Introduction to inflation and cosmological perturbation theory

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Plan of the talk

A few words on the conventions and notations

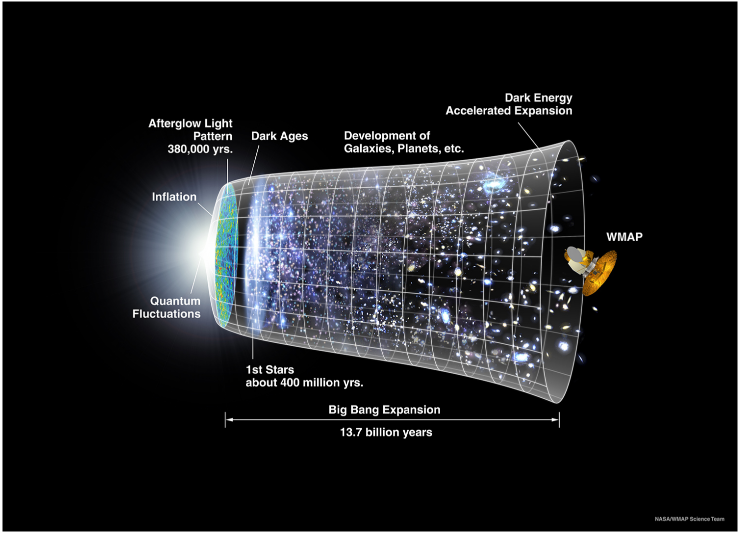
- We shall work in units such that $c = \hbar = 1$, and define the Planck mass to be $M_{_{\rm Pl}} = (8\,\pi\,G)^{-1/2}$.
- As is often done, particularly in the context of inflation, we shall assume the background universe to be described by the following spatially flat, Friedmann-Lemaître-Robertson-Walker (FLRW) line-element:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\boldsymbol{x}^2 = a^2(\eta)\,\left(-\mathrm{d}\eta^2 + \mathrm{d}\boldsymbol{x}^2\right),$$

  where t is the cosmic time, $a(t)$ is the scale factor and $\eta = \int \mathrm{d}t/a(t)$ denotes the conformal time coordinate.
- We shall denote differentiation with respect to the cosmic and the conformal times t and η by an overdot and an overprime, respectively.
- Moreover, as usual, $H = \dot{a}/a$ shall denote the Hubble parameter associated with the FLRW universe, and N shall denote the number of e-folds.

Plan of the talk

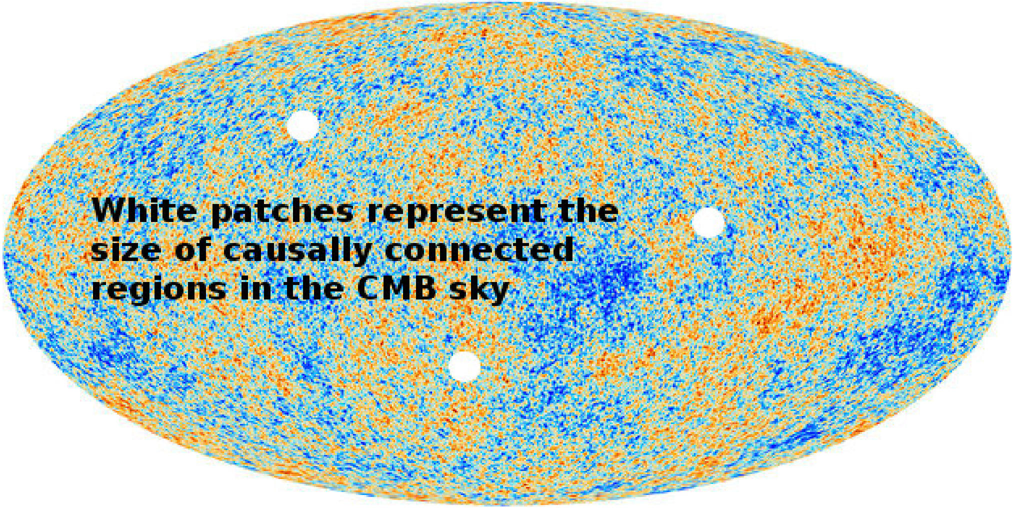
Timeline of the universe



A pictorial timeline of the universe[1].

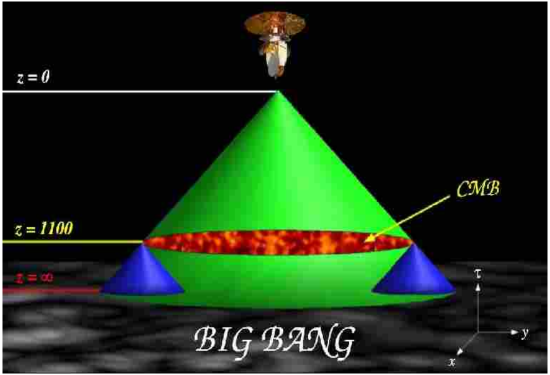
[1] Image from http://wmap.gsfc.nasa.gov/media/060915/060915_CMB_Timeline150.jpg.

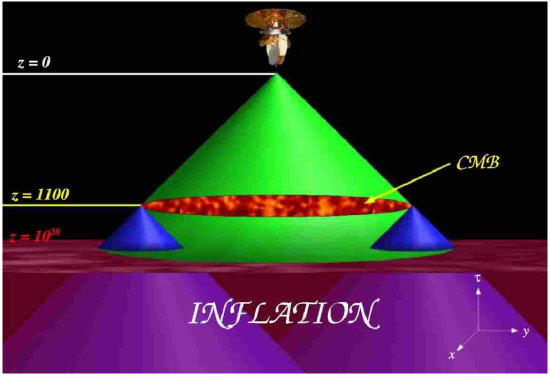
Horizon problem



The radiation from the CMB arriving at us from regions separated by more than the Hubble radius at the surface of last scattering, which subtends an angle of about 1° today, could not have interacted before decoupling.

Resolution of the horizon problem in the inflationary scenario

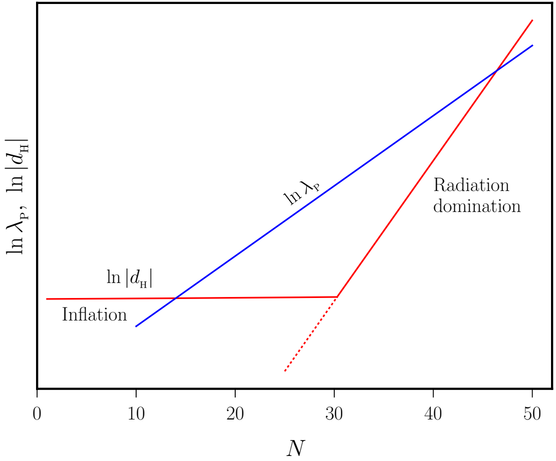




Another illustration of the horizon problem (on the left), and an illustration of its resolution (on the right) through an early and sufficiently long epoch of inflation[2].

[2]Images from W. Kinney, astro-ph/0301448.

Bringing the modes inside the Hubble radius



Evolution of comoving lengths

The physical wavelength $\lambda_{\rm P} \propto a$ (in blue) and the Hubble radius $d_{\rm H} = H^{-1}$ (in red) in the inflationary scenario[3]. The scale factor is expressed in terms of e-folds N as $a(N) \propto {\rm e}^{N}$.

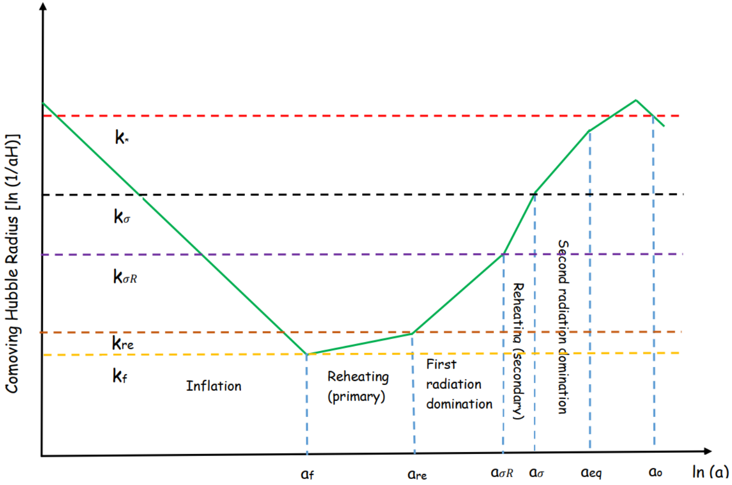
[3] See, for example, E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley Publishing Company, New York, 1990), Fig. 8.4.

Condition for inflation

If we require that $\lambda_{\mathrm{P}} < d_{\mathrm{H}}$ at a sufficiently early time, then we need to have an epoch wherein λ_{P} decreases faster than the Hubble scale *as we go back in time*, i.e. a regime during which

$$-\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{\lambda_{\mathrm{P}}}{d_{\mathrm{H}}}\right) < 0 \quad \Rightarrow \quad \ddot{a} > 0.$$

Behavior of the comoving wave number and Hubble radius



Behavior of the comoving wave number k (horizontal lines in different colors) and the comoving Hubble radius $d_{\rm H}/a = (a\,H)^{-1}$ (in green) across different epochs[4].

[4] Md. R. Haque, D. Maity, T. Paul and L. Sriramkumar, Phys. Rev. D **104**, 063513 (2021).

Scalar fields in a FLRW universe

The equations that govern the dynamics of the spatially flat, FLRW universe are given by

$$H^2 = \frac{8\,\pi\,G}{3}\,\rho, \quad \frac{\ddot{a}}{a} = -\frac{4\,\pi\,G}{3}(\rho + 3\,p),$$

where ρ and p denote the energy density and pressure associated with the source(s) that is (are) driving the expansion.

The energy density and pressure associated with a homogeneous scalar field in a FLRW universe are given by

$$\rho_\phi = \frac{1}{2}\,\dot{\phi}^2 + V(\phi), \quad p_\phi = \frac{1}{2}\,\dot{\phi}^2 - V(\phi),$$

where $V(\phi)$ is the potential describing the scalar field. From the equation governing the conservation of energy, viz.

$$\dot{\rho}_\phi + 3\,H\,(\rho_\phi + p_\phi) = 0,$$

the equation of motion governing the scalar field can be obtained to be

$$\ddot{\phi} + 3\,H\,\dot{\phi} + V_\phi = 0,$$

where $V_\phi = \mathrm{d}V/\mathrm{d}\phi$.

Driving inflation with scalar fields

From the Friedmann equations, for $\ddot{a} > 0$, we require that

$$(\rho + 3\,p) < 0.$$

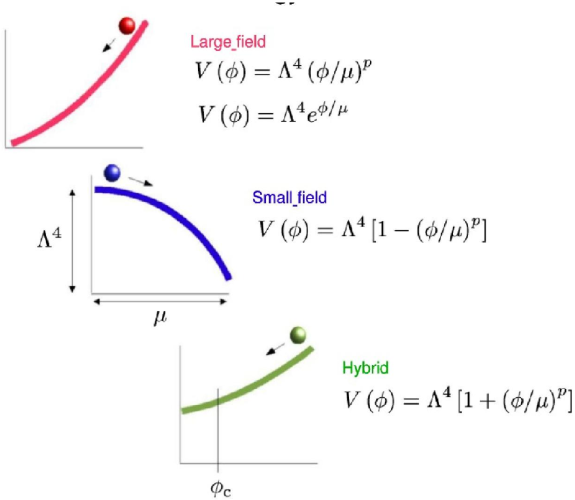
In the case of a canonical scalar field, this condition simplifies to

$$\dot{\phi}^2 < V(\phi).$$

This condition can be achieved if the scalar field ϕ is initially displaced from a minima of the potential, and inflation will end when the field approaches a minima with zero or negligible potential energy[5].

[5] See, for instance, B. A. Bassett, S. Tsujikawa and D. Wands, Rev. Mod. Phys. **78**, 537 (2006).

A variety of potentials to choose from



A variety of scalar field potentials have been considered to drive inflation[6]. Often, these potentials are classified as small field, large field and hybrid models.

[6]Image from W. Kinney, astro-ph/0301448.

Slow roll approximation and the potential slow roll parameters

In the slow roll approximation, we can write the first Friedmann equation as

$$H^2 \simeq \frac{V(\phi)}{3\,M_{_{\rm Pl}}^2}.$$

Also, if we assume that the acceleration of the field is small, we have

$$3\,H\,\dot{\phi} \simeq -V_\phi.$$

Given a potential $V(\phi)$, the slow roll approximation requires that the following potential slow roll parameters be small when compared to unity[7]:

$$\epsilon_{\rm V} = \frac{M_{_{\rm Pl}}^2}{2}\,\left(\frac{V_\phi}{V}\right)^2, \quad \eta_{\rm V} = M_{_{\rm Pl}}^2\,\left(\frac{V_{\phi\phi}}{V}\right), \quad \xi^2 = M_{_{\rm Pl}}^4\,\left(\frac{V_\phi\,V_{\phi\phi\phi}}{V^2}\right),$$

where $V_{\phi\phi} = {\rm d}^2V/{\rm d}\phi^2$ and $V_{\phi\phi\phi} = {\rm d}^3V/{\rm d}\phi^3$.

[7] See, for instance, A. R. Liddle, P. Parsons and J. D. Barrow, Phys. Rev. D **50**, 7222 (1994).

Hierarchy of slow roll parameters

Nowadays, it is more common to use the following hierarchy of slow roll parameters[8]:

$$\epsilon_1 = -\frac{\mathrm{d}\ln H}{\mathrm{d}N}, \quad \epsilon_{i+1} = \frac{\mathrm{d}\ln \epsilon_i}{\mathrm{d}N} \quad (i \geq 0).$$

We can express the first Friedmann equation and the equation of motion for the scalar field in terms of the first two slow roll parameters ϵ_1 and ϵ_2 as

$$H^2 = \frac{V(\phi)}{M_{\mathrm{Pl}}^2\,(3-\epsilon_1)}, \quad \left(3 - \epsilon_1 + \frac{\epsilon_2}{2}\right) H\,\dot{\phi} + V_\phi = 0.$$

In the slow roll approximation wherein $(\epsilon_1, \epsilon_2) \ll 1$, these two equations can be combined to arrive at

$$N - N_{\mathrm{i}} \simeq -\frac{1}{M_{\mathrm{Pl}}^2} \int_{\phi_{\mathrm{i}}}^{\phi} \mathrm{d}\phi\, \frac{V}{V_\phi}.$$

Given a potential $V(\phi)$, this equation can be integrated to obtain $\phi(N)$.

[8] D. J. Schwarz, C. A. Terrero-Escalante, A. A. Garcia, Phys. Lett. B **517**, 243 (2001); S. M. Leach, A. R. Liddle, J. Martin and D. J. Schwarz, Phys. Rev. D **66**, 023515 (2002).

