PH5870

INTRODUCTION TO GENERAL RELATIVITY

July–November 2021

Lecture schedule and meeting hours

- The course will consist of about 42 lectures, including about 8–10 tutorial sessions. However, note that there will be no separate tutorial sessions, and they will be integrated with the lectures.
- The duration of each lecture will be 50 minutes. We will be meeting on Google Meet. I will share the link over smail.
- The first lecture will be on Wednesday, August 4, and the last one will be on Friday, November 12.
- We will meet thrice a week. We shall meet during the following hours: 11:00–11:50 AM on Wednesdays, 9:00–9:50 AM on Thursdays, and 8:00–8:50 AM on Fridays.
- We may also meet during 4:50–5:40 PM on Tuesdays for either the quizzes or to make up for lectures that I may have to miss due to other unavoidable commitments. Changes in schedule, if any, will be notified sufficiently in advance.
- If you would like to discuss with me about the course outside the lecture hours, please send me an e-mail at `sriram@physics.iitm.ac.in`. We can converge on a mutually convenient time to meet and discuss online. I would request you to write to me from your smail addresses with the subject line containing the name of the course, i.e. PH5870: Introduction to General Relativity.

Information about the course

- All the information regarding the course such as the schedule of the lectures, the structure and the syllabus of the course, suitable textbooks and additional references will be available on the course's page on Moodle at the following URL:

  `https://courses.iitm.ac.in/`
- The exercise sheets and other additional material will also be made available on Moodle.
- A PDF file containing these information as well as completed quizzes will also be available at the link on this course at the following URL:

  `http://physics.iitm.ac.in/~sriram/professional/teaching/teaching.html`

  I will keep updating this file and the course's page on Moodle as we make progress.

Quizzes, end-of-semester exam and grading

- The grading will be based on three scheduled quizzes and an end-of-semester exam.
- I will consider the best two quizzes for grading, and the two will carry 25% weight each.
- The three quizzes will be held on September 3, September 28 and October 26. While September 3 is a Friday, the other two dates are Tuesdays, and the quizzes will be held during 5:00–6:30 PM.
- The end-of-semester exam will be held during 9:00 AM–12:00 NOON on Friday, November 26, and the exam will carry 50% weight.

Syllabus and structure

1. **Special theory of relativity** [$\sim$ 11 lectures]

   (a) The Michelson-Morley interferometric experiment – Postulates of special relativity
   (b) Lorentz transformations – The relativity of simultaneity – Length contraction and time dilation
   (c) Transformation of velocities and acceleration – Uniform acceleration – Doppler effect
   (d) Metric tensor – The light cone
   (e) Contravariant and covariant vectors – Action for the relativistic free particle – Charges in an electromagnetic field and the Lorentz force law
   (f) Conservation of relativistic energy and momentum
   (g) Infinitesimal generators of translations, rotations and boosts – Algebra of the generators – The Lorenz and the Poincare groups

Exercise sheets 1, 2, 3 and 4

2. **Theory of a real scalar field** [$\sim$ 6 lectures]

   (a) An illustrative example – Action formulation for a string
   (b) Action describing a real, canonical, scalar field – The Euler-Lagrange field equation
   (c) The conjugate momentum – Hamiltonian density – The stress-energy tensor – Physical interpretation

Exercise sheet 5

Quiz I

3. **The case of the complex scalar field** [$\sim$ 3 lectures]

   (a) Action governing the complex scalar field – Equations of motion
   (b) Global gauge invariance
   (c) Local gauge invariance and the need for the electromagnetic field

Exercise sheet 6

4. **Symmetries and conservation laws** [$\sim$ 4 lectures]

   (a) Noether's theorem
   (b) External symmetries – Symmetry under translations, rotations and Lorentz transformations – Conserved quantities
   (c) Dilatations – The conformal stress-energy tensor
   (d) Internal symmetries and gauge transformations

Exercise sheet 7

Additional exercises I

5. **The theory of the electromagnetic field** [$\sim$ 4 lectures]

   (a) The electromagnetic field tensor – The first pair of Maxwell's equations
   (b) The four current vector – The continuity equation – Charge conservation
   (c) Action governing the free electromagnetic field – Interaction of the electromagnetic field with charges and currents – The second pair of Maxwell's equations

(d) Gauge invariance of the electromagnetic field – The Lorentz and the Coulomb gauges
(e) Equations governing the free field – Electromagnetic waves – Polarization
(f) Energy density – Poynting vector – The stress-energy tensor of the electromagnetic field
(g) Lorentz transformation properties of the electric and the magnetic fields
(h) The retarded Green's function – The Lienard-Wiechart potentials – Radiation from moving charges

Exercise sheets 8 and 9

Quiz II

6. **Tensor algebra and tensor calculus** [$\sim$ 12 lectures]

(a) Manifolds and coordinates – Curves and surfaces
(b) Transformation of coordinates – Contravariant, covariant and mixed tensors – Elementary operations with tensors
(c) The partial derivative of a tensor – Covariant differentiation and the affine connection
(d) The metric – Geodesics
(e) Isometries – The Killing equation and conserved quantities
(f) The Riemann tensor – The equation of geodesic deviation
(g) The curvature and the Weyl tensors

Exercise sheets 10, 11, 12, 13 and 14

Additional exercises II

Quiz III

7. **Field equations of general relativity** [$\sim$ 6 lectures]

(a) The equivalence principle – The principle of general covariance – The principle of minimal gravitational coupling
(b) The vacuum Einstein equations
(c) Derivation of vacuum Einstein equations from the action – The Bianchi identities
(d) The stress-energy tensor – The cases of perfect fluid, scalar and electromagnetic fields
(e) Non-canonical scalar fields – Relation to relativistic fluids
(f) The structure of the Einstein equations

Exercise sheet 15

End-of-semester exam

Advanced problems

Note: The topics in red could not be covered for want of time.

Basic textbooks

1. L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Course of Theoretical Physics, Volume 2), Fourth Edition (Pergamon Press, New York, 1975).
2. B. F. Schutz, *A First Course in General Relativity* (Cambridge University Press, Cambridge, 1990).
3. R. d'Inverno, *Introducing Einstein's Relativity* (Oxford University Press, Oxford, 1992).
4. B. Felsager, *Geometry, Particles and Fields* (Springer, Heidelberg, 1998).
5. J. B. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (Pearson Education, Delhi, 2003).
6. S. Carroll, *Spacetime and Geometry* (Addison Wesley, New York, 2004).
7. M. P. Hobson, G. P. Efstathiou and A. N. Lasenby, *General Relativity: An Introduction for Physicists* (Cambridge University Press, Cambridge, 2006).
8. F. Scheck, *Classical Field Theory* (Springer, Heidelberg, 2012).

Additional references

1. S. Weinberg, *Gravitation and Cosmology* (John Wiley, New York, 1972).
2. A. P. Lightman, W. H. Press, R. H. Price and S. A. Teukolsky, *Problem Book in Relativity and Gravitation* (Princeton University Press, New Jersey, 1975).
3. L. H. Ryder, *Quantum Field Theory*, Second Edition (Cambridge University Press, Cambridge, England, 1996).
4. W. Rindler, *Relativity: Special, General and Cosmological* (Oxford University Press, Oxford, 2006).
5. T. Padmanabhan, *Gravitation: Foundation and Frontiers* (Cambridge University Press, Cambridge, 2010).
6. A. Zee, *Einstein Gravity in a Nutshell* (Princeton University Press, Princeton, New Jersey, 2013).

Advanced texts

1. S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge University Press, Cambridge, 1973).
2. C. W. Misner, K. S. Thorne and J. W. Wheeler, *Gravitation* (W. H. Freeman and Company, San Francisco, 1973).
3. R. M. Wald, *General Relativity* (The University of Chicago Press, Chicago, 1984).
4. E. Poisson, *A Relativist's Toolkit* (Cambridge University Press, Cambridge, 2004).

Exercise sheet 1

Lorentz transformations and some consequences

1. *Superluminal motion:* Consider a blob of plasma that is moving at a speed v along a direction that makes an angle θ with respect to the line of sight. Show that the *apparent* transverse speed of the source, projected on the sky, will be related to the actual speed v by the relation

$$v_{\mathrm{app}} = \frac{v \sin\theta}{1 - (v/c)\cos\theta}.$$

From this expression conclude that the apparent speed v_{app} can exceed the speed of light.

2. *Aberration of light:* Consider two inertial frames S and S', with the frame S' moving along the x-axis with a velocity v with respect to the frame S. Let the velocity of a particle in the frames S and S' be $\mathbf{u}$ and $\mathbf{u}'$, and let θ and θ' be the angles subtended by the velocity vectors with respect to the common x-axis, respectively.

   (a) Show that

$$\tan\theta = \frac{u' \sin\theta'}{\gamma\,[u'\cos\theta' + v]},$$

   where $\gamma = \left[1 - (v/c)^2\right]^{-1/2}$.

   (b) For $u = u' = c$, show that

$$\cos\theta = \frac{\cos\theta' + (v/c)}{1 + (v/c)\cos\theta'}$$

   and

$$\sin\theta = \frac{\sin\theta'}{\gamma\,[1 + (v/c)\cos\theta']}.$$

   (c) For $(v/c) \ll 1$, show that

$$\Delta\theta = (v/c)\,\sin\theta',$$

   where $\Delta\theta = (\theta' - \theta)$.

3. *Decaying muons:* Muons are unstable and decay according to the radioactive decay law $N = N_0\, \exp -(0.693\, t/t_{1/2})$, where N_0 and N are the number of muons at times $t = 0$ and t, respectively, while $t_{1/2}$ is the half life. The half life of the muons in their own rest frame is 1.52×10^{-6} s. Consider a detector on top of a $2,000$ m mountain which counts the number of muons traveling at the speed of $v = 0.98\, c$. Over a given period of time, the detector counts 10^3 muons. When the relativistic effects are taken into account, how many muons can be expected to reach the sea level?

4. *Binding energy:* As you may know, the deuteron which is the nucleus of deuterium, an isotope of hydrogen, consists of one proton and one neutron. Given that the mass of a proton and a neutron are $m_{\mathrm{p}} = 1.673 \times 10^{-27}$ kg and $m_{\mathrm{n}} = 1.675 \times 10^{-27}$ kg, while the mass of the deuteron is $m_{\mathrm{d}} = 3.344 \times 10^{-27}$ kg, show that the binding energy of the deuteron in about 2.225 MeV.

   Note: MeV refers to Million electron Volts, and an electron Volt is 1.602×10^{-19} J.

5. *Form invariance of the Minkowski line element:* Show that the following Minkowski line element is invariant under the Lorentz transformations:

$$\mathrm{d}s^2 = c^2\, \mathrm{d}t^2 - \mathrm{d}\boldsymbol{x}^2.$$

Exercise sheet 2

Working in terms of four vectors

1. *Compton effect using four vectors:* Consider the scattering between a photon of frequency ω and a relativistic electron with velocity $\mathbf{v}$ leading to a photon of frequency ω' and electron with velocity $\mathbf{v}'$. Such a scattering is known as Compton scattering. Let α be the angle between the incident and the scattered photon. Also, let θ and θ' be the angles subtended by the directions of propagation of the incident and the scattered photon with the velocity vector of the electron before the collision.

