PH5870

GENERAL RELATIVITY AND COSMOLOGY

January–May 2017

Lecture schedule and meeting hours

- The course will consist of about 42 lectures, including about 8–10 tutorial sessions. However, note that there will be no separate tutorial sessions, and they will be integrated with the lectures.
- The duration of each lecture will be 50 minutes. We will be meeting in HSB 317.
- The first lecture will be on Wednesday, January 11, and the last one will be on Thursday, April 20.
- We will meet thrice a week. We shall meet during the following hours: 11:00–11:50 AM on Wednesdays, 9:00–9:50 AM on Thursdays, and 8:00–8:50 AM on Fridays.
- We shall meet during 4:50–5:40 PM on Tuesdays for the quizzes.
- We may also meet on Tuesdays to make up for lectures that I may have to miss due to, say, travel. Changes in schedule, if any, will be notified sufficiently in advance.
- If you would like to discuss with me about the course outside the lecture hours, you are welcome to meet me at my office (HSB 202A) during 12:00–12:30 PM on Wednesdays. In case you are unable to find me in my office, please send me an e-mail at `sriram@physics.iitm.ac.in`.

Information about the course

- I will be distributing hard copies containing information such as the schedule of the lectures, the structure and the syllabus of the course, suitable textbooks and additional references at the start of the course. They will also be available on the course's page on Moodle at the following URL:
  `https://courses.iitm.ac.in/`
- The exercise sheets and other additional material will be made available on Moodle.
- A PDF file containing these information as well as completed quizzes will also made be available at the link on this course at the following URL:
  `http://www.physics.iitm.ac.in/~sriram/professional/teaching/teaching.html`
  I will keep updating this file and the course's page on Moodle as we make progress.

Quizzes, end-of-semester exam and grading

- The grading will be based on three scheduled quizzes and an end-of-semester exam.
- I will consider the best two quizzes for grading, and the two will carry 25% weight each.
- The three quizzes will be on February 7, March 7 and April 4. All these three dates are Tuesdays, and the quizzes will be held during 4:50–5:40 PM.
- The end-of-semester exam will be held during 9:00 AM – 12:00 NOON on Monday, May 1, and the exam will carry 50% weight.

Syllabus and structure

1. **Introduction** [$\sim$ 2 lectures]
   (a) The scope of the general theory of relativity
   (b) Geometry and physics
   (c) Space, time and gravity in Newtonian physics

2. **Spacetime and relativity** [$\sim$ 6 lectures]
   (a) The Michelson-Morley interferometric experiment – Postulates of special relativity
   (b) Lorentz transformations – The relativity of simultaneity – Length contraction and time dilation
   (c) Transformation of velocities and acceleration – Uniform acceleration – Doppler effect
   (d) Four vectors – Action for the relativistic free particle – Charges in an electromagnetic field and the Lorentz force law
   (e) Conservation of relativistic energy and momentum

Exercise sheets 1, 2 3, 4 and 5

Quiz I

3. **Tensor algebra and tensor calculus** [$\sim$ 14 lectures]
   (a) Manifolds and coordinates – Curves and surfaces
   (b) Transformation of coordinates – Contravariant, covariant and mixed tensors – Elementary operations with tensors
   (c) The partial derivative of a tensor – Covariant differentiation and the affine connection
   (d) The metric – Geodesics
   (e) Isometries – The Killing equation and conserved quantities
   (f) The Riemann tensor – The equation of geodesic deviation
   (g) The curvature and the Weyl tensors

Additional exercises I

Exercise sheets 6, 7, 8, 9 and 10

4. **Principles of general relativity** [$\sim$ 2 lectures]
   (a) The equivalence principle – The principle of general covariance – The principle of minimal gravitational coupling

5. **Field equations of general relativity** [$\sim$ 4 lectures]
   (a) The vacuum Einstein equations
   (b) Derivation of vacuum Einstein equations from the action – The Bianchi identities
   (c) The stress-energy tensor – The cases of perfect fluid, scalar and electromagnetic fields
   (d) The structure of the Einstein equations

Exercise sheet 11

Quiz II

6. **Schwarzschild solution, and black holes** [$\sim$ 6 Lectures]
   (a) The Schwarzschild solution – Properties of the metric – Symmetries and conserved quantities
   (b) Motion of particles in the Schwarzschild metric – Precession of the perihelion – Bending of light
   (c) Black holes – Event horizon, its properties and significance – Singularities
   (d) The Kruskal extension – Penrose diagrams

Exercise sheet 12

Quiz III

Additional exercises II

7. **Friedmann-Lemaître-Robertson-Walker (FLRW) universe** [$\sim$ 6 lectures]
   (a) Homogeneity and isotropy – The FLRW line-element
   (b) Friedmann equations – Solutions with different types of matter
   (c) Red-shift – Luminosity and angular diameter distances
   (d) The horizon problem – The inflationary scenario

Exercise sheets 13 and 14

8. **Gravitational waves** [$\sim$ 1 lecture]
   (a) The linearized Einstein equations – Solutions to the wave equation – Production of weak gravitational waves
   (b) Gravitational radiation from binary stars – The quadrupole formula for the energy loss

Exercise sheet 15

End-of-semester exam

Note: The topics in red could not be covered for want of time.

Basic textbooks

1. L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Course of Theoretical Physics, Volume 2), Fourth Edition (Pergamon Press, New York, 1975).
2. B. F. Schutz, *A First Course in General Relativity* (Cambridge University Press, Cambridge, 1990).
3. R. d'Inverno, *Introducing Einstein's Relativity* (Oxford University Press, Oxford, 1992).
4. J. B. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (Pearson Education, Delhi, 2003).
5. S. Carroll, *Spacetime and Geometry* (Addison Wesley, New York, 2004).
6. M. P. Hobson, G. P. Efstathiou and A. N. Lasenby, *General Relativity: An Introduction for Physicists* (Cambridge University Press, Cambridge, 2006).

Additional references

1. S. Weinberg, *Gravitation and Cosmology* (John Wiley, New York, 1972).
2. A. P. Lightman, W. H. Press, R. H. Price and S. A. Teukolsky, *Problem Book in Relativity and Gravitation* (Princeton University Press, New Jersey, 1975).
3. W. Rindler, *Relativity: Special, General and Cosmological* (Oxford University Press, Oxford, 2006).
4. T. Padmanabhan, *Gravitation: Foundation and Frontiers* (Cambridge University Press, Cambridge, 2010).
5. A. Zee, *Einstein Gravity in a Nutshell* (Princeton University Press, Princeton, New Jersey, 2013).

Advanced texts

1. S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge University Press, Cambridge, 1973).
2. C. W. Misner, K. S. Thorne and J. W. Wheeler, *Gravitation* (W. H. Freeman and Company, San Francisco, 1973).
3. R. M. Wald, *General Relativity* (The University of Chicago Press, Chicago, 1984).
4. E. Poisson, *A Relativist's Toolkit* (Cambridge University Press, Cambridge, 2004).

Exercise sheet 1

Lorentz transformations and some consequences

1. *Superluminal motion:* Consider a blob of plasma that is moving at a speed v along a direction that makes an angle θ with respect to the line of sight. Show that the *apparent* transverse speed of the source, projected on the sky, will be related to the actual speed v by the relation

$$v_{\rm app} = \frac{v\,\sin\theta}{1-(v/c)\cos\theta}.$$

From this expression conclude that the apparent speed $v_{\rm app}$ can exceed the speed of light.

2. *Aberration of light:* Consider two inertial frames K and K', with the frame K' moving along the common x-axis with a velocity v with respect to the frame K. Let the velocity of a particle in the frames K and K' be $\boldsymbol{u}$ and $\boldsymbol{u}'$, and let θ and θ' be the angles subtended by the velocity vectors with respect to the common x-axis, respectively.

   (a) Show that

$$\tan\theta = \frac{u'\sin\theta'}{\gamma\,(u'\cos\theta'+v)},$$

   where $\gamma = \left[1-(v/c)^2\right]^{-1/2}$.

   (b) For $u = u' = c$, show that

$$\cos\theta = \frac{\cos\theta' + (v/c)}{1+(v/c)\cos\theta'}$$

   and

$$\sin\theta = \frac{\sin\theta'}{\gamma\,[1+(v/c)\cos\theta']}.$$

   (c) For $v/c \ll 1$, show that

$$\Delta\theta = (v/c)\,\sin\theta',$$

   where $\Delta\theta = \theta' - \theta$.

3. *Decaying muons:* Muons are unstable and decay according to the radioactive decay law $N = N_0\,\exp-(0.693\,t/t_{1/2})$, where N_0 and N are the number of muons at times $t = 0$ and t, respectively, while $t_{1/2}$ is the half life. The half life of the muons in their own rest frame is 1.52×10^{-6} s. Consider a detector on top of a $2,000$ m mountain which counts the number of muons traveling at the speed of $v = 0.98\,c$. Over a given period of time, the detector counts 10^3 muons. When the relativistic effects are taken into account, how many muons can be expected to reach the sea level?

4. *Binding energy:* As you may know, the deuteron which is the nucleus of deuterium, an isotope of hydrogen, consists of one proton and one neutron. Given that the mass of a proton and a neutron are $m_{\rm p} = 1.673\times10^{-27}$ kg and $m_{\rm n} = 1.675\times10^{-27}$ kg, while the mass of the deuteron is $m_{\rm d} = 3.344\times10^{-27}$ kg, show that the binding energy of the deuteron in about 2.225 MeV.

   Note: MeV refers to Million electron Volts, and an electron Volt is 1.602×10^{-19} J.

5. *Form invariance of the Minkowski line-element:* Show that the following Minkowski line-element is invariant under the Lorentz transformations:

$${\rm d}s^2 = c^2\,{\rm d}t^2 - {\rm d}\boldsymbol{x}^2.$$

Exercise sheet 2

Spacetime diagrams

1. *Axes of inertial frames I:* Consider two inertial frames K and K', with K' moving at the velocity v along their common, positive x-direction. Let the two frames coincide at $t = t' = 0$, and let us ignore the y and the z-directions for simplicity.

   (a) In the plane of the spacetime coordinates $(c\,t, x)$, determine the $c\,t'$ axis.
   Hint: This is essentially given by the trajectory of an observer located at, say, $x' = 0$, in the K' frame.

   (b) Consider a beam of light emitted by a source located $x = 0$ and is being reflected by a mirror at, say, $x = a$. Evidently, if the source emits the beam of light at $c\,t = -a$, it will return to the source, after being reflected by the mirror, at $c\,t = a$. Using this method and the fact that the velocity of light is the same in all inertial frames of reference, determine the x' axis in the $(c\,t, x)$ plane.

2. *Axes of inertial frames II:* In the previous exercise, you had determined $c\,t'$ and x' axes in the $(c\,t, x)$ plane.

   (a) Determine the angles between the $c\,t$ and $c\,t'$ axes as well as the x and x' axes.

   (b) Draw the $c\,t$ and x axes in the plane of the spacetime coordinates $(c\,t', x')$ and determine the angles involved.

3. *Invariant hyperbolae:* Consider two Lorentz frames as discussed in the previous two exercises.

   (a) Draw hyperbolae corresponding to different possible values of the Lorentz invariant quantity $c^2\,t^2 - x^2$ in the $(c\,t, x)$ plane.

   (b) Using the invariant hyperbolae, calibrate the $c\,t'$ and x' axes with the respect to the values on the $c\,t$ and x axes.

   (c) A line of simultaneity at a given point, say, P, on the $c\,t$ axis in the spacetime diagram is, evidently, described by the tangent to the hyperbola passing through that point. Draw a line of simultaneity in the $(c\,t, x)$ plane and illustrate the same line of simultaneity (along with the point P and the original hyperbola passing through it) in the $(c\,t', x')$ plane.

   (d) Show that the event P can be shifted anywhere on the hyperbola by working in a suitable Lorentz frame.

   (e) Argue that the tangent to the hyperbola at any event P is the line of simultaneity of the Lorentz frame whose time axis joins P to the origin of the spacetime diagram.

4. *Lorentz contraction:* Consider a rod of a given length at rest in the K' frame. Let one end of the rod be at $x = 0$ at $t = 0$.

   (a) As you have done earlier, draw the $c\,t'$ and x' axes in the $(c\,t, x)$ plane. Also, draw the trajectory of the two ends of the rod in the $(c\,t, x)$ plane.

   (b) From the geometry, express the length of the rod at time $t = 0$ in the K frame, in terms of the actual length of the rod in the K' frame.
   Note: You will find that the length of the rod measured at a given time in the K frame is smaller than its original length. This is Lorentz contraction.