Solution in the slow roll approximation – Large field models

In the case of the potential

$$V(\phi) = V_0 \left(\frac{\phi}{M_{\rm Pl}}\right)^n,$$

note that

$$\epsilon_{\rm V} = \frac{M_{\rm Pl}^2}{2} \left(\frac{V_\phi}{V}\right)^2 = \frac{M_{\rm Pl}^2}{2} \left(\frac{n}{\phi}\right)^2 = 1,$$

leads to

$$\phi_{\rm e} = \frac{n}{\sqrt{2}}\, M_{\rm Pl}.$$

In such a case, we can easily integrate the equation of motion in the slow roll approximation to obain that

$$\phi^2(N) \simeq \phi_{\rm e}^2 + 2\, n\, (N_{\rm e} - N)\, M_{\rm Pl}^2.$$

For $n = 2$, $\phi_{\rm e} = \sqrt{2}\, M_{\rm Pl}$ and, if we choose, $N_{\rm e} - N_{\rm i} = 60$, we have $\phi(N_{\rm i}) \simeq 16\, M_{\rm Pl}$.

Solution in the slow roll approximation – Small field model

In the case of the potential

$$V(\phi) = V_0 \left[1 - \left(\frac{\phi}{\mu}\right)^q\right],$$

the condition

$$\epsilon_{\mathrm{V}} = \frac{M_{_{\mathrm{Pl}}}^2}{2\,\mu^2}\, \frac{q^2\,(\phi/\mu)^{2\,q-2}}{[1-(\phi/\mu)^q]^2} = 1,$$

leads to, for $q=2$ and $\mu \gg M_{_{\mathrm{Pl}}}$,

$$\phi_{\mathrm{e}} = \frac{1}{\sqrt{2}}\left(-1+\sqrt{1+\frac{2\,\mu^2}{M_{_{\mathrm{Pl}}}^2}}\right) M_{_{\mathrm{Pl}}} \simeq \mu.$$

Also, in the slow roll approximation, we can obain that

$$\mu^2\,\ln\left(\frac{\phi}{\phi_{\mathrm{e}}}\right) - \frac{1}{2}\,\left(\phi^2-\phi_{\mathrm{e}}^2\right) \simeq 2\,(N-N_{\mathrm{e}})\,M_{_{\mathrm{Pl}}}^2.$$

Solution in the slow roll approximation – Starobinsky model

In the Staorobinsky model described by the potential

$$V(\phi) = V_0 \left[1 - \exp\left(-\sqrt{\frac{2}{3}}\,\frac{\phi}{M_{\rm Pl}}\right)\right]^2,$$

note that

$$\epsilon_{\rm V} = \frac{M_{\rm Pl}^2}{2}\left(\frac{V_\phi}{V}\right)^2 = \frac{4}{3}\,\frac{1}{\left\{\exp\left[\sqrt{2/3}\,(\phi/M_{\rm Pl})\right] - 1\right\}^2} = 1,$$

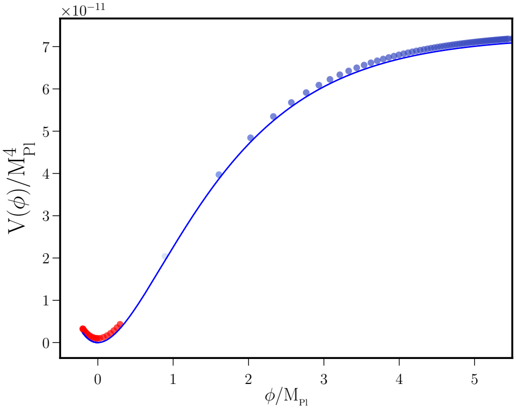
leads to

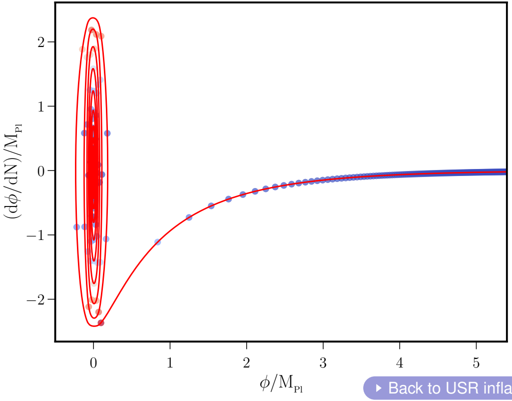
$$\phi_{\rm e} \simeq \sqrt{\frac{3}{2}}\,\ln\left(1 + \frac{2}{\sqrt{3}}\right) M_{\rm Pl}.$$

In this case, we find that

$$N - N_{\rm e} \simeq -\frac{3}{4}\left[\exp\left(\sqrt{\frac{2}{3}}\,\frac{\phi}{M_{\rm Pl}}\right) - \exp\left(\sqrt{\frac{2}{3}}\,\frac{\phi_{\rm e}}{M_{\rm Pl}}\right) - \sqrt{\frac{2}{3}}\left(\frac{\phi}{M_{\rm Pl}} - \frac{\phi_{\rm e}}{M_{\rm Pl}}\right)\right].$$

The inflationary attractor





Back to USR inflation

Evolution of the scalar field in the popular Starobinsky model, which leads to slow roll inflation, is indicated (as circles, in blue and red) at regular intervals of time (on the left). Illustration of the behavior of the scalar field in phase space (on the right)[9].

[9] Figure from H. V. Ragavendra, *Observational imprints of non-trivial inflationary dynamics over large and small, scales*, Ph.D. Thesis, Indian Institute of Technology Madras, Chennai, India (2022).

Plan of the talk

Nature of the perturbations

In a Friedmann universe, the perturbations in the metric and the matter can be classified according to their behavior with respect to local rotation of the spatial coordinates on hypersurfaces of constant time as follows[10]:

- ✦ Scalar perturbations – Density and pressure perturbations,
- ✦ Vector perturbations – Rotational velocity fields,
- ✦ Tensor perturbations – Gravitational waves.

The metric perturbations are related to the matter perturbations through the first order Einstein's equations.

Inflation does not produce any vector perturbations, while the tensor perturbations can be generated even in the absence of sources.

It is the fluctuations in the inflaton field ϕ that act as the seeds for the scalar perturbations that are primarily responsible for the anisotropies in the CMB and, eventually, the present day inhomogeneities.

[10] See, for instance, L. Sriramkumar, Curr. Sci. **97**, 868 (2009).

Perturbations in the metric tensor

The perturbations in the FLRW metric, say, $\delta g_{\mu\nu}$, can be split as[11]

$$\delta g_{\mu\nu} = \left(\delta g_{00}, \delta g_{0i}, \delta g_{ij}\right).$$

As it contains no running index, evidently, the perturbation $\delta g_{00} = \mathcal{A}$, say, is a scalar.

Since, one can always decompose a vector into a gradient of a scalar and a vector that is divergence free, we can write δg_{0i} as

$$\delta g_{0i} = \nabla_i\,\mathcal{B} + \mathcal{S}_i,$$

where $\mathcal{B}$ is a scalar, while $\mathcal{S}_i$ is a vector that satisfies the condition $\nabla_i\,\mathcal{S}^i = 0$.

A similar decomposition can be carried for the quantity δg_{ij} by repeating the above analysis on each of the two indices so that we have

$$\delta g_{ij} = \psi\,\delta_{ij} + \frac{1}{2}\left(\nabla_i\,\mathcal{F}_j + \nabla_j\,\mathcal{F}_i\right) + \left[\frac{1}{2}\left(\nabla_i\,\nabla_j + \nabla_j\,\nabla_i\right) - \frac{1}{3}\,\delta_{ij}\,\nabla^2\right]\mathcal{E} + \mathcal{H}_{ij},$$

where ψ and $\mathcal{E}$ are scalars, $\mathcal{F}_i$ (like $\mathcal{S}_i$) is a divergence free vector, and $\mathcal{H}_{ij}$ is a symmetric, traceless and transverse tensor that satisfies the conditions $\mathcal{H}^i_i = 0$ and $\nabla_i\,\mathcal{H}^{ij} = 0$.

[11] See, for instance, L. Sriramkumar, Curr. Sci. **97**, 868 (2009).

Counting the number of degrees of freedom

To describe $\delta g_{\mu\nu}$, we require the scalars $\mathcal{A}$, $\mathcal{B}$, ψ and $\mathcal{E}$, amounting to four degrees of freedom.

In $(D+1)$-spacetime dimensions, we also need the two divergence free spatial vectors $\mathcal{S}_i$ and $\mathcal{F}_i$ that add up to $2\,(D-1)$ degrees of freedom.

Moreover, after we impose the traceless and transverse conditions, the tensor $\mathcal{H}_{ij}$ has

$$\frac{1}{2}\,D\,(D+1)-(D+1)=\frac{1}{2}\,(D+1)\,(D-2)$$

degrees of freedom.

Upon adding all these scalar, vector and tensor degrees of freedom, we obtain

$$4+2\,(D-1)+\frac{1}{2}\,(D+1)\,(D-2)=\frac{1}{2}\,(D+1)\,(D+2),$$

which are the total number of degrees of freedom associated with the perturbed metric in $(D+1)$-spacetime dimensions.

Gauges or the coordinate degrees of freedom

In the FLRW background, the $(D+1)$ coordinate transformations that relate different coordinate systems that describe the perturbed metric can be expressed in terms of scalars, say, δt and δx, as follows:

$$t \to t + \delta t, \quad x^i \to x^i + \nabla^i\, (\delta x).$$

Similarly, we can construct coordinate transformations in terms of a divergence free vector, say, δx^i, as

$$t \to t, \quad x^i \to x^i + \delta x^i.$$

There are no coordinate degrees of freedom associated with the tensor perturbations.

The two scalar quantities δt and δx, and the divergence free vector δx^i, constitute the

$$2 + (D-1) = (D+1)$$

degrees of freedom associated with the coordinate transformations in $(D+1)$-spacetime dimensions.

Independent degrees of freedom

The scalar functions $(\mathcal{A}, \mathcal{B}, \psi, \mathcal{E})$ were required to describe the perturbed metric tensor. But, there were two scalar degrees of freedom, viz. δt and δx, associated with the choice of coordinates. Hence, the actual number of independent scalar degrees of freedom is $(4-2)=2$, which is true in any spacetime dimension.

We needed two divergence free vectors, $\mathcal{S}_i$ and $\mathcal{F}_i$, amounting to a total of $2\,(D-1)$ degrees of freedom, to describe the perturbed metric tensor. If we subtract the $(D-1)$ vector degrees of freedom corresponding to the coordinate transformations that can be achieved through δx^i from this number, we are left with $2\,(D-1)-(D-1)=(D-1)$ true vector degrees of freedom.

The tensor perturbations contained $(D+1)\,(D-2)/2$ independent degrees of freedom.

Upon adding these, we obtain that

$$2+(D-1)+\frac{1}{2}\,(D+1)\,(D-2)=\frac{1}{2}\,D\,(D+1)=\frac{1}{2}\,(D+1)\,(D+2)-(D+1),$$

which is the total number of *independent* degrees of freedom describing the perturbed metric tensor.

Decomposition theorem

The perturbations in the stress-energy tensor of matter fields, say, δT^{μ}_{ν}, can also be decomposed as scalars, vectors and tensors, just as in the same manner as the perturbations in the metric tensor.

At the *linear* order on perturbation theory, it can be shown that the scalar, vector and tensor perturbations evolve independently[12].

[12] See, for example, S. Weinberg, Cosmology (Oxford University Press, Oxford, England, 2008).

Plan of the talk

FLRW metric with scalar perturbations

As we discussed, in a FLRW universe, there are two independent degrees of freedom associated with the scalar perturbations. In the so-called logitudinal gauge, the FLRW line-element is given by

$$\mathrm{d}s^2 = -(1+2\,\Phi)\,\mathrm{d}t^2 + a^2(t)\,(1-2\,\Psi)\,\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j,$$

where Φ and Ψ are often referred to as the Bardeen potentials.

If we write the perturbations in the stress-energy tensor in terms of scalars as $\delta T^{\mu}_{\nu} = (-\delta\rho, \partial_i\delta q, \delta p\,\delta^i_j)$, then the Einstein's equations at the first order lead to

$$\begin{aligned} 3\,H\,\left(H\,\Phi + \dot{\Psi}\right) - \frac{1}{a^2}\,\nabla^2\Psi &= -\,(4\,\pi\,G)\;\delta\rho, \\ \partial_i\left(H\,\Phi + \dot{\Psi}\right) &= (4\,\pi\,G)\;\partial_i(\delta q), \\ \ddot{\Psi} + H\,\left(\dot{\Phi} + 3\,\dot{\Psi}\right) + \left(2\,\dot{H} + 3\,H^2\right)\,\Phi &= (4\,\pi\,G)\,\delta p, \\ \Phi - \Psi &= 0. \end{aligned}$$

Note that, in the absence of anisotropic stress, $\Phi = \Psi$.

Bardeen equation

We can combine the first and third of the above equations to arrive at the following differential equation that governs the Bardeen potential Φ:

$$\Phi'' + 3\,\mathcal{H}\,\left(1 + c_{_{\rm A}}^2\right)\,\Phi' - c_{_{\rm A}}^2\,\nabla^2\Phi + \left[2\,\mathcal{H}' + \left(1 + 3\,c_{_{\rm A}}^2\right)\,\mathcal{H}^2\right]\,\Phi = \left(4\,\pi\,G\,a^2\right)\,\delta p^{\rm NA},$$

where $\mathcal{H} = a'/a$ is the conformal Hubble parameter, $\delta p = c_{_{\rm A}}^2\,\delta\rho + \delta p^{\rm NA}$ and $c_{_{\rm A}}^2 = p'/\rho'$ denotes the adiabatic speed of the perturbations, while $\delta p^{\rm NA}$ represents the non-adiabatic pressure perturbation.

Back to Bardeen equation in the case of scalar field

Curvature perturbation

Consider the following combination of the Bardeen potential Φ and its time derivative

$$\mathcal{R} = \Phi + \frac{2\,\rho}{3\,\mathcal{H}}\,\left(\frac{\Phi' + \mathcal{H}\,\Phi}{\rho + p}\right),$$

a quantity that is referred to as the curvature perturbation[13].

On substituting this expression in the Bardeen equation and making use of the background equations, in Fourier space, we obtain that

$$\mathcal{R}_k' = \frac{\mathcal{H}}{\mathcal{H}^2 - \mathcal{H}'}\,\left[(4\,\pi\,G)\,a^2\,\delta p_k^{\mathrm{NA}} - c_{\mathrm{A}}^2\,k^2\,\Phi_k\right].$$

Note that, on super-Hubble scales wherein $(k/a\,H) \ll 1$, the term $\left(c_{\mathrm{A}}^2\,k^2\,\Phi_k\right)$ can be neglected. If one further assumes that no non-adiabatic pressure perturbations are present (i.e. $\delta p^{\mathrm{NA}} = 0$), say, as in the case of ideal fluids, then the above equation implies that $\mathcal{R}_k' \simeq 0$ on super-Hubble scales.