   (a) Using the conservation of four momentum, show that

   $$\frac{\omega'}{\omega} = \frac{1-(v/c)\cos\theta}{1-(v/c)\cos\theta' + (\hbar\,\omega/\gamma\, m_{\mathrm{e}}\, c^2)\,(1-\cos\alpha)},$$

   where $\gamma = \left[1-(v/c)^2\right]^{-1/2}$ and m_{e} is the mass of the electron.

   (b) When $\hbar\,\omega \ll \gamma\, m_{\mathrm{e}}\, c^2$, show that the frequency shift of the photon can be written as

   $$\frac{\Delta\omega}{\omega} = \frac{(v/c)\,\left(\cos\theta - \cos\theta'\right)}{1-(v/c)\cos\theta'},$$

   where $\Delta\omega = (\omega' - \omega)$.

2. *Creation of electron-positron pairs:* A purely relativistic process corresponds to the production of electron-positron pairs in a collision of two high energy gamma ray photons. If the energies of the photons are ϵ_1 and ϵ_2 and the relative angle between their directions of propagation is θ, then, by using the conservation of energy and momentum, show that the process can occur only if

   $$\epsilon_1\,\epsilon_2 > \frac{2\,m_{\mathrm{e}}^2\,c^4}{1-\cos\theta},$$

   where m_{e} is the mass of the electron.

3. *Transforming four vectors and invariance under Lorentz transformations:* Consider two inertial frames K and K', with K' moving with respect to K, say, along the common x-axis with a certain velocity.

   (a) Given a four vector A^μ in the K frame, construct the corresponding contravariant and covariant four vectors, say, $A^{\mu\prime}$ and A'_μ, in the K' frame.

   (b) Explicitly illustrate that the scalar product $A_\mu\, A^\mu$ is a Lorentz invariant quantity, i.e. show that $A_\mu\, A^\mu = A'_\mu\, A^{\mu\prime}$.

4. *Lorentz invariance of the wave equation:* Show that the following wave equation:

   $$\frac{1}{c^2}\,\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi = 0$$

   satisfied by, say, light, is invariant under the Lorentz transformations.

5. *Mirrors in motion:* A mirror moves with the velocity v in a direction perpendicular its plane. A ray of light of frequency ν_1 is incident on the mirror at an angle of incidence θ, and is reflected at an angle of reflection ϕ and frequency ν_2.

   (a) Show that

   $$\frac{\tan(\theta/2)}{\tan(\phi/2)} = \frac{c+v}{c-v} \qquad \text{and} \qquad \frac{\nu_2}{\nu_1} = \frac{c+v\cos\theta}{c-v\cos\phi}.$$

   (b) What happens if the mirror was moving parallel to its plane?

Exercise sheet 3

Tensors and transformations

1. *Four velocity and four acceleration:* The four acceleration of a relativistic particle is defined as $a^\mu = \mathrm{d}u^\mu/\mathrm{d}s$, where $u^\mu = \mathrm{d}x^\mu/\mathrm{d}s$ is the four velocity of the particle.

   (a) Express a^μ in terms of the three velocity $\mathbf{v}$ and the three acceleration $\mathbf{a} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t$ of the particle.

   (b) Evaluate $a^\mu\, u_\mu$ and $a^\mu\, a_\mu$ in terms of $\boldsymbol{v}$ and $\boldsymbol{a}$.

2. *The Lorentz force:* The action for a relativistic particle that is interacting with the electromagnetic field is given by

$$S[x^\mu(s)] = -m\,c \int \mathrm{d}s - \frac{e}{c} \int \mathrm{d}x_\mu\, A^\mu,$$

   where m is the mass of the particle, while e is its electric charge. The quantity $A^\mu = (\phi, \boldsymbol{A})$ is the four vector potential that describes the electromagnetic field, with, evidently, ϕ and $\mathbf{A}$ being the conventional scalar and three vector potentials.

   (a) Vary the above action with respect to x^μ to arrive at the following Lorentz force law:

$$m\,c\,\frac{\mathrm{d}u^\mu}{\mathrm{d}s} = \frac{e}{c}\, F^{\mu\nu}\, u_\nu,$$

   where u^μ is the four velocity of the particle and the electromagnetic field tensor $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} = \partial_\mu\, A_\nu - \partial_\nu\, A_\mu$$

   with $\partial_\mu \equiv \partial/\partial x^\mu$.

   (b) Show that the components of the field tensor $F_{\mu\nu}$ are given by

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix},$$

   where (E_x, E_y, E_z) and (B_x, B_y, B_z) are the components of the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ which are related to the components of the four vector potential by the following standard expressions:

$$\boldsymbol{E} = -\frac{1}{c}\,\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi \qquad \text{and} \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A}.$$

   (c) Express the above equation governing the motion of the charge in the more familiar three vector notation. What does the zeroth component of the equation describe?

3. *Reducing tensors to vectors:* If $X^\lambda{}_{\mu\nu}$ is a mixed tensor of rank $(1,2)$, show that the contracted quantity $Y_\mu = X^\nu{}_{\mu\nu}$ is a covariant vector.

4. *Transformation of electric and magnetic fields:* Consider two inertial frames, say, K and K', with the frame K' moving with a velocity v with respect to the frame K along the common x-axes.

   (a) Given the components of the electric and the magnetic fields, say, $\mathbf{E}$ and $\mathbf{B}$, in the frame K, using the transformation properties of the electromagnetic field tensor $F_{\mu\nu}$, construct the corresponding components in the frame K'.

   (b) Show that $|\boldsymbol{E}|^2 - |\boldsymbol{B}|^2$ is invariant under the Lorentz transformations.

   (c) Express the quantity $|\boldsymbol{E}|^2 - |\boldsymbol{B}|^2$ explicitly as a scalar in terms of the field tensor $F_{\mu\nu}$.

5. *The Lorentz invariant four volume:* Show that the differential spacetime volume $\mathrm{d}^4x = c\,\mathrm{d}t\,\mathrm{d}^3\boldsymbol{x}$ is a Lorentz invariant quantity.

Exercise sheet 4

Spacetime diagrams

1. *Axes of inertial frames I:* Consider two inertial frames K and K', with K' moving at the velocity v along their common, positive x-direction. Let the two frames coincide at $t = t' = 0$, and let us ignore the y and the z-directions for simplicity.

   (a) In the plane of the spacetime coordinates $(c\,t, x)$, determine the $c\,t'$ axis.
   Hint: This is essentially given by the trajectory of an observer located at, say, $x' = 0$, in the K' frame.

   (b) Consider a beam of light emitted by a source located $x = 0$ and is being reflected by a mirror at, say, $x = a$. Evidently, if the source emits the beam of light at $c\,t = -a$, it will return to the source, after being reflected by the mirror, at $c\,t = a$. Using this method and the fact that the velocity of light is the same in all inertial frames of reference, determine the x' axis in the $(c\,t, x)$ plane.

2. *Axes of inertial frames II:* In the previous exercise, you had determined $c\,t'$ and x' axes in the $(c\,t, x)$ plane.

   (a) Determine the angles between the $c\,t$ and $c\,t'$ axes as well as the x and x' axes.

   (b) Draw the $c\,t$ and x axes in the plane of the spacetime coordinates $(c\,t', x')$ and determine the angles involved.

3. *Invariant hyperbolae:* Consider two Lorentz frames as discussed in the previous two exercises.

   (a) Draw hyperbolae corresponding to different possible values of the Lorentz invariant quantity $c^2\,t^2 - x^2$ in the $(c\,t, x)$ plane.

   (b) Using the invariant hyperbolae, calibrate the $c\,t'$ and x' axes with the respect to the values on the $c\,t$ and x axes.

   (c) A line of simultaneity at a given point, say, P, on the $c\,t$ axis in the spacetime diagram is, evidently, described by the tangent to the hyperbola passing through that point. Draw a line of simultaneity in the $(c\,t, x)$ plane and illustrate the same line of simultaneity (along with the point P and the original hyperbola passing through it) in the $(c\,t', x')$ plane.

   (d) Show that the event P can be shifted anywhere on the hyperbola by working in a suitable Lorentz frame.

   (e) Argue that the tangent to the hyperbola at any event P is the line of simultaneity of the Lorentz frame whose time axis joins P to the origin of the spacetime diagram.

4. *Lorentz contraction:* Consider a rod of a given length at rest in the K' frame. Let one end of the rod be at $x = 0$ at $t = 0$.

   (a) As you have done earlier, draw the $c\,t'$ and x' axes in the $(c\,t, x)$ plane. Also, draw the trajectory of the two ends of the rod in the $(c\,t, x)$ plane.

   (b) From the geometry, express the length of the rod at time $t = 0$ in the K frame, in terms of the actual length of the rod in the K' frame.
   Note: You will find that the length of the rod measured at a given time in the K frame is smaller than its original length. This is Lorentz contraction.

5. *Global view of Minkowski spacetime:* Consider the following Minkowski line-element in $(1 + 1)$-spacetime dimensions:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}x^2.$$

(a) Show that, in terms of the null coordinates

$$u = c\,t - x, \quad v = c\,t + x,$$

the Minkowski line-element reduces to

$$\mathrm{d}s^2 = \mathrm{d}u\,\mathrm{d}v.$$

(b) Show that, if we perform the following coordinate transformation:

$$u' = 2\tan^{-1} u, \qquad v' = 2\tan^{-1} v,$$

where $-\pi \leq u' \leq \pi$ and $-\pi < v' \leq \pi$, then the Minkowski line-element can be expressed as

$$\mathrm{d}s^2 = \frac{1}{4}\sec^2(u'/2)\sec^2(v'/2)\,\mathrm{d}\bar{s}^2,$$

with $\mathrm{d}\bar{s}^2$ being given by

$$\mathrm{d}\bar{s}^2 = \mathrm{d}u'\,\mathrm{d}v'.$$

Note: The line-element $\mathrm{d}\bar{s}^2$ has the same structure as the original Minkowski line-element $\mathrm{d}s^2$. The above relation between $\mathrm{d}s^2$ and $\mathrm{d}\bar{s}^2$ is referred to as a conformal transformation. It is important to appreciate that it is *not* a coordinate transformation.

(c) Working with the line-element $\mathrm{d}\bar{s}^2$, identify the following points of the Minkowski coordinates in the u'-v' plane: (i) past and future time-like infinities ($t \to \mp\infty$, often referred to as i^- and i^+), (ii) space-like infinities ($x \to \mp\infty$ denoted as i^0), and (iii) past and future null-like infinities $[(u,v) \to -\infty$ and $(u,v) \to \infty$, denoted as $\mathcal{I}^-$ and $\mathcal{I}^+]$.

Note: The conformal transformation, even as it preserves the form of the Minkowski metric, brings in the infinities of the original Minkowski coordinates to finite values. This helps us create an image of the originally infinite Minkowski domain over a compact region. Such a diagram is known as a Penrose diagram.

(d) Do the the light cones in the Penrose diagram have the same shape as in the original Minkowski spacetime?

(e) Indicate how time-like trajectories behave in the diagram.

(f) Also, draw space-like surfaces corresponding to constant values for the time coordinate t.

