial x^a/\partial x'^b]$ and $[\partial x'^a/\partial x^b]$ in terms of the spherical polar coordinates.

   (c) Evaluate the corresponding Jacobians J and J'. Where is J' zero or infinite?

2. (a) Write down the transformations from the Cartesian coordinates $x^a = (x, y)$ to the plane polar coordinates $x'^a = (r, \phi)$ in $\mathbb{R}^2$.

   (b) Express the transformation matrix $[\partial x'^a/\partial x^b]$ in terms of the polar coordinates.

   (c) Consider the tangent vector to a circle of radius, say, a, that is centered at the origin. Find the components of the tangent vector in one of the two coordinate system, and use the transformation property of the vector to obtain the components in the other coordinate system.

3. Consider a scalar quantity ϕ. Show that, while the quantity $(\partial\phi/\partial x^a)$ is a vector, the quantity $\left(\partial^2\phi/\partial x^a\,\partial x^b\right)$ is *not* a tensor.

4. If $X^a{}_{bc}$ is a mixed tensor of rank $(1, 2)$, show that the contracted quantity $Y_c = X^a{}_{ac}$ is a covariant vector.

5. Evaluate the quantities δ^a_a and $\left(\delta^a_b\,\delta^b_a\right)$ on a n-dimensional manifold.

Additional problems I

Special relativity, electromagnetism and tensors

1. *Relative velocity between two inertial frames:* Consider two inertial frames that are moving with the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ with respect to, say, the laboratory frame. Show that the relative velocity v between the two frames can be expressed as

$$v^2 = \frac{(\mathbf{v}_1 - \mathbf{v}_2)^2 - (\mathbf{v}_1/c) \times (\mathbf{v}_2)^2}{[1 - (\mathbf{v}_1/c) \cdot \mathbf{v}_2/c^2)]^2}.$$

2. *The pole in the barn paradox:* An athlete carrying a pole 20 m long runs towards a barn of length 15 m at the speed of $0.8\,c$. A friend of the athlete watches the action, standing at rest by the door of the barn.

   (a) How long does the friend measure the length of the pole to be, as it approaches the barn?

   (b) The barn door is initially open, and immediately after the runner and the pole are inside the barn, the friend shuts the door. How long after the door is closed does the front of the pole hits the wall at the other end of the barn, as measured by the friend? Compute the interval between the events of the friend closing the door and the pole hitting the wall. Is it spacelike, null or timelike?

   (c) In the frame of the runner, what are the lengths of the barn and the pole?

   (d) Does the runner believe that the pole is entirely inside the barn, when its front hits the opposite wall? Can you explain why?

   (e) After the collision, the pole and the runner come to rest relative to the barn. From the friend's point of view, the 20 m pole is now inside a 15 m barn, since the barn door was shut before the pole stopped. How is this possible? Alternatively, from the runner's point of view, the collision should have occurred *before* the door was closed, so the door should not be closed at all. Was or was not the door closed with the pole inside?

3. *The twin paradox:* Alex and Bob are twins working on a space station located at a fixed position in deep space. Alex undertakes an extended return spaceflight to a distant star, while Bob stays on the station. Show that, on his return to the station, the proper time interval experienced by Alex must be less than that experienced by Bob, hence Bob is now the elder. How does Alex explain this age difference?

4. *Motion in a constant and uniform electric field:* Consider a particle that is moving in a constant and uniform electric field that is directed, say, along the positive x-axis. Let the relativistic three momentum of the particle at time $t = 0$ be $\mathbf{p} = (0, p_0, 0)$, where p_0 is a constant.

   (a) Solve the equation of motion to arrive at $x(t)$ and $y(t)$.

   (b) Determine the corresponding velocities along the two directions.

   (c) What is the velocity of the particle along the x-direction as $t \to \infty$?

   (d) Plot the trajectory of the particle in the $c\,t$-x plane.

   (e) What is the trajectory of the particle in the x-y plane?

   (f) Show that, in the non-relativistic limit, i.e. when $v/c \ll 1$, the trajectory in the x-y plane reduces to a parabola.

   Note: Recall that, a non-relativistic particle that is moving in a uniform field (such as the gravitational field on the surface of the Earth or in a constant electric field) describes a parabola.

5. *Three acceleration in terms of the electromagnetic fields:* Express the three acceleration (i.e. $\dot{\mathbf{v}}$) of a charged particle in terms of the electric and the magnetic fields $\mathbf{E}$ and $\mathbf{B}$.

   Note: The overdot denotes differentiation with respect to the coordinate time t.

6. *Motion in a constant and uniform magnetic field:* Consider a particle of mass m and charge e that is moving in a magnetic field of strength B that is directed, say, along the positive z-axis.

   (a) Show that the energy $\mathcal{E} = \gamma\, m\, c^2$ of the particle is a constant.

   (b) Determine the trajectory $\mathbf{x}(t)$ of the particle and show that, in the absence of any initial momentum along the z-direction, the particle describes a circular trajectory in the x-y plane with the angular frequency $\omega = e\, c\, B/\mathcal{E}$.

7. *Lorentz transformation of the electromagnetic field:* Evaluate how the electromagnetic field tensor $F_{\mu\nu}$ transforms under a Lorentz boost, say, along the positive x-direction. From the result, arrive at how the electric and magnetic fields transform under the Lorentz transformation.

8. (a) *Scalar invariant I:* Express the Lorentz invariant quantity $F_{\mu\nu}\, F^{\mu\nu}$ in terms of the electric and the magnetic fields $\mathbf{E}$ and $\mathbf{B}$.

   (b) *Scalar invariant II:* Show that $\mathbf{E} \cdot \mathbf{B}$ is a Lorentz invariant quantity.

9. *Equation of motion for a scalar field:* Consider the following action that describes a scalar field, say, ϕ, in Minkowski spacetime:

$$S[\phi] = \frac{1}{c} \int c\, \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, \left(\frac{1}{2}\, \eta_{\mu\nu}\, \partial^\mu\phi\, \partial^\nu\phi - \frac{1}{2}\, \sigma^2\, \phi^2\right),$$

   where $\eta_{\mu\nu}$ is the metric tensor in flat spacetime, while the quantity σ is related to the mass of the field. Vary the above action to arrive at the equation of motion for the scalar field.

   Note: The resulting equation of motion is called the Klein-Gordon equation.

10. *The Minkowski line element in a rotating frame:* In terms of the cylindrical polar coordinates, the Minkowski line element is given by

$$\mathrm{d}s^2 = c^2\, \mathrm{d}t^2 - \mathrm{d}\rho^2 - \rho^2\, \mathrm{d}\phi^2 - \mathrm{d}z^2.$$

   Consider a coordinate system that is rotating with an angular velocity Ω about the z-axis. The coordinates in the rotating frame, say, $(c\,t', \rho', \phi', z')$ are related to the standard Minkowski coordinates through the following relations:

$$c\,t = c\,t', \qquad \rho = \rho', \qquad \phi = \phi' + \Omega\, t' \qquad \text{and} \qquad z = z'.$$

   (a) Determine the line element in the rotating frame.

   (b) What happens to the line element when $\rho' \geq c/\Omega$?

Exercise sheet 5

The Christoffel symbols and the geodesic equation

1. *The metric of* $\mathbb{R}^3$*:* Evaluate the covariant and the contravariant components of the metric tensor describing the three-dimensional Euclidean space (usually denoted as $\mathbb{R}^3$) in the Cartesian, cylindrical polar and the spherical polar coordinates. Also, evaluate the determinant of the covariant metric tensor in each of these coordinate systems.

2. *Geodesics on a two sphere:* Evaluate the Christoffel symbols on $\mathbb{S}^2$, and solve the geodesic equation to show that the geodesics are the great circles.

3. *Important identities involving the metric tensor:* Establish the following identities that involve the metric tensor:

   (a) $g_{,c} = g\, g^{ab}\, g_{ab,c}$,

   (b) $g^{ab}\, g_{bc,d} = -g^{ab}{}_{,d}\, g_{bc}$,

   where the commas denote partial derivatives, while g is the determinant of the covariant metric tensor g_{ab}.

4. *Useful identities involving the Christoffel symbols:* Establish the following identities involving the Christoffel symbols:

   (a) $\Gamma^a_{ab} = \frac{1}{2}\,\partial_b \ln|g|$,

   (b) $g^{ab}\,\Gamma^c_{ab} = -\frac{1}{\sqrt{|g|}}\,\partial_d\left(\sqrt{|g|}\;g^{cd}\right)$,

   (c) $g^{ab}{}_{,c} = -\left(\Gamma^a_{cd}\, g^{bd} + \Gamma^b_{cd}\, g^{ad}\right)$,

   where the Christoffel symbol Γ^a_{bc} is given by

   $$\Gamma^a_{bc} = \frac{1}{2}\, g^{ad}\,\left(g_{db,c} + g_{dc,b} - g_{bc,d}\right).$$

5. *Invariant four volume:* Show that the spacetime volume $\sqrt{-g}\;\mathrm{d}^4x$ is invariant under arbitrary coordinate transformations.