Back to equation for $\mathcal{R}_k$ in the case of scalar field

[13] It is called so since it is proportional to the local three curvature on the spatial hypersurface.

Evolution of the Bardeen potential on super-Hubble scales

If we now define

$$\Phi = \frac{\mathcal{H}}{a^2\,\theta}\,\mathcal{U}, \quad \theta = \left[\frac{\mathcal{H}^2}{(\mathcal{H}^2 - \mathcal{H}')\,a^2}\right]^{1/2},$$

then, in Fourier space, the equation that governs the Bardeen potential reduces to

$$\mathcal{U}_k'' + \left(c_{\mathrm{A}}^2\,k^2 - \frac{\theta''}{\theta}\right)\mathcal{U}_k = \left(\frac{4\,\pi\,G\,a^4\,\theta}{\mathcal{H}}\right)\delta p_k^{\mathrm{NA}}.$$

When $\delta p_k^{\mathrm{NA}} = 0$, on super-Hubble scales, the Bardeen potential Φ_k can be expressed as

$$\Phi_k(\eta) \simeq C_{\mathrm{G}}(k)\,\frac{\mathcal{H}}{a^2(\eta)}\int^{\eta}\frac{\mathrm{d}\tilde{\eta}}{\theta^2(\tilde{\eta})} + C_{\mathrm{D}}(k)\,\frac{\mathcal{H}}{a^2(\eta)}.$$

In power law expansion wherein $a(\eta) = (-\bar{\mathcal{H}}\,\eta)^{\gamma+1}$, we find that

$$\Phi_k(\eta) \simeq C_{\mathrm{G}}(k)\,\frac{3\,(w+1)}{3\,w+5} + C_{\mathrm{D}}(k)\,\frac{2}{(3\,w+1)\,\bar{\mathcal{H}}^{2\,(\gamma+1)}\,\eta^{2\,\gamma+3}},$$

where $w = (1-\gamma)/[3\,(1+\gamma)]$.

Relation between Bardeen potential and curvature perturbation

Since Φ_k is a constant on super-Hubble scales, in this limit, the curvature perturbation $\mathcal{R}_k$ is given by

$$\mathcal{R}_k \simeq \frac{3\,w+5}{3\,(w+1)}\,\Phi_k \simeq C_{\mathrm{G}}(k).$$

Since $\mathcal{R}_k$ is conserved and Φ_k is a constant at super-Hubble scales in power law expansion, when the modes enter the Hubble radius during the radiation or the matter dominated epochs, the Bardeen potentials at entry are given by

$$\Phi_k^{\mathrm{r}} \simeq \frac{3\,(w_{\mathrm{r}}+1)}{3\,w_{\mathrm{r}}+5}\,\mathcal{R}_k = \frac{2}{3}\,C_{\mathrm{G}}(k), \quad \Phi_k^{\mathrm{m}} \simeq \frac{3\,(w_{\mathrm{m}}+1)}{3\,w_{\mathrm{m}}+5}\,\mathcal{R}_k = \frac{3}{5}\,C_{\mathrm{G}}(k),$$

where we have made use of the fact that $w_{\mathrm{r}} = 1/3$ and $w_{\mathrm{m}} = 0$. Note that $\Phi_k^{\mathrm{m}} = (9/10)\,\Phi_k^{\mathrm{r}}$.

Tensor perturbations in a FLRW universe

On including the tensor perturbations, the FLRW line-element can be expressed as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\,[\delta_{ij} + \gamma_{ij}(t, \boldsymbol{x})]\,\mathrm{d}x^i\,\mathrm{d}x^j,$$

where γ_{ij} is a symmetric, transverse and traceless tensor.

▸ Back to spectral energy density of GWs

The transverse and traceless conditions reduce the number of independent degrees of freedom of γ_{ij} to two. These two degrees of freedom correspond to the two types of polarization associated with the gravitational waves (GWs).

On imposing the transverse and the traceless conditions, the components of the perturbed Einstein tensor, say, δG^{μ}_{ν}, corresponding to the above line element simplify to be

$$\delta G^0_0 = \delta G^0_i = 0, \quad \delta G^i_j = -\frac{1}{2}\left(\ddot{\gamma}_{ij} + 3\,H\,\dot{\gamma}_{ij} - \frac{1}{a^2}\,\nabla^2\gamma_{ij}\right).$$

Therefore, in the absence of sources, the tensor perturbations in a FLRW universe satisfy the equation[14]

▸ Back to quantization of tensor perturbations

$$\ddot{\gamma}_{ij} + 3\,H\,\dot{\gamma}_{ij} - \frac{1}{a^2}\,\nabla^2\gamma_{ij} = 0.$$

[14] See, for example, S. Weinberg, Cosmology (Oxford University Press, Oxford, England 2008).

Plan of the talk

Perturbations in the stress-energy tensor of a scalar field

Let $\delta\phi$ denote the perturbations in the scalar field.

In the longitudinal gauge, the perturbations in the stress-energy tensor of a canonical scalar field are given by

$$\delta T^0_0 = -\delta\rho_\phi = -\dot{\phi}\,\dot{\delta\phi} + \Phi\,\dot{\phi}^2 - V_\phi\,\delta\phi,$$

$$\delta T^0_i = -\partial_i\,\delta q_\phi = -\partial_i\left(\dot{\phi}\,\delta\phi\right),$$

$$\delta T^i_j = \delta p_\phi\,\delta^i_j = \left(\dot{\phi}\,\dot{\delta\phi} - \Phi\,\dot{\phi}^2 - V_\phi\,\delta\phi\right)\,\delta^i_j.$$

These expressions for the perturbed stress-energy tensor lead to the following equation for the Bardeen potential:

$$\Phi'' + 3\,\mathcal{H}\,\left(1 + c_{\mathrm{A}}^2\right)\,\Phi' - c_{\mathrm{A}}^2\,\nabla^2\Phi + \left[2\,\mathcal{H}' + \left(1 + 3\,c_{\mathrm{A}}^2\right)\,\mathcal{H}^2\right]\,\Phi = \left(1 - c_{\mathrm{A}}^2\right)\,\nabla^2\Phi$$

so that

Bardeen equation in the case of fluid

$$\delta p^{\mathrm{NA}} = \left(\frac{1 - c_{\mathrm{A}}^2}{4\,\pi\,G\,a^2}\right)\,\nabla^2\Phi.$$

Equation of motion governing the curvature perturbation

For such a δp^{NA}, the equation that describes the evolution of the curvature perturbation simplifies to

▸ Equation for $\mathcal{R}_k$ in the case of fluid

$$\mathcal{R}_k' = -\left(\frac{\mathcal{H}}{\mathcal{H}^2 - \mathcal{H}'}\right) k^2\, \Phi_k.$$

On differentiating this equation with respect to time, and on using the background equations as well as the Bardeen equation, we obtain the following equation of motion governing the Fourier modes of the curvature perturbation induced by the scalar field

$$\mathcal{R}_k'' + 2\,\frac{z'}{z}\,\mathcal{R}_k' + k^2\,\mathcal{R}_k = 0,$$

where $z = a\,\dot{\phi}/H$.

Quantization of the scalar perturbations

On quantization, the operator $\hat{\mathcal{R}}(\eta, \boldsymbol{x})$ representing the scalar perturbations can be expressed in terms of the corresponding Fourier modes, say, f_k, as[15]

$$\hat{\mathcal{R}}(\eta, \boldsymbol{x}) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\,\pi)^{3/2}}\,\hat{\mathcal{R}}_{\boldsymbol{k}}(\eta)\,\mathrm{e}^{i\,\boldsymbol{k}\cdot\boldsymbol{x}} = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\,\pi)^{3/2}}\,\left[\hat{a}_{\boldsymbol{k}}\,f_k(\eta)\,\mathrm{e}^{i\,\boldsymbol{k}\cdot\boldsymbol{x}} + \hat{a}_{\boldsymbol{k}}^{\dagger}\,f_k^{*}(\eta)\,\mathrm{e}^{-i\,\boldsymbol{k}\cdot\boldsymbol{x}}\right],$$

where the operators $(\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}}^{\dagger})$ satisfy the standard commutation relations.

The Fourier mode functions f_k satisfy the following equation of motion:

$$f_k'' + 2\,\frac{z'}{z}\,f_k' + k^2\,f_k = 0.$$

[15] See, for instance, L. Sriramkumar, Curr. Sci. **97**, 868 (2009).

Scalar power spectrum

The dimensionless scalar power spectrum, say, $\mathcal{P}_{_{\rm S}}(k)$, is defined in terms of the correlation functions of the Fourier modes $\hat{\gamma}_{ij}^{\boldsymbol{k}}$ as follows:

$$\langle \hat{\mathcal{R}}_{\boldsymbol{k}}(\eta)\, \hat{\mathcal{R}}_{\boldsymbol{k}'}(\eta)\rangle = \frac{(2\,\pi)^2}{8\,k^3}\,\mathcal{P}_{_{\rm S}}(k)\;\delta^3\left(\boldsymbol{k}+\boldsymbol{k}'\right).$$

In the Bunch-Davies vacuum, say, $\vert 0\rangle$, which is defined as $\hat{a}_{\boldsymbol{k}}\vert 0\rangle = 0\;\forall\;\boldsymbol{k}$, we can express the scalar power spectrum in terms of the mode functions f_k as

$$\mathcal{P}_{_{\rm S}}(k) = \frac{k^3}{2\,\pi^2}\,\vert f_k\vert^2.$$

With the initial conditions imposed in the sub-Hubble domain, viz. when $k/(a\,H) \gg 1$, these spectra are to be evaluated on super-Hubble scales, i.e. as $k/(a\,H) \ll 1$.

Quantization of the tensor perturbations

On quantization, the operator $\hat{\gamma}_{ij}(\eta, \boldsymbol{x})$ representing the tensor perturbations can be expressed in terms of the corresponding Fourier modes, say, g_k, as[16]

$$\hat{\gamma}_{ij}(\eta, \boldsymbol{x}) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\,\pi)^{3/2}}\, \hat{\gamma}_{ij}^{\boldsymbol{k}}(\eta)\, \mathrm{e}^{i\,\boldsymbol{k}\cdot\boldsymbol{x}}$$

$$= \sum_{\lambda} \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\,\pi)^{3/2}}\, \varepsilon_{ij}^{\lambda}(\boldsymbol{k})\, \left[\hat{b}_{\boldsymbol{k}}^{\lambda}\, g_k(\eta)\, \mathrm{e}^{i\,\boldsymbol{k}\cdot\boldsymbol{x}} + \hat{b}_{\boldsymbol{k}}^{\lambda\dagger}\, g_k^{*}(\eta)\, \mathrm{e}^{-i\,\boldsymbol{k}\cdot\boldsymbol{x}}\right],$$

where $\lambda = (+, \times)$ denotes the two states of polarizations of the GWs.

In this decomposition, the operators $(\hat{b}_{\boldsymbol{k}}^{\lambda}, \hat{b}_{\boldsymbol{k}}^{\lambda\dagger})$ satisfy the standard commutation relations, while the quantity $\varepsilon_{ij}^{\lambda}(\boldsymbol{k})$, which is symmetric in the spatial indices, represents the transverse and traceless polarization tensor describing the GWs.

Back to spectral energy density of GWs

The Fourier mode functions g_k satisfy the following equation of motion:

Equation of motion

$$g_k'' + 2\,\mathcal{H}\, g_k' + k^2\, g_k = 0,$$

where, recall that, $\mathcal{H} = a\,H = a'/a$ is the conformal Hubble parameter.

[16] See, for instance, L. Sriramkumar, Curr. Sci. **97**, 868 (2009).

Polarization tensor

The transverse and traceless nature of GWs leads to the conditions

$$\delta^{ij}\, k_i\, \varepsilon_{jl}^{\lambda}(\boldsymbol{k}) = \delta^{ij}\, \varepsilon_{ij}^{\lambda}(\boldsymbol{k}) = 0.$$

Note that the polarization tensors $\varepsilon_{ij}^{+}(\boldsymbol{k})$ and $\varepsilon_{ij}^{\times}(\boldsymbol{k})$ can be expressed in terms of the set of orthonormal unit vectors $(\hat{e}(\boldsymbol{k}), \hat{\hat{e}}(\boldsymbol{k}), \hat{\boldsymbol{k}})$ in the following manner[17]:

$$\varepsilon_{ij}^{+}(\boldsymbol{k}) = \frac{1}{\sqrt{2}}\, \left[\hat{e}_i(\boldsymbol{k})\, \hat{e}_j(\boldsymbol{k}) - \hat{\hat{e}}_i(\boldsymbol{k})\, \hat{\hat{e}}_j(\boldsymbol{k})\right], \quad \varepsilon_{ij}^{\times}(\boldsymbol{k}) = \frac{1}{\sqrt{2}}\, \left[\hat{e}_i(\boldsymbol{k})\, \hat{\hat{e}}_j(\boldsymbol{k}) + \hat{\hat{e}}_i(\boldsymbol{k})\, \hat{e}_j(\boldsymbol{k})\right].$$

The orthonormal nature of the vectors $\hat{e}(\boldsymbol{k})$ and $\hat{\hat{e}}(\boldsymbol{k})$ lead to the normalization condition:

$$\delta^{il}\, \delta^{jm}\, \varepsilon_{ij}^{\lambda}(\boldsymbol{k})\, \varepsilon_{lm}^{\lambda'}(\boldsymbol{k}) = \delta^{\lambda\lambda'},$$

where λ and λ' can represent either of the two states of polarization $+$ or $\times$.

[17] M. Maggiore, Phys. Rep. **331**, 283 (2000).

Tensor power spectrum

The dimensionless tensor power spectrum, say, $\mathcal{P}_{_{\rm T}}(k)$, is defined in terms of the correlation functions of the Fourier modes $\hat{\gamma}_{ij}^{\boldsymbol{k}}$ as follows:

$$\langle \hat{\gamma}_{ij}^{\boldsymbol{k}}(\eta)\, \hat{\gamma}_{mn}^{\boldsymbol{k}'}(\eta)\rangle = \frac{(2\,\pi)^2}{8\,k^3}\, \Pi_{ij,mn}^{\boldsymbol{k}}\, \mathcal{P}_{_{\rm T}}(k)\, \delta^3\left(\boldsymbol{k}+\boldsymbol{k}'\right),$$

where

$$\Pi_{ij,mn}^{\boldsymbol{k}} = \sum_{\lambda} \varepsilon_{ij}^{\lambda}(\boldsymbol{k})\, \varepsilon_{mn}^{\lambda}(\boldsymbol{k}).$$

In the Bunch-Davies vacuum, say, $\vert 0\rangle$, which is defined as $\hat{b}_{\boldsymbol{k}}^{\lambda}\vert 0\rangle = 0\ \forall\ \boldsymbol{k}$ and λ, we can express the tensor power spectrum in terms of the mode functions g_k as

$$\mathcal{P}_{_{\rm T}}(k) = 4\,\frac{k^3}{2\,\pi^2}\,\vert g_k\vert^2.$$

With the initial conditions imposed in the sub-Hubble domain, viz. when $k/(a\,H) \gg 1$, these spectra are to be evaluated on super-Hubble scales, i.e. as $k/(a\,H) \ll 1$.