Exercise sheet 5

Theories of real scalar fields

1. *Solutions to the Klein-Gordon equation:* Consider a scalar field ϕ obeying the following Klein-Gordon equation:

$$\left(\Box + \sigma^2\right)\,\phi = 0.$$

   (a) Write the solution to the scalar field as

$$\phi(\tilde{x}) = q_{\boldsymbol{k}}(t)\,\exp\left(i\,\boldsymbol{k}\cdot\boldsymbol{x}\right)$$

   and show that $q_{\boldsymbol{k}}(t)$ satisfies the equation of motion of a simple harmonic oscillator.

   Note: For convenience in notation, here and hereafter, we shall use $\tilde{x}$ to denote $x^\mu = (c\,t, \boldsymbol{x})$.

   (b) Determine the relation between the frequency ω of the oscillator, the wave vector $\boldsymbol{k}$ and the quantity σ.

   (c) Since ϕ is a scalar, the solution has to be a Lorentz invariant quantity. Express the solution in an explicitly Lorentz invariant form.

   (d) As the Klein-Gordon equation is a linear equation, a superposition of the individual solutions will also be a solution. Write down the most general solution possible, and express it in an explicitly Lorentz invariant manner.

2. *Green's functions:* Consider a real scalar field ϕ that is sourced by a charge density ρ. Such a scalar field would be governed by the following equation of motion:

$$\left(\Box + \sigma^2\right)\,\phi = \alpha\,\rho,$$

   where α is a quantity of suitable dimensions. This inhomogeneous partial differential equation can be solved using the method of Green's functions as follows.

   (a) Show that the inhomogeneous solution to the above equation can be expressed as

$$\phi(\tilde{x}) = \alpha\int \mathrm{d}^4\tilde{x}\;G(\tilde{x},\tilde{x}')\;\rho(\tilde{x}),$$

   where the Green's function $G(\tilde{x},\tilde{x}')$ satisfies the differential equation

$$\left(\Box_{\tilde{x}} + \sigma^2\right)\,G(\tilde{x},\tilde{x}') = \delta^{(4)}\left(\tilde{x}-\tilde{x}'\right).$$

   (b) Express the Green's function as a Fourier transform as

$$G(\tilde{x},\tilde{x}') = \int \frac{\mathrm{d}^4\tilde{k}}{(2\,\pi)^4}\;G(\tilde{k})\;\exp\left[i\,k^\mu\left(x_\mu - x'_\mu\right)\right],$$

   and substitute it into the above equation to determine the form of $G(\tilde{k})$.

3. *The retarded Green's function for a massless field:* Using the form of $G(\tilde{k})$, evaluate the above integral to determine the Green's function $G(\tilde{x},\tilde{x}')$ for a field with $\sigma = 0$.

   Note: As we have discussed, in units wherein $c = \hbar = 1$, σ has dimensions of mass.

4. *Conservation of the stress-energy tensor and the equation of motion:* Show that demanding the conservation of the stress-energy tensor of a scalar field leads to its equation of motion.

5. *An unconventional scalar field:* Consider a scalar field, say, ϕ, that is governed by the following unconventional action:

$$S[\phi(\tilde{x})] = \frac{1}{c} \int \mathrm{d}^4\tilde{x} \left[\left(\frac{X}{\alpha^2}\right)^n - V(\phi)\right],$$

where $\mathrm{d}^4\tilde{x} = c\,\mathrm{d}t\,\mathrm{d}^3\boldsymbol{x}$, the quantity X is given by

$$X = \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi,$$

while $V(\phi)$ denotes the potential describing the scalar field and α is a constant of suitable dimensions.

(a) Determine the dimension of ϕ/α.

(b) What is the equation of motion governing the scalar field?

(c) What is the corresponding stress-energy tensor?

A request

As you know, I will be conducting the first online quiz of my course tomorrow. In this regard, I have a request to all of you who are attending the course and will be taking the quiz. My request is simple: *kindly do not copy.*

Let me elaborate. In all my years of teaching, I have always adopted the following two methods for the quizzes and exams I have conducted: (1) I have allowed the students to keep their own handwritten notes with them, and (2) I have never been strict with the time limit. These methods have been motivated by my belief that the quizzes and exams should free of stress, allowing the students to think clearly and perform at their best. I recall an instance when a set of Ph.D. students took nine hours to complete an exam!

I would like to adopt the same methods for my online quiz as well. During the quiz, you are welcome to look up your own handwritten notes or materials such as the recorded lectures and the solutions to exercises that are available on Moodle. I would request you not to look up books or the internet. Needless to add, I would request you not to consult your classmates or anyone else for that matter. Evidently, it will not be possible for me to monitor all of you remotely. I do not intend to get you to sign an honor code or warn you of dire consequences. I would prefer to trust all of you. I am hoping that you will not break my trust.

Let us not lose sight of the fact that we are here to learn. I should mention that I too learn as I teach. You stand to benefit in your understanding of the topic by working through the problems in the quiz. I will be available online on our regular online platform during the period of the quiz. If you are having any difficulty, you are welcome to speak to me. I will be happy to assist you to the extent that it does not unduly benefit you when compared to the other students.

I am also hoping that you will be able to finish the quiz within the stipulated time. If many of you are having difficulty in completing the quiz in time, I will be glad to permit a little more time to complete it.

To repeat my request, *kindly do not copy* either from books, the internet or your classmates during the quiz.

Good luck.

Quiz I

Special relativity

1. *Repeated Lorentz boosts:* Consider a series of inertial observers, with each observer seeing the preceding observer moving away along the positive x-axis with the speed u. Let the speed of a particle (that is moving along the positive x-axis) in the frame of the zeroth observer be v_0, and let v_n be the speed of the particle as observed by the n-th observer.

   (a) Determine the relation between v_{n+1} and v_n. $\boxed{\text{3 marks}}$

   (b) What is speed v_n as $n \to \infty$? $\boxed{\text{7 marks}}$

2. (a) *Radial Doppler effect:* Consider a source of photons that is moving radially away with a velocity v from an observer who is at rest. Let ω_{E} and ω_{O} denote the frequency of the photons in the frames of the source and the observer, respectively. Obtain the relation between ω_{E} and ω_{O} in terms of the velocity v. $\boxed{\text{5 marks}}$

   (b) *Transverse Doppler effect:* What is the relation between ω_{E} and ω_{O} if the source is moving transversely to the direction of the photon, as it occurs, say, when the source is on a circular trajectory about the observer? $\boxed{\text{5 marks}}$

3. *Electron in an electric field:* An electron moving relativistically enters a region of constant electric field that is pointed along the positive y-axis. Let the relativistic three-momentum of the electron as it enters the region of the electric field at time, say, $t = 0$, be $\boldsymbol{p} = (p_x^0, p_y^0, 0)$.

   (a) Integrate the equation of motion describing the electron to determine p_x and p_y as function of time. $\boxed{\text{2 marks}}$

   (b) Express the energy of the electron in terms of $p_x(t)$ and $p_y(t)$. $\boxed{\text{2 marks}}$

   (c) From the equation governing the conservation of energy of the electron and the above expression for energy, arrive at the expression for v_y in terms of time. $\boxed{\text{2 marks}}$

   (d) Using the above results, arrive at the expression for v_x in terms of time. $\boxed{\text{2 marks}}$

   (e) Determine the asymptotic (i.e. the large time) behavior of v_x and v_y. $\boxed{\text{2 marks}}$

4. *Motion in a constant and uniform magnetic field:* Consider a particle of mass m and charge e that is moving in a magnetic field of strength B that is directed, say, along the positive z-axis.

   (a) Show that the energy $\mathcal{E} = \gamma\, m\, c^2$ of the particle is a constant. $\boxed{\text{3 marks}}$

   (b) Determine the trajectory $\boldsymbol{x}(t)$ of the particle and show that, in the absence of any initial momentum along the z-direction, the particle describes a circular trajectory in the x-y plane with the angular frequency $\omega = e\, c\, B/\mathcal{E}$. $\boxed{\text{7 marks}}$

5. *Nature of tensors:* Show that, if a tensor is symmetric or anti-symmetric in a coordinate system, it is so in all coordinate systems. $\boxed{\text{5+5 marks}}$

Exercise sheet 6

The case of the complex scalar field

1. *Two real scalar fields:* Consider the following action that describes two real scalar fields, say, ϕ_1 and ϕ_2:
$$S[\phi_1(\tilde{x}), \phi_2(\tilde{x})] = \frac{1}{c} \sum_{s=1,2} \int \mathrm{d}^4\tilde{x} \, \left(\frac{1}{2}\,\eta_{\mu\nu}\,\partial^\mu\phi_s\,\partial^\nu\phi_s - \frac{1}{2}\,\sigma^2\,\phi_s^2\right).$$
   (a) Vary the action with respect to ϕ_1 and ϕ_2 to arrive at the equations of motion.
   (b) Evaluate the stress-energy tensor of the complete system.

2. *Variation of the action governing a complex scalar field:* As we have discussed, the above system involving two real scalar fields can be described in terms of a single *complex* scalar field, say, ϕ, that is defined as
$$\phi = (\phi_1 + i\,\phi_2)\,.$$
   (a) Express the above action for the fields ϕ_1 and ϕ_2 in terms of the complex field ϕ.
   (b) Vary the action to arrive at the equations of motions describing ϕ and ϕ^*.

3. *From the conservation of the stress-energy tensor to the equations of motion:* Consider a complex scalar field that is described by the canonical kinetic term, but is governed by an arbitrary potential energy density of the following form: $V(|\phi|^2)$, where $|\phi|^2 = \phi\,\phi^*$.
   (a) Obtain the equations of motion for the field ϕ and its complex conjugate in such a case.
   (b) Construct the stress-energy tensor associated with the complex scalar field and show that the conservation of the stress-energy tensor also leads to these equations of motion.

4. *Non-linear Schrodinger equation, the Madelung transformation and superfluids:* Consider a complex wavefunction $\psi(t, \boldsymbol{x})$ that describes a non-relativistic, quantum mechanical system that is governed by a non-linear Schrodinger equation of the following form:
$$i\,\hbar\,\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2}\,\nabla^2\psi + \frac{\partial V\left(|\psi|^2\right)}{\partial\psi^*},$$
where the quantity $V\left(|\psi|^2\right)$ describes self-interactions whose nature can be very similar to those in the case of the complex scalar field discussed in the previous exercise.
   (a) Express the wavefunction $\psi(t, \boldsymbol{x})$ as
$$\psi(t, \boldsymbol{x}) = \sqrt{\rho(t, \boldsymbol{x})}\,\exp\left[i\,\chi(t, \boldsymbol{x})/\hbar\right]$$
   and show that the imaginary part of the above non-linear Schrodinger equation reduces to the following continuity equation that describes a non-relativistic fluid with density ρ and velocity $\boldsymbol{v}$:
$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\,\boldsymbol{v}) = 0,$$
   where $\boldsymbol{v} = \nabla\chi$.
   Note: The functions $\rho(t, \boldsymbol{x})$ and $\chi(t, \boldsymbol{x})$ are real quantities.
   (b) From the real part of the equation, also arrive at the following Euler equation governing the fluid:
$$\frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v} + \nabla f(\rho) = \frac{\hbar^2}{2}\,\nabla\left(\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right),$$
   where $f(\rho) = \mathrm{d}V(\rho)/\mathrm{d}\rho$.