5. *Global view of Minkowski spacetime:* Consider the following Minkowski line-element in $(1 + 1)$-spacetime dimensions:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}x^2.$$

(a) Show that, in terms of the null coordinates

$$u = c\,t - x, \quad v = c\,t + x,$$

the Minkowski line-element reduces to

$$\mathrm{d}s^2 = \mathrm{d}u\,\mathrm{d}v.$$

(b) Show that, if we perform the following coordinate transformation:

$$u' = 2\tan^{-1} u, \qquad v' = 2\tan^{-1} v,$$

where $-\pi \le u' \le \pi$ and $-\pi < v' \le \pi$, then the Minkowski line-element can be expressed as

$$\mathrm{d}s^2 = \frac{1}{4}\sec^2(u'/2)\sec^2(v'/2)\,\mathrm{d}\bar{s}^2,$$

with $\mathrm{d}\bar{s}^2$ being given by

$$\mathrm{d}\bar{s}^2 = \mathrm{d}u'\,\mathrm{d}v'.$$

Note: The line-element $\mathrm{d}\bar{s}^2$ has the same structure as the original Minkowski line-element $\mathrm{d}s^2$. The above relation between $\mathrm{d}s^2$ and $\mathrm{d}\bar{s}^2$ is referred to as a conformal transformation. It is important to appreciate that it is *not* a coordinate transformation.

(c) Working with the line-element $\mathrm{d}\bar{s}^2$, identify the following points of the Minkowski coordinates in the u'-v' plane: (i) past and future time-like infinities ($t \to \mp\infty$, often referred to as i^- and i^+), (ii) space-like infinities ($x \to \mp\infty$ denoted as i^0), and (iii) past and future null-like infinities [$(u, v) \to -\infty$ and $(u, v) \to \infty$, denoted as $\mathcal{I}^-$ and $\mathcal{I}^+$].

Note: The conformal transformation, even as it preserves the form of the Minkowski metric, brings in the infinities of the original Minkowski coordinates to finite values. This helps us create an image of the originally infinite Minkowski domain over a compact region. Such a diagram is known as a Penrose diagram.

(d) Do the the light cones in the Penrose diagram have the same shape as in the original Minkowski spacetime?

(e) Indicate how time-like trajectories behave in the diagram.

(f) Also, draw space-like surfaces corresponding to constant values for the time coordinate t.

Exercise sheet 3

Extremal principles

1. *The hanging chain:* A chain of uniform thickness and density, supported only at its two ends, is hanging under its own weight. Given its length and tension, determine the shape of the chain.

   Note: The resulting curve is known as a catenary.

2. *The brachistochrone problem:* Consider a particle that is moving in a constant force field starting at rest from some point to a lower point. Determine the path that allows the particle to accomplish the transit in the least possible time.

   Note: The resulting curve is referred to as the brachistochrone, i.e. the curve of the fastest descent, say, in a uniform gravitational field.

3. *Variation involving higher derivatives:* Recall that the standard Euler equation governing a function, say, $y(x)$, is arrived at by extremizing an integral of the following form:

$$J[y(x)] = \int_{x_1}^{x_2} \mathrm{d}x\, f(y, y_x, x),$$

   where $y_x = \mathrm{d}y/\mathrm{d}x$.

   (a) Now, consider an integral of the form

$$J[y(x)] = \int_{x_1}^{x_2} \mathrm{d}x\, f(y, y_x, y_{xx}, x),$$

   where $y_{xx} = \mathrm{d}^2y/\mathrm{d}x^2$. Identify the boundary conditions that need to assumed in order to extremize this integral.

   (b) Obtain the Euler equation corresponding to the above integral.

4. *Fermat's principle and Snell's law of refraction:* Two homogeneous media of refractive indices n_1 and n_2 are placed adjacent to each other. A ray of light propagates from a point in the first medium to a point in the second medium. According to the Fermat's principle, the light ray will follow a path that minimizes the transit time between the two points. Use Fermat's principle to derive the Snell's law of refraction, viz. that

$$n_1 \sin\theta_1 = n_2 \sin\theta_2,$$

   where θ_1 and θ_2 are the angles of incidence and refraction at the interface.

   Note: Actually, since the complete path is not differentiable at the interface, the problem is not an Euler equation problem.

5. *Is a cylinder truly curved?* Consider an ant on the *outer* surface of a cylindrical glass at a depth d from the rim. The ant is trying to reach a drop of honey located at a diametrically opposite point on the glass, but on the *inner* surface. The drop of honey is at the same depth as the ant from the rim. If the radius of the cylinder is, say, R, determine the shortest distance that the ant can take to reach the drop of honey.

   Note: The solution illustrates the fact that the cylinder has no intrinsic curvature.

Exercise sheet 4

Working in terms of four vectors

1. *Compton effect using four vectors:* Consider the scattering between a photon of frequency ω and a relativistic electron with velocity $\boldsymbol{v}$ leading to a photon of frequency ω' and electron with velocity $\boldsymbol{v}'$. Such a scattering is known as Compton scattering. Let α be the angle between the incident and the scattered photon. Also, let θ and θ' be the angles subtended by the directions of propagation of the incident and the scattered photon with the velocity vector of the electron before the collision.

   (a) Using the conservation of four momentum, show that
   $$\frac{\omega'}{\omega} = \frac{1-(v/c)\cos\theta}{1-(v/c)\cos\theta' + (\hbar\,\omega/\gamma\, m_{\rm e}\, c^2)\,(1-\cos\alpha)},$$
   where $\gamma = \left[1-(v/c)^2\right]^{-1/2}$ and $m_{\rm e}$ is the mass of the electron.

   (b) When $\hbar\,\omega \ll \gamma\, m_{\rm e}\, c^2$, show that the frequency shift of the photon can be written as
   $$\frac{\Delta\omega}{\omega} = \frac{(v/c)\ (\cos\theta - \cos\theta')}{1-(v/c)\cos\theta'},$$
   where $\Delta\omega = \omega' - \omega$.

2. *Creation of electron-positron pairs:* A purely relativistic process corresponds to the production of electron-positron pairs in a collision of two high energy gamma ray photons. If the energies of the photons are ϵ_1 and ϵ_2 and the relative angle between their directions of propagation is θ, then, by using the conservation of energy and momentum, show that the process can occur only if
   $$\epsilon_1\,\epsilon_2 > \frac{2\, m_{\rm e}^2\, c^4}{1-\cos\theta},$$
   where $m_{\rm e}$ is the mass of the electron.

3. *Transforming four vectors and invariance under Lorentz transformations:* Consider two inertial frames K and K', with K' moving with respect to K, say, along the common x-axis with a certain velocity.

   (a) Given a four vector A^μ in the K frame, construct the corresponding contravariant and covariant four vectors, say, $A^{\mu\prime}$ and A'_μ, in the K' frame.

   (b) Explicitly illustrate that the scalar product $A_\mu\, A^\mu$ is a Lorentz invariant quantity, i.e. show that $A_\mu\, A^\mu = A'_\mu\, A^{\mu\prime}$.

4. *Lorentz invariance of the wave equation:* Show that the following wave equation:
   $$\frac{1}{c^2}\,\frac{\partial^2\phi}{\partial t^2} - \boldsymbol{\nabla}^2\phi = 0$$
   satisfied by, say, light, is invariant under the Lorentz transformations.

5. *Mirrors in motion:* A mirror moves with the velocity v in a direction perpendicular its plane. A ray of light of frequency ν_1 is incident on the mirror at an angle of incidence θ, and is reflected at an angle of reflection ϕ and frequency ν_2.

   (a) Show that
   $$\frac{\tan(\theta/2)}{\tan(\phi/2)} = \frac{c+v}{c-v} \qquad \text{and} \qquad \frac{\nu_2}{\nu_1} = \frac{c+v\cos\theta}{c-v\cos\phi}.$$

   (b) What happens if the mirror was moving parallel to its plane?

Exercise sheet 5

Electromagnetism in tensorial notation

1. *The Lorentz force:* In Minkowski spacetime, the action for a relativistic particle that is interacting with the electromagnetic field is given by

$$S[x^\mu(s)] = -m\,c \int \mathrm{d}s - \frac{e}{c} \int \mathrm{d}x_\mu\, A^\mu,$$

where m is the mass of the particle, while e is its electric charge. The quantity $A^\mu = (\phi, \boldsymbol{A})$ is the four vector potential that describes the electromagnetic field, with, evidently, ϕ and $\boldsymbol{A}$ being the conventional scalar and three-vector potentials.

   (a) Vary the above action with respect to x^μ to arrive at the following Lorentz force law:

$$m\,c\,\frac{\mathrm{d}u^\mu}{\mathrm{d}s} = \frac{e}{c}\,F^{\mu\nu}\,u_\nu,$$

   where $u^\mu = \mathrm{d}x^\mu/\mathrm{d}s$ is the four velocity of the particle and the electromagnetic field tensor $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} = \partial_\mu\, A_\nu - \partial_\nu\, A_\mu$$

   with $\partial_\mu \equiv \partial/\partial x^\mu$.

   (b) Show that the components of the field tensor $F_{\mu\nu}$ are given by

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix},$$

   where (E_x, E_y, E_z) and (B_x, B_y, B_z) are the components of the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ which are related to the components of the four vector potential by the following standard expressions:

$$\boldsymbol{E} = -\frac{1}{c}\,\frac{\partial \boldsymbol{A}}{\partial t} - \boldsymbol{\nabla}\phi \qquad \text{and} \qquad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}.$$

   (c) Establish that the spatial components of the above equation of motion for the charge can be written as

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = e\,\boldsymbol{E} + \frac{e}{c}\,(\boldsymbol{v} \times \boldsymbol{B}),$$

   with $\boldsymbol{p} = \gamma\, m\, \boldsymbol{v}$ being the relativistic three momentum of the particle.

   Note: This equation reduces to the familiar equation of motion for a charge driven by the Lorentz force in the non-relativistic limit [i.e. when terms of order (v^2/c^2) can be ignored] wherein $\mathbf{p} \simeq m\,\mathbf{v}$.

   (d) Show that the time component of the the above equation of motion for the charge reduces to

$$\frac{\mathrm{d}E_{\mathrm{KE}}}{\mathrm{d}t} = e\,(\boldsymbol{v} \cdot \boldsymbol{E}),$$

   where $E_{\mathrm{KE}} = \gamma\, m\, c^2$ is the kinetic energy of the particle.

2. *The first pair of Maxwell's equations:* Show that the above definition of $F_{\mu\nu}$ leads to the following Maxwell's equations in flat spacetime:

$$\frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} = 0.$$

Also, show that these equations correspond to the following two source free Maxwell's equations:

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{1}{c}\,\frac{\partial \boldsymbol{B}}{\partial t} \qquad \text{and} \qquad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0.$$

3. *The Lorentz invariant four volume:* Show that the spacetime volume $\mathrm{d}^4\tilde{x} = c\,\mathrm{d}t\,\mathrm{d}^3\boldsymbol{x}$ is a Lorentz invariant quantity.

4. *The second pair of Maxwell's equations:* Let the four current $j^\mu = (\rho\, c, \boldsymbol{j})$ represent the charge density ρ and the three current $\boldsymbol{j}$ that source the electric and the magnetic fields. In flat spacetime, the action describing the electromagnetic field that is sourced by the four current j^μ is given by

$$S[A^\mu(\tilde{x})] = -\frac{1}{c^2}\int \mathrm{d}^4\tilde{x}\; A_\mu\, j^\mu - \frac{1}{16\,\pi\, c}\int \mathrm{d}^4\tilde{x}\; F^{\mu\nu}\, F_{\mu\nu}.$$

   (a) Vary this action with respect to the vector potential A^μ and arrive at the following Maxwell's equations:

$$\partial_\nu\, F^{\mu\nu} = -\frac{4\,\pi}{c}\, j^\mu.$$

   (b) Show that these equations correspond to the following two Maxwell's equations with sources:

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = 4\,\pi\,\rho \qquad \text{and} \qquad \boldsymbol{\nabla}\times\boldsymbol{B} = 4\,\pi\,\boldsymbol{j} + \frac{1}{c}\,\frac{\partial\boldsymbol{E}}{\partial t}.$$

5. *The continuity equation:* Show that, from the second pair of Maxwell's equations above, one can arrive at the continuity equation, viz.

$$\partial_\mu\, j^\mu = \frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0.$$

Quiz I

Special relativity

1. *Successive Lorentz transformations:* Consider two successive Lorentz transformations along a given direction with speeds, say, v_1 and v_2.

   (a) Show that the two Lorentz transformations are equivalent to a single Lorentz transformation with a speed, say, v, along the same direction. **5 marks**

   (b) Express the speed v in terms of the speeds v_1 and v_2. **5 marks**

2. *Relative speed:* In the laboratory frame, two particles are moving with speed v along directions which subtend an angle θ with respect to each other. Determine the speed of one of these particles with respect to the other. **10 marks**

3. (a) *Colliding particles:* A particle with energy E and mass m collides elastically with an identical particle that is at rest. After the collision, the two particles scatter at the same angle θ with respect to the direction of the incident particle.

   i. Determine the angle θ in terms of E and m. **3 marks**

   ii. What is θ in the extreme relativistic (i.e. when $E \gg m\,c^2$) and non-relativistic (i.e. when $E \simeq m\,c^2$) limits? **2 marks**

   (b) *Decaying particle:* A particle of mass m and energy E that is moving along the positive x-direction decays into two identical particles. Let the decay products be moving in the x-y-plane in the laboratory frame. While one of the decay products is emitted along the positive y-direction, the other is found to be moving at an angle $\theta < 0$ with respect to the x-axis. Determine the energies of the two decay products in terms of E and m. **6 marks**

4. *Transforming electromagnetic fields:* Recall that the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are related to the scalar and vector potentials ϕ and $\boldsymbol{A}$ through the relations

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \boldsymbol{\nabla}\phi, \qquad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}.$$

   The electromagnetic field tensor $F_{\mu\nu}$ is described in terms of the four potential $A^\mu = (\phi, \boldsymbol{A})$ as follows:

$$F_{\mu\nu} = \partial_\mu\, A_\nu - \partial_\nu\, A_\mu$$

   where $\partial_\mu = \partial/\partial x^\mu$.