Exercise sheet 6

Killing vectors and conserved quantities

1. *Killing vectors in $\mathbb{R}^3$:* Construct all the Killing vectors in the three dimensional Euclidean space $\mathbb{R}^3$ by solving the Killing's equation.

2. *Killing vectors on $\mathbb{S}^2$:* Construct the most generic Killing vectors on a two sphere.

3. *Killing vectors in Minkowski spacetime:* Solve the Killing's equation in flat spacetime, and construct all the independent Killing vectors. What do these different Killing vectors correspond to?

4. *The line element and the conserved quantities around a cosmic string:* The spacetime around a cosmic string is described by the line-element

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}\rho^2 - \alpha^2\,\rho^2\,\mathrm{d}\phi^2 - \mathrm{d}z^2,$$

   where α is a constant that is called the deficit angle.

   (a) List the components of the momentum of a relativistic particle on geodesic motion in this spacetime that are conserved.

   (b) Consider a particle of mass m that is moving along a time-like geodesic in the spacetime of a cosmic string. Using the relation $p^\mu\,p_\mu = m^2\,c^2$ and the conserved momenta, obtain the (first order) differential equation for $\mathrm{d}\rho/\mathrm{d}t$ of the particle in terms of *all* the conserved components of its momenta.

5. *Conserved quantities in the Schwarzschild spacetime:* The spacetime around a central mass M is described by the following Schwarzschild line element:

$$\mathrm{d}s^2 = c^2\left(1-\frac{2\,G\,M}{c^2\,r}\right)\mathrm{d}t^2 - \left(1-\frac{2\,G\,M}{c^2\,r}\right)^{-1}\mathrm{d}r^2 - r^2\,\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right).$$

   Identify the Killing vectors and the corresponding conserved quantities in such a static and spherically symmetric spacetime.

Exercise sheet 7

The Riemann and the Ricci tensors and the scalar curvature

1. *Algebraic identity involving the Riemann tensor:* Recall that, the Riemann tensor is defined as

$$R^{a}{}_{bcd} = \Gamma^{a}_{bd,c} - \Gamma^{a}_{bc,d} + \Gamma^{a}_{ec}\,\Gamma^{e}_{bd} - \Gamma^{a}_{ed}\,\Gamma^{e}_{bc}.$$

Using this expression, establish that

$$R^{a}{}_{bcd} + R^{a}{}_{dbc} + R^{a}{}_{cdb} = 0.$$

2. *The number of independent components of the Riemann tensor:* Show that, on a n-dimensional manifold, the number of independent components of the Riemann tensor are $\left(n^2/12\right)\,\left(n^2-1\right)$.

3. *The flatness of the cylinder:* Calculate the Riemann tensor of a cylinder of constant radius, say, R, in three dimensional Euclidean space. What does the result you find imply?

Note: The surface of the cylinder is actually two-dimensional.

4. *The curvature of a two sphere:* Calculate all the components of the Riemann and the Ricci tensors, and also the corresponding scalar curvature associated with the two sphere.

Note: Given the Riemann tensor $R^{a}{}_{bcd}$, the Ricci tensor R_{ab} and the Ricci scalar R are defined as

$$R_{ab} = R^{c}{}_{acb} \qquad \text{and} \qquad R = g^{ab}\,R_{ab}.$$

5. *Identities involving the covariant derivative and the Riemann tensor:* Establish the following relations:

   (a) $\nabla_c\,\nabla_b\,A_a - \nabla_b\,\nabla_c\,A_a = R^{d}{}_{abc}\,A_d$,

   (b) $\nabla_d\,\nabla_c\,A_{ab} - \nabla_c\,\nabla_d\,A_{ab} = R^{e}{}_{bcd}\,A_{ae} + R^{e}{}_{acd}\,A_{eb}$.

Quiz II

Tensor algebra and tensor calculus

1. *Conformally flat metrics:* Given that, a $(3+1)$ dimensional spacetime is described by the following metric:
$$g_{\mu\nu} = \eta_{\mu\nu}\, \exp\, \psi(x^\lambda),$$
where the quantity ψ is an arbitrary function of the coordinates x^λ, compute the corresponding $g^{\mu\nu}$, g as well as $\Gamma^\mu_{\nu\lambda}$. $\boxed{2+3+5 \text{ marks}}$

   Note: Such spacetimes are said to be conformally flat.

2. *Geodesics on $\mathbb{R}^2$:* Solve the geodesic equation in the polar coordinates to arrive at the geodesics on a plane. $\boxed{10 \text{ marks}}$

   Note: You are expected to solve the geodesic equation involving the Christoffel symbols, and not the more familiar variation of the problem!

3. *Another derivation of the geodesic equation:* Consider the following action:
$$S[x^c(\lambda)] = \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda\, \frac{1}{2}\, g_{ab}\, \dot{x}^a\, \dot{x}^b,$$
where λ is an affine parameter, and the overdots denote differentiation with respect to λ. Show that the Euler-Lagrange equation corresponding to this action leads to the geodesic equation. $\boxed{10 \text{ marks}}$

4. *Klein-Gordon equation in curved spacetime:* A scalar field of 'mass' m (in suitable units) satisfies the following Klein-Gordon equation in a curved spacetime:
$$\phi^{;\mu}{}_{;\mu} + m^2\, \phi = 0.$$
Show that this equation can be written as $\boxed{10 \text{ marks}}$
$$\frac{1}{\sqrt{-g}}\, \partial_\mu \left(\sqrt{-g}\; g^{\mu\nu}\, \partial_\nu\right)\, \phi + m^2\, \phi = 0.$$

5. *Properties of Killing vectors:* If ξ^a is a Killing vector, show that
   (a) $\xi_{a;b;c} = R_{dcba}\, \xi^d$, $\boxed{7 \text{ marks}}$
   (b) $\xi_{a;b}{}^{;b} + R_{ac}\, \xi^c = 0.$ $\boxed{3 \text{ marks}}$

Quiz II – Again

Tensor algebra and tensor calculus

1. *Behavior of the Christoffel symbols under conformal transformations:* Consider the following transformation of the metric tensor:

$$g_{ab} \to \tilde{g}_{ab} = \Omega^2(x^c)\, g_{ab},$$

where $\Omega(x^c)$ is an arbitrary function of the coordinates. Express the Christoffel symbols associated with the metric tensor $\tilde{g}_{ab}$ in terms of the Christoffel symbols corresponding to the metric tensor g_{ab}. $\boxed{\text{10 marks}}$

Note: Transformations of the metric tensor as above are known as conformal transformations. It is important to note that conformal transformations are *not* coordinate transformations.

2. *Geodesics on a cylinder:* Solve the geodesic equation to determine the geodesics on a two dimensional cylindrical surface. $\boxed{\text{10 marks}}$

3. *Some properties involving covariant derivatives:* Prove that

   (a) For any second rank tensor A^{ab}, $\boxed{\text{5 marks}}$

$$A^{ab}{}_{;ab} = A^{ab}{}_{;ba}.$$

   (b) For an anti-symmetric tensor $F_{\mu\nu}$, $\boxed{\text{5 marks}}$

$$F^{\mu\nu}{}_{;\nu} = \frac{1}{\sqrt{-g}}\, \partial_\nu \left(\sqrt{-g}\, F^{\mu\nu}\right).$$

4. *Conserved currents:* If ξ_a is a Killing vector and T_{ab} is the stress energy tensor, show that $j^a = T^{ab}\, \xi_b$ is a conserved current, i.e. $j^a{}_{;a} = 0$. $\boxed{\text{10 marks}}$

5. *Spaces of constant curvature:* Consider a space that is described by the following Riemann tensor:

$$R_{abcd} = \kappa\, \left(g_{ac}\, g_{bd} - g_{ad}\, g_{bc}\right).$$

   (a) Evaluate the Ricci tensor and the scalar curvature associated with this Riemann tensor. $\boxed{\text{3+3 marks}}$

   (b) Show that the above Riemann tensor describes spaces of constant scalar curvature. $\boxed{\text{4 marks}}$
   Hint: Initially assume that κ is dependent on the coordinates and make use of the Bianchi identity.