Note: The transformation above which helps in reducing the non-linear Schrodinger equation to the hydrodynamical form is referred to as the Madelung transformation. It is essentially due to this reason that systems such as superfluids can be described by the above non-linear Schrodinger equation, which, for suitable forms of $V\left(|\psi|^2\right)$, is known as the Gross-Pitaevskii equation.

5. *A complex scalar field in an external electric field:* Consider a complex scalar field, say, ϕ, that is propagating in a constant and uniform electric field background. Let the strength of the electric field be E, and let it be pointed towards the positive x-direction. Such an electric field can be described by either the vector potential $A_1^\mu = (-E\,x, 0, 0, 0)$ or by the potential $A_2^\mu = (0, -E\,c\,t, 0, 0)$.

   (a) Obtain the equation of motion that governs the scalar field in the two gauges.

   (b) Using the method of separation of variables to solve a partial differential equation, arrive at the differential equations governing the modes along the time coordinate and the three spatial directions.

   (c) Since the gauge A_1^μ is independent of the t, y and the z coordinates, the modes along these directions can be easily arrived at. Express these modes in terms of simple functions. Can you identify the nature of the modes along the x-direction?

   (d) Carry out the corresponding exercise in the gauge A_2^μ.

Additional exercises I

From special relativity to the case of the complex scalar field

1. *Algebra of the infinitesimal generators of the Lorentz group:* Recall that the infinitesimal generators of rotation and the Lorentz transformations, viz. (J_x, J_y, J_z) and (K_x, K_y, K_z), respectively, can be written in a matrix form as follows:

$$J_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$K_x = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_y = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_z = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}.$$

Utilizing these representations, establish the following commutation relations:

$$[J_i, J_j] = i\,\epsilon_{ijk}\,J_k, \quad [K_i, K_j] = -i\,\epsilon_{ijk}\,J_k \quad \text{and} \quad [J_i, K_j] = i\,\epsilon_{ijk}\,K_k,$$

where ϵ_{ijk} represents the completely anti-symmetric tensor and, as is our convention, the Latin indices i, j, k, take values $(1, 2, 3)$.

2. *The generators as differential operators:* As we had discussed, the generators (J_x, J_y, J_z) and (K_x, K_y, K_z) can also be represented in the differential form as follows:

$$J_x = -i\left(y\,\frac{\partial}{\partial z} - z\,\frac{\partial}{\partial y}\right), \quad J_y = -i\left(z\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial z}\right), \quad J_z = -i\left(x\,\frac{\partial}{\partial y} - y\,\frac{\partial}{\partial x}\right),$$

and

$$K_x = i\left(c\,t\,\frac{\partial}{\partial x} + \frac{x}{c}\,\frac{\partial}{\partial t}\right), \quad K_y = i\left(c\,t\,\frac{\partial}{\partial y} + \frac{y}{c}\,\frac{\partial}{\partial t}\right), \quad K_z = i\left(c\,t\,\frac{\partial}{\partial z} + \frac{z}{c}\,\frac{\partial}{\partial t}\right).$$

Use these representations to establish all the commutation relations listed in the previous exercise.

3. *The Lorentz group in terms of Pauli matrices:* Construct a representation of the Lorentz group, consisting of the six infinitesimal generators (J_x, J_y, J_z) and (K_x, K_y, K_z), in terms of the following Pauli matrices and the unit matrix:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note: This is known as the complex uni-modular matrix representation of the Lorentz group.

4. *Thomas rotation:* Show that two successive, arbitrary, Lorentz boosts, say, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, is equivalent to a pure boost, say, $\boldsymbol{v}_3$, followed by a pure rotation, say, $\theta\,\hat{n}$, where $\hat{n}$ is the unit vector along the axis of rotation. Determine the angle θ in terms of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, and also establish that $\hat{n}\cdot\boldsymbol{v}_3 = 0$.

Hint: The easiest way to solve this problem would be to make use of the uni-modular representation of the Lorentz group discussed in the previous exercise.

Note: The rotation involved is known as Thomas (or, occasionally, Wigner) rotation.

5. *The Poincare group:* As we have discussed, the Poincare group consists of the operators (J_x, J_y, J_z) and (K_x, K_y, K_z), which generate rotations and Lorentz transformations, as well as the generators of translations, which can be represented as $P_\mu \equiv i\,\partial_\mu$. Let us represent the six elements of the Lorentz group as the components of an anti-symmetric tensor $J_{\mu\nu}$ as follows:

$$J_{\mu\nu} = \begin{cases} J_{0i} &= -J_{i0} = K_i, \\ J_{ij} &= -J_{ji} = i\,\epsilon_{ijk}\,J_k. \end{cases}$$

Using the representations of $J_{\mu\nu}$ and P_μ in terms of differential operators, establish the following commutation relations:

$$\begin{aligned} [J_{\mu\nu}, J_{\rho\sigma}] &= i\,(\eta_{\nu\rho}\,J_{\mu\sigma} - \eta_{\mu\rho}\,J_{\nu\sigma} + \eta_{\mu\sigma}\,J_{\nu\rho} - \eta_{\nu\sigma}\,J_{\mu\rho})\,, \\ [P_\mu, J_{\rho\sigma}] &= i\,(\eta_{\mu\rho}\,P_\sigma - \eta_{\mu\sigma}\,P_\rho)\,. \end{aligned}$$

6. *The retarded and the advanced Green's functions:* Recall that, the Green's function associated with a massless field can be expressed as

$$G(\tilde{x}, \tilde{x}') = \int \frac{\mathrm{d}^4\tilde{k}}{(2\,\pi)^4}\, G(\tilde{k})\, \exp -\left[i\,k^\mu\,(x_\mu - x'_\mu)\right],$$

with $G(\tilde{k})$ being given by

$$G(\tilde{k}) = -\frac{1}{k^\mu\,k_\mu}.$$

It is important to notice that the phase factor in the exponential now contains an overall minus sign in contrast to the expression which we had worked with before [see Exercise sheet 5, Exercise 3 (b)]. Both the representations are indeed correct, and they are simply a matter of convention.

(a) Show that, in such a case, when $t > t'$, the poles on the real k^0 axis (at $k = \pm|\boldsymbol{k}|$) now need to be provided a small and *negative* imaginary part (i.e. they have to be pushed downwards instead of upwards as we had done earlier) and the contour has to be closed in the lower half of the the complex k^0-plane to arrive at the retarded Green's function, say, $D_{\mathrm{ret}}(\tilde{x}, \tilde{x}')$.

Note: As we had discussed, the spacetime coordinates $\tilde{x}$ denote the point of observation, while the $\tilde{x}'$ represent the position of the source. Hence, the condition $t > t'$ essentially implies causality. Also, it should pointed out that the Green's functions associated with massless fields are often denoted as $D(\tilde{x}, \tilde{x}')$.

(b) Assuming $t < t'$ and pushing the poles upward by providing them with a small and *positive* imaginary part, arrive at the so-called advanced Green's function $D_{\mathrm{adv}}(\tilde{x}, \tilde{x}')$ by closing the contour in the upper half of the complex k^0-plane.

Note: You will find that the Green's function $D_{\mathrm{adv}}(\tilde{x}, \tilde{x}')$ is non-zero only along the *past* light cone.

(c) Establish that the difference between the retarded and the advanced Green's functions satisfies the homogeneous wave equation.

7. *Retarded Green's function in the presence of a boundary:* Consider a free, real and massless scalar field, say, ϕ. Let the field vanish on a boundary that is located on the $x = 0$ plane. i.e. $\phi(c\,t, x = 0, y, z) = 0$ for all t, y and z. Determine the retarded Green's function associated with the field in such a case.

8. *The retarded Green's function in terms of spherical polar coordinates:* Consider a situation wherein you are required to work in terms of the spherical polar coordinates instead of the more conventional and, not to mention, convenient, Cartesian coordinates. Express the Green's function as a suitable integral and sum over the modes associated with the d'Alembertian in the spherical polar coordinates, and carry out the sums and integrals involved to arrive at the standard result for the retarded Green's function of a massless field.

9. *Complex scalar fields and gauge transformations:* We had earlier considered a complex scalar field propagating in a constant and uniform electric field background described by either the vector potential $A_1^\mu = (-E\,x, 0, 0, 0)$ or by the potential $A_2^\mu = (0, -E\,c\,t, 0, 0)$.

   (a) Construct the gauge transformation that takes one from the gauge A_1^μ to the gauge A_2^μ.

   (b) Determine how the scalar field transforms as one moves from one gauge to the other.

   (c) Explicitly check that the transformed solution indeed satisfies the equation of motion in the new gauge.

10. *Multiple scalar fields:* Consider a system involving N scalar fields that is described by the canonical kinetic term and a potential. Write down the action of such a system assuming it is invariant under the global transformations of the $SO(N)$ group.

Exercise sheet 7

Symmetries and conservation laws

1. *Conserved current in the presence of the electromagnetic field:* We had earlier derived the conserved current associated with the symmetry of a complex scalar field, say, ϕ, under global gauge transformations. Recall that, the action governing a massive, complex scalar field that is interacting with the electromagnetic field described by the vector potential A^μ can be written as

$$S[\phi(\tilde{x}), A^\mu(\tilde{x})] = \frac{1}{c} \int \mathrm{d}^4\tilde{x}\ \left[(D_\mu\phi)\ (D^\mu\phi)^* - \sigma^2\,\phi\,\phi^*\right],$$

where the quantity $D_\mu\phi$ is given by

$$D_\mu\phi = \partial_\mu\phi + i\,e\,A_\mu\,\phi,$$

with e denoting the coupling constant.

   (a) Show that the above action is invariant under local gauge transformations of the form

$$\phi(\tilde{x}) \to \exp - [i\,e\,\Lambda(\tilde{x})]\ \phi(\tilde{x}) \qquad \text{and} \qquad A_\mu(\tilde{x}) \to A_\mu(\tilde{x}) + \partial_\mu\Lambda(\tilde{x}).$$

   (b) Determine the conserved current associated with this local symmetry.

2. *Scale invariance:* Consider a real scalar field ϕ that is governed by the action

$$S[\phi(\tilde{x})] = \frac{1}{c} \int \mathrm{d}^4\tilde{x}\ \left(\frac{1}{2}\,\partial^\mu\phi\,\partial_\mu\phi - \lambda\,\phi^4\right).$$

   (a) Show that this action is invariant under the following scale transformations:

$$x^\mu \to b\,x^\mu \quad \text{and} \quad \phi \to \phi/b,$$

   where b is a constant.

   (b) What is the conserved current associated with this symmetry?

   (c) Explicitly show that the four divergence of the conserved current vanishes.

   (d) Can you identify the reason for the existence of such a symmetry?