   (a) Express the field tensor $F_{\mu\nu}$ in terms of the components of the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$. **2 marks**

   (b) Let $\boldsymbol{E}$ and $\boldsymbol{B}$ be the electric and magnetic fields in the laboratory frame. Determine the corresponding fields, say, $\boldsymbol{E}'$ and $\boldsymbol{B}'$, in an inertial frame that is moving at a speed v with respect to the laboratory frame in the positive direction along the common x-axis. **5 marks**

   (c) Show that, when $v/c \ll 1$, the relations between the electric and magnetic fields in the two frames can be expressed in a vector form as **3 marks**

$$\boldsymbol{E} = \boldsymbol{E}' - \frac{1}{c}\,(\boldsymbol{v} \times \boldsymbol{B}'), \qquad \boldsymbol{H} = \boldsymbol{H}' + \frac{1}{c}\,(\boldsymbol{v} \times \boldsymbol{E}').$$

5. *Three and four acceleration:* The four acceleration of a relativistic particle is defined as $a^\mu = \mathrm{d}^2x^\mu/\mathrm{d}s^2$.

   (a) Express the four acceleration a^μ in terms of the three velocity $\boldsymbol{v} = \mathrm{d}\boldsymbol{x}/\mathrm{d}t$ and the three acceleration $\boldsymbol{a} = \mathrm{d}^2\boldsymbol{x}/\mathrm{d}t^2$. **2 marks**

(b) Consider a particle moving with a constant speed v around the circle $x^2 + y^2 = r^2$ in the $z = 0$ plane. What are the three and four acceleration of the particle at the instant it crosses the negative y-axis? $\boxed{\text{4 marks}}$

(c) What are the corresponding three and four acceleration in the instantaneous rest frame of the particle? $\boxed{\text{4 marks}}$

Exercise sheet 6

Manifolds, coordinates and geometry

1. *Non-degenerate coordinate patches for $\mathbb{S}^2$:* Recall that, the usual angular coordinates, viz. θ and ϕ, that describe the two-sphere $\mathbb{S}^2$ in three-dimensional Euclidean space are pathological at the poles, since the metric coefficients vanish at these points. Usually, the sphere is covered with the aid of two coordinate patches arrived through a stereographic projection. In such a projection, one assigns coordinates, say, (ρ, ϕ), to each point on the sphere, with ϕ being the standard azimuthal angle. In one of the coordinate patches, the coordinate ρ of each point is arrived at by drawing a straight line in three dimensions from the south pole of the sphere through the point in question and extending the line until it intersects the tangent plane to the north pole of the sphere. The ρ-coordinate is then the distance in the tangent plane from the north pole to the point of intersection.

   (a) Show that the line-element describing the surface of the sphere in terms of these coordinates is given by
   $$\mathrm{d}\ell^2 = \frac{1}{[1+\rho^2/(4\,R^2)]^2}\,\left(\mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\phi^2\right),$$
   where R is the radius of the sphere. At what point(s) on the sphere are these coordinates degenerate?

   (b) What is the line-element of the sphere if, instead of working with the ρ and ϕ coordinates, one works with the Cartesian coordinates, say, x and y, in the tangent plane at the north pole? Are there any point(s) on the sphere at which these new coordinates are degenerate?

   (c) Construct the coordinates of the second patch in order to cover the sphere completely.

2. *Mercator's projection:* Consider the surface of the Earth, which we shall assume, for simplicity, to be a two-sphere of radius, say, R. In terms of the standard polar coordinates (θ, ϕ), the longitude of a point, in radians, rather than the usual degrees, is simply ϕ (measured eastwards from the Greenwich meridian), whereas its latitude is $\lambda = \pi/2 - \theta$ radians.

   (a) Show that the line-element on the Earth's surface in these coordinates is given by
   $$\mathrm{d}\ell^2 = R^2\,\left(\mathrm{d}\lambda^2 + \cos^2\lambda\,\mathrm{d}\phi^2\right).$$

   (b) In order to make a map of the Earth's surface, let us introduce the functions $x = x(\lambda, \phi)$ and $y = y(\lambda, \phi)$ and use them as Cartesian coordinates on a plane. The Mercator projection is defined as follows:
   $$x = \frac{W\,\phi}{2\,\pi} \quad \text{and} \quad y = \frac{H}{2\,\pi}\,\ln\,\left[\tan\left(\frac{\pi}{4} + \frac{\lambda}{2}\right)\right],$$
   where W and H denote the width and the height of the map, respectively. Determine the line-element on the plane.

3. *The Rindler and the Milne coordinates:* Consider the following non-linear transformations of the Minkowski coordinates $(c\,t, x, y, z)$ to the coordinates $(c\,\tau, \xi, y', z')$:
   $$c\,t = \xi\,\sinh\,(g\,\tau/c), \quad x = \xi\,\cosh\,(g\,\tau/c), \quad y = y' \quad \text{and} \quad z = z'.$$
   The set of coordinates $(c\,\tau, \xi, y', z')$ are referred to as the Rindler coordinates.

   (a) Draw lines of constant τ and ξ in the $c\,t$-x plane, and show that the coordinates $(c\,\tau, \xi)$ cover only the right wedge of the light cone centered at the origin.

   (b) Construct similar coordinates to cover the wedge to the left of the light cone.

(c) Arrive at the set of coordinates that can cover the past and future wedges of the light cone in a similar fashion.

Note: These new set of coordinates that cover the past and the future wedges are known as the Milne coordinates.

(d) Determine the form of the Minkowski line-element in the Rindler and the Milne coordinates.

4. *Embedding a three-sphere in four dimensions:* Recall that a two-sphere of radius, say, R, is a surface which is subject to the constraint $x^2+y^2+z^2=R^2$ in three-dimensional Euclidean space described by the Cartesian coordinates (x,y,z). In a similar manner, we can define a three sphere as the surface that is subject to the constraint $x^2+y^2+z^2+w^2=R^2$ in the four-dimensional Euclidean space characterized by the Cartesian coordinates, say, (x,y,z,w).

(a) Using the constraint equation to eliminate w in terms of the other three variables and the standard Euclidean line-element in four dimensions, show that the geometry of the three-sphere can be expressed as

$$\mathrm{d}\ell^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + d\,z^2 + \frac{(x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z)^2}{R^2 - (x^2+y^2+z^2)}.$$

(b) Upon transforming into the spherical polar coordinates using the conventional relations, show that the above line-element is given by

$$\mathrm{d}\ell^2 = \frac{R^2}{R^2 - r^2}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2.$$

(c) Further show that this line-element can be written as

$$\mathrm{d}\ell^2 = R^2\,\left(\mathrm{d}\chi^2 + \sin^2\chi\,\mathrm{d}\theta^2 + \sin^2\chi\,\sin^2\theta\,\mathrm{d}\phi^2\right),$$

where (χ,θ,ϕ) are the three angular coordinates that are required to cover the three-sphere.

(d) Construct the transformations from the original Cartesian coordinates in the four dimensional Euclidean space, viz. (x,y,z,w), to the angular coordinates (χ,θ,ϕ).

(e) What are allowed ranges of the angular coordinates (χ,θ,ϕ)?

5. *Reducing to the Minkowski line-element:* Show that the following spacetime line-element:

$$\mathrm{d}s^2 = \left(c^2 - a^2\,\tau^2\right)\,\mathrm{d}\tau^2 - 2\,a\,\tau\,\mathrm{d}\tau\,\mathrm{d}\xi - \mathrm{d}\xi^2 - \mathrm{d}y^2 - \mathrm{d}z^2,$$

where a is a constant, can be reduced to the Minkowski line-element by a suitable coordinate transformation.

Exercise sheet 7

Tensors and transformations

1. *Onto the spherical polar coordinates:* Consider the transformation from the Cartesian coordinates $x^a = (x, y, z)$ to the spherical polar coordinates $x'^a = (r, \theta, \phi)$ in $\mathbb{R}^3$.

   (a) Write down the transformation as well as its inverse.

   (b) Express the transformation matrices $[\partial x^a/\partial x'^b]$ and $[\partial x'^a/\partial x^b]$ in terms of the spherical polar coordinates.

   (c) Evaluate the corresponding Jacobians J and J'. Where is J' zero or infinite?

2. *Transformation of vectors:* Consider the transformation from the Cartesian coordinates $x^a = (x, y)$ to the plane polar coordinates $x'^a = (\rho, \phi)$ in $\mathbb{R}^2$.

   (a) Express the transformation matrix $[\partial x'^a/\partial x^b]$ in terms of the polar coordinates.

   (b) Consider the tangent vector to a circle of radius, say, a, that is centered at the origin. Find the components of the tangent vector in one of the two coordinate systems, and use the transformation property of the vector to obtain the components in the other coordinate system.

3. *Properties of partial derivatives:* Consider a scalar quantity ϕ. Show that, while the quantity $(\partial\phi/\partial x^a)$ is a vector, the quantity $\left(\partial^2\phi/\partial x^a\, \partial x^b\right)$ is *not* a tensor.

4. *Transforming tensors:* If $X^a{}_{bc}$ is a mixed tensor of rank $(1, 2)$, show that the contracted quantity $Y_c = X^a{}_{ac}$ is a covariant vector.

5. *Symmetric and anti-symmetric nature of tensors:* Show that, a tensor, if it is symmetric or anti-symmetric in one coordinate system, it remains so in any other coordinate system.

Additional exercises I

Special relativity, electromagnetism and tensors

1. *Simultaneity in a new inertial frame:* Two events, say, A and B, with the spacetime coordinates $(c\,t_{\rm A}, \boldsymbol{x}_{\rm A})$ and $(c\,t_{\rm B}, \boldsymbol{x}_{\rm B})$, are found to be separated by a spacelike interval in a particular inertial frame. Determine the velocity of a new inertial frame (with respect to the original frame) wherein the two events can be found to occur simultaneously.

2. *Relative velocity between two inertial frames:* Consider two inertial frames that are moving with the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ with respect to, say, the laboratory frame. Show that the relative velocity v between the two frames can be expressed as

$$v^2 = \frac{(\boldsymbol{v}_1 - \boldsymbol{v}_2)^2 - (\boldsymbol{v}_1/c) \times (\boldsymbol{v}_2)^2}{[1 - (\boldsymbol{v}_1/c) \cdot \boldsymbol{v}_2/c^2)]^2}.$$

3. *The pole in the barn paradox:* An athlete carrying a pole 20 m long runs towards a barn of length 15 m at the speed of $0.8\,c$. A friend of the athlete watches the action, standing at rest by the door of the barn.

   (a) How long does the friend measure the length of the pole to be, as it approaches the barn?

   (b) The barn door is initially open, and immediately after the runner and the pole are inside the barn, the friend shuts the door. How long after the door is closed does the front of the pole hits the wall at the other end of the barn, as measured by the friend? Compute the interval between the events of the friend closing the door and the pole hitting the wall. Is it spacelike, null or timelike?

   (c) In the frame of the runner, what are the lengths of the barn and the pole?

   (d) Does the runner believe that the pole is entirely inside the barn, when its front hits the opposite wall? Can you explain why?

   (e) After the collision, the pole and the runner come to rest relative to the barn. From the friend's point of view, the 20 m pole is now inside a 15 m barn, since the barn door was shut before the pole stopped. How is this possible? Alternatively, from the runner's point of view, the collision should have occurred *before* the door was closed, so the door should not be closed at all. Was or was not the door closed with the pole inside?

4. *The twin paradox:* Alex and Bob are twins working on a space station located at a fixed position in deep space. Alex undertakes an extended return spaceflight to a distant star, while Bob stays on the station. Show that, on his return to the station, the proper time interval experienced by Alex must be less than that experienced by Bob, hence Bob is now the elder. How does Alex explain this age difference?

5. *Three acceleration in terms of the electromagnetic fields:* Consider a charged particle that is moving under the influence of the electric and the magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$. Express the three acceleration (i.e. $\dot{\boldsymbol{v}}$) of the particle in terms of the electric and the magnetic fields.

   Note: The overdot denotes differentiation with respect to the coordinate time t.

6. *Motion in a constant and uniform electric field:* Consider a particle that is moving in a constant and uniform electric field that is directed, say, along the positive x-axis. Let the relativistic three momentum $\boldsymbol{p}$ of the particle at the time, say, $t = 0$, be zero.