Exercise sheet 8

The stress energy tensor and the Einstein's equations

1. *The Bianchi identity:* Recall that, the Riemann tensor is defined as

$$R_{abcd} = g_{ae}\, R^{e}{}_{bcd} = g_{ae}\, \left(\Gamma^{e}_{bd,c} - \Gamma^{e}_{bc,d} + \Gamma^{e}_{fc}\,\Gamma^{f}_{bd} - \Gamma^{e}_{fd}\,\Gamma^{f}_{bc}\right).$$

Also, note that, given the Riemann tensor $R^{a}{}_{bcd}$, the Ricci tensor R_{ab} and the Ricci scalar R are defined as

$$R_{ab} = R^{c}{}_{acb} \qquad \text{and} \qquad R = g^{ab}\, R_{ab}.$$

Further, the Einstein tensor is given by

$$G_{ab} = R_{ab} - \frac{1}{2}\, R\, g_{ab}.$$

(a) Using the expression for the Riemann tensor, establish the following Bianchi identity:

$$\nabla_e\, R_{abcd} + \nabla_d\, R_{abec} + \nabla_c\, R_{abde} = 0.$$

Note: It will be a lot more convenient to use a different version of the Riemann tensor and work in the local coordinates, where the Christoffel symbols vanish, but their derivatives do not.

(b) Using the above identity, show that

$$\nabla_b\, G^{b}_{a} = 0.$$

2. *The stress energy tensor of an ideal fluid:* Consider an ideal fluid described by the energy density $\rho\, c^2$ (with ρ being the mass density) and pressure p. Further, assume that the fluid does not possess any anisotropic stress.

(a) Argue that, in the comoving frame, the stress energy tensor of the fluid is given by

$$T^{\mu}_{\nu} = \text{diag.}\, \left(\rho\, c^2, -p, -p, -p\right).$$

(b) Further, show that, in a general frame, the stress energy tensor of the fluid can be written as

$$T^{\mu}_{\nu} = (\rho\, c^2 + p)\, u^{\mu}\, u_{\nu} - p\, \delta^{\mu}_{\nu},$$

where u^{μ} is the four velocity of the fluid.

(c) Using the law governing the conservation of the stress energy tensor, arrive at the equations of motion that describe an ideal fluid in Minkowski spacetime.

3. *The stress energy tensor of a scalar field:* Recall that, given an action that describes a matter field, the stress energy tensor associated with the matter field is given by the variation of the action with respect to the metric tensor as follows:

$$\delta S = \frac{1}{2\,c} \int \mathrm{d}^4x\, \sqrt{-g}\, T_{\mu\nu}\, \delta g^{\mu\nu} = -\frac{1}{2\,c} \int \mathrm{d}^4x\, \sqrt{-g}\, T^{\mu\nu}\, \delta g_{\mu\nu}.$$

Consider a scalar field ϕ that is governed by the following action:

$$S[\phi] = \frac{1}{c} \int \mathrm{d}^4x\, \sqrt{-g}\, \left[\frac{1}{2}\, g^{\mu\nu}\, \partial_\mu\phi\, \partial_\nu\phi - V(\phi)\right],$$

where $V(\phi)$ is the potential describing the scalar field.

(a) Upon varying this action with respect to the metric tensor, arrive at the stress energy tensor of the scalar field.

(b) Show that the conservation of the stress energy tensor leads to the equation of motion of the scalar field.

4. *The stress energy tensor of the electromagnetic field:* In a curved spacetime, the action describing the electromagnetic field is given by

$$S[A^\mu] = -\frac{1}{16\,\pi\,c}\int \mathrm{d}^4x\,\sqrt{-g}\,F_{\mu\nu}\,F^{\mu\nu},$$

where

$$F_{\mu\nu} = A_{\mu;\nu} - A_{\nu;\mu} = A_{\mu,\nu} - A_{\nu,\mu}.$$

(a) Construct the stress energy tensor associated with the electromagnetic field.

(b) What are the time-time and the time-space components of the stress energy tensor of the electromagnetic field in flat spacetime?

5. *The nature of a worm hole:* The spacetime of a worm hole is described by the line-element

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}r^2 - \left(b^2 + r^2\right)\,\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right),$$

where b is a constant with the dimensions of length that reflects the size of the 'traversable' region. Show that the energy density of matter has to be negative to sustain such a spacetime.

Exercise sheet 9

The Schwarzschild metric

1. *Spherically symmetric spacetimes:* Consider the following line element that describes spherically symmetric spacetimes in $(3+1)$-dimensions:

$$\mathrm{d}s^2 = c^2\, e^{\Phi(t,r)}\, \mathrm{d}t^2 - e^{\Psi(t,r)}\, \mathrm{d}r^2 - r^2\, \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right),$$

where $\Phi(t,r)$ and $\Psi(t,r)$ are arbitrary functions of the coordinates t and r.

   (a) Find $g_{\mu\nu}$ and $g^{\mu\nu}$ corresponding to this line element.

   (b) Evaluate the resulting $\Gamma^{\alpha}_{\mu\nu}$.

   (c) Also, calculate the corresponding $R_{\mu\nu}$ and R.

2. *Utilizing the Bianchi identities:* Compute the Einstein tensor corresponding to the above line element and show that its non-zero components are given by

$$\begin{aligned}
G^t_t &= \left(\frac{\Psi'}{r} - \frac{1}{r^2}\right)\, e^{-\Psi} + \frac{1}{r^2},\\
G^r_t &= -\frac{\dot{\Psi}}{r\, e^{\Psi}} = -G^t_r\, e^{(\Psi-\Phi)},\\
G^r_r &= -\left(\frac{\Phi'}{r} + \frac{1}{r^2}\right)\, e^{-\Psi} + \frac{1}{r^2},\\
G^\theta_\theta &= G^\phi_\phi = \frac{1}{2}\, \left(\frac{\Phi'\,\Psi'}{2} + \frac{\Psi'}{r} - \frac{\Phi'}{r} - \frac{\Phi'^2}{2} - \Phi''\right)\, e^{-\Psi} + \frac{1}{2}\, \left(\ddot{\Psi} + \frac{\dot{\Psi}^2}{2} - \frac{\dot{\Phi}\,\dot{\Psi}}{2}\right)\, e^{-\Phi},
\end{aligned}$$

where the overdots and the overprimes denote differentiation with respect to $c\,t$ and r, respectively. Show that the contracted Bianchi identities, viz. $\nabla_\mu\, G^\mu_\nu = 0$, imply that the last of the above equations vanishes, if the remaining three equations vanish.

3. *Spherically symmetric vacuum solution of the Einstein's equations:* In the absence of any sources, the above components of the Einstein tensor should vanish. Integrate the equations to arrive at the following Schwarzschild line element:

$$\mathrm{d}s^2 = c^2\, \left(1 - \frac{2\,G\,M}{c^2\,r}\right)\, \mathrm{d}t^2 - \left(1 - \frac{2\,G\,M}{c^2\,r}\right)^{-1}\, \mathrm{d}r^2 - r^2\, \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right),$$

where M is a constant of integration that denotes the mass of the central object that is responsible for the gravitational field.

4. *The precession of the perihelion of Mercury:* Consider a particle of mass m propagating in the above Schwarzschild spacetime.

   (a) Using the relation $p^\mu\, p_\mu = m^2\, c^2$ and the conserved momenta, arrive at the following differential equation describing the orbital motion of massive particles:

$$\frac{\mathrm{d}^2\, u}{\mathrm{d}\phi^2} + u = \frac{G\,M}{\tilde{L}^2} + \frac{3\,G\,M}{c^2}\, u^2,$$

   where $u = 1/r$, while $\tilde{L} = L/m = r^2\, (\mathrm{d}\phi/d\tau)$, with L being the angular momentum of the particle and τ its proper time.

(b) The second term on the right hand side of the above equation would have been absent in the case of the conventional, non-relativistic, Kepler problem. Treating the term as a small perturbation, show that the orbits are no more closed, and the perihelion precesses by the angle

$$\Delta\phi \simeq \frac{6\,\pi\,(G\,M)^2}{\tilde{L}^2\,c^2} = \frac{6\,\pi\,G\,M}{a\,(1-e^2)\,c^2}\ \text{radians/revolution},$$

where e and a are the eccentricity and the semi-major axis of the original closed, Keplerian elliptical orbit.

(c) For the case of the planet Mercury, $a = 5.8 \times 10^{10}$ m, while $e = 0.2$. Also, the period of the Mercury's orbit around the Sun is 88 days. Further, the mass of the Sun is $M_\odot = 2 \times 10^{30}$ kg. Use these information to determine the angle by which the perihelion of Mercury would have shifted in a century.