3. *The conserved charges associated with scalar fields interacting with the Yang-Mills field:* Consider a three component scalar field, say, $\boldsymbol{\phi} \equiv (\phi_1, \phi_2, \phi_3)$, that is interacting with the Yang-Mills field described by the gauge potential $\boldsymbol{W}_\mu$, and is governed by the action

$$S[\boldsymbol{\phi}(\tilde{x}), \boldsymbol{W}_\mu(\tilde{x})] = \frac{1}{c} \int \mathrm{d}^4\tilde{x}\ \left(\frac{1}{2}\,D_\mu\boldsymbol{\phi}\cdot D^\mu\boldsymbol{\phi} - \frac{\sigma^2}{2}\,\boldsymbol{\phi}\cdot\boldsymbol{\phi}\right),$$

where $D_\mu\boldsymbol{\phi}$ is given by

$$D_\mu\boldsymbol{\phi} = \partial_\mu\boldsymbol{\phi} + g\,\boldsymbol{W}_\mu \times \boldsymbol{\phi},$$

with g being the coupling constant. As we have discussed, the above action is invariant under local rotations (i.e. when the extent of rotation, say, $\boldsymbol{\Lambda}$, is dependent on the spacetime coordinates) in the internal field space and the following transformations of the gauge potential $\boldsymbol{W}_\mu$:

$$\boldsymbol{W}_\mu \to \boldsymbol{W}_\mu - \boldsymbol{\Lambda} \times \boldsymbol{W}_\mu + \frac{1}{g}\,\partial_\mu\boldsymbol{\Lambda}.$$

   (a) Construct the conserved current associated with the symmetry.

   (b) How many conserved charges are associated with the symmetry?

Note: Actually, $\boldsymbol{\Lambda} = \Lambda\,\hat{n}$, where Λ denotes the angle of rotation, while $\hat{n}$ is the unit vector along the axis of rotation.

4. *Gauge invariance and electromagnetism:* As you may know, the action describing a free electromagnetic field (which we would also formally discuss in due course) is given by

$$S[A^\mu(\tilde{x})] = -\frac{1}{16\,\pi\,c} \int \mathrm{d}^4\tilde{x}\; F_{\mu\nu}\,F^{\mu\nu},$$

where $F_{\mu\nu}$ is the field tensor defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

As should be evident, the electromagnetic field tensor $F_{\mu\nu}$ is invariant under gauge transformations of the form

$$A_\mu \to A_\mu + \partial_\mu \Lambda.$$

(a) Determine the conserved current associated with this gauge symmetry.

(b) What are the corresponding conserved charges?

5. *Traceless stress-energy tensors:* Suppose that the action describing a field ϕ is found to be invariant under spacetime translations as well as the following dilatations:

$$x^\mu \to b\,x^\mu \quad \text{and} \quad \phi \to \phi,$$

where b is a constant. Show that, in such a case, the trace of the corresponding stress-energy tensor vanishes.

Note: The exercise in red could not be discussed for want of time.

Quiz II

From real and complex scalar fields to symmetries and conservation laws

1. <u>*Fields as a collection of oscillators:*</u> Consider a real scalar field ϕ described by the action

$$S[\phi(\tilde{x})] = \frac{1}{c} \int \mathrm{d}^4\tilde{x} \left[\frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}\,\sigma^2\,\phi^2\right].$$

Let us write

$$\phi(\tilde{x}) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\,\pi)^{3/2}}\, q_{\boldsymbol{k}}(t)\,\mathrm{e}^{i\,\boldsymbol{k}\cdot\boldsymbol{x}} = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\,\pi)^{3/2}} \left[q_{\boldsymbol{k}}^{\mathrm{R}}(t) + i\,q_{\boldsymbol{k}}^{\mathrm{I}}(t)\right]\,\mathrm{e}^{i\,\boldsymbol{k}\cdot\boldsymbol{x}},$$

where, evidently, we have set $q_{\boldsymbol{k}}(t) = q_{\boldsymbol{k}}^{\mathrm{R}}(t) + i\,q_{\boldsymbol{k}}^{\mathrm{I}}(t)$, with $q_{\boldsymbol{k}}^{\mathrm{R}}(t)$ and $q_{\boldsymbol{k}}^{\mathrm{I}}(t)$ being real quantities.

(a) Substitute such a Fourier decomposition of the scalar field in the above action and express the action in terms of $q_{\boldsymbol{k}}$ and its time derivative. **[5 marks]**

(b) Using the fact that the scalar field is a real quantity, determine the relation between the quantities $q_{\boldsymbol{k}}$ and $q_{-\boldsymbol{k}}$. **[3 marks]**

(c) Vary the action to obtain the equations of motion satisfied by $q_{\boldsymbol{k}}^{\mathrm{R}}$ and $q_{\boldsymbol{k}}^{\mathrm{I}}$. **[2 marks]**

2. <u>*Components of the stress-energy tensor:*</u> Consider a canonical, real scalar field, say, ϕ, of mass σ (in suitable units). Recall that, such a scalar field can be decomposed in terms of the normal modes $u_{\boldsymbol{k}}(\tilde{x})$ as follows:

$$\phi(\tilde{x}) = \int \mathrm{d}^3\boldsymbol{k}\,\left[a_{\boldsymbol{k}}\,u_{\boldsymbol{k}}(\tilde{x}) + a_{\boldsymbol{k}}^{*}\,u_{\boldsymbol{k}}^{*}(\tilde{x})\right],$$

where the $a_{\boldsymbol{k}}$'s are, in general, $\boldsymbol{k}$-dependent constants and the modes $u_{\boldsymbol{k}}(\tilde{x})$ are given by

$$u_{\boldsymbol{k}}(\tilde{x}) = \frac{1}{\sqrt{(2\,\pi)^3\,(2\,\omega)}}\,\exp-(i\,k_\mu\,x^\mu) = \frac{1}{\sqrt{(2\,\pi)^3\,(2\,\omega)}}\,\exp-\left[i\,(\omega\,t - \boldsymbol{k}\cdot\boldsymbol{x})\right]$$

with, evidently, $k^\mu = (\omega/c, \boldsymbol{k})$. Moreover, note that, in such a case, $\omega/c = \left(|\boldsymbol{k}|^2 + \sigma^2\right)^{1/2}$, with ω being assumed to be a positive definite quantity.

Let $a_{\boldsymbol{k}} = \alpha\,\delta^{(3)}\,(\boldsymbol{k} - \boldsymbol{p})$, where α is real constant, while $\boldsymbol{p}$ is a constant vector.

(a) Express the stress-energy tensor of the scalar field associated with the above $a_{\boldsymbol{k}}$ completely in terms of the four vector p^μ. **[4 marks]**

(b) Explicitly write down the various components of the stress-energy tensor in terms of the components of the four vector p^μ. **[6 marks]**

3. <u>*Green's functions for a damped oscillator:*</u> Consider a damped, forced, one-dimensional simple harmonic oscillator satisfying the equation

$$\ddot{x} + 2\,\gamma\,\dot{x} + \omega_0^2\,x = f(t),$$

where ω_0 is the frequency of the oscillator, γ is the damping constant, and $f(t)$ is the forcing term.

(a) Determine the Green's function for the following situations: (i) under-damped case wherein $\gamma < \omega_0$, (ii) over-damped case wherein $\gamma > \omega_0$ and (iii) the critically damped case wherein $\gamma = \omega_0$. **[2+2+3 marks]**

(b) Plot these Green's functions. **[1+1+1 marks]**

4. *Equation of motion for a Galileon field:* Consider a scalar field ϕ described by the action

$$S[\phi(\tilde{x})] = \frac{1}{c}\int \mathrm{d}^4\tilde{x}\,\mathcal{L}(\phi, \partial_\mu\phi, \partial_\mu\partial_\nu\phi).$$

Note that, apart from the usual dependence on the scalar field ϕ and its first derivative $\partial_\mu\phi$, the Lagrangian density $\mathcal{L}$ depends on the second derivatives $\partial_\mu\partial_\nu\phi$.

(a) Derive the Euler-Lagrange equation for the system. 6 marks

Note: In order to obtain the Euler-Lagrange equation, the variation of the field *as well as* its first derivative need to be set to zero at the initial and final hypersurfaces.

(b) Obtain the equation of motion for a system described by the following Lagrangian density: 4 marks

$$\mathcal{L} = \partial_\mu\phi\,\partial^\mu\phi\,\partial_\nu\partial^\nu\phi.$$

Note: You will find that the equation of motion does not contain derivatives of the field higher than the second, despite the fact that the Lagrangian density itself involves second derivatives of the field. Such scalar fields are referred to as Galileons.

5. *Multiple scalar fields:* Consider a system involving N real scalar fields that is described by the canonical kinetic term and a potential. Let the N-dimensional vector $\boldsymbol{\phi}$ denote the scalar fields.

(a) Write down the action of such a system (in terms of $\boldsymbol{\phi}$) assuming that it is invariant under global rotations in the internal space. 3 marks

(b) Obtain the equations of motion of the scalar fields. 3 marks

(c) Determine the conserved current associated with the global internal symmetry. How many independent conserved currents are associated with the internal symmetry 4 marks

Exercise sheet 8

The theory of the electromagnetic field I

1. *Equivalence of actions under gauge transformations:* Recall that the action governing the electromagnetic field described by the vector potential A_μ that is interacting with the four current j^μ is given by
$$S[A^\mu(\tilde{x})] = -\frac{1}{c^2} \int \mathrm{d}^4\tilde{x}\, j^\mu\, A_\mu - \frac{1}{16\,\pi\,c} \int \mathrm{d}^4\tilde{x}\, F_{\mu\nu}\, F^{\mu\nu},$$
where $F_{\mu\nu}$ is the field tensor defined as
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$
Since $F_{\mu\nu}$ is explicitly invariant under the gauge transformation
$$A_\mu \to A_\mu + \partial_\mu \Lambda,$$
evidently, the second term in the above action is invariant as well. Determine if the first term transforms to an equivalent action under the gauge transformation.

2. *The spatial components of the stress-energy tensor of the free electromagnetic field:* We had arrived at the forms of the time-time and the time-space components of the stress-energy tensor of the free electromagnetic field in terms of the components of the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$. Arrive at the corresponding expressions for the purely spatial components of the stress-energy tensor.

   Note: These components are usually referred to as the Maxwell stress tensor.

3. *From source free Maxwell's equations to the conservation of the stress-energy tensor:* Establish that the source free Maxwell's equations imply that the stress-energy tensor of the free electromagnetic field is conserved.

4. *Conservation of the stress-energy tensor in the presence of sources:* The above exercise had involved the electromagnetic field in the absence of charges. If the charges are also present, then it is the sum of the stress-energy tensors of the charges as well as the field that will be conserved. The stress-energy tensor of a collection of mutually non-interacting particles can be written as
$$T^{\mu\nu}_{\mathrm{P}} = \mu\, c\, u^\mu\, u^\nu\, \frac{\mathrm{d}s}{\mathrm{d}t},$$
where μ is the mass density associated with the particles, while u^μ denotes the four velocity of the particles.

   Note: The above expression for the stress-energy tensor for a collection of mutually non-interacting particles is equivalent to a pressureless relativistic fluid. Often, such a system is referred to as 'dust'.