   (a) Solve the equation of motion to arrive at $x(t)$.
   Hint: It is useful to note that we can write $\boldsymbol{v} = {\rm d}\boldsymbol{x}/{\rm d}t = \boldsymbol{p}\,c^2/\mathcal{E}$, where $\mathcal{E}/c = \sqrt{\boldsymbol{p}^2 + m^2\,c^2}$.

   (b) Plot the trajectory of the particle in the $c\,t$-x plane.

7. *Motion in a constant and uniform magnetic field:* Consider a particle of mass m and charge e that is moving in a magnetic field of strength B that is directed, say, along the positive z-axis.

   (a) Show that the energy $\mathcal{E} = \gamma\, m\, c^2$ of the particle is a constant.

   (b) Determine the trajectory $\boldsymbol{x}(t)$ of the particle and show that, in the absence of any initial momentum along the z-direction, the particle describes a circular trajectory in the x-y plane with the angular frequency $\omega = e\, c\, B/\mathcal{E}$.

8. *Equation of motion for a scalar field:* Consider the following action that describes a scalar field, say, ϕ, in Minkowski spacetime:

$$S[\phi] = \frac{1}{c} \int c\, \mathrm{d}t\, \mathrm{d}^3\boldsymbol{x} \left(\frac{1}{2}\, \eta_{\mu\nu}\, \partial^\mu \phi\, \partial^\nu \phi - \frac{1}{2}\, \sigma^2\, \phi^2\right),$$

   where $\eta_{\mu\nu}$ is the metric tensor in flat spacetime, while the quantity σ is related to the mass of the field. Vary the above action to arrive at the equation of motion for the scalar field.

   Note: The resulting equation of motion is called the Klein-Gordon equation.

9. *The Minkowski line-element in a rotating frame:* In terms of the cylindrical polar coordinates, the Minkowski line element is given by

$$\mathrm{d}s^2 = c^2\, \mathrm{d}t^2 - \mathrm{d}\rho^2 - \rho^2\, \mathrm{d}\phi^2 - \mathrm{d}z^2.$$

   Consider a coordinate system that is rotating with an angular velocity Ω about the z-axis. The coordinates in the rotating frame, say, $(c\,t', \rho', \phi', z')$, are related to the standard Minkowski coordinates through the following relations:

$$c\,t = c\,t', \qquad \rho = \rho', \qquad \phi = \phi' + \Omega\, t' \qquad \text{and} \qquad z = z'.$$

   (a) Determine the line element in the rotating frame.

   (b) What happens to the line-element when $\rho' \geq c/\Omega$?

10. *The Kronecker delta:* Evaluate the quantities δ^a_a and $\delta^a_b\, \delta^b_a$ on a n-dimensional manifold.

Exercise sheet 8

Christoffel symbols and the geodesic equation

1. *The metric of* $\mathbb{R}^3$*:* Evaluate the covariant and the contravariant components of the metric tensor describing the three-dimensional Euclidean space (usually denoted as $\mathbb{R}^3$) in the Cartesian, cylindrical polar and the spherical polar coordinates. Also, evaluate the determinant of the covariant metric tensor in each of these coordinate systems.

2. *Geodesics on a two sphere:* Evaluate the Christoffel symbols on $\mathbb{S}^2$, and solve the geodesic equation to show that the geodesics are the great circles.

3. *Important identities involving the metric tensor:* Establish the following identities that involve the metric tensor:

   (a) $g_{,c} = g\, g^{ab}\, g_{ab,c}$,

   (b) $g^{ab}\, g_{bc,d} = -g^{ab}{}_{,d}\, g_{bc}$,

   where the commas denote partial derivatives, while g is the determinant of the covariant metric tensor g_{ab}.

4. *Useful identities involving the Christoffel symbols:* Establish the following identities involving the Christoffel symbols:

   (a) $\Gamma^a_{ab} = \frac{1}{2}\,\partial_b \ln|g|$,

   (b) $g^{ab}\,\Gamma^c_{ab} = -\frac{1}{\sqrt{|g|}}\,\partial_d\left(\sqrt{|g|}\; g^{cd}\right)$,

   (c) $g^{ab}{}_{,c} = -\left(\Gamma^a_{cd}\, g^{bd} + \Gamma^b_{cd}\, g^{ad}\right)$,

   where the Christoffel symbol Γ^a_{bc} is given by

$$\Gamma^a_{bc} = \frac{1}{2}\, g^{ad}\,\left(g_{db,c} + g_{dc,b} - g_{bc,d}\right).$$

5. *Invariant four volume:* Show that the spacetime volume $\sqrt{-g}\;\mathrm{d}^4\tilde{x}$ is invariant under arbitrary coordinate transformations.

Exercise sheet 9

Killing vectors and conserved quantities

1. *Killing vectors in $\mathbb{R}^3$:* Construct all the Killing vectors in the three dimensional Euclidean space $\mathbb{R}^3$ by solving the Killing's equation.

2. *Killing vectors on $\mathbb{S}^2$:* Construct the most generic Killing vectors on a two sphere.

3. *Killing vectors in Minkowski spacetime:* Solve the Killing's equation in flat spacetime, and construct all the independent Killing vectors. What do these different Killing vectors correspond to?

4. *The line element and the conserved quantities around a cosmic string:* The spacetime around a cosmic string is described by the line-element

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}\rho^2 - \alpha^2\,\rho^2\,\mathrm{d}\phi^2 - \mathrm{d}z^2,$$

where α is a constant that is called the deficit angle.

   (a) List the components of the momentum of a relativistic particle on geodesic motion in this spacetime that are conserved.

   (b) Consider a particle of mass m that is moving along a time-like geodesic in the spacetime of a cosmic string. Using the relation $p^\mu\,p_\mu = m^2\,c^2$ and the conserved momenta, obtain the (first order) differential equation for $\mathrm{d}\rho/\mathrm{d}t$ of the particle in terms of *all* the conserved components of its momenta.

5. *Conserved quantities in the Schwarzschild spacetime:* The spacetime around a central mass M is described by the following Schwarzschild line element:

$$\mathrm{d}s^2 = c^2\,\left(1-\frac{2\,G\,M}{c^2\,r}\right)\,\mathrm{d}t^2 - \left(1-\frac{2\,G\,M}{c^2\,r}\right)^{-1}\,\mathrm{d}r^2 - r^2\,\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right),$$

where G is the Newton's gravitational constant. Identify the Killing vectors and the corresponding conserved quantities in such a static and spherically symmetric spacetime.

Exercise sheet 10

Riemann, Ricci tensors and scalar curvature

1. *Algebraic identity involving the Riemann tensor:* Recall that, the Riemann tensor is defined as

$$R^{a}{}_{bcd} = \Gamma^{a}_{bd,c} - \Gamma^{a}_{bc,d} + \Gamma^{a}_{ec}\,\Gamma^{e}_{bd} - \Gamma^{a}_{ed}\,\Gamma^{e}_{bc}.$$

Using this expression, establish that

$$R^{a}{}_{bcd} + R^{a}{}_{dbc} + R^{a}{}_{cdb} = 0.$$

2. *The number of independent components of the Riemann tensor:* Show that, on a n-dimensional manifold, the number of independent components of the Riemann tensor are $\left(n^2/12\right)\,\left(n^2-1\right)$.

3. *The flatness of the cylinder:* Calculate the Riemann tensor of a cylinder of constant radius, say, R, in three dimensional Euclidean space. What does the result you find imply?

Note: The surface of the cylinder is actually two-dimensional.

4. *The curvature of the two-sphere:* Calculate all the components of the Riemann and the Ricci tensors, and also the corresponding scalar curvature associated with the two sphere.

Note: Given the Riemann tensor $R^{a}{}_{bcd}$, the Ricci tensor R_{ab} and the Ricci scalar R are defined as

$$R_{ab} = R^{c}{}_{acb} \qquad \text{and} \qquad R = g^{ab}\,R_{ab}.$$

5. *Identities involving the covariant derivative and the Riemann tensor:* Establish the following relations:

   (a) $\nabla_c\,\nabla_b\,A_a - \nabla_b\,\nabla_c\,A_a = R^{d}{}_{abc}\,A_d,$

   (b) $\nabla_d\,\nabla_c\,A_{ab} - \nabla_c\,\nabla_d\,A_{ab} = R^{e}{}_{bcd}\,A_{ae} + R^{e}{}_{acd}\,A_{eb}.$

Quiz II

Tensor algebra, tensor calculus, geodesics and Killing vectors

1. *Behavior of the Christoffel symbols under conformal transformations:* Consider the following transformation of the metric tensor:
$$g_{ab} \to \tilde{g}_{ab} = \Omega^2(x^c)\, g_{ab},$$
where $\Omega(x^c)$ is an arbitrary function of the coordinates. Express the Christoffel symbols associated with the metric tensor $\tilde{g}_{ab}$ in terms of the Christoffel symbols corresponding to the metric tensor g_{ab}. **10 marks**

   Note: Transformations of the metric tensor as above are known as conformal transformations. It is important to note that conformal transformations are *not* coordinate transformations.

2. *Klein-Gordon equation in a curved spacetime:* Consider the following action that describes a scalar field, say, ϕ, in a generic curved spacetime:
$$S[\phi] = \frac{1}{c} \int \mathrm{d}^4x\, \sqrt{-g}\, \left(\frac{1}{2}\, g_{\mu\nu}\, \partial^\mu\phi\, \partial^\nu\phi - \frac{1}{2}\, \sigma^2\, \phi^2\right),$$
where $g_{\mu\nu}$ is the metric tensor that describes the curved spacetime, while the quantity σ is related to the mass of the field. Also, the quantity g denotes the determinant of the metric tensor $g_{\mu\nu}$.

   (a) Vary the above action to arrive at the equation of motion for the scalar field. **5 marks**

   (b) Show that equation of motion of the scalar field can be written as **5 marks**
$$\nabla_\mu \nabla^\mu \phi + \sigma^2 \phi \equiv \phi^{;\mu}{}_{;\mu} + \sigma^2\, \phi = 0.$$

3. *Energy of photons in a Friedmann universe:* The spatially flat Friedmann universe is described by the line element
$$\mathrm{d}s^2 = c^2\, \mathrm{d}t^2 - a^2(t)\, \left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right),$$
where $a(t)$ is known as the scale factor that characterizes the expansion of the universe.

   (a) Obtain the Christoffel symbols corresponding to this line element. **2 marks**

   (b) Explicitly write down the time and the spatial components of the geodesic equation governing a photon in the spatially flat Friedmann universe. **4 marks**

   (c) Solve the geodesic equation and show that the energy, say, E, of the photon behaves as $E \propto 1/a$. **4 marks**

   Note: Recall that the time component of the four momentum of a particle represents its energy. The above result implies that the energy of photons constantly decreases as the universe expands, a phenomenon that is known as cosmological redshift.

4. *Geodesics on a cone:* Consider a cone with a semi-vertical angle α.

   (a) Determine the line element on the cone. **2 marks**

   (b) Obtain the equations governing the geodesics on the cone. **4 marks**

   (c) Solve the equations to arrive at the geodesics. **4 marks**

5. *Killing vectors of a plane in polar coordinates:* Consider the two dimensional Euclidean plane described in terms of the polar coordinates.

   (a) What is the line element of the Euclidean plane in terms of the polar coordinates? **1 mark**

   (b) Evaluate all the Christoffel symbols associated with the line element. **2 marks**

   (c) Write down the equations describing the Killing vectors in the polar coordinates. **3 marks**

   (d) Obtain all the Killing vectors by solving the equations and interpret the solutions. **4 marks**

Quiz II – Again

Tensor algebra, tensor calculus, geodesics and Killing vectors

1. *Some properties involving covariant derivatives:* Prove that

   (a) For any second rank tensor A^{ab}, $\boxed{\text{5 marks}}$

   $$A^{ab}{}_{;a;b} = A^{ab}{}_{;b;a},$$

   where the semi-colons, as usual, denote covariant differentiation.

   (b) For an anti-symmetric tensor $F_{\mu\nu}$, $\boxed{\text{5 marks}}$

   $$F^{\mu\nu}{}_{;\nu} = \frac{1}{\sqrt{-g}}\,\partial_\nu\left(\sqrt{-g}\,F^{\mu\nu}\right).$$

2. *First pair of Maxwell's equations:* Recall that, in Minkowski spacetime, the first pair of Maxwell's equations are described by the equation

   $$F_{\mu\nu,\lambda} + F_{\nu\lambda,\mu} + F_{\lambda\mu,\nu} = 0,$$

   where the field tensor $F_{\mu\nu}$ is defined as

   $$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}.$$

   This equation is purely a consequence of the anti-symmetric nature of the field tensor. Show that, in a generic curved spacetime, this equation generalizes to $\boxed{\text{10 marks}}$

   $$F_{\mu\nu;\lambda} + F_{\nu\lambda;\mu} + F_{\lambda\mu;\nu} = 0.$$

3. *Geodesics on a plane:* Working in the polar coordinates, arrive at the geodesic equations on the plane. Solve the equations to show that the geodesics are straight lines. $\boxed{\text{5+5 marks}}$

   Note: You are expected to solve the geodesic equation involving the Christoffel symbols, and not the more familiar variation of the problem!