Plan of the talk

Scalar Mukhanov-Sasaki variable and Bunch-Davies initial condition

If we write $f_k = v_k/z$, then the equation describing the scalar mode functions reduces to the form

$$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0.$$

The scalar power spectrum $\mathcal{P}_{\rm S}(k)$ can then be expressed in terms of the modes functions f_k and the corresponding Mukhanov-Sasaki variable v_k as follows:

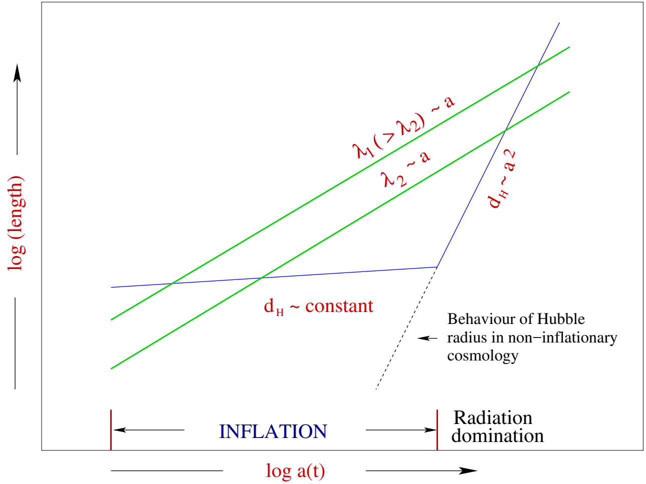
$$\mathcal{P}_{\rm S}(k) = \frac{k^3}{2\,\pi^2}\,|f_k|^2 = \frac{k^3}{2\,\pi^2}\,\left(\frac{|v_k|}{z}\right)^2.$$

The assumption that the scalar perturbations are in the vacuum state then requires that the corresponding Mukhanov-Sasaki variable v_k behave as positive frequency modes at sub-Hubble scales, i.e. they have the asymptotic form[18]

$$\lim_{[k/(a\,H)]\to\infty} v_k(\eta) \to \frac{1}{\sqrt{2\,k}}\,{\rm e}^{-i\,k\,\eta}.$$

[18] T. S. Bunch and P. C. W. Davies, Proc. Roy. Soc. Lond. A **360**, 117 (1978).

From sub-Hubble to super-Hubble scales



The initial conditions are imposed in the sub-Hubble regime when the modes are well inside the Hubble radius [viz. when $k/(a\,H) \gg 1$] and the power spectrum is evaluated when they are sufficiently outside [i.e. when $k/(a\,H) \ll 1$].

Solution to the scalar Mukhanov-Sasaki variable in slow roll inflation

Back to ultra slow roll inflation

It can be shown that

$$\frac{z''}{z} = \mathcal{H}^2 \left(2 - \epsilon_1 + \frac{3\,\epsilon_2}{2} - \frac{\epsilon_1\,\epsilon_2}{2} + \frac{\epsilon_2^2}{4} + \frac{\epsilon_2\,\epsilon_3}{2}\right),$$

which is an exact relation. Also, we can write

$$\eta = -\frac{1}{(1-\epsilon_1)\,\mathcal{H}} - \int \frac{\epsilon_1\,\epsilon_2}{(1-\epsilon_1)^3}\,\mathrm{d}\left(\frac{1}{\mathcal{H}}\right).$$

At the leading order in the slow roll approximation, the second term can be ignored and we obtain that

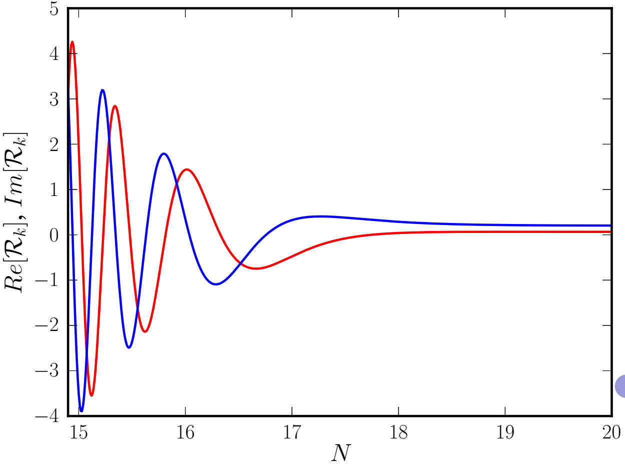
$$\mathcal{H} \simeq -\frac{1}{(1-\epsilon_1)\,\eta}, \quad \frac{z''}{z} = \left(2 + 3\,\epsilon_1 + \frac{3\,\epsilon_2}{2}\right)\frac{1}{\eta^2}.$$

The solution to the scalar Mukhanov-Sasaki variable v_k, which satisfies the Bunch-Davies initial condition, is given by

$$v_k(\eta) = \sqrt{\frac{-\pi\,\eta}{4}}\,\mathrm{e}^{i\,[\nu_{\mathrm{T}}+1/2]\,\pi/2}\,H_{\nu_{\mathrm{S}}}^{(1)}(-k\,\eta),$$

where $H_\nu^{(1)}(z)$ denotes Hankel function of the first kind and $\nu_{\mathrm{S}} = (3/2) + \epsilon_1 + (\epsilon_2/2)$.

Typical evolution of the perturbations



Back to evolution in ultra slow roll

Typical evolution of the real and the imaginary parts of the scalar modes during slow roll inflation. The mode considered here leaves the Hubble radius at about 18 e-folds[19].

[19]Figure from V. Sreenath, *Computation and characteristics of inflationary three-point functions*, Ph.D. Thesis, Indian Institute of Technology Madras, Chennai, India (2015).

Scalar power spectrum in slow roll inflation

The scalar power spectrum evaluated in the super-Hubble limit, i.e. when $(-k\,\eta) \ll 1$, can be obtained to be[20]

▸ Tensor power spectrum in slow roll inflation

$$\mathcal{P}_{_{\rm S}}(k) = \frac{H_{_{\rm I}}^2}{8\,\pi^2\,M_{_{\rm Pl}}^2\,\epsilon_1}\,\left[1 - 2\,(C+1)\,\epsilon_1 - C\,\epsilon_2 - (2\,\epsilon_1 + \epsilon_2)\,\ln\,(-k\,\eta)\right],$$

where $C = (\gamma_{_{\rm E}} - 2 + \ln 2)$ with $\gamma_{_{\rm E}}$ being the Euler constant.

The scalar spectral index $n_{_{\rm S}}$ is defined as

$$n_{_{\rm S}} - 1 = \frac{{\rm d}\ln \mathcal{P}_{_{\rm S}}}{{\rm d}\ln k}$$

and, in slow roll inflation, we have

$$n_{_{\rm S}} = 1 - 2\,\epsilon_1 - \epsilon_2.$$

[20] See, for instance, J. Martin and R. Brandenberger, Phys. Rev. D **68**, 063513 (2003) [hep-th/0305161].

Tensor Mukhanov-Sasaki variable and Bunch-Davies initial condition

If we write $g_k = (\sqrt{2}/M_{_{\rm Pl}})\,(u_k/a)$, then the equation describing the tensor mode functions reduces to the form

Back to solution to during radiation domination

$$u_k'' + \left(k^2 - \frac{a''}{a}\right)\, u_k = 0.$$

The tensor power spectrum $\mathcal{P}_{_{\rm T}}(k)$ can then be expressed in terms of the modes functions g_k and the Mukhanov-Sasaki variable u_k as follows:

$$\mathcal{P}_{_{\rm T}}(k) = 4\,\frac{k^3}{2\,\pi^2}\,\vert g_k\vert^2 = \frac{8}{M_{_{\rm Pl}}^2}\,\frac{k^3}{2\,\pi^2}\,\left(\frac{\vert u_k\vert}{a}\right)^2.$$

The assumption that the tensor perturbations are in the vacuum state then requires that the Mukhanov-Sasaki variable u_k behave as positive frequency modes at sub-Hubble scales, i.e. they have the asymptotic form[21]

$$\lim_{[k/(a\,H)]\to\infty} u_k(\eta) \to \frac{1}{\sqrt{2\,k}}\,{\rm e}^{-i\,k\,\eta}.$$

[21] T. S. Bunch and P. C. W. Davies, Proc. Roy. Soc. Lond. A **360**, 117 (1978).

Solution to the tensor Mukhanov-Sasaki variable in slow roll inflation

From the definition of ϵ_1, it can be established that

$$\frac{a''}{a} = \mathcal{H}^2\,(2-\epsilon_1).$$

Also, we can write

$$\eta = -\frac{1}{(1-\epsilon_1)\,\mathcal{H}} - \int \frac{\epsilon_1\,\epsilon_2}{(1-\epsilon_1)^3}\,\mathrm{d}\left(\frac{1}{\mathcal{H}}\right).$$

At the leading order in the slow roll approximation, the second term can be ignored and, at the same order, one can assume ϵ_1 to be a constant. Therefore, we have

$$\mathcal{H} \simeq -\frac{1}{(1-\epsilon_1)\,\eta}, \quad \frac{a''}{a} \simeq (2+3\,\epsilon_1)\,\frac{1}{\eta^2}$$

and the solution which satisfies the Bunch-Davies initial condition is given by

$$u_k(\eta) = \sqrt{\frac{-\pi\,\eta}{4}}\,\mathrm{e}^{i\,[\nu_{\mathrm{T}}+1/2]\,\pi/2}\,H_{\nu_{\mathrm{T}}}^{(1)}(-k\,\eta),$$

where $H_\nu^{(1)}(z)$ denotes Hankel function of the first kind and $\nu_{\mathrm{T}} = (3/2)+\epsilon_1$.

Tensor power spectrum in slow roll inflation

The tensor power spectrum evaluated in the super-Hubble limit, i.e. when $(-k\,\eta) \ll 1$, can be obtained to be[22]

Back to scalar power spectrum in slow roll inflation

$$\mathcal{P}_{_{\rm T}}(k) = \frac{2\,H_{_{\rm I}}^2}{\pi^2\,M_{_{\rm Pl}}^2}\,\left[1 - 2\,(C+1)\,\epsilon_1\; - 2\,\epsilon_1\,\ln\,(-k\,\eta)\right],$$

where $C = (\gamma_{_{\rm E}} - 2 + \ln 2)$ with $\gamma_{_{\rm E}}$ being the Euler constant.

The tensor spectral index $n_{_{\rm T}}$ is defined as

$$n_{_{\rm T}} = \frac{{\rm d}\ln \mathcal{P}_{_{\rm T}}}{{\rm d}\ln k}$$

and, in slow roll inflation, we have

$$n_{_{\rm T}} = -2\,\epsilon_1.$$

Also, in slow roll inflation, the tensor-to-scalar ratio is given by

$$r = 16\,\epsilon_1 = -8\,n_{_{\rm T}}.$$

[22] See, for instance, J. Martin and R. Brandenberger, Phys. Rev. D **68**, 063513 (2003) [hep-th/0305161].

Tensor power spectrum in de Sitter inflation

In the case of de Sitter inflation, which corresponds to $\epsilon_1 = 0$, the scale factor is given by

$$a(\eta) = -\frac{1}{H_{\rm I}\,\eta},$$

where $H_{\rm I}$ is a constant.

In such a case, we have

$$\frac{a''}{a} = \frac{2}{\eta^2}$$

and the solution to the Mukhanov-Sasaki variable u_k has the following simple form:

$$u_k(\eta) = \frac{1}{\sqrt{k\,\eta}}\left(1 - \frac{i}{k\,\eta}\right)\,{\rm e}^{-i\,k\,\eta}.$$

The tensor power spectrum at late times, i.e. as $(-k\,\eta) \ll 1$, can be easily evaluated to be

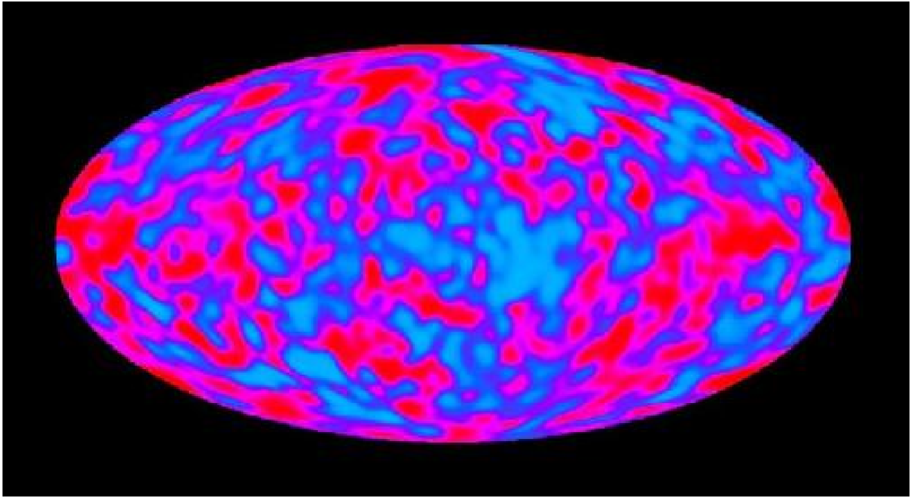
$$\mathcal{P}_{\rm T}(k) = \frac{2\,H_{\rm I}^2}{\pi^2\,M_{\rm Pl}^2},$$

which is strictly scale invariant.