Note: The measured precession of the perihelion of the planet Mercury proves to be $5599''.7 \pm 0''.4$ per century, but a large part of it is caused due to the influences of the other planets. When the other contributions have been subtracted, the precession of the perihelion of the planet Mercury due to the purely relativistic effects amounts to 43.1 ± 0.5 seconds of arc per century.

5. *Gravitational bending of light:* Consider the propagation of photons in the Schwarzschild spacetime.

(a) Using the relation $p^\mu\,p_\mu = 0$ and the conserved momenta, arrive at the following differential equation describing the orbital motion of photons in the spacetime:

$$\frac{\mathrm{d}^2\,u}{\mathrm{d}\phi^2} + u = \frac{3\,G\,M}{c^2}\,u^2,$$

(b) Establish that, in the absence of the term on the right hand side, the photons will travel in straight lines.

(c) As in the previous case, treating the term on the right hand side as a small perturbation, show that it leads to a deflection of a photon's trajectory by the angle

$$\Delta\phi \simeq \frac{4\,G\,M}{c^2\,b},$$

where b is the impact parameter of the photon (i.e. the distance of the closest approach of the photon to the central mass).

(d) Given that the radius of the Sun is 6.96×10^8 m, determine the deflection angle $\Delta\phi$ for a ray of light that grazes the Sun.

Note: The famous 1919 eclipse expedition led by Eddington led to two sets of results, viz.

$$\Delta\phi = 1''.98 \pm 0''.16 \quad \text{and} \quad \Delta\phi = 1''.61 \pm 0''.4,$$

both of which happen to be consistent with the theory.

Additional problems II

Tensor algebra, calculus and general relativity

1. *Metric on a three sphere:* The three sphere $\mathbb{S}^3$ is a three-dimensional spherical surface in the four-dimensional Euclidean space $\mathbb{R}^4$. Let (x, y, z, w) be the coordinates of $\mathbb{R}^4$, while R is the radius of the three sphere. The three sphere can be then described by the constraint

$$x^2 + y^2 + z^2 + w^2 = R^2.$$

   (a) Show that the following equations:

$$x = R \sin\chi \sin\theta \cos\phi, \quad y = R \sin\chi \sin\theta \sin\phi, \quad z = R \sin\chi \cos\theta \quad \text{and} \quad w = R \cos\chi,$$

   which relate the coordinates (θ, ϕ, χ) on the three-sphere to the Euclidean coordinates (x, y, z, w) satisfy the above constraint.

   Note: These generalize the familiar spherical polar coordinates to a higher dimension.

   (b) Evaluate the metric of the three sphere in terms of the coordinates (θ, ϕ, χ).

2. *Parallel transporting a vector on $\mathbb{S}^2$:* The components of a vector A^a on the two sphere $\mathbb{S}^2$ are found to be $(1, 0)$ at $(\theta = \theta_0, \phi = 0)$, where θ_0 is a constant. The vector is parallel transported around the circle $\theta = \theta_0$. Determine the vector when it returns to the original point.

3. *Rewriting the Riemann tensor:* Recall that, the Riemann tensor is defined as

$$R_{abcd} = g_{ae}\, R^{e}{}_{bcd} = g_{ae}\, \left(\Gamma^e_{bd,c} - \Gamma^e_{bc,d} + \Gamma^e_{fc}\,\Gamma^f_{bd} - \Gamma^e_{fd}\,\Gamma^f_{bc}\right).$$

   Show that this can be rewritten as

$$R_{abcd} = \frac{1}{2}\, \left(g_{ad,bc} + g_{bc,ad} - g_{ac,bd} - g_{bd,ac}\right) + g_{ef}\, \left(\Gamma^e_{bc}\,\Gamma^f_{ad} - \Gamma^e_{bd}\,\Gamma^f_{ac}\right),$$

   an expression which reflects the symmetries of the Riemann tensor more easily.

4. *Geodesic deviation:* Consider two nearby geodesics, say, $x^a(\lambda)$ and $\bar{x}^a(\lambda)$, where λ is an affine parameter. Let $\xi^a(\lambda)$ denote a 'small vector' that connects these two geodesics. Working in the locally geodesic coordinates, show that ξ^a satisfies the differential equation

$$\frac{\mathrm{D}^2\xi^a}{\mathrm{D}\lambda^2} + R^a_{bcd}\, \dot{x}^b\, \xi^c\, \dot{x}^d = 0,$$

   where

$$\frac{\mathrm{D}^2\xi^a}{\mathrm{D}\lambda^2} \equiv \left(\dot{\xi}^a + \Gamma^a_{bc}\,\xi^b\,\dot{x}^c\right)^{\cdot},$$

   while the overdots denote differentiation with respect to λ.

   Note: This implies that a non-zero Riemann tensor R^a_{bcd} will lead to a situation where geodesics, in general, will not remain parallel as, for instance, on the surface of the two sphere $\mathbb{S}^2$.

5. *Gravity in two dimensions:* Consider an arbitrary spacetime in two dimensions that is described by the metric g_{ab}.

   (a) Argue that, in such a case, the Riemann tensor can be expressed as follows:

$$R_{abcd} = \kappa\, \left(g_{ac}\, g_{bd} - g_{ad}\, g_{bc}\right),$$

   where κ is a scalar that is, in general, a function of the coordinates.

   Note: It is useful to recall that, in n-dimensions, the number of independent components of the Riemann tensor is $n^2\,(n^2 - 1)/12$.

(b) Using this result, show that the Einstein tensor vanishes *identically* in two dimensions.

6. *Conformal transformation:* Show that, under the conformal transformation,

$$g_{ab}(x^c) \to \Omega^2(x^c)\, g_{ab}(x^c),$$

the Christoffel symbols Γ^a_{bc}, the Ricci tensor R^a_b, and the scalar curvature R of a n-dimensional manifold are modified as follows:

$$\begin{aligned}
\Gamma^a_{bc} &\to \Gamma^a_{bc} + \Omega^{-1}\, \left(\delta^a_b\, \Omega_{;c} + \delta^a_c\, \Omega_{;b} - g_{bc}\, g^{ad}\, \Omega_{;d}\right), \\
R^a_b &\to \Omega^{-2}\, R^a_b - (n-2)\, \Omega^{-1}\, g^{ac}\, \left(\Omega^{-1}\right)_{;bc} + \frac{1}{n-2}\, \Omega^{-n}\, \delta^a_b\, g^{cd}\, \left[\Omega^{(n-2)}\right]_{;cd}, \\
R &\to \Omega^{-2}\, R + 2\,(n-1)\, \Omega^{-3}\, g^{ab}\, \Omega_{;ab} + (n-1)\,(n-4)\, \Omega^{-4}\, g^{ab}\, \Omega_{;a}\, \Omega_{;b}.
\end{aligned}$$

7. *Conformal invariance of the electromagnetic action:* Recall that, in a curved spacetime, the dynamics of the source free electromagnetic field is governed by the action

$$S[A^\mu] = -\frac{1}{16\,\pi\, c} \int \mathrm{d}^4x\, \sqrt{-g}\, F_{\mu\nu}\, F^{\mu\nu},$$

where

$$F_{\mu\nu} = A_{\mu;\nu} - A_{\nu;\mu} = A_{\mu,\nu} - A_{\nu,\mu}$$

and the semi-colons denote covariant differentiation, while the commas denote partial derivatives. Show that this action is invariant under the following conformal transformation:

$$x^\mu \to x^\mu, \quad A_\mu \to A_\mu \quad \text{and} \quad g_{\mu\nu} \to \Omega^2\, g_{\mu\nu}.$$

8. *k-essence:* Consider a generic scalar field ϕ that is described by the action

$$S[\phi] = \frac{1}{c} \int \mathrm{d}^4x\, \sqrt{-g}\, \mathcal{L}(X, \phi),$$

where X denotes the kinetic energy of the scalar field and is given by

$$X = \frac{1}{2}\, g^{\mu\nu}\, \partial_\mu\phi\, \partial_\nu\phi.$$

Let the Lagrangian density $\mathcal{L}$ be an arbitrary function of the kinetic term X and the field ϕ. Vary the above action with respect to the metric tensor, and show that the corresponding stress energy tensor can be written as

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial X}\, \partial^\mu\phi\, \partial_\nu\phi - \delta^\mu_\nu\, \mathcal{L}.$$

Note: Such scalar fields are often referred to as k-essence.