   (a) Show that, upon using the second pair Maxwell's equations, in the presence of sources, the stress-energy of the electromagnetic field, say, $T^{\mu\nu}_{\mathrm{F}}$, satisfies the equation
$$\partial_\mu T^{\mu\nu}_{\mathrm{F}} = -\frac{1}{c}\, F^{\nu\lambda}\, j_\lambda.$$

   (b) As in the case of charges, the continuity equation corresponding to the mass flow can be expressed as follows:
$$\partial_\mu \left(\mu\, \frac{\mathrm{d}x^\mu}{\mathrm{d}t}\right) = 0.$$

Using this equation and the following Lorentz force law:

$$\mu\, c\, \frac{\mathrm{d}u^\mu}{\mathrm{d}s} = \frac{\rho}{c}\, F^{\mu\nu}\, u_\nu,$$

where ρ denotes the charge density of the particles, show that

$$\partial_\mu T^{\mu\nu}_{\mathrm{P}} = \frac{1}{c}\, F^{\mu\lambda}\, j_\lambda$$

so that the total stress-energy tensor of the system, viz. $T^{\mu\nu} = T^{\mu\nu}_{\mathrm{P}} + T^{\mu\nu}_{\mathrm{F}}$, is conserved, as required.

5. *Traceless nature of the stress-energy tensor of the electromagnetic field:* Show that the trace of the stress-energy tensor of the electromagnetic field vanishes. Can you identify the reason behind the vanishing trace?

Note: The exercise in red could not be discussed for want of time.

Exercise sheet 9

The theory of the electromagnetic field II

1. *The Coulomb gauge and the degrees of freedom of the electromagnetic field:* Recall that the Lorenz gauge was determined by the covariant condition $\partial_\mu A^\mu = 0$. However, even the Lorenz condition does not uniquely fix the gauge. Further gauge transformations of the form $A_\mu \to A_\mu + \partial_\mu \Lambda$ are possible, provided Λ satisfies the condition $\Box \Lambda = 0$. In such a situation, often, one breaks Lorentz covariance and works in the so-called Coulomb gauge wherein A^t is set to zero, so that the Lorenz condition reduces to $\nabla \cdot \mathbf{A} = 0$.

   (a) Show that this implies that the free electromagnetic field possesses two independent degrees of freedom.

   (b) What do these two degrees of freedom correspond to?

2. *The massive vector field:* Consider the following action that governs a massive vector field $\mathcal{A}_\mu$:

$$S[\mathcal{A}^\mu(\tilde{x})] = \frac{1}{16\,\pi\,c} \int \mathrm{d}^4\tilde{x}\, \left(-\mathcal{F}_{\mu\nu}\,\mathcal{F}^{\mu\nu} + 2\,\sigma^2\,\mathcal{A}^\mu\,\mathcal{A}_\mu\right),$$

   where $\mathcal{F}_{\mu\nu}$ represents the field tensor defined in the usual form, viz.

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu,$$

   and σ has dimensions of mass in suitable units.

   (a) Obtain the equation of motion governing the field $\mathcal{A}_\mu$.

   (b) Show that the Lorentz condition, viz. $\partial_\mu \mathcal{A}^\mu = 0$, *has* to be satisfied by the field apart from satisfying the equation of motion.

   (c) Is the action invariant under gauge transformations of the form $\mathcal{A}_\mu \to \mathcal{A}_\mu + \partial_\mu \Lambda$?

   (d) How many independent degrees of freedom does the massive field $\mathcal{A}^\mu$ possess?

   Note: The massive vector field $\mathcal{A}^\mu$ is known as the Proca field.

3. *The Lienard-Wiechart potentials:* Consider a point particle with charge e that is moving along the trajectory $r^\mu(\tau)$, where τ is the proper time in the frame of the charge. The four current associated with the charge is given by

$$j^\mu(\tilde{x}) = e\,c^2 \int \mathrm{d}\tau\; u^\mu\; \delta^{(4)}\left[\tilde{x} - \tilde{r}(\tau)\right],$$

   where $\tilde{r}(\tau) \equiv r^\mu(\tau)$ is the trajectory of the charge and $u^\mu = \mathrm{d}r^\mu/\mathrm{d}s$ is its four velocity, so that the corresponding charge and current densities are given by

$$\rho(\tilde{x}) = e\;c\;\delta^{(3)}[\boldsymbol{x} - \boldsymbol{r}(t)] \quad \text{and} \quad \boldsymbol{j}(\tilde{x}) = e\;\boldsymbol{v}(t)\;\delta^{(3)}[\boldsymbol{x} - \boldsymbol{r}(t)]$$

   with $\boldsymbol{v}(t) = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$, as required. In the Lorenz gauge, the electromagnetic vector potential A^μ satisfies the equation

$$\Box A^\mu = \frac{4\,\pi}{c}\,j^\mu.$$

   (a) Using the retarded Green's function for a massless field that we had obtained earlier, solve the above equation to arrive at the following expression for the vector potential A^μ:

$$A^\mu(\tilde{x}) = \frac{e\;u^\mu}{R_\mu\,u^\mu},$$

   where $R^\mu = x^\mu - r^\mu$ and $R_\mu\,R^\mu = 0$.

(b) Show that the above vector potential A^μ can be written in the three dimensional form as

$$\phi(\tilde{x}) = \frac{e}{R - (\boldsymbol{v}\cdot\boldsymbol{R})/c} \quad \text{and} \quad \boldsymbol{A}(\tilde{x}) = \frac{e\,\boldsymbol{v}/c}{R - (\boldsymbol{v}\cdot\boldsymbol{R})/c},$$

where $\boldsymbol{R} = \boldsymbol{x} - \boldsymbol{r}$ and $R = |\boldsymbol{R}|$, with the right hand sides evaluated at the so-called retarded time determined by the condition $R^\mu R_\mu = 0$.

Note: These are known as the Lienard-Wiechart potentials.

4. *The radiation field:* Using the above Lienard-Wiechart potentials, obtain the following expressions for the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ generated by a point charge that is moving along an arbitrary trajectory:

$$\begin{aligned}
\boldsymbol{E} &= \frac{e}{\gamma^2\,\mu^3\,R^2}\,[\hat{\boldsymbol{n}} - (\boldsymbol{v}/c)] + \frac{e}{c^2\,\mu^3\,R}\,[\hat{\boldsymbol{n}} \times ([\hat{\boldsymbol{n}} - (\boldsymbol{v}/c)] \times \boldsymbol{a})]\,,\\
\boldsymbol{B} &= \hat{\boldsymbol{n}} \times \boldsymbol{E},
\end{aligned}$$

where γ is the standard Lorentz factor, $\boldsymbol{a} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t$ is the acceleration of the charge, while the quantity μ is given by

$$\mu = \left(1 - \frac{\boldsymbol{v}\cdot\hat{\boldsymbol{n}}}{c}\right)^{-1},$$

with $\hat{\boldsymbol{n}} = \boldsymbol{R}/R$.

Note: The contribution to the electric and the magnetic fields above which depends on the acceleration of the charge and behaves as $1/R$ with distance is known as the radiation field.

5. *Relativistic beaming:* Recall that the flux of energy being carried by electromagnetic radiation is described by the Poynting vector, viz.

$$\boldsymbol{S} = \frac{c}{4\,\pi}\,(\boldsymbol{E} \times \boldsymbol{B})\,.$$

When $\boldsymbol{B} = \hat{\boldsymbol{n}} \times \boldsymbol{E}$, the amount of energy, say, $\mathrm{d}\mathcal{E}$, that is propagating into a solid angle $\mathrm{d}\Omega$ in unit time is then given by

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}\Omega\,\mathrm{d}t} = |\boldsymbol{S}|\,R^2 = \frac{c\,|\boldsymbol{E}|^2\,R^2}{4\,\pi}.$$

(a) Upon using the above expressions for the radiative component of the electric field, show that the energy emitted by a point charge per unit time within a unit solid angle can be written as

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t\,\mathrm{d}\Omega} = \frac{e^2}{4\,\pi\,c^3}\left[2\,\mu^5\,(\hat{\boldsymbol{n}}\cdot\boldsymbol{a})\,(\boldsymbol{v}\cdot\boldsymbol{a}/c) + \mu^4\,a^2 - \mu^6\,\gamma^{-2}\,(\hat{\boldsymbol{n}}\cdot\boldsymbol{a})^2\right].$$

(b) Clearly, the intensity of the radiation is the largest along directions wherein $\mu \gg 1$. Show that, if θ is the angle between $\boldsymbol{v}$ and $\hat{\boldsymbol{n}}$, then, for $\theta \ll 1$ and $|\boldsymbol{v}| \simeq c$, we can write

$$\mu = \frac{2\,\gamma^2}{1 + \gamma^2\,\theta^2}.$$

(c) Argue that, for $\gamma \gg 1$, this expression is sharply peaked around $\theta = 0$, with a width $\Delta\theta \simeq \gamma^{-1}$.

Note: This effect, where most of the intensity is pointed along the direction of velocity of the charge, is known as relativistic beaming.

Note: The exercises in red could not be discussed for want of time.

Exercise sheet 10

Manifolds, coordinates and geometry

1. *Non-degenerate coordinate patches for $\mathbb{S}^2$:* Recall that, the usual angular coordinates, viz. θ and ϕ, that describe the two-sphere $\mathbb{S}^2$ in three-dimensional Euclidean space are pathological at the poles, since the metric coefficients vanish at these points. Usually, the sphere is covered with the aid of two coordinate patches arrived through a stereographic projection. In such a projection, one assigns coordinates, say, (ρ, ϕ), to each point on the sphere, with ϕ being the standard azimuthal angle. In one of the coordinate patches, the coordinate ρ of each point is arrived at by drawing a straight line in three dimensions from the south pole of the sphere through the point in question and extending the line until it intersects the tangent plane to the north pole of the sphere. The ρ-coordinate is then the distance in the tangent plane from the north pole to the point of intersection.

   (a) Show that the line-element describing the surface of the sphere in terms of these coordinates is given by
   $$\mathrm{d}\ell^2 = \frac{1}{[1+\rho^2/(4\,R^2)]^2}\,\left(\mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\phi^2\right),$$
   where R is the radius of the sphere. At what point(s) on the sphere are these coordinates degenerate?

   (b) What is the line-element of the sphere if, instead of working with the ρ and ϕ coordinates, one works with the Cartesian coordinates, say, x and y, in the tangent plane at the north pole? Are there any point(s) on the sphere at which these new coordinates are degenerate?

   (c) Construct the coordinates of the second patch in order to cover the sphere completely.

2. *Mercator's projection:* Consider the surface of the Earth, which we shall assume, for simplicity, to be a two-sphere of radius, say, R. In terms of the standard polar coordinates (θ, ϕ), the longitude of a point, in radians, rather than the usual degrees, is simply ϕ (measured eastwards from the Greenwich meridian), whereas its latitude is $\lambda = \pi/2 - \theta$ radians.

   (a) Show that the line-element on the Earth's surface in these coordinates is given by
   $$\mathrm{d}\ell^2 = R^2\,\left(\mathrm{d}\lambda^2 + \cos^2\lambda\,\mathrm{d}\phi^2\right).$$

   (b) In order to make a map of the Earth's surface, let us introduce the functions $x = x(\lambda, \phi)$ and $y = y(\lambda, \phi)$ and use them as Cartesian coordinates on a plane. The Mercator projection is defined as follows:
   $$x = \frac{W\,\phi}{2\,\pi} \quad \text{and} \quad y = \frac{H}{2\,\pi}\,\ln\left[\tan\left(\frac{\pi}{4}+\frac{\lambda}{2}\right)\right],$$
   where W and H denote the width and the height of the map, respectively. Determine the line-element on the plane.