4. *Geodesics in a FLRW universe:* The Friedmann-Lemaître-Robertson-Walker (FLRW) universe is described by the line-element

   $$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\,\left[\frac{\mathrm{d}r^2}{1-\kappa\,r^2} + r^2\,\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right],$$

   where the function $a(t)$ is referred to as the scale factor and $\kappa = 0, \pm 1$.

   (a) Determine all the Christoffel symbols associated with the metric. $\boxed{\text{3 marks}}$

   (b) Explicitly write down the geodesic equations governing massive particles in the FLRW universe. $\boxed{\text{3 marks}}$

   (c) Solve the geodesic equations suitably to show that the magnitude of the three momentum of the particle decreases (as inversely proportional to the scale factor) with the expansion of the universe. $\boxed{\text{4 marks}}$

5. *A property of Killing vectors:* If ξ^a is a Killing vector, show that $\boxed{\text{10 marks}}$

   $$\xi_{a;b;c} = R_{dcba}\,\xi^d.$$

Exercise sheet 11

Stress-energy tensor and Einstein's equations

1. *The Bianchi identity:* Recall that, the Riemann tensor is defined as

$$R_{abcd} = g_{ae}\, R^{e}{}_{bcd} = g_{ae}\, \left(\Gamma^{e}_{bd,c} - \Gamma^{e}_{bc,d} + \Gamma^{e}_{fc}\,\Gamma^{f}_{bd} - \Gamma^{e}_{fd}\,\Gamma^{f}_{bc}\right).$$

Also, note that, given the Riemann tensor $R^{a}{}_{bcd}$, the Ricci tensor R_{ab} and the Ricci scalar R are defined as

$$R_{ab} = R^{c}{}_{acb} \qquad \text{and} \qquad R = g^{ab}\, R_{ab}.$$

Further, the Einstein tensor is given by

$$G_{ab} = R_{ab} - \frac{1}{2}\, R\, g_{ab}.$$

(a) Using the expression for the Riemann tensor, establish the following Bianchi identity:

$$\nabla_e\, R_{abcd} + \nabla_d\, R_{abec} + \nabla_c\, R_{abde} = 0.$$

Note: It will be a lot more convenient to use a different version of the Riemann tensor and work in the local coordinates, where the Christoffel symbols vanish, but their derivatives do not.

(b) Using the above identity, show that

$$\nabla_b\, G^{b}_{a} = 0.$$

2. *The stress-energy tensor of an ideal fluid:* Consider an ideal fluid described by the energy density $\rho\, c^2$ (with ρ being the mass density) and pressure p. Further, assume that the fluid does not possess any anisotropic stress.

(a) Argue that, in the comoving frame, the stress energy tensor of the fluid is given by

$$T^{\mu}_{\nu} = \text{diag.}\, \left(\rho\, c^2, -p, -p, -p\right).$$

(b) Further, show that, in a general frame, the stress energy tensor of the fluid can be written as

$$T^{\mu}_{\nu} = (\rho\, c^2 + p)\, u^{\mu}\, u_{\nu} - p\, \delta^{\mu}_{\nu},$$

where u^{μ} is the four velocity of the fluid.

(c) Using the law governing the conservation of the stress energy tensor, arrive at the equations of motion that describe an ideal fluid in Minkowski spacetime.

3. *The stress-energy tensor of a scalar field:* Recall that, given an action that describes a matter field, the stress-energy tensor associated with the matter field is given by the variation of the action with respect to the metric tensor as follows:

$$\delta S = \frac{1}{2\, c} \int \mathrm{d}^4x\, \sqrt{-g}\, T_{\mu\nu}\, \delta g^{\mu\nu} = -\frac{1}{2\, c} \int \mathrm{d}^4x\, \sqrt{-g}\, T^{\mu\nu}\, \delta g_{\mu\nu}.$$

Consider a scalar field ϕ that is governed by the following action:

$$S[\phi] = \frac{1}{c} \int \mathrm{d}^4x\, \sqrt{-g}\, \left[\frac{1}{2}\, g^{\mu\nu}\, \partial_\mu\phi\, \partial_\nu\phi - V(\phi)\right],$$

where $V(\phi)$ is the potential describing the scalar field.

(a) Upon varying this action with respect to the metric tensor, arrive at the stress energy tensor of the scalar field.

(b) Show that the conservation of the stress-energy tensor leads to the equation of motion of the scalar field.

4. *The stress-energy tensor of the electromagnetic field:* In a curved spacetime, the action describing the electromagnetic field is given by

$$S[A^\mu] = -\frac{1}{16\,\pi\,c} \int \mathrm{d}^4x\, \sqrt{-g}\, F_{\mu\nu}\, F^{\mu\nu},$$

where

$$F_{\mu\nu} = A_{\mu;\nu} - A_{\nu;\mu} = A_{\mu,\nu} - A_{\nu,\mu}.$$

(a) Construct the stress-energy tensor associated with the electromagnetic field.

(b) What are the time-time and the time-space components of the stress energy tensor of the electromagnetic field in flat spacetime?

5. *The nature of a worm hole:* The spacetime of a worm hole is described by the line-element

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}r^2 - \left(b^2 + r^2\right)\,\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right),$$

where b is a constant with the dimensions of length that reflects the size of the 'traversable' region. Show that the energy density of matter has to be negative to sustain such a spacetime.

Exercise sheet 12

Schwarzschild spacetime

1. *Spherically symmetric spacetimes:* Consider the following line element that describes spherically symmetric spacetimes in $(3+1)$-dimensions:

$$\mathrm{d}s^2 = c^2\, e^{\Phi(t,r)}\, \mathrm{d}t^2 - e^{\Psi(t,r)}\, \mathrm{d}r^2 - r^2\, \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right),$$

where $\Phi(t,r)$ and $\Psi(t,r)$ are arbitrary functions of the coordinates t and r.

   (a) Find $g_{\mu\nu}$ and $g^{\mu\nu}$ corresponding to this line element.

   (b) Evaluate the resulting $\Gamma^{\alpha}_{\mu\nu}$.

   (c) Also, calculate the corresponding $R_{\mu\nu}$ and R.

2. *Utilizing the Bianchi identities:* Compute the Einstein tensor corresponding to the above line element and show that its non-zero components are given by

$$\begin{aligned}
G^t_t &= \left(\frac{\Psi'}{r} - \frac{1}{r^2}\right) e^{-\Psi} + \frac{1}{r^2},\\
G^r_t &= -\frac{\dot{\Psi}}{r\, e^{\Psi}} = -G^t_r\, e^{(\Psi-\Phi)},\\
G^r_r &= -\left(\frac{\Phi'}{r} + \frac{1}{r^2}\right) e^{-\Psi} + \frac{1}{r^2},\\
G^\theta_\theta &= G^\phi_\phi = \frac{1}{2}\left(\frac{\Phi'\,\Psi'}{2} + \frac{\Psi'}{r} - \frac{\Phi'}{r} - \frac{\Phi'^2}{2} - \Phi''\right) e^{-\Psi} + \frac{1}{2}\left(\ddot{\Psi} + \frac{\dot{\Psi}^2}{2} - \frac{\dot{\Phi}\,\dot{\Psi}}{2}\right) e^{-\Phi},
\end{aligned}$$

where the overdots and the overprimes denote differentiation with respect to $c\,t$ and r, respectively. Show that the contracted Bianchi identities, viz. $\nabla_\mu\, G^\mu_\nu = 0$, imply that the last of the above equations vanishes, if the remaining three equations vanish.

3. *Spherically symmetric vacuum solution of the Einstein's equations:* In the absence of any sources, the above components of the Einstein tensor should vanish. Integrate the equations to arrive at the following Schwarzschild line element:

$$\mathrm{d}s^2 = c^2\left(1 - \frac{2\,G\,M}{c^2\,r}\right) \mathrm{d}t^2 - \left(1 - \frac{2\,G\,M}{c^2\,r}\right)^{-1} \mathrm{d}r^2 - r^2\, \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right),$$

where M is a constant of integration that denotes the mass of the central object that is responsible for the gravitational field.

4. *The precession of the perihelion of Mercury:* Consider a particle of mass m propagating in the above Schwarzschild spacetime.

   (a) Using the relation $p^\mu\, p_\mu = m^2\, c^2$ and the conserved momenta, arrive at the following differential equation describing the orbital motion of massive particles:

$$\frac{\mathrm{d}^2\, u}{\mathrm{d}\phi^2} + u = \frac{G\,M}{\tilde{L}^2} + \frac{3\,G\,M}{c^2}\, u^2,$$

   where $u = 1/r$, while $\tilde{L} = L/m = r^2\,(\mathrm{d}\phi/d\tau)$, with L being the angular momentum of the particle and τ its proper time.

(b) The second term on the right hand side of the above equation would have been absent in the case of the conventional, non-relativistic, Kepler problem. Treating the term as a small perturbation, show that the orbits are no more closed, and the perihelion precesses by the angle

$$\Delta\phi \simeq \frac{6\,\pi\,(G\,M)^2}{\tilde{L}^2\,c^2} = \frac{6\,\pi\,G\,M}{a\,(1-e^2)\,c^2} \text{ radians/revolution,}$$

where e and a are the eccentricity and the semi-major axis of the original closed, Keplerian elliptical orbit.

(c) For the case of the planet Mercury, $a = 5.8 \times 10^{10}$ m, while $e = 0.2$. Also, the period of the Mercury's orbit around the Sun is 88 days. Further, the mass of the Sun is $M_\odot = 2 \times 10^{30}$ kg. Use these information to determine the angle by which the perihelion of Mercury would have shifted in a century.

Note: The measured precession of the perihelion of the planet Mercury proves to be $5599''.7 \pm 0''.4$ per century, but a large part of it is caused due to the influences of the other planets. When the other contributions have been subtracted, the precession of the perihelion of the planet Mercury due to the purely relativistic effects amounts to 43.1 ± 0.5 seconds of arc per century.

5. *Gravitational bending of light:* Consider the propagation of photons in the Schwarzschild spacetime.

(a) Using the relation $p^\mu\,p_\mu = 0$ and the conserved momenta, arrive at the following differential equation describing the orbital motion of photons in the spacetime:

$$\frac{\mathrm{d}^2\,u}{\mathrm{d}\phi^2} + u = \frac{3\,G\,M}{c^2}\,u^2,$$

(b) Establish that, in the absence of the term on the right hand side, the photons will travel in straight lines.

(c) As in the previous case, treating the term on the right hand side as a small perturbation, show that it leads to a deflection of a photon's trajectory by the angle

$$\Delta\phi \simeq \frac{4\,G\,M}{c^2\,b},$$

where b is the impact parameter of the photon (i.e. the distance of the closest approach of the photon to the central mass).

(d) Given that the radius of the Sun is 6.96×10^8 m, determine the deflection angle $\Delta\phi$ for a ray of light that grazes the Sun.

Note: The famous 1919 eclipse expedition led by Eddington led to two sets of results, viz.

$$\Delta\phi = 1''.98 \pm 0''.16 \quad \text{and} \quad \Delta\phi = 1''.61 \pm 0''.4,$$

both of which happen to be consistent with the theory.

Quiz III

Einstein's equations and Schwarzschild spacetime

1. *Gravitation in two dimensions:* Consider an arbitrary spacetime in two dimensions that is described by the metric g_{ab}.

   (a) Argue that, in such a case, the Riemann tensor can be expressed as follows:

   $$R_{abcd} = \kappa\ (g_{ac}\, g_{bd} - g_{ad}\, g_{bc}),$$

   where κ is a scalar that is, in general, a function of the coordinates. $\boxed{\text{6 marks}}$

   Note: It is useful to recall that, in n-dimensions, the number of independent components of the Riemann tensor is $n^2\,(n^2-1)/12$.

   (b) Using this result, show that the Einstein tensor vanishes *identically* in two dimensions. $\boxed{\text{4 marks}}$

2. *Generic scalar fields:* Consider a generic scalar field ϕ that is described by the action

   $$S[\phi] = \frac{1}{c}\int \mathrm{d}^4x\ \sqrt{-g}\ \mathcal{L}(X,\phi),$$

   where X denotes the kinetic energy of the scalar field and is given by

   $$X = \frac{1}{2}\, g^{\mu\nu}\, \partial_\mu\phi\, \partial_\nu\phi.$$

   (a) Let the Lagrangian density $\mathcal{L}$ be an arbitrary function of the kinetic term X and the field ϕ. Vary the above action with respect to the metric tensor and obtain the corresponding stress-energy tensor. $\boxed{\text{4 marks}}$

   Note: Such scalar fields are often referred to as k-essence.