▸ Back to solution in radiation domination

Plan of the talk

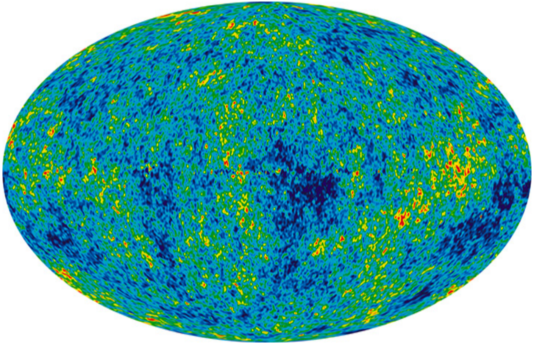
Anisotropies in the CMB as observed by COBE

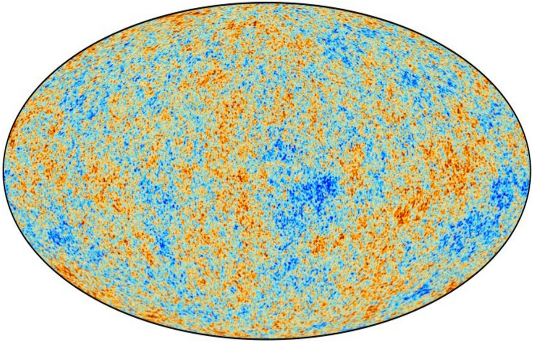


The fluctuations in the temperature of the CMB as observed by COBE[23]. The amplitude of the fluctuations are of the order of one part in 10^5.

[23] Image from http://aether.lbl.gov/www/projects/cobe/COBE_Home/DMR_Images.html.

Anisotropies in the CMB as observed by WMAP and Planck





Left: All-sky map of the anisotropies in the CMB created from nine years of Wilkinson Microwave Anisotropy Probe (WMAP) data[24].
Right: CMB intensity map derived from the joint analysis of Planck, WMAP, and $408\,\mathrm{MHz}$ observations[25]. The above images show temperature variations (as color differences) of the order of $200\,\mu\mathrm{K}$.

[24] Image from http://wmap.gsfc.nasa.gov/media/121238/index.html.

[25] P. A. R. Ade *et al.*, arXiv:1502.01582 [astro-ph.CO].

Spectral indices and tensor-to-scalar ratio

While comparing with the observations, for convenience, one often uses the following power law template for the primordial scalar and the tensor spectra:

$$\mathcal{P}_{\mathrm{S}}(k) = A_{\mathrm{S}}\, \left(\frac{k}{k_*}\right)^{n_{\mathrm{S}}-1}, \qquad \mathcal{P}_{\mathrm{T}}(k) = A_{\mathrm{T}}\, \left(\frac{k}{k_*}\right)^{n_{\mathrm{T}}},$$

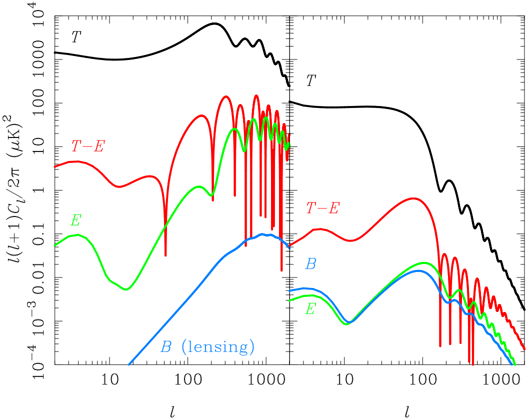
with the spectral indices n_{S} and n_{T} assumed to be constant.

The tensor-to-scalar ratio r is defined as

$$r(k) = \frac{\mathcal{P}_{\mathrm{T}}(k)}{\mathcal{P}_{\mathrm{S}}(k)}$$

and it is usual to further set $r = -8\, n_{\mathrm{T}}$, viz. the so-called consistency relation, which is valid during slow roll inflation.

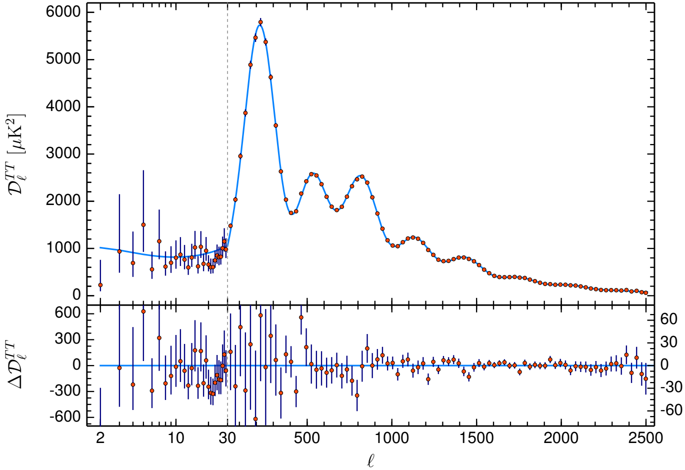
Theoretical angular power spectra[26]



The *theoretically* computed, angular power and cross-correlation spectra of the CMB arising due to scalars (on the left) and tensors (on the right) corresponding to a tensor-to-scalar ratio of $r = 0.24$. The B-mode spectrum induced by weak gravitational lensing has also been shown (in blue) in the panel on the left.

[26] Figure from, A. Challinor, arXiv:1210.6008 [astro-ph.CO].

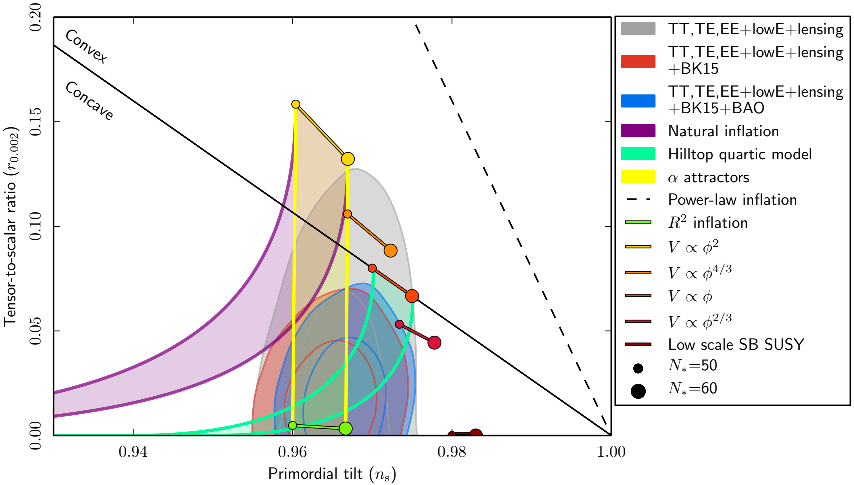
CMB angular power spectrum from Planck



The CMB TT angular power spectrum from the Planck 2018 data (red dots with error bars) and the best fit ΛCDM model with a power law primordial spectrum (solid blue curve)[27]

[27] Planck Collaboration (N. Aghanim *et al.*), Astron. Astrophys. **641**, A6 (2020).

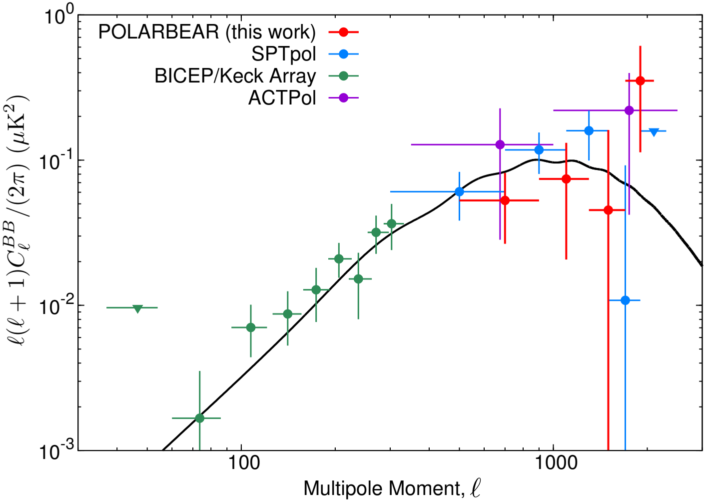
Performance of inflationary models in the $n_{\rm s}$-r plane



Joint constraints on $n_{\rm s}$ and $r_{0.002}$ from Planck in combination with other data sets, compared to the theoretical predictions of some of the popular inflationary models[28].

[28] Planck Collaboration (Y. Akrami *et al.*), Astron. Astrophys. **641**, A10 (2020).

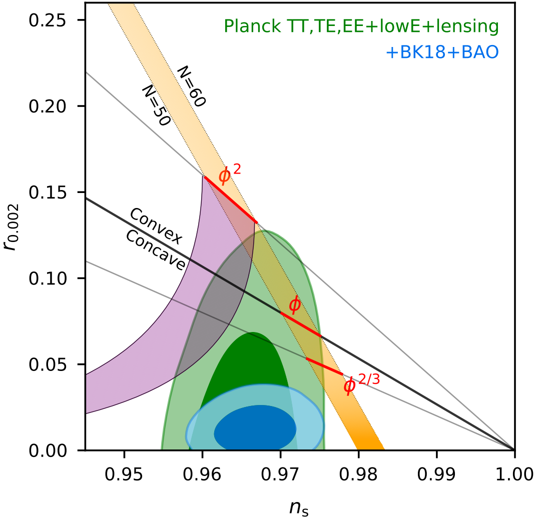
Constraints on the B-mode polarization of the CMB



Constraints on the B-mode, angular power spectrum of the CMB from two years of POLARBEAR data[29].

[29] POLARBEAR Collaboration (P. A. R. Ade *et al.*), Ap. J. **848**, 141, (2017).

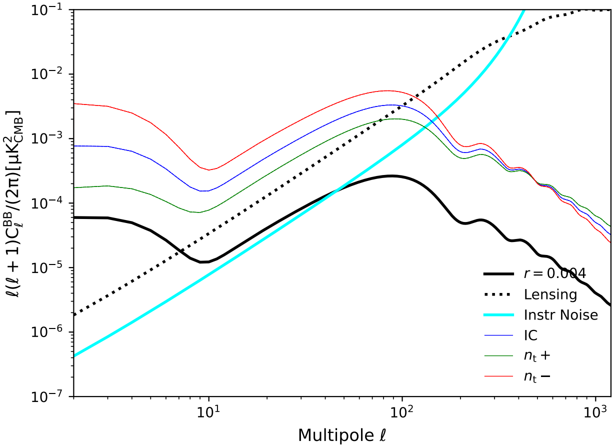
Latest constraints on the tensor-to-scalar ratio r



Latest constraints on the tensor-to-scalar ratio r from the BICEP/Keck telescopes[30].

[30] BICEP/Keck Collaboration (P. A. R. Ade *et al.*), arXiv:2203.16556 [astro-ph.CO].

Prospects of observing the imprints of the tensor perturbations

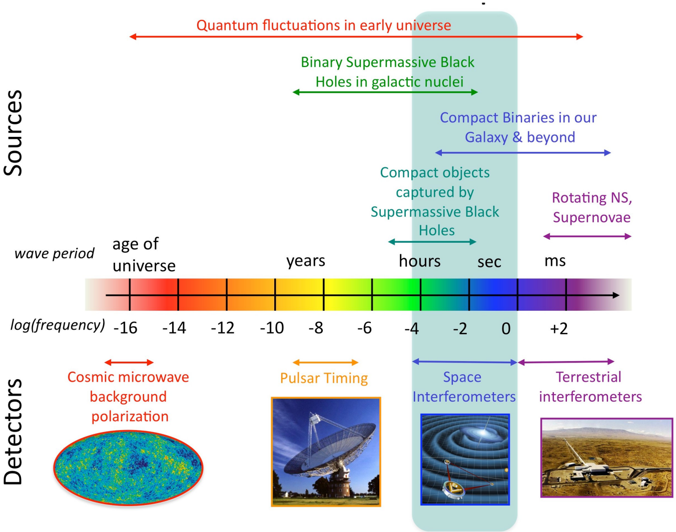


The B-mode angular power spectra of the CMB resulting from the primordial tensor perturbations for three models with $r_{0.05} = 0.05$ have been plotted, along with the CMB lensing signal and the instrumental noise of a LiteBIRD-like configuration[31].

[31] D. Paoletti, F. Finelli, J. Valiviita and M. Hazumi, Phys. Rev. D **106**, 083528 (2022).

Plan of the talk

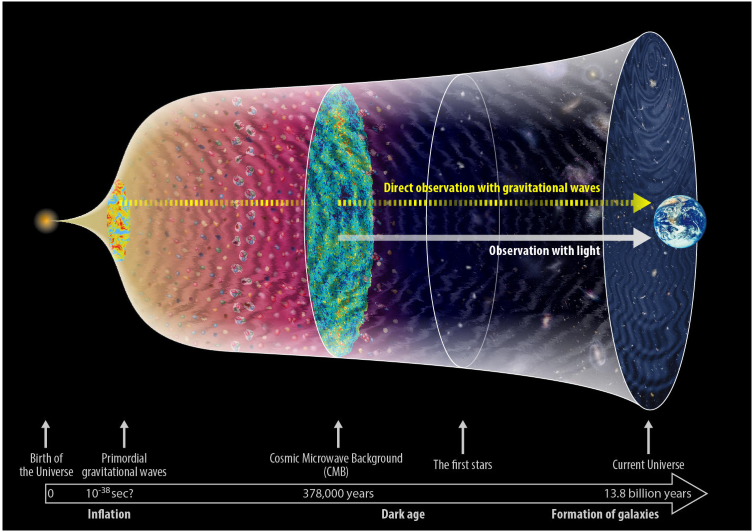
The spectrum of GWs



Different sources of GWs and corresponding detectors[32].

[32] J. B. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (Pearson Education, Delhi, 2003).

Probing the primordial universe through GWs



GWs provide a unique window to probe the primordial universe[33].

[33] Image from https://gwpo.nao.ac.jp/en/gallery/000061.html.

Stress-energy tensor associated with GWs

Consider a line-element which includes the tensor perturbations at the linear order, say, γ_{ij}, in a given background.

The stress-energy tensor associated with the GWs can be calculated using the Einstein's equation, viz.

$$\delta T_{\mu\nu} = -M_{_{\rm Pl}}^2\, \delta G_{\mu\nu}^{(2)},$$

where the superscript (2) on the right hand side indicates that the quantity has to be evaluated at the quadratic order in γ_{ij}.

It can be shown that, in a spatially flat FLRW background, the energy density of primary GWs is given by[34]

▸ FLRW line-element with tensor perturbations

$$\rho = \delta G_{00}^{(2)} = -\frac{1}{8}\,\dot{\gamma}^{ij}\,\dot{\gamma}_{ij} + \frac{H}{2}\,\dot{\gamma}^{ij}\,\gamma_{ij} + \frac{1}{2}\,\gamma^{ij}\,\ddot{\gamma}_{ij} + \frac{3}{8}\,(\partial^l\gamma^{ij})\,(\partial_l\gamma_{ij}) - \frac{1}{4}\,(\partial^l\gamma^{ij})\,(\partial_j\gamma_{il}).$$

[34] See, for instance, F. Finelli, G. Marozzi, G. P. Vacca and G. Venturi, Phys. Rev. D **71**, 023522 (2005).