3. *The Rindler and the Milne coordinates:* Consider the following non-linear transformations of the Minkowski coordinates $(c\,t, x, y, z)$ to the coordinates $(c\,\tau, \xi, y', z')$:
   $$c\,t = \xi\,\sinh\,(g\,\tau/c), \quad x = \xi\,\cosh\,(g\,\tau/c), \quad y = y' \quad \text{and} \quad z = z'.$$
   The set of coordinates $(c\,\tau, \xi, y', z')$ are referred to as the Rindler coordinates.

   (a) Draw lines of constant τ and ξ in the $c\,t$-x plane, and show that the coordinates $(c\,\tau, \xi)$ cover only the right wedge of the light cone centered at the origin.

   (b) Construct similar coordinates to cover the wedge to the left of the light cone.

(c) Arrive at the set of coordinates that can cover the past and future wedges of the light cone in a similar fashion.

Note: These new set of coordinates that cover the past and the future wedges are known as the Milne coordinates.

(d) Determine the form of the Minkowski line-element in the Rindler and the Milne coordinates.

4. *Embedding a three-sphere in four dimensions:* Recall that a two-sphere of radius, say, R, is a surface which is subject to the constraint $x^2 + y^2 + z^2 = R^2$ in three-dimensional Euclidean space described by the Cartesian coordinates (x, y, z). In a similar manner, we can define a three sphere as the surface that is subject to the constraint $x^2 + y^2 + z^2 + w^2 = R^2$ in the four-dimensional Euclidean space characterized by the Cartesian coordinates, say, (x, y, z, w).

(a) Using the constraint equation to eliminate w in terms of the other three variables and the standard Euclidean line-element in four dimensions, show that the geometry of the three-sphere can be expressed as

$$\mathrm{d}\ell^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + d\,z^2 + \frac{(x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z)^2}{R^2 - (x^2 + y^2 + z^2)}.$$

(b) Upon transforming into the spherical polar coordinates using the conventional relations, show that the above line-element is given by

$$\mathrm{d}\ell^2 = \frac{R^2}{R^2 - r^2}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2.$$

(c) Further show that this line-element can be written as

$$\mathrm{d}\ell^2 = R^2\,\left(\mathrm{d}\chi^2 + \sin^2\chi\,\mathrm{d}\theta^2 + \sin^2\chi\sin^2\theta\,\mathrm{d}\phi^2\right),$$

where (χ, θ, ϕ) are the three angular coordinates that are required to cover the three-sphere.

(d) Construct the transformations from the original Cartesian coordinates in the four dimensional Euclidean space, viz. (x, y, z, w), to the angular coordinates (χ, θ, ϕ).

(e) What are allowed ranges of the angular coordinates (χ, θ, ϕ)?

5. *Reducing to the Minkowski line-element:* Show that the following spacetime line-element:

$$\mathrm{d}s^2 = \left(c^2 - a^2\,\tau^2\right)\,\mathrm{d}\tau^2 - 2\,a\,\tau\,\mathrm{d}\tau\,\mathrm{d}\xi - \mathrm{d}\xi^2 - \mathrm{d}y^2 - \mathrm{d}z^2,$$

where a is a constant, can be reduced to the Minkowski line-element by a suitable coordinate transformation.

Exercise sheet 11

Tensors and transformations

1. *Onto the spherical polar coordinates:* Consider the transformation from the Cartesian coordinates $x^a = (x, y, z)$ to the spherical polar coordinates $x'^a = (r, \theta, \phi)$ in $\mathbb{R}^3$.

   (a) Write down the transformation as well as its inverse.

   (b) Express the transformation matrices $[\partial x^a/\partial x'^b]$ and $[\partial x'^a/\partial x^b]$ in terms of the spherical polar coordinates.

   (c) Evaluate the corresponding Jacobians J and J'. Where is J' zero or infinite?

2. *Transformation of vectors:* Consider the transformation from the Cartesian coordinates $x^a = (x, y)$ to the plane polar coordinates $x'^a = (\rho, \phi)$ in $\mathbb{R}^2$.

   (a) Express the transformation matrix $[\partial x'^a/\partial x^b]$ in terms of the polar coordinates.

   (b) Consider the tangent vector to a circle of radius, say, a, that is centered at the origin. Find the components of the tangent vector in one of the two coordinate systems, and use the transformation property of the vector to obtain the components in the other coordinate system.

3. *Properties of partial derivatives:* Consider a scalar quantity ϕ. Show that, while the quantity $(\partial\phi/\partial x^a)$ is a vector, the quantity $\left(\partial^2\phi/\partial x^a\, \partial x^b\right)$ is *not* a tensor.

4. *Transforming tensors:* If $X^a{}_{bc}$ is a mixed tensor of rank $(1, 2)$, show that the contracted quantity $Y_c = X^a{}_{ac}$ is a covariant vector.

5. *Symmetric and anti-symmetric nature of tensors:* Show that, a tensor, if it is symmetric or anti-symmetric in one coordinate system, it remains so in any other coordinate system.

Quiz III

From symmetries and conservation laws to tensor algebra

1. Conserved energy for a Lagrangian involving a second time derivative: To begin with, consider the following conventional action describing a non-relativistic particle that is moving in one dimension:

$$S[q(t)] = \int \mathrm{d}t\, L(q, \dot{q}).$$

   (a) Since the Lagrangian is not explicitly dependent on time, the energy of the system should be conserved. Arrive at the standard form of the energy in terms of $\dot{q}$ and the Lagrangian L. $\boxed{\text{2 marks}}$

   Hint: You can arrive at the required result by following the procedure that we had adopted to arrive at the form of the stress-energy tensor. Differentiate the Lagrangian with respect to the independent variable in this case, viz. time, and assuming that the system satisfies the Euler-Lagrange equation of motion, identify the quantity whose total time derivative is zero.

   (b) Now, consider a system that is governed by the action

$$S[q(t)] = \int \mathrm{d}t\, L(q, \dot{q}, \ddot{q}).$$

   Assuming that, apart from the position q, the corresponding velocity $\dot{q}$ is also fixed at the end points, arrive at the Euler-Lagrange equation of motion governing the system. $\boxed{\text{3 marks}}$

   (c) Since the Lagrangian in the latter case is also not explicitly dependent on time, the corresponding 'energy' of the system will be conserved as well. Construct the form of the conserved energy in terms of $\dot{q}$, $\ddot{q}$ and the Lagrangian L. $\boxed{\text{5 marks}}$

2. Shift symmetry: Consider a system involving N real scalar fields that can be described by the N-dimensional vector $\boldsymbol{\phi}$. Let the scalar field be described by the action

$$S[\boldsymbol{\phi}] = \frac{1}{c} \int \mathrm{d}^4\tilde{x}\, \frac{1}{2}\, \partial^\mu \boldsymbol{\phi} \cdot \partial_\mu \boldsymbol{\phi}.$$

   (a) Show that the action is invariant under the following shift: $\boldsymbol{\phi} \to \boldsymbol{\phi} + \boldsymbol{\alpha}$, where $\boldsymbol{\alpha}$ is a constant N-dimensional vector in field space. $\boxed{\text{2 marks}}$

   (b) What is the conserved current associated with such a symmetry? $\boxed{\text{4 marks}}$

   (c) Explicitly show that the current is conserved. $\boxed{\text{4 marks}}$

3. The dual field tensor in electromagnetism: Consider the so-called dual field tensor $\widetilde{F}^{\mu\nu}$ which is defined in terms of the standard electromagnetic field tensor $F_{\mu\nu}$ as follows:

$$\widetilde{F}^{\mu\nu} = \frac{1}{2}\, \varepsilon^{\mu\nu\alpha\beta}\, F_{\alpha\beta},$$

   with $\varepsilon^{\mu\nu\alpha\beta}$ denoting the completely anti-symmetric Levi-Civita tensor.

   (a) Express the components of the quantity $\widetilde{F}^{\mu\nu}$ in terms of the components of the electric and the magnetic fields. $\boxed{\text{3 marks}}$

   (b) Evaluate the quantity $\widetilde{F}^{\mu\nu}\, F_{\mu\nu}$ in terms of the components of the electric and the magnetic fields. $\boxed{\text{3 marks}}$

   (c) Recall that the first pair of Maxwell's equations, viz. the source free equations, are given by

$$\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0.$$

   Show that these equations can be written as $\boxed{\text{4 marks}}$

$$\partial_\mu \widetilde{F}^{\mu\nu} = 0.$$

4. <u>*Action for the free electromagnetic field:*</u> Motivated by the fact that we require a Lorentz invariant action that contains no more than the first derivative of the vector potential A_μ and is quadratic in the potential, we had considered an action of the electromagnetic field involving the quantity $F_{\mu\nu}\, F^{\mu\nu}$. Another quantity that satisfies the conditions demanded above would be $\widetilde{F}_{\mu\nu}\, F^{\mu\nu}$, where $\widetilde{F}_{\mu\nu}$ is the dual field tensor defined as $\widetilde{F}_{\mu\nu} = (1/2)\, \varepsilon_{\mu\nu\alpha\beta}\, F^{\alpha\beta}$, which we encountered in the previous problem. Identify the reason as to why $\widetilde{F}_{\mu\nu}\, F^{\mu\nu}$ is not a suitable quantity to be considered in an action. $\boxed{\text{10 marks}}$

5. <u>*The Minkowski line-element in a rotating frame:*</u> In terms of the cylindrical polar coordinates, the Minkowski line-element is given by

$$\mathrm{d}s^2 = c^2\, \mathrm{d}t^2 - \mathrm{d}\rho^2 - \rho^2\, \mathrm{d}\phi^2 - \mathrm{d}z^2.$$

Consider a coordinate system that is rotating with an angular velocity Ω about the z-axis. The coordinates in the rotating frame, say, $(c\,t', \rho', \phi', z')$, are related to the standard Minkowski coordinates through the following relations:

$$c\,t = c\,t', \qquad \rho = \rho', \qquad \phi = \phi' + \Omega\, t' \qquad \text{and} \qquad z = z'.$$

(a) Determine the line-element in the rotating frame. $\boxed{\text{7 marks}}$

(b) What happens to the line-element when $\rho' \geq c/\Omega$? $\boxed{\text{3 marks}}$

Exercise sheet 12

Christoffel symbols and the geodesic equation

1. *The metric of* $\mathbb{R}^3$*:* Evaluate the covariant and the contravariant components of the metric tensor describing the three-dimensional Euclidean space (usually denoted as $\mathbb{R}^3$) in the Cartesian, cylindrical polar and the spherical polar coordinates. Also, evaluate the determinant of the covariant metric tensor in each of these coordinate systems.

2. *Geodesics on a two sphere:* Evaluate the Christoffel symbols on $\mathbb{S}^2$, and solve the geodesic equation to show that the geodesics are the great circles.