9. *Charged and rotating black holes:* Use the given `Mathematica` file to evaluate the metric connections, the Riemann, the Ricci, and the Einstein tensors as well as the Ricci scalar around the charged Reissner-Nordstrom and the rotating Kerr black holes that are described by the following line elements:

$$\mathrm{d}s^2 = c^2\, \left(1 - \frac{2\,\mu}{r} + \frac{q^2}{r^2}\right)\, \mathrm{d}t^2 - \left(1 - \frac{2\,\mu}{r} + \frac{q^2}{r^2}\right)^{-1}\, \mathrm{d}r^2 - r^2\, \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right),$$

where

$$\mu = \frac{G\, M}{c^2} \quad \text{and} \quad q^2 = \frac{G\, Q^2}{4\,\pi\, c^4},$$

and

$$\mathrm{d}s^2 = c^2\, \frac{\rho^2\,\Delta}{\Sigma^2}\, \mathrm{d}t^2 - \frac{\Sigma^2 \sin^2\theta}{\rho^2}\, (\mathrm{d}\phi - \omega\, \mathrm{d}t)^2 - \frac{\rho^2}{\Delta}\, \mathrm{d}r^2 - \rho^2\, \mathrm{d}\theta^2,$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta,\;\; \Delta = r^2 - 2\,\mu\, r + a^2,\;\; \Sigma^2 = (r^2+a^2)^2 - a^2\,\Delta\,\sin^2\theta,\;\; \omega = \frac{2\,\mu\, c\, r\, a}{\Sigma^2}\;\; \text{and}\;\; a = \frac{J}{M\,c}.$$

The quantities M, Q and J are constants that denote the mass, the electric charge and the angular momentum associated with the black holes, respectively.

10. *The Newtonian limit and the Poisson equation:* Recall that, in the non-relativistic limit, the metric corresponding to the Newtonian potential ϕ is given by

$$\mathrm{d}s^2 = c^2\left[1 + \frac{2\,\phi(\mathbf{x})}{c^2}\right] \mathrm{d}t^2 - \mathrm{d}\mathbf{x}^2.$$

Let the energy density of the matter field that is giving rise to the Newtonian potential ϕ be $\rho\, c^2$. Show that, in such a case, the time-time component of the Einstein's equations reduces to the conventional Poisson equation in the limit of large c.

Note: As I had mentioned during the lectures, it is this Newtonian limit that determines the overall constant in the Einstein-Hilbert action.

Quiz III

The Einstein's equations and the Schwarzschild metric

1. *Scalar curvature in two dimensions:* Consider the following $(1+1)$-dimensional line element:

$$\mathrm{d}s^2 = f^2(\eta, \xi)\,\left(\mathrm{d}\eta^2 - d\xi^2\right),$$

where $f(\eta, \xi)$ is an arbitrary function of the coordinates η and ξ. Show that the scalar curvature associated with this line element can be expressed as $\boxed{\text{10 marks}}$

$$R = -\nabla_\mu \nabla^\mu \ln f^2 = -\Box \ln f^2.$$

Note: In $(1+1)$-dimensions, any metric can be reduced to the above conformally flat form.

2. *Einstein's equations for a homogeneous and isotropic spacetime:* A homogeneous and isotropic spacetime can be described by the following line element:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\,\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right),$$

where a is a function of the time coordinate t.

(a) Evaluate the Christoffel symbols corresponding to this line element. $\boxed{\text{4 marks}}$

(b) Calculate the corresponding Ricci and the Einstein tensors. $\boxed{\text{4 marks}}$

(c) In such a line element, the stress energy tensor of a perfect fluid reduces to the following simple form:

$$T^\mu_\nu = \mathrm{diag.}\,\left(\rho\, c^2, -p, -p, -p\right),$$

where the mass density ρ and the pressure p are only functions of the time t. Arrive at the Einstein's equations assuming that the above line element is driven by such a source. $\boxed{\text{2 marks}}$

3. *Effective potential for massive particles in the Schwarzschild metric:* Consider a particle of mass m which is moving in the Schwarzschild metric. The trajectory of the particle can be described by an equation of the following form:

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + V_{\mathrm{eff}}(r) = \frac{c^2}{2}\left[\left(\frac{E}{m\,c^2}\right)^2 - 1\right],$$

where V_{eff} is the 'effective potential' which governs the motion of the relativistic particle, E is its energy, while τ denotes the proper time as measured in the frame of the particle.

(a) Obtain the form of the effective potential V_{eff}. $\boxed{\text{5 marks}}$

(b) Show that the potential admits two circular orbits. Also, determine the radii of the orbits. $\boxed{\text{3 marks}}$

(c) Arrive at the condition on the angular momentum of the particle that leads to a situation wherein these two circular orbits merge into one. Further, determine the radius of the corresponding orbit. $\boxed{\text{2 marks}}$

4. *Circular orbits of photons in the Schwarzschild spacetime:* Recall that the orbital trajectory of a photon in the Schwarzschild metric is governed by the differential equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = \frac{3\,G\,M}{c^2}\,u^2,$$

where $u = (1/r)$.

(a) Does this equation admit circular orbits? If it does, utilize the equation to arrive at the radii of the circular orbits. $\boxed{\text{7 marks}}$

(b) Determine if these circular orbits are stable or unstable. $\boxed{\text{3 marks}}$

5. *Radial motion of particles and photons in the Schwarzschild metric:* Consider particles and photons which are traveling radially in the Schwarzschild spacetime.

(a) Let a particle fall radially from rest at radius r_0 to a radius r $(< r_0)$. Show that the proper time taken by the particle to travel from the larger radius to the smaller one is given by $\boxed{\text{5 marks}}$

$$\tau = \frac{2}{3}\left[\sqrt{\frac{r_0^3}{2\,G\,M}} - \sqrt{\frac{r^3}{2\,G\,M}}\right].$$

(b) Show that the trajectories of radially outgoing and ingoing photons can be expressed as $\boxed{\text{5 marks}}$

$$c\,t = r + \frac{2\,G\,M}{c^2}\,\ln\left|\frac{c^2\,r}{2\,G\,M} - 1\right| + \text{constant}$$

and

$$c\,t = -r - \frac{2\,G\,M}{c^2}\,\ln\left|\frac{c^2\,r}{2\,G\,M} - 1\right| + \text{constant},$$

respectively.

Exercise sheet 10

The kinematics of the Friedmann model

1. *Spaces of constant curvature:* Consider spaces of constant curvature that are described by the metric tensor g_{ab}.

   (a) Argue that, the Riemann tensor associated with such a space can be expressed in terms of the metric g_{ab} as follows:
   $$R_{abcd} = \kappa\ (g_{ac}\, g_{bd} - g_{ad}\, g_{bc})\,,$$
   where κ is a constant.

   (b) Show that the Ricci tensor corresponding to the above Riemann tensor is given by
   $$R_{ab} = 2\,\kappa\, g_{ab}.$$

   Note: Examples of spacetimes with a constant scalar curvature are the Einstein static universe, the de Sitter and the anti de Sitter spacetimes.

2. *Visualizing the Friedmann metric:* The Friedmann universe is described by the line-element
   $$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\,\mathrm{d}\ell^2,$$
   where
   $$\mathrm{d}\ell^2 = \frac{\mathrm{d}r^2}{(1-\kappa\, r^2)} + r^2\ \left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)$$
   and $\kappa = 0, \pm 1$.

   (a) Let us define a new coordinate χ as follows:
   $$\chi = \int \frac{dr}{\sqrt{1-\kappa\, r^2}}.$$
   Show that in terms of the coordinate χ the spatial line element $\mathrm{d}\ell^2$ reduces to
   $$\mathrm{d}\ell^2 = \mathrm{d}\chi^2 + S_\kappa^2(\chi)\ \left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right),$$
   where
   $$S_\kappa(\chi) = \begin{cases} \sin\chi & \text{for} \quad \kappa = 1, \\ \chi & \text{for} \quad \kappa = 0, \\ \sinh\chi & \text{for} \quad \kappa = -1. \end{cases}$$

   (b) Show that, for $\kappa = 1$, the spatial line-element $\mathrm{d}\ell^2$ can be described as the spherical surface
   $$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$$
   embedded in an Euclidean space described by the line-element
   $$\mathrm{d}\ell^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \mathrm{d}x_4^2.$$

   (c) Show that, for $\kappa = -1$, the spatial line-element $\mathrm{d}\ell^2$ can be described as the hyperbolic surface
   $$x_1^2 + x_2^2 + x_3^2 - x_4^2 = -1$$
   embedded in a Lorentzian space described by the line-element
   $$\mathrm{d}\ell^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 - \mathrm{d}x_4^2.$$

3. *Geodesic equations in a Friedmann universe:* Obtain the following non-zero components of the Christoffel symbols for the Friedmann line element:

$$\Gamma^t_{ij} = \frac{a\,\dot{a}}{c}\,\sigma_{ij},$$

where σ_{ij} denotes the spatial metric defined through the relation $\mathrm{d}\ell^2 = \sigma_{ij}\ \mathrm{d}x^i\,\mathrm{d}x^j$. Use these Christoffel symbols to arrive at the geodesic equations corresponding to the t coordinate for massive as well as massless particles in a Friedmann universe.