Spectral energy density of GWs

If we now substitute the decomposition of $\hat{\gamma}_{ij}$ in the above expression for the energy density of GWs, we obtain that

▸ Fourier decomposition of the tensor perturbations in a FLRW universe

$$\rho_{\mathrm{GW}}(\eta) = M_{\mathrm{Pl}}^2 \sum_{\lambda} \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\,\pi)^3} \left[\frac{1}{4}\,|\dot{g}_k|^2 + \frac{k^2}{4\,a^2}\,|g_k|^2 + H\,\left(\dot{g}_k\, g_k^\ast + g_k\,\dot{g}_k^\ast\right)\right].$$

The spectral energy density of GWs, say, $\rho_{\mathrm{GW}}(k,\eta)$, is defined through the relation

$$\rho_{\mathrm{GW}}(\eta) = \int_0^\infty \mathrm{d}\ln k\; \rho_{\mathrm{GW}}(k,\eta).$$

From the above expression for $\rho_{\mathrm{GW}}(\eta)$, we obtain that

$$\rho_{\mathrm{GW}}(k,\eta) = \frac{M_{\mathrm{Pl}}^2}{a^2}\,\frac{k^3}{2\,\pi^2}\,\left[\frac{1}{2}\,|g_k'|^2 + \frac{k^2}{2}\,|g_k|^2 + 2\,\mathcal{H}\,\left(g_k'\,g_k^\ast + g_k^{\ast\prime}\,g_k\right)\right].$$

Dimensionless spectral energy density of GWs today

The dimensionless spectral energy density of GWs, say, $\Omega_{\mathrm{GW}}(k,\eta)$, is defined as

$$\Omega_{\mathrm{GW}}(k,\eta) = \frac{\rho_{\mathrm{GW}}(k,\eta)}{\rho_{\mathrm{c}}(\eta)} = \frac{\rho_{\mathrm{GW}}(k,\eta)}{3\,H^2\,M_{\mathrm{Pl}}^2},$$

where $\rho_{\mathrm{c}} = 3\,H^2\,M_{\mathrm{Pl}}^2$ is the critical density at time η.

During the late stages of radiation domination, the wave numbers such that $k \gg k_{\mathrm{eq}}$ are well inside the Hubble radius. Therefore, the spectral energy density of GWs will fall in the same manner as the energy density of radiation (i.e. as a^{-4}).

Utilizing this behavior, we can express the dimensionless spectral energy density of GWs *today*, i.e. $\Omega_{\mathrm{GW}}(k)$, in terms of $\Omega_{\mathrm{GW}}(k,\eta)$ above in the following manner:

$$\Omega_{\mathrm{GW}}(k)\,h^2 = \frac{g_k}{g_0}\left(\frac{g_{s,0}}{g_{s,k}}\right)^{4/3}\Omega_{\mathrm{r}}\,h^2\,\Omega_{\mathrm{GW}}(k,\eta) \simeq \left(\frac{g_0}{g_k}\right)^{1/3}\Omega_{\mathrm{r}}\,h^2\,\Omega_{\mathrm{GW}}(k,\eta)$$

$$\simeq 1.38\times 10^{-5}\left(\frac{g_{*,k}}{106.75}\right)^{-1/3}\left(\frac{\Omega_{\mathrm{r}}\,h^2}{4.16\times 10^{-5}}\right)\Omega_{\mathrm{GW}}(k,\eta),$$

where $\Omega_{\mathrm{r}}\,h^2$ denotes the present day dimensionless energy density of radiation.

Transition from de Sitter inflation to radiation domination

During radiation domination, the scale factor can be expressed as

$$a(\eta) = a_{\mathrm{r}}\,(\eta - \eta_{\mathrm{r}}),$$

where $a_{\mathrm{r}} = -a_{\mathrm{e}}/\eta_{\mathrm{e}}$ and $\eta_{\mathrm{r}} = 2\,\eta_{\mathrm{e}}$, which are arrived at by matching the scale factor and its time derivative at the end of inflation corresponding to the time η_{e} and scale factor a_{e}.

Since $a'' = 0$ during radiation domination, it is straightforward to see that the general solution to the Mukhanov-Sasaki variable can be written as ▸ Mukhanov-Sasaki equation for tensors

$$u_k(\eta) = \frac{1}{\sqrt{2\,k}}\left[A_{\mathrm{r}}(k)\,\mathrm{e}^{-i\,k\,(\eta-\eta_{\mathrm{r}})} + B_{\mathrm{r}}(k)\,\mathrm{e}^{i\,k\,(\eta-\eta_{\mathrm{r}})}\right].$$

On matching the Mukhanov-Sasaki variable u_k and its time derivative at η_{e}, we obtain the coefficients $A_{\mathrm{r}}(k)$ and $B_{\mathrm{r}}(k)$ to be ▸ Solution in de Sitter

$$A_{\mathrm{r}}(k) = \frac{1}{2\,k^2\,\eta_{\mathrm{e}}^2}\,\left(-1 - 2\,i\,k\,\eta_{\mathrm{e}} + 2\,k^2\,\eta_{\mathrm{e}}^2\right)\,\mathrm{e}^{-2\,i\,k\,\eta_{\mathrm{e}}}, \quad B_{\mathrm{r}}(k) = \frac{1}{2\,k^2\,\eta_{\mathrm{e}}^2}.$$

It is useful to note that $|A_{\mathrm{r}}(k)|^2 - |B_{\mathrm{r}}(k)|^2 = 1$, which is the Wronskian condition.

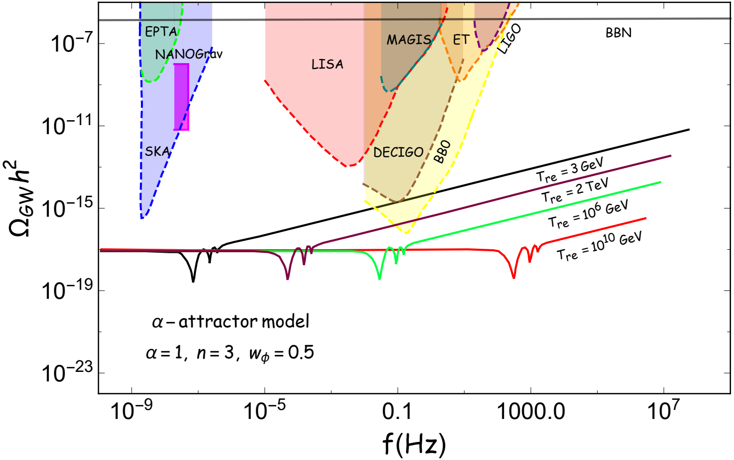
Relation between wave number and frequency

Note that the frequency f and the wave number k are related as follows:

$$f = \frac{k}{2\,\pi} = 1.55 \times 10^{-15} \left(\frac{k}{1\,\mathrm{Mpc}^{-1}}\right)\,\mathrm{Hz}.$$

For instance, $k \simeq 10^{6}\,\mathrm{Mpc}^{-1}$ corresponds to $f \simeq 10^{-9}\,\mathrm{Hz} = 1\,\mathrm{nHz}$.

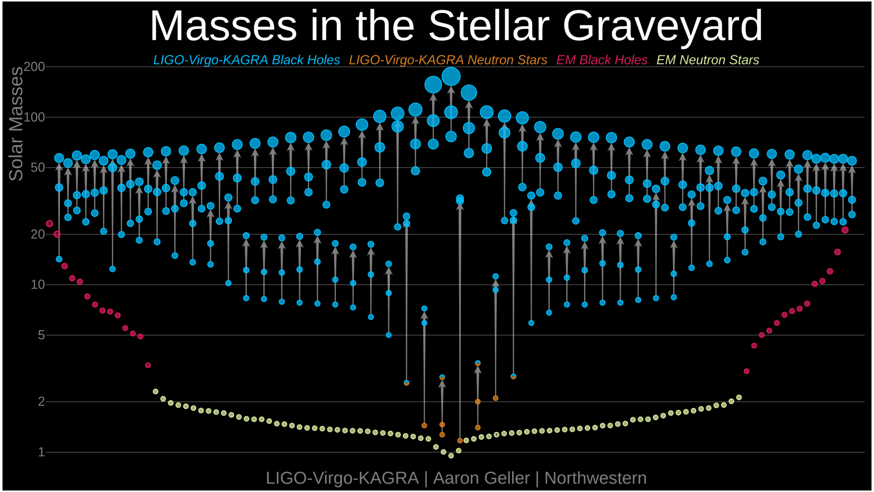
Effects on $\Omega_{\rm GW}(f)$ due to reheating



The behavior of the dimensionless spectral energy density of primary GWs today, viz. $\Omega_{\rm GW}$, has been plotted, over a wide range of frequency f, for different reheating temperatures (in red, green, brown and black)[35].

[35] Md. R. Haque, D. Maity, T. Paul and L. Sriramkumar, Phys. Rev. D **104**, 063513 (2021).

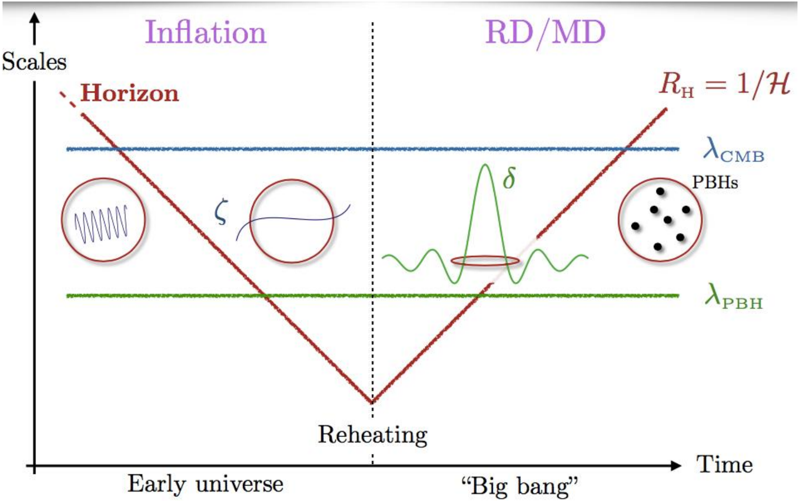
Coalescence of compact binaries observed by LIGO



The third GW Transient Catalog of mergers involving black holes and neutron stars observed by the LIGO-Virgo-KAGRA collaboration[36].

[36] Image from https://www.ligo.caltech.edu/LA/image/ligo20211107a.

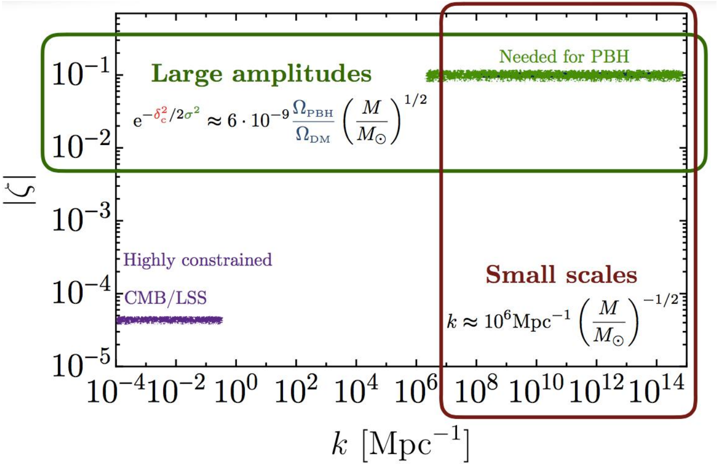
Formation of primordial black holes (PBHs)



BHs can form in the primordial universe when perturbations with significant amplitudes on small scales reenter the Hubble radius during the radiation dominated epoch[37].

[37] Figure from G. Franciolini, arXiv:2110.06815 [astro-ph.CO].

Amplitude required to form significant number of PBHs



In order to form significant number of black holes, the amplitude of the perturbations on small scales has to be large enough such that the dimensionless amplitude of the scalar perturbation is close to unity[38].

[38] Figure credit G. Franciolini.

Point of inflection in the potential and ultra slow roll inflation

Recall that the canonical scalar field ϕ satisfies the following equation in a FLRW universe:

$$\ddot{\phi} + 3\,H\,\dot{\phi} + V_\phi = 0.$$

Consider a potential $V(\phi)$ that contains a point of inflection. Since $V_\phi \simeq 0$ near the point, the above equation can be integrated immediately to arrive at

$$\dot{\phi} \propto \frac{1}{a^3}.$$

In such a situation, as we can assume H to be constant, we have

$$\epsilon_1 = \frac{\dot{\phi}^2}{2\,M_{_{\rm Pl}}^2\,H^2} \propto \frac{1}{a^6}.$$

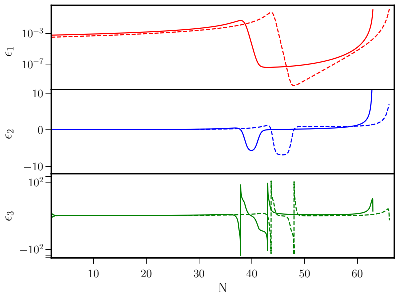
Also, in such a situation

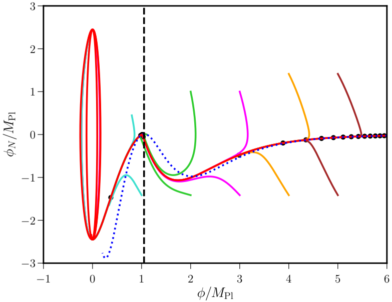
$$\epsilon_2 = \frac{{\rm d}\ln\epsilon_1}{{\rm d}N} = \frac{\dot{\epsilon}_1}{H\,\epsilon_1} = -6.$$

Such a period of inflation is referred to as ultra slow roll inflation[39]

[39] N. C. Tsamis and R. P. Woodard, Phys. Rev. D **69**, 084005 (2004) [astro-ph/0307463];
W. H. Kinney, Phys. Rev. D **72**, 023515 (2005) [gr-qc/0503017].

Single-field models admitting ultra slow roll inflation





Potentials leading to ultra slow roll inflation (with $x = \phi/v$, v being a constant)[40]: ▸ Inflationary attractor

$$\mathrm{USR1}: V(\phi) = V_0 \, \frac{6\,x^2 - 4\,\alpha\,x^3 + 3\,x^4}{(1+\beta\,x^2)^2},$$

$$\mathrm{USR2}: V(\phi) = V_0 \left\{\tanh\left(\frac{\phi}{\sqrt{6}\,M_{_{\rm Pl}}}\right) + A\,\sin\left[\frac{\tanh\left[\phi/\left(\sqrt{6}\,M_{_{\rm Pl}}\right)\right]}{f_\phi}\right]\right\}^2.$$

[40] C. Germani and T. Prokopec, Phys. Dark Univ. **18**, 6 (2017);
J. Garcia-Bellido and E. R. Morales, Phys. Dark Univ. **18**, 47 (2017);
I. Dalianis, A. Kehagias and G. Tringas, JCAP **01**, 037 (2019).