   (b) Assuming $\mathcal{L} = X - V(\phi)$, where $V(\phi)$ is the potential describing the scalar field, determine the corresponding stress-energy tensor. From the conservation of the stress-energy tensor, arrive at the equation of motion governing the scalar field for the case wherein $V(\phi) = \sigma^2\,\phi^2/2$. $\boxed{\text{3+3 marks}}$

3. *Massive particles in the Schwarzschild metric:* Consider a particle of mass m which is moving in the Schwarzschild metric. The trajectory of the particle can be described by an equation of the following form:

   $$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + V_{\mathrm{eff}}(r) = \frac{c^2}{2}\left[\left(\frac{E}{m\,c^2}\right)^2 - 1\right],$$

   where V_{eff} is the 'effective potential' which governs the motion of the relativistic particle, E is its energy, while τ denotes the proper time as measured in the frame of the particle.

   (a) Obtain the form of the effective potential $V_{\mathrm{eff}}(r)$. $\boxed{\text{5 marks}}$

   (b) Consider a radially infalling particle which starts at rest from infinity. Determine the behavior of the radial coordinate r as a function of the time coordinate t, when the particle is very close to the Schwarzschild radius $r_{\mathrm{S}} = 2\,G\,M/c^2$. When will the particle reach $r = r_{\mathrm{S}}$ in terms of the coordinate time t? $\boxed{\text{5 marks}}$

4. *Circular orbits of photons in the Schwarzschild spacetime:* Recall that the orbital trajectory of a photon in the Schwarzschild metric is governed by the differential equation

   $$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = \frac{3\,G\,M}{c^2}\,u^2,$$

   where $u = 1/r$.

(a) Does this equation admit circular orbits? If it does, utilize the equation to arrive at the radii of the circular orbits. $\boxed{\text{5 marks}}$

(b) Use the above equation to determine if these circular orbits are stable or unstable. $\boxed{\text{5 marks}}$

5. *Radial motion of particles and photons in the Schwarzschild metric:* Consider particles and photons which are traveling radially in the Schwarzschild spacetime.

   (a) Let a particle fall radially from rest at infinity. Determine the proper time taken by the particle to travel from a larger radius, say, r_0, to a smaller one, say, r. $\boxed{\text{5 marks}}$

   (b) Determine the trajectories of radially outgoing and ingoing photons in the standard Schwarzschild coordinates. $\boxed{\text{5 marks}}$

   Hint: Obtain t as a function of r.

Additional exercises II

Tensor algebra, calculus, general relativity and black holes

1. *Parallel transporting a vector on $\mathbb{S}^2$:* The components of a vector A^a on the two sphere $\mathbb{S}^2$ are found to be $(1, 0)$ at $(\theta = \theta_0, \phi = 0)$, where θ_0 is a constant. The vector is parallel transported around the circle $\theta = \theta_0$. Determine the vector when it returns to the original point.

2. *Rewriting the Riemann tensor:* Recall that, the Riemann tensor is defined as
$$R_{abcd} = g_{ae}\, R^{e}{}_{bcd} = g_{ae}\, \left(\Gamma^{e}_{bd,c} - \Gamma^{e}_{bc,d} + \Gamma^{e}_{fc}\,\Gamma^{f}_{bd} - \Gamma^{e}_{fd}\,\Gamma^{f}_{bc}\right).$$
Show that this can be rewritten as
$$R_{abcd} = \frac{1}{2}\, \left(g_{ad,bc} + g_{bc,ad} - g_{ac,bd} - g_{bd,ac}\right) + g_{ef}\, \left(\Gamma^{e}_{bc}\,\Gamma^{f}_{ad} - \Gamma^{e}_{bd}\,\Gamma^{f}_{ac}\right),$$
an expression which reflects the symmetries of the Riemann tensor more easily.

3. *Geodesic deviation:* Consider two nearby geodesics, say, $x^a(\lambda)$ and $\bar{x}^a(\lambda)$, where λ is an affine parameter. Let $\xi^a(\lambda)$ denote a 'small vector' that connects these two geodesics. Working in the locally geodesic coordinates, show that ξ^a satisfies the differential equation
$$\frac{\mathrm{D}^2\xi^a}{\mathrm{D}\lambda^2} + R^{a}_{bcd}\,\dot{x}^b\,\xi^c\,\dot{x}^d = 0,$$
where
$$\frac{\mathrm{D}^2\xi^a}{\mathrm{D}\lambda^2} \equiv \left(\dot{\xi}^a + \Gamma^{a}_{bc}\,\xi^b\,\dot{x}^c\right)^{\bullet},$$
while the overdots denote differentiation with respect to λ.

   Note: This implies that a non-zero Riemann tensor R^{a}_{bcd} will lead to a situation where geodesics, in general, will not remain parallel as, for instance, on the surface of the two sphere $\mathbb{S}^2$.

4. *Scalar curvature in two dimensions:* Consider the following $(1+1)$-dimensional line element:
$$\mathrm{d}s^2 = f^2(\eta, \xi)\, \left(\mathrm{d}\eta^2 - d\xi^2\right),$$
where $f(\eta, \xi)$ is an arbitrary function of the coordinates η and ξ. Show that the scalar curvature associated with this line-element can be expressed as
$$R = -\nabla_\mu \nabla^\mu \ln f^2 = -\Box\, \ln f^2.$$
Note: In $(1+1)$-dimensions, any metric can be reduced to the above, so-called conformally flat form.

5. *Tachyons:* Consider a scalar field T that is described by the action
$$S[T] = -\frac{1}{c} \int \mathrm{d}^4x\, \sqrt{-g}\, V(T)\, \sqrt{1 - \alpha^2\, \partial_\mu T\, \partial^\mu T},$$
where α is a constant of suitable dimensions.

   Note: The field T is often referred to as the tachyon.

   (a) Vary the action with respect to the scalar field T to arrive at the equation of motion governing the field in a curved spacetime.

   (b) Vary the action with respect to the metric tensor and obtain the corresponding stress-energy tensor.

(c) Show that the conservation of the stress-energy tensor leads to the equation of motion governing the field T.

6. *Conformal transformations:* Show that, under the conformal transformation,

$$g_{ab}(x^c) \to \Omega^2(x^c)\, g_{ab}(x^c),$$

the Christoffel symbols Γ^a_{bc}, the Ricci tensor R^a_b, and the scalar curvature R of a n-dimensional manifold are modified as follows:

$$\begin{aligned}
\Gamma^a_{bc} &\to \Gamma^a_{bc} + \Omega^{-1}\, \left(\delta^a_b\, \Omega_{;c} + \delta^a_c\, \Omega_{;b} - g_{bc}\, g^{ad}\, \Omega_{;d}\right),\\
R^a_b &\to \Omega^{-2}\, R^a_b - (n-2)\, \Omega^{-1}\, g^{ac}\, \left(\Omega^{-1}\right)_{;bc} + \frac{1}{n-2}\, \Omega^{-n}\, \delta^a_b\, g^{cd}\, \left[\Omega^{(n-2)}\right]_{;cd},\\
R &\to \Omega^{-2}\, R + 2\,(n-1)\, \Omega^{-3}\, g^{ab}\, \Omega_{;ab} + (n-1)\,(n-4)\, \Omega^{-4}\, g^{ab}\, \Omega_{;a}\, \Omega_{;b}.
\end{aligned}$$

7. *Conformal invariance of the electromagnetic action:* Recall that, in a curved spacetime, the dynamics of the source free electromagnetic field is governed by the action

$$S[A^\mu] = -\frac{1}{16\,\pi\, c} \int \mathrm{d}^4x\, \sqrt{-g}\, F_{\mu\nu}\, F^{\mu\nu},$$

where

$$F_{\mu\nu} = A_{\mu;\nu} - A_{\nu;\mu} = A_{\mu,\nu} - A_{\nu,\mu},$$

while the commas and semi-colons, as usual, represent partial and covariant differentiation, respectively. Show that this action is invariant under the following conformal transformation:

$$x^\mu \to x^\mu, \quad A_\mu \to A_\mu \quad \text{and} \quad g_{\mu\nu} \to \Omega^2\, g_{\mu\nu}.$$

8. *The Schwarzschild singularity:* With the help of the given `Mathematica` code, evaluate the curvature invariant $R_{\alpha\beta\gamma\delta}\, R^{\alpha\beta\gamma\delta}$ for the case of the Schwarzschild metric. Show that, whereas the quantity is finite at the Schwarzschild radius $r_{\mathrm{S}} = 2\, G\, M/c^2$, it diverges at the origin.

Note: This implies that, while the Schwarzschild radius is a coordinate singularity (which can be avoided with a better choice of coordinates to describe the spacetime), the singularity at the origin is an unavoidable, physical one.

9. *Charged and rotating black holes:* Use the given `Mathematica` code to evaluate the Christoffel symbols, the Riemann, the Ricci, and the Einstein tensors as well as the Ricci scalar around the charged Reissner-Nordstrom and the rotating Kerr black holes that are described by the following line elements:

$$\mathrm{d}s^2 = c^2\, \left(1 - \frac{2\,\mu}{r} + \frac{q^2}{r^2}\right)\, \mathrm{d}t^2 - \left(1 - \frac{2\,\mu}{r} + \frac{q^2}{r^2}\right)^{-1}\, \mathrm{d}r^2 - r^2\, \left(\mathrm{d}\theta^2 + \sin^2\!\theta\, \mathrm{d}\phi^2\right),$$

where

$$\mu = \frac{G\,M}{c^2} \quad \text{and} \quad q^2 = \frac{G\,Q^2}{4\,\pi\, c^4},$$

and

$$\mathrm{d}s^2 = c^2\, \frac{\rho^2\, \Delta}{\Sigma^2}\, \mathrm{d}t^2 - \frac{\Sigma^2 \sin^2\theta}{\rho^2}\, (\mathrm{d}\phi - \omega\, \mathrm{d}t)^2 - \frac{\rho^2}{\Delta}\, \mathrm{d}r^2 - \rho^2\, \mathrm{d}\theta^2,$$

where

$$\rho^2 = r^2 + a^2 \cos^2\theta,\ \ \Delta = r^2 - 2\,\mu\, r + a^2,\ \ \Sigma^2 = (r^2+a^2)^2 - a^2\,\Delta\, \sin^2\theta,\ \ \omega = \frac{2\,\mu\, c\, r\, a}{\Sigma^2} \ \text{and}\ a = \frac{J}{M\,c}.$$

The quantities M, Q and J are constants that denote the mass, the electric charge and the angular momentum associated with the black holes, respectively.

10. *The Newtonian limit and the Poisson equation:* Recall that, in the non-relativistic limit, the metric corresponding to the Newtonian potential ϕ is given by

$$\mathrm{d}s^2 = c^2 \left[1 + \frac{2\,\phi(\boldsymbol{x})}{c^2}\right] \mathrm{d}t^2 - \mathrm{d}\boldsymbol{x}^2.$$

Let the energy density of the matter field that is giving rise to the Newtonian potential ϕ be $\rho\, c^2$. Show that, in such a case, the time-time component of the Einstein's equations reduces to the conventional Poisson equation in the limit of large c.

Note: As I had mentioned during the lectures, it is this Newtonian limit that determines the overall constant in the Einstein-Hilbert action.

Exercise sheet 13

Kinematics of the FLRW universe

1. *Spaces of constant curvature:* Consider spaces of constant curvature that are described by the metric tensor g_{ab}.

   (a) Argue that, the Riemann tensor associated with such a space can be expressed in terms of the metric g_{ab} as follows:
   $$R_{abcd} = \kappa\, \left(g_{ac}\, g_{bd} - g_{ad}\, g_{bc}\right),$$
   where κ is a constant.

   (b) Show that the Ricci tensor corresponding to the above Riemann tensor is given by
   $$R_{ab} = 2\, \kappa\, g_{ab}.$$

   Note: Examples of spacetimes with a constant scalar curvature are the Einstein static universe, the de Sitter and the anti de Sitter spacetimes.

2. *Visualizing the Friedmann metric:* The Friedmann universe is described by the line-element
   $$\mathrm{d}s^2 = c^2\, \mathrm{d}t^2 - a^2(t)\, \mathrm{d}\ell^2,$$
   where
   $$\mathrm{d}\ell^2 = \frac{\mathrm{d}r^2}{(1-\kappa\, r^2)} + r^2\, \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right)$$
   and $\kappa = 0, \pm 1$.