4. *Weyl tensor and conformal invariance:* In $(3+1)$-spacetime dimensions, the Weyl tensor $C_{\alpha\beta\gamma\delta}$ is defined as follows:

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + \frac{1}{2}\left(g_{\alpha\delta}\,R_{\beta\gamma} + g_{\beta\gamma}\,R_{\alpha\delta} - g_{\alpha\gamma}\,R_{\beta\delta} - g_{\beta\delta}\,R_{\alpha\gamma}\right) + \frac{1}{6}\left(g_{\alpha\gamma}\,g_{\delta\beta} - g_{\alpha\delta}\,g_{\gamma\beta}\right)\,R.$$

   (a) Show that the Weyl tensor vanishes for the Friedmann metric.

   (b) The vanishing Weyl tensor implies that there exists a coordinate system in which the Friedmann metric (for all κ) is conformal to the Minkowski metric. It is straightforward to check that the metric of the $\kappa = 0$ (i.e. the spatially flat) Friedmann universe can be expressed in the following form:

$$g_{\mu\nu} = a^2(\eta)\,\eta_{\mu\nu},$$

   where η is the conformal time coordinate defined by the relation

$$\eta = \int \frac{\mathrm{d}t}{a(t)},$$

   and $\eta_{\mu\nu}$ denotes the flat spacetime metric. Construct the coordinate systems in which the metrics corresponding to the $\kappa = \pm 1$ Friedmann universes can be expressed in a form wherein they are conformally related to flat spacetime.

5. *Consequences of conformal invariance:* As we have seen, the action of the electromagnetic field in a curved spacetime is invariant under the conformal transformation.

   (a) Utilizing the conformal invariance of the electromagnetic action, show that the electromagnetic waves in the spatially flat Friedmann universe can be written in terms of the conformal time coordinate η as follows:

$$A_\mu \propto \exp -(i\,k\,\eta) = \exp -\left[i\,k\,\int dt/a(t)\right].$$

   (b) Since the time derivative of the phase defines the instantaneous frequency $\omega(t)$ of the wave, conclude that $\omega(t) \propto a^{-1}(t)$.

Exercise sheet 11

The dynamics of the Friedmann model

1. *The Friedmann equations:* Recall that the Friedmann universe is described by the line element

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-\kappa\,r^2} + r^2\,\left(\mathrm{d}\theta^2 + \sin^2\!\theta\,\mathrm{d}\phi^2\right)\right],$$

where $\kappa = 0, \pm 1$.

(a) Arrive at the following expressions for the Ricci tensor R^μ_ν, the scalar curvature R, and the Einstein tensor G^μ_ν for the above Friedmann metric:

$$\begin{aligned} R^t_t &= -\frac{3\,\ddot{a}}{c^2\,a}, \\ R^i_j &= -\left[\frac{\ddot{a}}{c^2\,a} + 2\left(\frac{\dot{a}}{c\,a}\right)^2 + \frac{2\,\kappa}{a^2}\right]\delta^i_j, \\ R &= -6\left[\frac{\ddot{a}}{c^2\,a} + \left(\frac{\dot{a}}{c\,a}\right)^2 + \frac{\kappa}{a^2}\right], \\ G^t_t &= 3\left[\left(\frac{\dot{a}}{c\,a}\right)^2 + \frac{\kappa}{a^2}\right], \\ G^i_j &= \left[\frac{2\,\ddot{a}}{c^2\,a} + \left(\frac{\dot{a}}{c\,a}\right)^2 + \frac{\kappa}{a^2}\right]\delta^i_j, \end{aligned}$$

where the overdots denote differentiation with respect to the cosmic time t.

(b) Consider a fluid described by the stress energy tensor

$$T^\mu_\nu = \mathrm{diag.}\left(\rho\,c^2, -p, -p, -p\right),$$

where ρ and p denote the mass density and the pressure associated with the fluid. In a smooth Friedmann universe, the quantities ρ and p depend only on time. Using the above Einstein tensor, obtain the following Friedmann equations for such a source:

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa\,c^2}{a^2} &= \frac{8\,\pi\,G}{3}\,\rho, \\ \frac{2\,\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa\,c^2}{a^2} &= -\,\frac{8\,\pi\,G}{c^2}\,p. \end{aligned}$$

(c) Show that these two Friedmann equations lead to the equation

$$\frac{\ddot{a}}{a} = -\,\frac{4\,\pi\,G}{3}\,\left(\rho + \frac{3\,p}{c^2}\right).$$

Note: This relation implies that $\ddot{a} > 0$, i.e. the universe will undergo accelerated expansion, only when $\left(\rho\,c^2 + 3\,p\right) < 0$.

2. *Conservation of the stress energy tensor in a Friedmann universe:* Recall that the conservation of the stress energy tensor is described by the equation $T^\mu_{\nu;\mu} = 0$.

(a) Show that the time component of the stress energy tensor conservation law leads to the following equation in a Friedmann universe:

$$\dot{\rho} + 3\,H\,\left(\rho + \frac{p}{c^2}\right) = 0,$$

where $H = \dot{a}/a$, a quantity that is known as the Hubble parameter.

(b) Also arrive at this equation from the two Friedmann equations obtained above.

(c) Show that the above equation can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\, a^3\right) = -\frac{p}{c^2}\left(\frac{\mathrm{d}a^3}{\mathrm{d}t}\right).$$

3. <u>*Evolution of energy density in a Friedmann universe:*</u> The different types of matter that are present in the universe are often described by an equation of state, i.e. the relation between the density and the pressure associated the matter. Consider the following equation of state $p = w\,\rho\, c^2$, where w is a constant.

   (a) Using the above equation which governs the evolution of ρ in a Friedmann universe, show that, in such a case,
   $$\rho \propto a^{-3\,(1+w)}.$$

   (b) While the quantity w vanishes for pressure free non-relativistic matter (such as baryons and cold dark matter), $w = 1/3$ for relativistic particles (such as photons and the nearly massless neutrinos). Note that the energy density does not change with time when $w = -1$ or, equivalently, when $p = -\rho\, c^2$. Such a type of matter is known as the cosmological constant. Utilizing the above result, express the total density of a universe filled with non-relativistic (NR) and relativistic (R) matter as well as the cosmological constant (Λ) as follows:
   $$\rho(a) = \rho_{\mathrm{NR}}^0\,\left(\frac{a_0}{a}\right)^3 + \rho_{\mathrm{R}}^0\,\left(\frac{a_0}{a}\right)^4 + \rho_{\Lambda},$$
   where ρ_{NR}^0 and ρ_{R}^0 denote the density of non-relativistic and relativistic matter today (i.e. at, say, $t = t_0$, corresponding to the scale factor $a = a_0$).

   (c) Also, further rewrite the above expression as
   $$\rho(a) = \rho_{\mathrm{C}}\,\left[\Omega_{\mathrm{NR}}\,\left(\frac{a_0}{a}\right)^3 + \Omega_{\mathrm{R}}\,\left(\frac{a_0}{a}\right)^4 + \Omega_{\Lambda}\right] = \rho_{\mathrm{C}}\,\left[\Omega_{\mathrm{NR}}\,(1+z)^3 + \Omega_{\mathrm{R}}\,(1+z)^4 + \Omega_{\Lambda}\right],$$
   where $\Omega_{\mathrm{NR}} = \rho_{\mathrm{NR}}^0/\rho_{\mathrm{C}}$, $\Omega_{\mathrm{R}} = \rho_{\mathrm{R}}^0/\rho_{\mathrm{C}}$ and $\Omega_{\Lambda} = \rho_{\Lambda}/\rho_{\mathrm{C}}$, while ρ_{C} is the so-called critical density defined as
   $$\rho_{\mathrm{C}} = \frac{3\,H_0^2}{8\,\pi\,G},$$
   with the quantity H_0 being the Hubble parameter (referred to as the Hubble constant) today. Note: The quantities H_0, Ω_{NR}, Ω_{R} and Ω_{Λ} are cosmological parameters that are to be determined by observations.