Understanding the effect of ultra slow roll

Consider a situation wherein there arises a transtion from slow roll inflation to a phase of ultra slow roll at the time $\eta_1 = -1/k_1$. In the first regime, on super-Hubble scales, the mode function governing the curvature perturbation, say, f_k^{I}, can be expressed as

$$f_k^{\mathrm{I}}(\eta) = C_k + \frac{D_k}{2}\,\eta^2.$$

During the phase of ultra slow roll, for modes that are already on super-Hubble scales, the mode function, say, f_k^{II}, can be expressed as

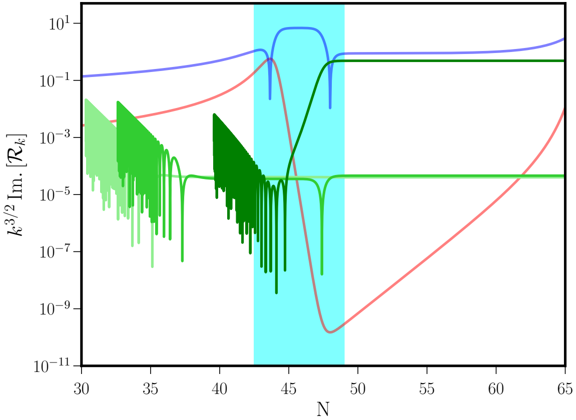
$$f_k^{\mathrm{II}}(\eta) = A_k + B_k\left(\frac{1}{\eta^3} - \frac{1}{\eta_1^3}\right).$$

At late times, say, at $\eta_2 = -1/k_2$, the power spectrum can be evaluated to be

$$\mathcal{P}_{_{\mathrm{S}}}(k) = \frac{H_{_{\mathrm{I}}}^2}{288\,\pi^2\,M_{_{\mathrm{Pl}}}^2\,\epsilon_{1\mathrm{i}}}\left[6 + \frac{k^2}{k_1^2}\left(5 + 2\,\frac{k_2^3}{k_1^3}\right)\right]^2.$$

For $k \ll k_1$, we find that $\mathcal{P}_{_{\mathrm{S}}}(k) \propto k^0$, whereas for $k_1 < k < k_2$, we have $\mathcal{P}_{_{\mathrm{S}}}(k) \propto k^4$.

Evolution of the curvature perturbation during ultra slow roll



The evolution of the amplitudes of the imaginary parts of the curvature perturbation in the USR2 model are plotted for three different wave numbers (in light, lime and dark green)[41]. The evolution of the first and the second slow roll parameters (ϵ_1 and ϵ_2) are also illustrated (in red and blue).

▸ Evolution in slow roll inflation

[41] H. V. Ragavendra, P. Saha, L. Sriramkumar and J. Silk, Phys. Rev. D **103**, 083510 (2021).

Analytical modelling of a brief phase of ultra slow roll inflation

Let us assume the following behavior of the first slow roll parameter:

$$\epsilon_1(a) = \begin{cases} \epsilon_{1\mathrm{i}}, & \text{for } \eta_\mathrm{i} < \eta < \eta_1, \\ \epsilon_{1\mathrm{i}}\,(a_1/a)^{-6}, & \text{for } \eta_1 < \eta < \eta_2, \\ \epsilon_{1\mathrm{a}}\,(a/a_2)^2, & \text{for } \eta_2 < \eta < \eta_\mathrm{e}, \end{cases}$$

The corresponding second slow roll parameter behaves as

$$\epsilon_2 \simeq \begin{cases} 0, & \text{for } \eta_\mathrm{i} < \eta < \eta_1, \\ -6, & \text{for } \eta_1 < \eta < \eta_2, \\ 2, & \text{for } \eta_2 < \eta < \eta_\mathrm{e}. \end{cases}$$

Note that a period of ultra slow roll inflation follows an epoch of slow roll inflation before ϵ_1 rises leading to the end of inflation.

It can be shown that, in the three phases, the quantity z''/z behaves as

z''/z

$$\frac{z''}{z} \simeq \frac{2}{\eta^2}, \quad \frac{z''}{z} \simeq \frac{2}{\eta^2}, \quad \frac{z''}{z} \simeq \frac{6}{\eta^2}.$$

Solutions to the Mukhanov-Sasaki variable in the three phases

The solutions to the Mukhanov-Sasaki variable in the three phases can be expressed as

$$v_k^{\rm I}(\eta) = \frac{1}{\sqrt{2\,k}}\,\left(1-\frac{i}{k\,\eta}\right)\,{\rm e}^{-i\,k\,\eta},$$

$$v_k^{\rm II}(\eta) = \frac{\gamma_k}{\sqrt{2\,k}}\,\left(1-\frac{i}{k\,\eta}\right)\,{\rm e}^{-i\,k\,\eta} + \frac{\delta_k}{\sqrt{2\,k}}\,\left(1+\frac{i}{k\,\eta}\right)\,{\rm e}^{i\,k\,\eta},$$

$$v_k^{\rm III}(\eta) = \frac{\alpha_k}{\sqrt{2\,k}}\,\left(1-\frac{3\,i}{k\,\eta}-\frac{3}{k^2\,\eta^2}\right)\,{\rm e}^{-i\,k\,\eta} + \frac{\beta_k}{\sqrt{2\,k}}\,\left(1+\frac{3\,i}{k\,\eta}-\frac{3}{k^2\,\eta^2}\right)\,{\rm e}^{i\,k\,\eta}.$$

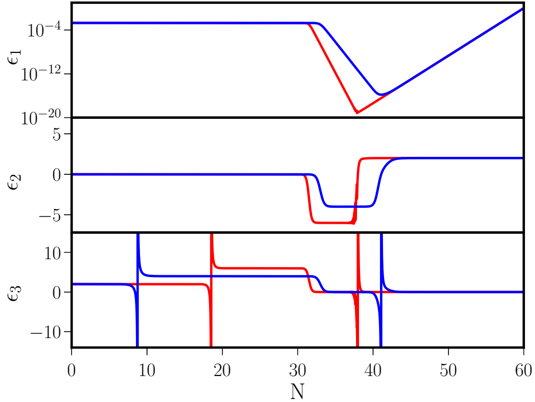
The coefficients $(\alpha_k, \beta_k, \gamma_k, \delta_k)$ can be obtained to be

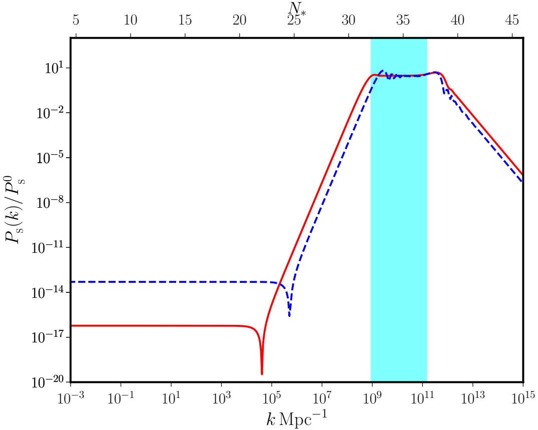
$$\gamma_k = 1+\frac{3\,i}{2\,k\,\eta_1}+\frac{3\,i}{2\,k^3\,\eta_1^3}, \quad \delta_k = \left(-\frac{3\,i}{2\,k\,\eta_1}-\frac{3}{k^2\,\eta_1^2}+\frac{3\,i}{2\,k^3\,\eta_1^3}\right)\,{\rm e}^{-2\,i\,k\,\eta_1},$$

$$\alpha_k = \left(1+\frac{2}{k^2\,\eta_2^2}+\frac{9}{2\,k^4\,\eta_2^4}\right)\,\gamma_k - \left(\frac{2\,i}{k\,\eta_2}-\frac{7}{k^2\,\eta_2^2}-\frac{9\,i}{k^3\,\eta_2^3}+\frac{9}{2\,k^4\,\eta_2^4}\right)\,\delta_k\,{\rm e}^{2\,i\,k\,\eta_2},$$

$$\beta_k = \left(1+\frac{2}{k^2\,\eta_2^2}+\frac{9}{2\,k^4\,\eta_2^4}\right)\,\delta_k + \left(\frac{2\,i}{k\,\eta_2}+\frac{7}{k^2\,\eta_2^2}-\frac{9\,i}{k^3\,\eta_2^3}-\frac{9}{2\,k^4\,\eta_2^4}\right)\,\gamma_k\,{\rm e}^{-2\,i\,k\,\eta_2}.$$

Behavior of the slow roll parameters and the scalar power spectrum

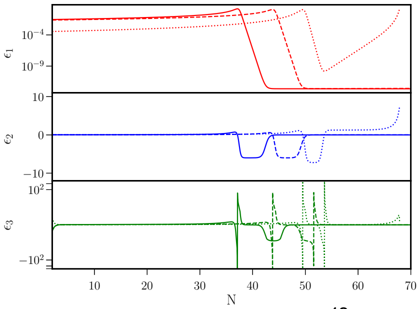


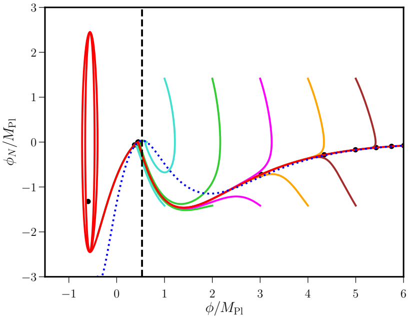


The behavior of the first three slow roll parameters $(\epsilon_1, \epsilon_2\, \epsilon_3)$ (on the left) and the resulting scalar power spectrum $\mathcal{P}_{\rm S}(k)$ (on the right). In addition to the analytically evaluated quantities (plotted in blue), we have also plotted the numerical results (in red). On small scales, the power spectrum rises as k^4 and falls as k^{-2} after the peak[42].

[42] C. T. Byrnes, P. S. Cole, and S. P. Patil, JCAP **06**, 028 (2019).

Single-field models permitting punctuated inflation





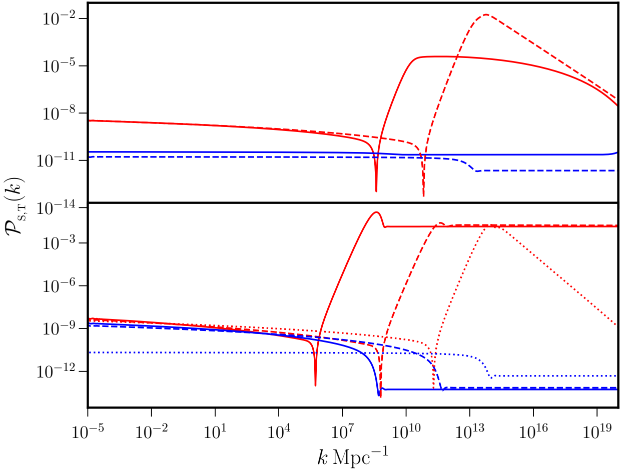
Potentials admitting punctuated inflation[43]:

$$\mathrm{PI1}: V(\phi) = V_0\,\left(1 + B\,\phi^4\right), \quad \mathrm{PI2}: V(\phi) = \frac{m^2}{2}\,\phi^2 - \frac{2\,m^2}{3\,\phi_0}\,\phi^3 + \frac{m^2}{4\,\phi_0^2}\,\phi^4,$$

$$\mathrm{PI3}: V(\phi) = V_0\,\left[c_0 + c_1\,\tanh\left(\frac{\phi}{\sqrt{6\,\alpha}\,M_{\rm Pl}}\right) + c_2\,\tanh^2\left(\frac{\phi}{\sqrt{6\,\alpha}\,M_{\rm Pl}}\right) + c_3\,\tanh^3\left(\frac{\phi}{\sqrt{6\,\alpha}\,M_{\rm Pl}}\right)\right]^2.$$

[43] D. Roberts, A. R. Liddle and D. H. Lyth, Phys. Rev. D **51**, 4122 (1995);
R. K. Jain, P. Chingangbam, J.-O. Gong, L. Sriramkumar and T. Souradeep, JCAP **01**, 009 (2009);
I. Dalianis, A. Kehagias and G. Tringas, JCAP **01**, 037 (2019).

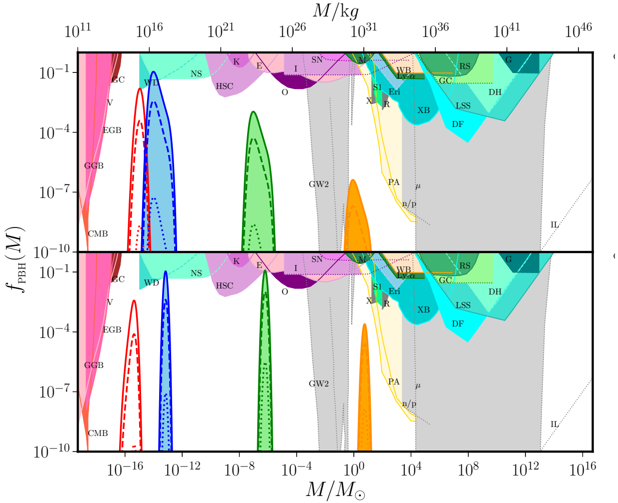
Power spectra in ultra slow roll and punctuated inflation



Scalar (in red) and the tensor (in blue) power spectra arising in various single field models that permit a period of ultra slow roll (on top) and punctuated inflation (at the bottom)[44].

[44] H. V. Ragavendra, P. Saha, L. Sriramkumar and J. Silk, Phys. Rev. D **103**, 083510 (2021);
Also see H. V. Ragavendra and L. Sriramkumar, Galaxies **11**, 34 (2023).

$f_{\rm PBH}(M)$ in ultra slow roll and punctuated inflation



The fraction of PBHs contributing to the cold dark matter density today $f_{\rm PBH}(M)$ has been plotted for different models, viz. USR2 (on top, in red) and PI3 (at the bottom, in red)[45].

[45] H. V. Ragavendra, P. Saha, L. Sriramkumar and J. Silk, Phys. Rev. D **103**, 083510 (2021).

Tensor perturbations at the second order

In order to compute the second-order tensor perturbations induced by the first order scalar perturbations, we can start with the following line-element:

$$\mathrm{d}s^2 = -\left(1 + 2\,\Phi^{(1)} + 2\,\Phi^{(2)}\right)\,\mathrm{d}t^2 + 2\,V_i^{(2)}\,a(t)\,\mathrm{d}t\,\mathrm{d}x^i + a^2(t)\,\left[\left(1 - 2\,\Psi^{(1)} - 2\,\Psi^{(2)}\right)\delta_{ij} + \frac{1}{2}\,h_{ij}^{(2)}\right]\mathrm{d}x^i\,\mathrm{d}x^j,$$

where the superscripts (1) and (2) denote the perturbations at the first and the second order perturbations.