   (a) Let us define a new coordinate χ as follows:
   $$\chi = \int \frac{dr}{\sqrt{1-\kappa\, r^2}}.$$
   Show that in terms of the coordinate χ the spatial line element $\mathrm{d}\ell^2$ reduces to
   $$\mathrm{d}\ell^2 = \mathrm{d}\chi^2 + S_\kappa^2(\chi)\, \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right),$$
   where
   $$S_\kappa(\chi) = \begin{cases} \sin\chi & \text{for} \quad \kappa = 1, \\ \chi & \text{for} \quad \kappa = 0, \\ \sinh\chi & \text{for} \quad \kappa = -1. \end{cases}$$

   (b) Show that, for $\kappa = 1$, the spatial line-element $\mathrm{d}\ell^2$ can be described as the spherical surface
   $$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$$
   embedded in an Euclidean space described by the line-element
   $$\mathrm{d}\ell^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \mathrm{d}x_4^2.$$

   (c) Show that, for $\kappa = -1$, the spatial line-element $\mathrm{d}\ell^2$ can be described as the hyperbolic surface
   $$x_1^2 + x_2^2 + x_3^2 - x_4^2 = -1$$
   embedded in a Lorentzian space described by the line-element
   $$\mathrm{d}\ell^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 - \mathrm{d}x_4^2.$$

3. *Geodesic equations in a FLRW universe:* Obtain the following non-zero components of the Christoffel symbols for the FLRW line element:

$$\Gamma^t_{ij} = \frac{a\,\dot{a}}{c}\,\sigma_{ij},$$

where σ_{ij} denotes the spatial metric defined through the relation $\mathrm{d}\ell^2 = \sigma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$. Use these Christoffel symbols to arrive at the geodesic equations corresponding to the t coordinate for massive as well as massless particles in a FLRW universe.

4. *Weyl tensor and conformal invariance:* In $(3+1)$-spacetime dimensions, the Weyl tensor $C_{\alpha\beta\gamma\delta}$ is defined as follows:

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + \frac{1}{2}\,\left(g_{\alpha\delta}\,R_{\beta\gamma} + g_{\beta\gamma}\,R_{\alpha\delta} - g_{\alpha\gamma}\,R_{\beta\delta} - g_{\beta\delta}\,R_{\alpha\gamma}\right) + \frac{1}{6}\,\left(g_{\alpha\gamma}\,g_{\delta\beta} - g_{\alpha\delta}\,g_{\gamma\beta}\right)\,R.$$

   (a) Show that the Weyl tensor vanishes for the FLRW metric.

   (b) The vanishing Weyl tensor implies that there exists a coordinate system in which the FLRW metric (for all κ) is conformal to the Minkowski metric. It is straightforward to check that the metric of the $\kappa = 0$ (i.e. the spatially flat) FLRW universe can be expressed in the following form:

$$g_{\mu\nu} = a^2(\eta)\,\eta_{\mu\nu},$$

   where η is the conformal time coordinate defined by the relation

$$\eta = \int \frac{\mathrm{d}t}{a(t)},$$

   and $\eta_{\mu\nu}$ denotes the flat spacetime metric. Construct the coordinate systems in which the metrics corresponding to the $\kappa = \pm 1$ FLRW universes can be expressed in a form wherein they are conformally related to flat spacetime.

5. *Consequences of conformal invariance:* As we have seen, the action of the electromagnetic field in a curved spacetime is invariant under the conformal transformation.

   (a) Utilizing the conformal invariance of the electromagnetic action, show that the electromagnetic waves in the spatially flat FLRW universe can be written in terms of the conformal time coordinate η as follows:

$$A_\mu \propto \exp -(i\,k\,\eta) = \exp - \left[i\,k\,\int dt/a(t)\right].$$

   (b) Since the time derivative of the phase defines the instantaneous frequency $\omega(t)$ of the wave, conclude that $\omega(t) \propto a^{-1}(t)$.

Exercise sheet 14

Dynamics of the FLRW universe

1. *The Friedmann equations:* Recall that the FLRW universe is described by the line element

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-\kappa\,r^2} + r^2\,\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right],$$

where $\kappa = 0, \pm 1$.

(a) Arrive at the following expressions for the Ricci tensor R^μ_ν, the scalar curvature R, and the Einstein tensor G^μ_ν for the above Friedmann metric:

$$\begin{aligned}
R^t_t &= -\frac{3\,\ddot{a}}{c^2\,a},\\
R^i_j &= -\left[\frac{\ddot{a}}{c^2\,a} + 2\left(\frac{\dot{a}}{c\,a}\right)^2 + \frac{2\,\kappa}{a^2}\right]\delta^i_j,\\
R &= -6\left[\frac{\ddot{a}}{c^2\,a} + \left(\frac{\dot{a}}{c\,a}\right)^2 + \frac{\kappa}{a^2}\right],\\
G^t_t &= 3\left[\left(\frac{\dot{a}}{c\,a}\right)^2 + \frac{\kappa}{a^2}\right],\\
G^i_j &= \left[\frac{2\,\ddot{a}}{c^2\,a} + \left(\frac{\dot{a}}{c\,a}\right)^2 + \frac{\kappa}{a^2}\right]\delta^i_j,
\end{aligned}$$

where the overdots denote differentiation with respect to the cosmic time t.

(b) Consider a fluid described by the stress energy tensor

$$T^\mu_\nu = \mathrm{diag.}\left(\rho\,c^2, -p, -p, -p\right),$$

where ρ and p denote the mass density and the pressure associated with the fluid. In a smooth Friedmann universe, the quantities ρ and p depend only on time. Using the above Einstein tensor, obtain the following Friedmann equations for such a source:

$$\begin{aligned}
\left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa\,c^2}{a^2} &= \frac{8\,\pi\,G}{3}\,\rho,\\
\frac{2\,\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa\,c^2}{a^2} &= -\,\frac{8\,\pi\,G}{c^2}\,p.
\end{aligned}$$

(c) Show that these two Friedmann equations lead to the equation

$$\frac{\ddot{a}}{a} = -\,\frac{4\,\pi\,G}{3}\,\left(\rho + \frac{3\,p}{c^2}\right).$$

Note: This relation implies that $\ddot{a} > 0$, i.e. the universe will undergo accelerated expansion, only when $\left(\rho\,c^2 + 3\,p\right) < 0$.

2. *Conservation of the stress energy tensor in a FLRW universe:* Recall that the conservation of the stress energy tensor is described by the equation $T^\mu_{\nu;\mu} = 0$.

(a) Show that the time component of the stress energy tensor conservation law leads to the following equation in a Friedmann universe:

$$\dot{\rho} + 3\,H\,\left(\rho + \frac{p}{c^2}\right) = 0,$$

where $H = \dot{a}/a$, a quantity that is known as the Hubble parameter.

(b) Also arrive at this equation from the two Friedmann equations obtained above.

(c) Show that the above equation can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\, a^3\right) = -\frac{p}{c^2}\left(\frac{\mathrm{d}a^3}{\mathrm{d}t}\right).$$

3. *Evolution of energy density in a FLRW universe:* The different types of matter that are present in the universe are often described by an equation of state, i.e. the relation between the density and the pressure associated with the matter. Consider the following equation of state $p = w\,\rho\,c^2$, where w is a constant.

   (a) Using the above equation which governs the evolution of ρ in a FLRW universe, show that, in such a case,
   $$\rho \propto a^{-3\,(1+w)}.$$

   (b) While the quantity w vanishes for pressure free non-relativistic matter (such as baryons and cold dark matter), $w = 1/3$ for relativistic particles (such as photons and the nearly massless neutrinos). Note that the energy density does not change with time when $w = -1$ or, equivalently, when $p = -\rho\, c^2$. Such a type of matter is known as the cosmological constant. Utilizing the above result, express the total density of a universe filled with non-relativistic (NR) and relativistic (R) matter as well as the cosmological constant (Λ) as follows:
   $$\rho(a) = \rho_{_{\rm NR}}^0\left(\frac{a_0}{a}\right)^3 + \rho_{_{\rm R}}^0\left(\frac{a_0}{a}\right)^4 + \rho_{_\Lambda},$$
   where $\rho_{_{\rm NR}}^0$ and $\rho_{_{\rm R}}^0$ denote the density of non-relativistic and relativistic matter today (i.e. at, say, $t = t_0$, corresponding to the scale factor $a = a_0$).

   (c) Also, further rewrite the above expression as
   $$\rho(a) = \rho_{_{\rm C}}\left[\Omega_{_{\rm NR}}\left(\frac{a_0}{a}\right)^3 + \Omega_{_{\rm R}}\left(\frac{a_0}{a}\right)^4 + \Omega_{_\Lambda}\right] = \rho_{_{\rm C}}\left[\Omega_{_{\rm NR}}\,(1+z)^3 + \Omega_{_{\rm R}}\,(1+z)^4 + \Omega_{_\Lambda}\right],$$
   where $\Omega_{_{\rm NR}} = \rho_{_{\rm NR}}^0/\rho_{_{\rm C}}$, $\Omega_{_{\rm R}} = \rho_{_{\rm R}}^0/\rho_{_{\rm C}}$ and $\Omega_{_\Lambda} = \rho_{_\Lambda}/\rho_{_{\rm C}}$, while $\rho_{_{\rm C}}$ is the so-called critical density defined as
   $$\rho_{_{\rm C}} = \frac{3\,H_0^2}{8\,\pi\,G},$$
   with the quantity H_0 being the Hubble parameter (referred to as the Hubble constant) today. Note: The quantities H_0, $\Omega_{_{\rm NR}}$, $\Omega_{_{\rm R}}$ and $\Omega_{_\Lambda}$ are cosmological parameters that are to be determined by observations.

   (d) Observations suggest that $H_0 \simeq 72$ km s^{-1} Mpc^{-1}. Evaluate the corresponding numerical value of the critical density $\rho_{\rm c}$.
   Note: A parsec (pc) corresponds to 3.26 light years, and a Mega parsec (Mpc) amounts to 10^6 parsecs.

4. *The Cosmic Microwave Background:* It is found that we are immersed in a perfectly thermal and nearly isotropic distribution of radiation, which is referred to as Cosmic Microwave Background (CMB), as it energy density peaks in the microwave region of the electromagnetic spectrum. The CMB is a relic of an earlier epoch when the universe was radiation dominated, and it provides the dominant contribution to the relativistic energy density in the universe.

   (a) Given that the temperature of the CMB today is $T \simeq 2.73$ K, show that one can write
   $$\Omega_{_{\rm R}}\, h^2 \simeq 2.56 \times 10^{-5},$$
   where h is related to the Hubble constant H_0 as follows:
   $$H_0 \simeq 100\, h \text{ km s}^{-1}\text{ Mpc}^{-1}.$$

(b) Show that the redshift $z_{\rm eq}$ at which the energy density of matter and radiation were equal is given by

$$1 + z_{\rm eq} = \frac{\Omega_{\rm NR}}{\Omega_{\rm R}} \simeq 3.9 \times 10^4 \, \left(\Omega_{\rm NR}\, h^2\right).$$

(c) Also, show that the temperature of the radiation at this epoch is given by

$$T_{\rm eq} \simeq 9.24 \, \left(\Omega_{\rm NR}\, h^2\right) \, {\rm eV}.$$

5. *Solutions to the Friedmann equations:* We had discussed the solutions to Friedmann equations in the presence of a single component when the universe is spatially flat (i.e. when $\kappa = 0$). It proves to be difficult to obtain analytical solutions for the scale factor when all the three components of matter (viz. non-relativistic and relativistic matter as well as the cosmological constant) are simultaneously present. However, the solutions can be obtained for the cases wherein two of the components are present.

(a) Integrate the first Friedmann equation for a $\kappa = 0$ universe with matter and radiation to obtain that

$$a(\eta) = \sqrt{\Omega_{\rm R}\, a_0^4}\, \left(H_0\, \eta\right) + \frac{\Omega_{\rm NR}\, a_0^3}{4}\, \left(H_0\, \eta\right)^2,$$

where η is the conformal time coordinate. Show that, at early (i.e. for small η) and late times (i.e. for large η), this solution reduces to the behavior in the radiation and matter dominated epochs, respectively, as required.

Note: In obtaining the above result, it has been assumed that $a = 0$ at $\eta = 0$.

(b) Integrate the Friedmann equation for a $\kappa = 0$ universe with matter and cosmological constant to obtain that

$$\frac{a(t)}{a_0} = \left(\frac{\Omega_{\rm NR}}{\Omega_\Lambda}\right)^{1/3} \sinh^{2/3}\left(3\, \sqrt{\Omega_\Lambda}\, H_0\, t/2\right).$$

Also, show that, at early times, this solution simplifies to $a \propto t^{2/3}$, while at late times, it behaves as $a \propto \exp\,(\Omega_\Lambda^{3/2}\, H_0\, t/\Omega_{\rm NR})$, as expected.

Exercise sheet 15

Gravitational waves

1. *The linearized metric I:* Consider a small perturbation to flat spacetime so that the standard Minkowski metric can be expressed as

$$g_{\mu\nu} \simeq \eta_{\mu\nu} + \epsilon\, h_{\mu\nu},$$

where ϵ is a small dimensionless quantity. Show that, at the same order in ϵ, the corresponding contravariant metric tensor and the Christoffel symbols are given by

$$g^{\mu\nu} \simeq \eta^{\mu\nu} + \epsilon\, h^{\mu\nu}$$

and

$$\Gamma^{\alpha}_{\beta\gamma} \simeq \frac{\epsilon}{2}\left(h^{\alpha}_{\gamma,\beta} + h^{\alpha}_{\beta,\gamma} - h_{\beta\gamma}{}^{,\alpha}\right),$$

respectively.