   (d) Observations suggest that $H_0 \simeq 72$ km s^{-1} Mpc^{-1}. Evaluate the corresponding numerical value of the critical density ρ_{c}.
   Note: A parsec (pc) corresponds to 3.26 light years, and a Mega parsec (Mpc) amounts to 10^6 parsecs.

4. <u>*The Cosmic Microwave Background:*</u> It is found that we are immersed in a perfectly thermal and nearly isotropic distribution of radiation, which is referred to as Cosmic Microwave Background (CMB), as it energy density peaks in the microwave region of the electromagnetic spectrum. The CMB is a relic of an earlier epoch when the universe was radiation dominated, and it provides the dominant contribution to the relativistic energy density in the universe.

   (a) Given that the temperature of the CMB today is $T \simeq 2.73$ K, show that one can write
   $$\Omega_{\mathrm{R}}\, h^2 \simeq 2.56 \times 10^{-5},$$
   where h is related to the Hubble constant H_0 as follows:
   $$H_0 \simeq 100\, h \text{ km s}^{-1}\text{ Mpc}^{-1}.$$

(b) Show that the redshift $z_{\rm eq}$ at which the energy density of matter and radiation were equal is given by

$$1+z_{\rm eq} = \frac{\Omega_{\rm NR}}{\Omega_{\rm R}} \simeq 3.9\times 10^4\,\left(\Omega_{\rm NR}\,h^2\right).$$

(c) Also, show that the temperature of the radiation at this epoch is given by

$$T_{\rm eq} \simeq 9.24\,\left(\Omega_{\rm NR}\,h^2\right)\ {\rm eV}.$$

5. <u>*Solutions to the Friedmann equations:*</u> We had discussed the solutions to Friedmann equations in the presence of a single component when the universe is spatially flat (i.e. when $\kappa = 0$). It proves to be difficult to obtain analytical solutions for the scale factor when all the three components of matter (viz. non-relativistic and relativistic matter as well as the cosmological constant) are simultaneously present. However, the solutions can be obtained for the cases wherein two of the components are present.

   (a) Integrate the first Friedmann equation for a $\kappa = 0$ universe with matter and radiation to obtain that

   $$a(\eta) = \sqrt{\Omega_{\rm R}\,a_0^4}\,\left(H_0\,\eta\right) + \frac{\Omega_{\rm NR}\,a_0^3}{4}\,\left(H_0\,\eta\right)^2,$$

   where η is the conformal time coordinate. Show that, at early (i.e. for small η) and late times (i.e. for large η), this solution reduces to the behavior in the radiation and matter dominated epochs, respectively, as required.

   Note: In obtaining the above result, it has been assumed that $a = 0$ at $\eta = 0$.

   (b) Integrate the Friedmann equation for a $\kappa = 0$ universe with matter and cosmological constant to obtain that

   $$\frac{a(t)}{a_0} = \frac{\Omega_{\rm NR}}{\Omega_{\Lambda}}\,\sinh^{2/3}\left(\frac{3\,\Omega_{\Lambda}^{3/2}\,H_0\,t}{2\,\Omega_{\rm NR}}\right).$$

   Also, show that, at early times, this solution simplifies to $a \propto t^{2/3}$, while at late times, it behaves as $a \propto \exp\,(\Omega_{\Lambda}^{3/2}\,H_0\,t/\Omega_{\rm NR})$, as expected.

Exercise sheet 12

Gravitational waves

1. *The linearized metric I:* Consider a small perturbation to flat spacetime so that the standard Minkowski metric can be expressed as

$$g_{\mu\nu} \simeq \eta_{\mu\nu} + \epsilon\, h_{\mu\nu},$$

where ϵ is a small dimensionless quantity. Show that, at the same order in ϵ, the corresponding contravariant metric tensor and the Christoffel symbols are given by

$$g^{\mu\nu} \simeq \eta^{\mu\nu} + \epsilon\, h^{\mu\nu}$$

and

$$\Gamma^{\alpha}_{\beta\gamma} \simeq \frac{\epsilon}{2}\left(h^{\alpha}_{\gamma,\beta} + h^{\alpha}_{\beta,\gamma} - h_{\beta\gamma}{}^{,\alpha}\right),$$

respectively.

2. *The linearized metric II:* Let us now turn to the evaluation of the curvature and the Einstein tensors corresponding to the above metric.

(a) Show that, at the linear order, the Riemann and the Ricci tensors and the scalar curvature are given by

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &\simeq \frac{\epsilon}{2}\left(h_{\alpha\delta,\beta\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta} - h_{\beta\delta,\alpha\gamma}\right), \\ R_{\beta\delta} &\simeq \frac{\epsilon}{2}\left(h^{\gamma}_{\delta,\beta\gamma} + h^{\alpha}_{\beta,\alpha\delta} - h_{,\beta\delta} - \eta^{\alpha\gamma}\, h_{\beta\delta,\alpha\gamma}\right) \end{aligned}$$

and

$$R = \epsilon\left(h^{\alpha\beta}{}_{,\alpha\beta} - \Box\, h\right),$$

where $\Box$ is the d'Alembertian corresponding to the Minkowski metric $\eta_{\mu\nu}$, while $h = \eta^{\mu\nu}\, h_{\mu\nu}$ denotes the trace of the perturbation $h_{\mu\nu}$.

(b) Finally, show that the corresponding Einstein tensor can be expressed as

$$G_{\alpha\beta} = \frac{\epsilon}{2}\left(h^{\gamma}_{\beta,\alpha\gamma} + h^{\gamma}_{\alpha,\beta\gamma} - \Box\, h_{\alpha\beta} - h_{,\alpha\beta} - \eta_{\alpha\beta}\, h^{\gamma\delta}{}_{,\gamma\delta} + \eta_{\alpha\beta}\, \Box\, h\right).$$

3. *Gauge transformations:* Consider the following 'small' coordinate transformations:

$$x^{\mu} \to x'^{\mu} \simeq x^{\mu} + \epsilon\, \xi^{\mu},$$

which are of the same amplitude as the perturbation $h_{\mu\nu}$. Show that under such a transformation, the perturbation $h_{\mu\nu}$ transforms as follows:

$$h_{\mu\nu} \to h'_{\mu\nu} \simeq h_{\mu\nu} - \left(\xi_{\mu,\nu} + \xi_{\nu,\mu}\right).$$

Note: Such a 'small' transformation is known as a gauge transformation.

4. *The de Donder gauge:* Let us define a new set of variables $\psi_{\mu\nu}$, which are related to the metric perturbation $h_{\mu\nu}$ as follows:

$$\psi_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\, h.$$

(a) Show that, in terms of $\psi_{\mu\nu}$, the above Einstein tensor is given by

$$G_{\alpha\beta} = \frac{\epsilon}{2}\left(\psi^{\gamma}_{\alpha,\beta\gamma} + \psi^{\gamma}_{\beta,\alpha\gamma} - \Box\, \psi_{\alpha\beta} - \eta_{\alpha\beta}\, \psi^{\gamma\delta}{}_{,\gamma\delta}\right).$$

(b) Show that, under the above-mentioned gauge transformations, the variables $\psi_{\mu\nu}$ transform as

$$\psi_{\mu\nu} \to \psi'_{\mu\nu} \simeq \psi_{\mu\nu} - (\xi_{\mu,\nu} + \xi_{\nu,\mu}) + \eta_{\mu\nu}\, \xi^{\lambda}_{,\lambda}.$$

(c) If we now impose the condition

$$\psi^{\alpha}_{\beta,\alpha} = 0,$$

show that, this corresponds to

$$\psi'^{\alpha}_{\beta,\alpha} = \psi^{\alpha}_{\beta,\alpha} - \Box\, \xi_{\beta}.$$

Note: These conditions correspond to four equations, which can be achieved using the gauge functions ξ_{μ}. A gauge wherein the condition is satisfied is known as the de Donder gauge.

(d) Also, show that the above condition corresponds to the following condition on $h_{\alpha\beta}$:

$$h^{\alpha}_{\beta,\alpha} - \frac{1}{2}\, h_{,\beta} = 0.$$

5. *The wave equation:* In the absence of sources, one has $G_{\alpha\beta} = 0$.

(a) Show that, in a gauge wherein $\psi^{\alpha}_{\beta,\alpha} = 0$, the vacuum Einstein's equations simplify to

$$\Box\, \psi_{\alpha\beta} = 0.$$

(b) Show that, in terms of $h_{\alpha\beta}$, this equation corresponds to the equation

$$\Box\, h_{\alpha\beta} = 0,$$

along with the additional condition

$$\Box\, h = 0.$$

Note: The solutions to these equations describe propagating gravitational waves in flat spacetime.