Note that $(\Phi^{(1)}, \Psi^{(1)}, \Phi^{(2)}, \Psi^{(2)})$ denote the scalar perturbations, $V_i^{(2)}$ is the vector perturbation and $h_{ij}^{(2)}$ represents the tensor perturbation of our interest[46].

[46] D. Baumann, P. J. Steinhardt, K. Takahashi and K. Ichiki, Phys. Rev. D **76**, 084019 (2007).

GWs sourced by second order scalar perturbations I

At the second order in the perturbations, one finds that the equation governing the tensor modes, say, $h_{\boldsymbol{k}}$, can be written as[47]

$$h_{\boldsymbol{k}}^{\lambda\,\prime\prime} + 2\,\mathcal{H}\,h_{\boldsymbol{k}}^{\lambda\,\prime} + k^2\,h_{\boldsymbol{k}}^{\lambda} = S_{\boldsymbol{k}}^{\lambda}$$

with the source term $S_{\boldsymbol{k}}^{\lambda}$ being given by

$$S_{\boldsymbol{k}}^{\lambda}(\eta) = 4\int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\,\pi)^{3/2}}\,e^{\lambda}(\boldsymbol{k},\boldsymbol{p})\,\biggl\{2\,\Phi_{\boldsymbol{p}}(\eta)\,\Phi_{\boldsymbol{k}-\boldsymbol{p}}(\eta) + \frac{4}{3\,(1+w)\,\mathcal{H}^2}\,\left[\Phi_{\boldsymbol{p}}'(\eta) + \mathcal{H}\,\Phi_{\boldsymbol{p}}(\eta)\right]\,\left[\Phi_{\boldsymbol{k}-\boldsymbol{p}}'(\eta) + \mathcal{H}\,\Phi_{\boldsymbol{k}-\boldsymbol{p}}(\eta)\right]\biggr\},$$

where $\Phi_{\boldsymbol{k}}$ represents the Fourier modes of the Bardeen potential, while $\mathcal{H}$ and w denote the conformal Hubble parameter and the EoS parameter describing the universe at the conformal time η. Also, $e^{\lambda}(\boldsymbol{k},\boldsymbol{p}) = e_{ij}^{\lambda}(\boldsymbol{k})\,p^i\,p^j$, with $e_{ij}^{\lambda}(\boldsymbol{k})$ representing the polarization of the tensor perturbations.

[47] K. N. Ananda, C. Clarkson and D. Wands, Phys. Rev. D **75**, 123518 (2007);
D. Baumann, P. J. Steinhardt, K. Takahashi and K. Ichiki, Phys. Rev. D **76**, 084019 (2007).

GWs sourced by second order scalar perturbations II

During radiation domination, we can express the Fourier modes $\Phi_{\boldsymbol{k}}$ of the Bardeen potential in terms of the inflationary Fourier modes $\mathcal{R}_{\boldsymbol{k}}$ of the curvature perturbations generated during inflation through the relation

$$\Phi_{\boldsymbol{k}}(\eta) = \frac{2}{3}\,\mathcal{T}(k\,\eta)\,\mathcal{R}_{\boldsymbol{k}},$$

where $\mathcal{T}(k\,\eta)$ is the transfer function given by

$$\mathcal{T}(k\,\eta) = \frac{9}{(k\,\eta)^2}\left[\frac{\sin\left(k\,\eta/\sqrt{3}\right)}{k\,\eta/\sqrt{3}} - \cos\left(k\,\eta/\sqrt{3}\right)\right].$$

Particular solution to the inhomogeneous equation

If we make use of the Green's function corresponding to the tensor modes during radiation domination, we find that we can express the inhomogeneous contribution to $h_{\boldsymbol{k}}^{\lambda}$ as[48]

$$h_{\boldsymbol{k}}^{\lambda}(\eta) = \frac{4}{9\,k^3\,\eta} \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\,\pi)^{3/2}}\, e^{\lambda}(\boldsymbol{k},\boldsymbol{p})\, \mathcal{R}_{\boldsymbol{k}}\, \mathcal{R}_{\boldsymbol{k}-\boldsymbol{p}}$$
$$\times\, \left[\mathcal{I}_c\left(p/k, |\boldsymbol{k}-\boldsymbol{p}|/k\right)\, \cos\left(k\,\eta\right) + \mathcal{I}_s\left(p/k, |\boldsymbol{k}-\boldsymbol{p}|/k\right)\, \sin\left(k\,\eta\right)\right],$$

where the kernels $\mathcal{I}_c(v,u)$ and $\mathcal{I}_s(v,u)$ are described by the integrals (with $\mathcal{T}_z = \mathrm{d}\mathcal{T}/\mathrm{d}z$)

$$\mathcal{I}_c(v,u) = -4\int_0^{\infty} \mathrm{d}\tau\,\tau\,\sin\tau\, \Big\{2\,\mathcal{T}(v\,\tau)\,\mathcal{T}(u\,\tau)$$
$$+ \left[\mathcal{T}(v\,\tau) + v\,\tau\,\mathcal{T}_{v\tau}(v\,\tau)\right]\, \left[\mathcal{T}(u\,\tau) + u\,\tau\,\mathcal{T}_{u\tau}(u\,\tau)\right]\Big\},$$

$$\mathcal{I}_s(v,u) = 4\int_0^{\infty} \mathrm{d}\tau\,\tau\,\cos\tau\, \Big\{2\,\mathcal{T}(v\,\tau)\,\mathcal{T}(u\,\tau)$$
$$+ \left[\mathcal{T}(v\,\tau) + v\,\tau\,\mathcal{T}_{v\tau}(v\,\tau)\right]\, \left[\mathcal{T}(u\,\tau) + u\,\tau\,\mathcal{T}_{u\tau}(u\,\tau)\right]\Big\}.$$

[48] J. R. Espinosa, D. Racco and A. Riotto, JCAP, **1809**, 012 (2018).

Forms of the kernels

Upon utilizing the transfer function $\mathcal{T}(k\,\eta)$ the kernels $\mathcal{I}_c(v,u)$ and $\mathcal{I}_s(v,u)$ can be calculated analytically to obtain that[49]

$$\mathcal{I}_c(v,u) = -\frac{27\,\pi}{4\,v^3\,u^3}\,\Theta\left(v+u-\sqrt{3}\right)\,(v^2+u^2-3)^2,$$
$$\mathcal{I}_s(v,u) = -\frac{27}{4\,v^3\,u^3}\,(v^2+u^2-3)\,\left[4\,v\,u+(v^2+u^2-3)\,\log\left|\frac{3-(v-u)^2}{3-(v+u)^2}\right|\right],$$

where $\Theta(z)$ denotes the theta function. Note that $\mathcal{I}_{c,s}(v,u) = \mathcal{I}_{c,s}(u,v)$.

[49] K. Kohri and T. Terada, Phys. Rev. D **97**, 123532 (2018);
J. R. Espinosa, D. Racco and A. Riotto, JCAP, **1809**, 012 (2018).

The secondary tensor power spectrum

The power spectrum of the secondary GWs, say, $\mathcal{P}_h(k,\eta)$, generated due to the second order scalar perturbations can be defined through the relation

$$\langle h_{\boldsymbol{k}}^{\lambda}(\eta)\, h_{\boldsymbol{k}'}^{\lambda'}(\eta)\rangle = \frac{2\,\pi^2}{k^3}\,\mathcal{P}_h(k,\eta)\,\delta^{(3)}(\boldsymbol{k}+\boldsymbol{k}')\,\delta^{\lambda\lambda'}.$$

If we assume that the Fourier modes of the curvature perturbations are Gaussian random variables, using Wick's theorem, we can express the four-point function involving $\mathcal{R}_{\boldsymbol{k}}$ in terms of the inflationary scalar power spectrum $\mathcal{P}_{_{\rm S}}(k)$ to arrive at the following expression for $\mathcal{P}_h(k,\eta)$:

$$\mathcal{P}_h(k,\eta) = 481\,k^2\,\eta^2 \int_0^\infty {\rm d}v \int_{|1-v|}^{1+v} {\rm d}u\, \left[\frac{4\,v^2-(1+v^2-u^2)^2}{4\,u\,v}\right]^2 \mathcal{P}_{_{\rm S}}(k\,v)\,\mathcal{P}_{_{\rm S}}(k\,u)$$
$$\times\,\left[\mathcal{I}_c(u,v)\cos\,(k\,\eta)+\mathcal{I}_s(u,v)\sin\,(k\,\eta)\right]^2.$$

The trigonometric functions in this expression arise because of the form of the transfer function $\mathcal{T}(k\,\eta)$.

Averaging over the oscillations

On averaging the secondary tensor power spectrum $\mathcal{P}_h(k,\eta)$ over small time scales, we can replace the trigonometric functions in the above expression by their average over a time period This leads to

$$\overline{\mathcal{P}_h(k,\eta)} = \frac{2}{81\,k^2\,\eta^2}\int_0^\infty \mathrm{d}v \int_{|1-v|}^{1+v} \mathrm{d}u\,\left[\frac{4\,v^2-(1+v^2-u^2)^2}{4\,u\,v}\right]^2 \mathcal{P}_{\mathrm{S}}(k\,v)\,\mathcal{P}_{\mathrm{S}}(k\,u)$$
$$\times\,\left[\mathcal{I}_c^2(u,v)+\mathcal{I}_s^2(u,v)\right],$$

where the line over $\mathcal{P}_h(k,\eta)$ implies the average over small time scales.

The spectral energy density of GWs at a time η is given by

$$\rho_{\mathrm{GW}}(k,\eta) = \frac{M_{\mathrm{Pl}}^2}{8}\left(\frac{k}{a}\right)^2\overline{\mathcal{P}_h(k,\eta)}.$$

We can define the corresponding dimensionless density parameter $\Omega_{\mathrm{GW}}(k,\eta)$ in terms of the critical density $\rho_{\mathrm{cr}}(\eta)$ as

$$\Omega_{\mathrm{GW}}(k,\eta) = \frac{\rho_{\mathrm{GW}}(k,\eta)}{\rho_{\mathrm{cr}}(\eta)} = \frac{1}{24}\left(\frac{k}{\mathcal{H}}\right)^2\overline{\mathcal{P}_h(k,\eta)} = \frac{k^2\,\eta^2}{24}\,\overline{\mathcal{P}_h(k,\eta)}.$$

Spectral energy density of secondary GWs today

The dimensionless spectral energy density of GWs $\Omega_{\rm GW}(k,\eta)$, when evaluated at late times during the radiation dominated epoch, can be expressed as[50]

$$\Omega_{\rm GW}(k,\eta) = \frac{\rho_{\rm GW}(k,\eta)}{\rho_{\rm cr}(\eta)} = \frac{1}{972}\int_0^\infty {\rm d}v \int_{|1-v|}^{1+v} {\rm d}u \left[\frac{4\,v^2-(1+v^2-u^2)^2}{4\,u\,v}\right]^2 \mathcal{P}_{\rm S}(k\,v)\,\mathcal{P}_{\rm S}(k\,u) \times \left[\mathcal{I}_c^2(u,v)+\mathcal{I}_s^2(u,v)\right],$$

where the quantities $\mathcal{I}_c(u,v)$ and $\mathcal{I}_s(u,v)$ are determined by the transfer function $\mathcal{T}(k,\eta)$ for the scalar perturbations.

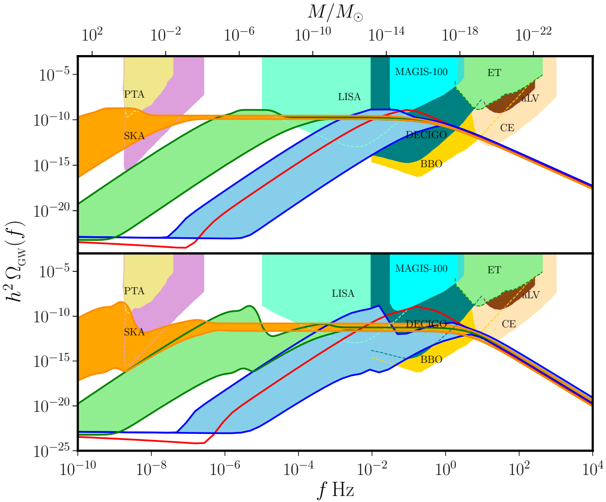
Recall that, we can express $\Omega_{\rm GW}(k)$ today in terms of the above $\Omega_{\rm GW}(k,\eta)$ as follows:

$$h^2\,\Omega_{\rm GW}(k) \simeq 1.38\times 10^{-5}\left(\frac{g_k}{106.75}\right)^{-1/3}\left(\frac{\Omega_r\,h^2}{4.16\times 10^{-5}}\right)\Omega_{\rm GW}(k,\eta),$$

where $\Omega_{\rm r}$ denotes the dimensionless energy density of radiation today, while g_k represents the number of relativistic degrees of freedom at reentry.

[50] K. Kohri and T. Terada, Phys. Rev. D **97**, 123532 (2018);
J. R. Espinosa, D. Racco and A. Riotto, JCAP **09**, 012 (2018).

$\Omega_{\rm GW}(f)$ in ultra slow roll inflation



The dimensionless spectral density of GWs today $\Omega_{\rm GW}(f)$ arising in single field models that permit a brief epoch of ultra slow roll inflation[51].

[51] H. V. Ragavendra, P. Saha, L. Sriramkumar and J. Silk, Phys. Rev. D **103**, 083510 (2021).

Plan of the talk

Summary

- If one of the future CMB missions—such as LiteBIRD (Lite, Light satellite for the studies of B-mode polarization and Inflation from cosmic background Radiation Detection), Primordial Inflation Explorer (PIXIE) or Exploring Cosmic History and Origin (ECHO, a proposed Indian effort)—detect the signatures of the primordial GWs, it will help us arrive at strong constraints on the dynamics during inflation and reheating.
- The observations by LIGO are a culmination of almost fifty years of effort to detect GWs. They have opened up a new window to observe the universe.
- The observations by the PTAs and their possible implications for the stochastic GW background offer a wonderful opportunity to understand the physics operating over a wider range of scales in the early universe.
- During the coming decades, GW observatories such as the Laser Interferometer Space Antenna, Einstein Telescope and Cosmic Explorer, can be expected to provide us with an unhindered view of the primordial universe.

Plan of the talk

References

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- L. Sriramkumar, *On the generation and evolution of perturbations during inflation and reheating*, in Vignettes in Gravitation and Cosmology, Eds. L. Sriramkumar and T. R. Seshadri (World Scientific, Singapore, 2012), pp. 207–249.
- H. V. Ragavendra and L. Sriramkumar, *Observational imprints of enhanced scalar power on small scales in ultra slow roll inflation and associated non-Gaussianities*, Galaxies **11**, 34 (2023) [arXiv:2301.08887 [astro-ph.CO]].

Thank you for your attention