2. *The linearized metric II:* Let us now turn to the evaluation of the curvature and the Einstein tensors corresponding to the above metric.

   (a) Show that, at the linear order, the Riemann and the Ricci tensors and the scalar curvature are given by

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &\simeq \frac{\epsilon}{2}\left(h_{\alpha\delta,\beta\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta} - h_{\beta\delta,\alpha\gamma}\right),\\ R_{\beta\delta} &\simeq \frac{\epsilon}{2}\left(h^{\gamma}_{\delta,\beta\gamma} + h^{\alpha}_{\beta,\alpha\delta} - h_{,\beta\delta} - \eta^{\alpha\gamma}\, h_{\beta\delta,\alpha\gamma}\right) \end{aligned}$$

   and

$$R = \epsilon\left(h^{\alpha\beta}{}_{,\alpha\beta} - \Box\, h\right),$$

   where $\Box$ is the d'Alembertian corresponding to the Minkowski metric $\eta_{\mu\nu}$, while $h = \eta^{\mu\nu}\, h_{\mu\nu}$ denotes the trace of the perturbation $h_{\mu\nu}$.

   (b) Finally, show that the corresponding Einstein tensor can be expressed as

$$G_{\alpha\beta} = \frac{\epsilon}{2}\left(h^{\gamma}_{\beta,\alpha\gamma} + h^{\gamma}_{\alpha,\beta\gamma} - \Box\, h_{\alpha\beta} - h_{,\alpha\beta} - \eta_{\alpha\beta}\, h^{\gamma\delta}{}_{,\gamma\delta} + \eta_{\alpha\beta}\,\Box\, h\right).$$

3. *Gauge transformations:* Consider the following 'small' coordinate transformations:

$$x^{\mu} \to x'^{\mu} \simeq x^{\mu} + \epsilon\, \xi^{\mu},$$

which are of the same amplitude as the perturbation $h_{\mu\nu}$. Show that under such a transformation, the perturbation $h_{\mu\nu}$ transforms as follows:

$$h_{\mu\nu} \to h'_{\mu\nu} \simeq h_{\mu\nu} - \left(\xi_{\mu,\nu} + \xi_{\nu,\mu}\right).$$

Note: Such a 'small' transformation is known as a gauge transformation.

4. *The de Donder gauge:* Let us define a new set of variables $\psi_{\mu\nu}$, which are related to the metric perturbation $h_{\mu\nu}$ as follows:

$$\psi_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\, h.$$

   (a) Show that, in terms of $\psi_{\mu\nu}$, the above Einstein tensor is given by

$$G_{\alpha\beta} = \frac{\epsilon}{2}\left(\psi^{\gamma}_{\alpha,\beta\gamma} + \psi^{\gamma}_{\beta,\alpha\gamma} - \Box\,\psi_{\alpha\beta} - \eta_{\alpha\beta}\,\psi^{\gamma\delta}{}_{,\gamma\delta}\right).$$

(b) Show that, under the above-mentioned gauge transformations, the variables $\psi_{\mu\nu}$ transform as

$$\psi_{\mu\nu} \to \psi'_{\mu\nu} \simeq \psi_{\mu\nu} - (\xi_{\mu,\nu} + \xi_{\nu,\mu}) + \eta_{\mu\nu}\, \xi^{\lambda}_{,\lambda}.$$

(c) If we now impose the condition

$$\psi^{\alpha}_{\beta,\alpha} = 0,$$

show that, this corresponds to

$$\psi'^{\alpha}_{\beta,\alpha} = \psi^{\alpha}_{\beta,\alpha} - \Box\, \xi_{\beta}.$$

Note: These conditions correspond to four equations, which can be achieved using the gauge functions ξ_{μ}. A gauge wherein the condition is satisfied is known as the de Donder gauge.

(d) Also, show that the above condition corresponds to the following condition on $h_{\alpha\beta}$:

$$h^{\alpha}_{\beta,\alpha} - \frac{1}{2}\, h_{,\beta} = 0.$$

5. *The wave equation:* In the absence of sources, one has $G_{\alpha\beta} = 0$.

(a) Show that, in a gauge wherein $\psi^{\alpha}_{\beta,\alpha} = 0$, the vacuum Einstein's equations simplify to

$$\Box\, \psi_{\alpha\beta} = 0.$$

(b) Show that, in terms of $h_{\alpha\beta}$, this equation corresponds to the equation

$$\Box\, h_{\alpha\beta} = 0,$$

along with the additional condition

$$\Box\, h = 0.$$

Note: The solutions to these equations describe propagating gravitational waves in flat spacetime.

End-of-semester exam

From special relativity to gravitational waves

1. <u>*Repeated Lorentz boosts:*</u> Consider a series of inertial observers, with each observer seeing the preceding observer moving away along the positive x-axis with the speed u. Let the speed of a particle (that is moving along the positive x-axis) in the frame of the zeroth observer be v_0, and let v_n be the speed of the particle as observed by the n-th observer.

   (a) Determine the relation between v_{n+1} and v_n. **3 marks**

   (b) What is speed v_n as $n \to \infty$? **7 marks**

2. <u>*Motion in a constant and uniform magnetic field:*</u> Consider a particle of mass m and charge e that is moving in a magnetic field of strength B that is directed, say, along the positive z-axis.

   (a) Show that the energy $\mathcal{E} = \gamma\, m\, c^2$ of the particle is a constant. **3 marks**

   (b) Determine the trajectory $\boldsymbol{x}(t)$ of the particle and show that, in the absence of any initial momentum along the z-direction, the particle describes a circular trajectory in the x-y plane with the angular frequency $\omega = e\, c\, B/\mathcal{E}$. **7 marks**

3. <u>*Some tensor algebra:*</u>

   (a) If t_{ab} are the components of a symmetric tensor and v_a the components of a vector, show that if **6 marks**

   $$v_a\, t_{bc} + v_c\, t_{ab} + v_b\, t_{ca} = 0,$$

   then either $t_{ab} = 0$ or $v_a = 0$.

   (b) If the tensor t_{abcd} satisfies $t_{abcd}\, v^a\, w^b\, v^c\, w^d = 0$ for arbitrary vectors v^a and w^a, show that **4 marks**

   $$t_{abcd} + t_{cdab} + t_{cbad} + t_{adcb} = 0.$$

4. <u>*Geodesics in a Poincaré half plane:*</u> Consider the so-called Poincaré half plane described by the line-element

   $$\mathrm{d}l^2 = \frac{a^2}{y^2}\,\left(\mathrm{d}x^2 + \mathrm{d}y^2\right),$$

   where $-\infty < x < \infty$, while $0 < y < \infty$. Determine the trajectory $y(x)$ of geodesics in this geometry. **10 marks**

5. <u>*Is the spacetime curved?*</u> Recall that the FLRW universe is described by the line-element

   $$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\,\left[\frac{\mathrm{d}r^2}{1-\kappa\, r^2} + r^2\,\left(\mathrm{d}\theta^2 + \sin^2\!\theta\,\mathrm{d}\phi^2\right)\right],$$

   where the function $a(t)$ is referred to as the scale factor and $\kappa = 0, \pm 1$.

   (a) Consider the case wherein $a(t) = c\, t$ and $\kappa = -1$. Is this a solution to the Friedmann equations? **2 marks**

   (b) What are energy density and pressure that drive the expansion? **2 marks**

   (c) What does the metric describe? Is it a curved spacetime? **2 marks**

Note: Recall that, for instance, the non-zero components of the Ricci tensor and scalar curvature are given by

$$\begin{aligned}
R^t_t &= -\frac{3\,\ddot{a}}{c^2\,a}, \\
R^i_j &= -\left[\frac{\ddot{a}}{c^2\,a} + 2\left(\frac{\dot{a}}{c\,a}\right)^2 + \frac{2\,\kappa}{a^2}\right]\delta^i_j, \\
R &= -6\left[\frac{\ddot{a}}{c^2\,a} + \left(\frac{\dot{a}}{c\,a}\right)^2 + \frac{\kappa}{a^2}\right].
\end{aligned}$$

(d) Can you construct a coordinate transformation that reduces the FLRW line-element with $a(t) = c\,t$ and $\kappa = -1$ to the Minkowskian form? [4 marks]

6. *Properties of Killing vectors:* If ξ^a is a Killing vector, show that

   (a) $\xi_{a;b;c} = R_{dcba}\,\xi^d$, [7 marks]

   (b) $\xi_{a;b}{}^{;b} + R_{ac}\,\xi^c = 0$. [3 marks]

7. Painlevé-Gullstrand coordinates: Recall that the Schwarzschild line-element is given by

$$\mathrm{d}s^2 = c^2\left(1 - \frac{r_\mathrm{S}}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{r_\mathrm{S}}{r}\right)^{-1}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right),$$

   where $r_\mathrm{S} = 2\,G\,M/c^2$ is the Schwarzschild radius. Let

$$c\,\mathrm{d}t = c\,\mathrm{d}T - h(r)\,\mathrm{d}r,$$

   where

$$h(r) = \frac{\sqrt{r_\mathrm{S}/r}}{1 - (r_\mathrm{S}/r)}.$$

   (a) Determine the Schwarzschild line-element in terms of the coordinates (T, r, θ, ϕ). [8 marks]
   Note: The coordinates (T, r, θ, ϕ) are known as the Painlevé-Gullstrand coordinates.

   (b) Does the Schwarzschild line-element expressed in the Painlevé-Gullstrand coordinates exhibit any singular behavior at $r = r_\mathrm{S}$? [2 marks]

8. *Numbers describing our universe:* Various observations indicate the value of the Hubble constant to be $H_0 \simeq 72$ km s^{-1} Mpc^{-1}. Given this information,

   (a) Evaluate the corresponding time scale H_0^{-1} in terms of billions of years. [3 marks]

   (b) Estimate the resulting distance $c\,H_0^{-1}$ in units of Mpc. [3 marks]

   (c) Determine the corresponding critical density of the universe today in units of kg/m^3. [4 marks]

   Note: A parsec (pc) corresponds to 3.26 light years, and a Mega parsec (Mpc) amounts to 10^6 parsecs. The value of the Newton's gravitational constant G is 6.673×10^{-11} m^3 kg^{-1} s^{-2}.

9. *Solutions to the Friedmann equations:* We had discussed the solutions to the Friedmann equations in the presence of a single component when the universe is spatially flat (i.e. when $\kappa = 0$). It proves to be difficult to obtain analytical solutions for the scale factor when all the three components of matter (viz. non-relativistic and relativistic matter as well as the cosmological constant) are simultaneously present. However, the solutions can be obtained for the cases wherein two of the components are present.

(a) Integrate the first Friedmann equation for a $\kappa = 0$ universe with matter and radiation to obtain that

$$a(\eta) = \sqrt{\Omega_{\rm R}\, a_0^4}\; (H_0\, \eta) + \frac{\Omega_{\rm NR}\, a_0^3}{4}\; (H_0\, \eta)^2\,,$$

where η is the conformal time coordinate. Show that, at early (i.e. for small η) and late (i.e. for large η) times, this solution reduces to the behavior in the radiation and matter dominated epochs, respectively, as required. $\boxed{\text{5 marks}}$

Note: To arrive at the above result, it is to be assumed that $a = 0$ at $\eta = 0$.

(b) Integrate the Friedmann equation for a $\kappa = 0$ universe with matter and cosmological constant to obtain that

$$\frac{a(t)}{a_0} = \left(\frac{\Omega_{\rm NR}}{\Omega_{\Lambda}}\right)^{1/3}\, \sinh^{2/3}\left(3\,\sqrt{\Omega_\Lambda}\, H_0\, t/2\right).$$

Also, determine the behavior of the solution at early and late times. $\boxed{\text{5 marks}}$

10. *Homogeneous scalar field in a FLRW universe:* Consider a scalar field ϕ that is governed by the action

$$S[\phi] = \frac{1}{c}\int {\rm d}^4x\, \sqrt{-g}\, \left[\frac{1}{2}\, g^{\mu\nu}\, \partial_\mu\phi\, \partial_\nu\phi - V(\phi)\right],$$

where $V(\phi)$ is the potential describing the scalar field.

(a) Upon varying this action with respect to the metric tensor, arrive at the stress energy-tensor of the scalar field. $\boxed{\text{3 marks}}$

(b) Recall that the spatially flat FLRW line-element is described by the line-element

$${\rm d}s^2 = c^2\, {\rm d}t^2 - a^2(t)\, \left({\rm d}x^2 + {\rm d}y^2 + {\rm d}z^2\right),$$

where $a(t)$ denotes the scale factor. Assuming the scalar field to be homogeneous (i.e. only dependent on time and *independent* of the spatial coordinates), determine the non-zero components of the stress-energy tensor. $\boxed{\text{3 marks}}$

(c) Comparing the stress-energy tensor of the scalar field with that of a fluid (in the comoving frame) in a FLRW universe, identify the energy density and pressure associated with the scalar field. Then, using the equation governing the conservation of energy of a fluid in a FLRW universe, arrive at the equation of motion describing the scalar field. $\boxed{\text{4 marks}}$