End-of-semester exam

From special relativity to gravitational waves

1. *Relative velocity between two inertial frames:* Consider two inertial frames that are moving with the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ with respect to, say, the laboratory frame. Show that the relative velocity v between the two frames can be expressed as $\boxed{\text{10 marks}}$

$$v^2 = \frac{(\mathbf{v}_1 - \mathbf{v}_2)^2 - (\mathbf{v}_1 \times \mathbf{v}_2/c)^2}{[1 - (\mathbf{v}_1 \cdot \mathbf{v}_2/c^2)]^2}.$$

2. *Lorentz transformation of the electromagnetic field:* Consider a charge e that is moving along, say, the positive x-axis at the constant velocity v with respect to the laboratory frame.

   (a) Write down the electric field in the frame of the charge. $\boxed{\text{3 marks}}$

   (b) Show that the electric field in the laboratory frame can be written as $\boxed{\text{5 marks}}$

$$\mathbf{E} = \frac{e\,\mathbf{R}}{R^3}\,\left[\frac{1 - v^2/c^2}{1 - (v^2/c^2)\,\sin^2\theta}\right],$$

   where $\mathbf{R}$ is the radius vector from the location of the charge to the point of observation, while the quantity θ is the angle between the direction of motion of the charge and the radius vector $\mathbf{R}$ in the laboratory frame.

   (c) What is the value of θ for which the strength of the electric field is the maximum? $\boxed{\text{2 marks}}$

3. *Is it three or is it two?* A space purports to be three dimensional, with coordinates x, y and z, and the metric

$$\mathrm{d}\ell^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - \left(\frac{3}{13}\,\mathrm{d}x + \frac{4}{13}\,\mathrm{d}y + \frac{12}{13}\,\mathrm{d}z\right)^2.$$

   (a) Show that it is actually a two dimensional space. $\boxed{\text{7 marks}}$

   (b) Is a constant z surface two dimensional? $\boxed{\text{3 marks}}$

4. *Klein-Gordon equation in a spatially flat Friedmann universe:* Recall that a scalar field, say, ϕ, satisfies the following Klein-Gordon equation in a curved spacetime:

$$\frac{1}{\sqrt{-g}}\,\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\,\partial_\nu\right)\,\phi + m^2\,\phi = 0.$$

   Assuming that the field ϕ to be dependent only on the cosmic time t, arrive at the equation of motion of the field in a spatially flat, Friedmann universe. $\boxed{\text{10 marks}}$

   Note: It would be to convenient to work in the Cartesian coordinates to describe the spatial part of the spatially flat Friedmann universe.

5. *Rewriting the Riemann tensor:* Recall that, the Riemann tensor is defined as

$$R_{abcd} = g_{ae}\,R^e{}_{bcd} = g_{ae}\,\left(\Gamma^e_{bd,c} - \Gamma^e_{bc,d} + \Gamma^e_{fc}\,\Gamma^f_{bd} - \Gamma^e_{fd}\,\Gamma^f_{bc}\right).$$

   Show that this can be rewritten as

$$R_{abcd} = \frac{1}{2}\,\left(g_{ad,bc} + g_{bc,ad} - g_{ac,bd} - g_{bd,ac}\right) + g_{ef}\,\left(\Gamma^e_{bc}\,\Gamma^f_{ad} - \Gamma^e_{bd}\,\Gamma^f_{ac}\right),$$

   an expression which reflects the symmetries of the Riemann tensor more easily. $\boxed{\text{10 marks}}$

6. *Scalar curvature corresponding to a gravitational wave:* Recall that a gravitational wave propagating in a flat spacetime can be described as a perturbation about the Minkowski metric $\eta_{\mu\nu}$ as follows:
$$g_{\mu\nu} \simeq \eta_{\mu\nu} + \epsilon\, h_{\mu\nu}$$
with ϵ being a small dimensionless quantity. Show that, at the order ϵ, the scalar curvature associated with the above metric can be expressed as
$$R \simeq \epsilon\, \left(h^{\alpha\beta}{}_{,\alpha\beta} - \Box\, h\right),$$
where $\Box$ is the d'Alembertian corresponding to the Minkowski metric $\eta_{\mu\nu}$, while $h = \eta^{\mu\nu}\, h_{\mu\nu}$ denotes the trace of the perturbation $h_{\mu\nu}$. 10 marks

7. *Schwarzschild metric in the Eddington-Finkelstein coordinates:* Recall that, in a Schwarzschild spacetime, the trajectory of radially ingoing photons is given by
$$c\,t = -r - \frac{2\,G\,M}{c^2}\,\ln\left|\frac{c^2\,r}{2\,G\,M} - 1\right| + \text{constant}.$$
Define a new time coordinate $\bar{t}$ that is related to the coordinates t and r through the following relation for $r > 2\,G\,M/c^2$:
$$c\,\bar{t} = c\,t + r + \frac{2\,G\,M}{c^2}\,\ln\left(\frac{c^2\,r}{2\,G\,M} - 1\right).$$
Show that, in terms of the new time coordinate $\bar{t}$, the Schwarzschild line-element reduces to 10 marks
$$\mathrm{d}s^2 = c^2\,\left(1 - \frac{2\,G\,M}{c^2\,r}\right)\,\mathrm{d}\bar{t}^2 - 2\,c\,\mathrm{d}\bar{t}\,\mathrm{d}r - r^2\,\left(\mathrm{d}\theta^2 + \sin^2\!\theta\,\mathrm{d}\phi^2\right).$$
Note: The new coordinates $(\bar{t}, r, \theta, \phi)$ are known as the Eddington-Finkelstein coordinates.

8. *Numbers describing our universe:* Various observations indicate the value of the Hubble constant H_0 to be
$$H_0 \simeq 72\ \text{km s}^{-1}\ \text{Mpc}^{-1}.$$
Given this information,

   (a) Evaluate the corresponding time scale H_0^{-1} in terms of billions of years. 3 marks

   (b) Estimate the resulting distance $c\,H_0^{-1}$ in units of Mpc. 3 marks

   (c) Determine the corresponding critical density of the universe, viz. 4 marks
$$\rho_{\mathrm{C}} = \frac{3\,H_0^2}{8\,\pi\,G}$$
   in units of kg/m^3.

   Note: A parsec (pc) corresponds to 3.26 light years, and a Mega parsec (Mpc) amounts to 10^6 parsecs. The value of the Newton's gravitational constant G is $6.673 \times 10^{-11}\ \text{m}^3\ \text{kg}^{-1}\ \text{s}^{-2}$.

9. *An empty Friedmann universe:* Consider a Friedmann universe without any matter, i.e. $\rho = 0$ and $p = 0$.

   (a) For what values of the spatial curvature κ do the Friedmann equations lead to realistic and non-trivial solutions for the scale factor. What are the corresponding solutions for the scale factor? 3 marks

(b) Evaluate the Ricci tensor and the scalar curvature associated with the scale factors. 3 marks

(c) Can you identify the resulting spacetime, one which is completely devoid of matter? Construct a coordinate transformation that reduces the non-trivial line-element to its more familiar form. 4 marks

Note: It will be convenient to express the Friedmann line-element in the terms of the coordinate χ which is related to r through the relation $\mathrm{d}\chi = \mathrm{d}r/\sqrt{1-\kappa\, r^2}$.

10. *Behavior of the Hubble radius and the luminosity distance during matter domination:* Consider a spatially flat Friedmann universe. Recall that, in such a case, the luminosity distance $d_{\mathrm{L}}(z)$ can be expressed as

$$d_{\mathrm{L}}(z) = r_{\mathrm{em}}(z)\,(1+z),$$

where the quantity $r_{\mathrm{em}}(z)$ is described by the integral

$$r_{\mathrm{em}}(z) = \int\limits_0^z dz\; d_{\mathrm{H}}(z),$$

while $d_{\mathrm{H}}(z)$ denotes the Hubble radius which is defined as

$$d_{\mathrm{H}} = c\,H^{-1} = c\,\left(\frac{\dot{a}}{a}\right)^{-1}.$$

Show that, when the spatially flat universe is dominated by non-relativistic matter, the Hubble radius and the luminosity distance can be expressed in terms of the red-shift as follows: 7+3 marks

$$\begin{aligned} d_{\mathrm{H}}(z) &= c\,H_0^{-1}\,(1+z)^{-3/2}, \\ d_{\mathrm{L}}(z) &= 2\,c\,H_0^{-1}\,(1+z)\left[1-(1+z)^{-1/2}\right], \end{aligned}$$

where H_0 is the value of the Hubble parameter *today*.