3. *Important identities involving the metric tensor:* Establish the following identities that involve the metric tensor:

   (a) $g_{,c} = g\, g^{ab}\, g_{ab,c}$,

   (b) $g^{ab}\, g_{bc,d} = -g^{ab}{}_{,d}\, g_{bc}$,

   where the commas denote partial derivatives, while g is the determinant of the covariant metric tensor g_{ab}.

4. *Useful identities involving the Christoffel symbols:* Establish the following identities involving the Christoffel symbols:

   (a) $\Gamma^a_{ab} = \frac{1}{2}\, \partial_b \ln |g|$,

   (b) $g^{ab}\, \Gamma^c_{ab} = -\frac{1}{\sqrt{|g|}}\, \partial_d \left(\sqrt{|g|}\, g^{cd}\right)$,

   (c) $g^{ab}{}_{,c} = -\left(\Gamma^a_{cd}\, g^{bd} + \Gamma^b_{cd}\, g^{ad}\right)$,

   where the Christoffel symbol Γ^a_{bc} is given by

   $$\Gamma^a_{bc} = \frac{1}{2}\, g^{ad}\, \left(g_{db,c} + g_{dc,b} - g_{bc,d}\right).$$

5. *Invariant four volume:* Show that the spacetime volume $\sqrt{-g}\ \mathrm{d}^4\tilde{x}$ is invariant under arbitrary coordinate transformations.

Exercise sheet 13

Killing vectors and conserved quantities

1. *Killing vectors in $\mathbb{R}^3$:* Construct all the Killing vectors in the three dimensional Euclidean space $\mathbb{R}^3$ by solving the Killing's equation.

2. *Killing vectors on $\mathbb{S}^2$:* Construct the most generic Killing vectors on a two sphere.

3. *Killing vectors in Minkowski spacetime:* Solve the Killing's equation in flat spacetime, and construct all the independent Killing vectors. What do these different Killing vectors correspond to?

4. *The line element and the conserved quantities around a cosmic string:* The spacetime around a cosmic string is described by the line-element

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}\rho^2 - \alpha^2\,\rho^2\,\mathrm{d}\phi^2 - \mathrm{d}z^2,$$

   where α is a constant that is called the deficit angle.

   (a) List the components of the momentum of a relativistic particle on geodesic motion in this spacetime that are conserved.

   (b) Consider a particle of mass m that is moving along a time-like geodesic in the spacetime of a cosmic string. Using the relation $p^\mu\,p_\mu = m^2\,c^2$ and the conserved momenta, obtain the (first order) differential equation for $\mathrm{d}\rho/\mathrm{d}t$ of the particle in terms of *all* the conserved components of its momenta.

5. *Conserved quantities in the Schwarzschild spacetime:* The spacetime around a central mass M is described by the following Schwarzschild line element:

$$\mathrm{d}s^2 = c^2\,\left(1-\frac{2\,G\,M}{c^2\,r}\right)\,\mathrm{d}t^2 - \left(1-\frac{2\,G\,M}{c^2\,r}\right)^{-1}\,\mathrm{d}r^2 - r^2\,\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right),$$

   where G is the Newton's gravitational constant. Identify the Killing vectors and the corresponding conserved quantities in such a static and spherically symmetric spacetime.

Exercise sheet 14

Riemann, Ricci tensors and scalar curvature

1. *Algebraic identity involving the Riemann tensor:* Recall that, the Riemann tensor is defined as

$$R^a{}_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{ec}\,\Gamma^e_{bd} - \Gamma^a_{ed}\,\Gamma^e_{bc}.$$

   Using this expression, establish that

$$R^a{}_{bcd} + R^a{}_{dbc} + R^a{}_{cdb} = 0.$$

2. *The number of independent components of the Riemann tensor:* Show that, on a n-dimensional manifold, the number of independent components of the Riemann tensor are $\left(n^2/12\right)\,\left(n^2-1\right)$.

3. *The flatness of the cylinder:* Calculate the Riemann tensor of a cylinder of constant radius, say, R, in three dimensional Euclidean space. What does the result you find imply?

   Note: The surface of the cylinder is actually two-dimensional.

4. *The curvature of the two-sphere:* Calculate all the components of the Riemann and the Ricci tensors, and also the corresponding scalar curvature associated with the two sphere.

   Note: Given the Riemann tensor $R^a{}_{bcd}$, the Ricci tensor R_{ab} and the Ricci scalar R are defined as

$$R_{ab} = R^c{}_{acb} \qquad \text{and} \qquad R = g^{ab}\,R_{ab}.$$

5. *Identities involving the covariant derivative and the Riemann tensor:* Establish the following relations:

   (a) $\nabla_c\,\nabla_b\,A_a - \nabla_b\,\nabla_c\,A_a = R^d{}_{abc}\,A_d$,

   (b) $\nabla_d\,\nabla_c\,A_{ab} - \nabla_c\,\nabla_d\,A_{ab} = R^e{}_{bcd}\,A_{ae} + R^e{}_{acd}\,A_{eb}$.

End-of-semester exam

From special relativity to tensor algebra and tensor calculus

1. *Relative velocity between two inertial frames:* Consider two inertial frames that are moving with the velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ with respect to, say, the laboratory frame in Minkowski spacetime. Show that the relative velocity v between the two frames can be expressed as

$$v^2 = \frac{(\boldsymbol{v}_1 - \boldsymbol{v}_2)^2 - [(\boldsymbol{v}_1/c) \times \boldsymbol{v}_2]^2}{[1 - (\boldsymbol{v}_1/c) \cdot \boldsymbol{v}_2/c^2)]^2}.$$

2. *Scale invariance and the conformal stress-energy tensor:* Earlier, we had obtained the conserved current, say, j_μ, upon demanding that the action describing a canonical and real scalar field ϕ is invariant under spacetime translations as well as the scale transformations in Minkowski spacetime

$$x^\mu \to b\, x^\mu \quad \text{and} \quad \phi \to \phi/b,$$

where b is a constant. Let us express the conserved current as

$$j_\mu = -x^\nu\, \widetilde{T}_{\mu\nu},$$

where the new stress-energy tensor $\widetilde{T}_{\mu\nu}$ is a quantity that is not only conserved (i.e. $\partial_\mu \widetilde{T}^{\mu\nu} = 0$), but is also traceless (i.e. $\widetilde{T}^\mu_\mu = 0$). Writing $\widetilde{T}^{\mu\nu} = T^{\mu\nu} + \theta^{\mu\nu}$, show that $\theta^{\mu\nu}$ has the following form: $\boxed{\text{10 marks}}$

$$\theta^{\mu\nu} = \frac{1}{6}\, (\eta^{\mu\nu}\, \Box - \partial^\mu\, \partial^\nu)\; \phi^2.$$

Note: The quantity $\widetilde{T}_{\mu\nu}$ is known as the conformal stress-energy tensor.

3. *Reparametrization invariance and Hamiltonians:* Recall that the action governing a relativistic free particle is given by

$$S[x^\mu(s)] = -\, m\, c \int \mathrm{d}s.$$

Instead of the quantity s, let us describe the trajectory in terms of another parameter, say, λ, as $x^\mu(\lambda)$.

   (a) Show that the above action then reduces to $\boxed{\text{2 marks}}$

$$S[x^\mu(\lambda)] = -\, m\, c \int \mathrm{d}\lambda\, \sqrt{\dot{x}^\mu\, \dot{x}_\mu},$$

   where $\dot{x}^\mu \equiv \mathrm{d}x^\mu/\mathrm{d}\lambda$.

   Note: The above action is invariant under reparametrizations of the form $\lambda \to f(\lambda)$.

   (b) Determine the momentum p_μ that is conjugate to $\dot{x}^\mu$, i.e. $p_\mu = (\partial L/\partial \dot{x}^\mu)$. $\boxed{\text{3 marks}}$

   (c) Evaluate the corresponding Hamiltonian, viz. $H = p_\mu\, \dot{x}^\mu - L$. $\boxed{\text{5 marks}}$

4. *The Kalb-Ramond field:* The Kalb-Ramond field $H_{\alpha\beta\gamma}$ is defined in terms of the anti-symmetric tensor $B_{\alpha\beta}$ through the relation in Minkowski spacetime

$$H_{\alpha\beta\gamma} = \partial_\alpha B_{\beta\gamma} + \partial_\gamma B_{\alpha\beta} + \partial_\beta B_{\gamma\alpha}.$$

   (a) Show that the Kalb-Ramond field is invariant under the gauge transformation

$$B_{\alpha\beta} \to B_{\alpha\beta} + \partial_\alpha \epsilon_\beta - \partial_\beta \epsilon_\alpha,$$

   where ϵ_α is an arbitrary four vector. $\boxed{\text{2 marks}}$

(b) The Kalb-Ramond field is described by the action

$$S[B_{\alpha\beta}(\tilde{x})] = \frac{1}{c}\int \mathrm{d}^4\tilde{x}\,\left(-\frac{1}{6\,\kappa^2}\,H_{\alpha\beta\gamma}\,H^{\alpha\beta\gamma} - j^{\alpha\beta}\,B_{\alpha\beta}\right).$$

Vary the action with respect to $B_{\alpha\beta}$ to arrive at the equation of motion governing the Kalb-Ramond field $H^{\alpha\beta\gamma}$. 4 marks

(c) Is the current $j^{\alpha\beta}$ symmetric or anti-symmetric in its indices? 2 marks

(d) Show that the current $j^{\alpha\beta}$ is conserved. 2 marks

5. *Parallel transporting a vector on a sphere:* Consider the parallel transport of a vector around the curve corresponding to $\theta = \theta_0$ on a two dimensional sphere of fixed radius R. The curve can be parameterized in terms of ϕ as $x^a = (\theta_0, \phi)$.

(a) Determine the normalized tangent vector, say, u^a, to the curve. 1 marks

(b) Write down the equation for parallel transporting a vector v^a along the tangent u^a. 2 marks

(c) Assuming that the initial values of v^a are (v_0^θ, v_0^ϕ), solve the differential equation to arrive at the values of $v^\theta(\phi)$ and $v^\phi(\phi)$ along the curve. 5 marks

(d) Using the solutions, show that the circle corresponding to $\theta_0 = \pi/2$ (i.e. the equator) is a geodesic. 2 marks

6. *Geodesics on a cone:* Consider a cone with a semi-vertical angle α.

(a) Determine the line element on the cone. 2 marks

(b) Obtain the equations governing the geodesics on the cone. 4 marks

(c) Solve the equations to arrive at the geodesics. 4 marks

7. *Killing vectors of a plane in polar coordinates:* Consider the two dimensional Euclidean plane described in terms of the polar coordinates.

(a) What is the line element of the Euclidean plane in terms of the polar coordinates? 1 mark

(b) Evaluate all the Christoffel symbols associated with the line element. 2 marks

(c) Write down the equations describing the Killing vectors in the polar coordinates. 3 marks

(d) Obtain all the Killing vectors by solving the equations and interpret the solutions. 4 marks

8. *Maxwell's equations in a curved spacetime:* Typically, the equations governing fields in a curved spacetime can be arrived at by replacing the partial derivatives encountered in the Minkowski spacetime by the corresponding covariant ones.

(a) Show that 2 marks

$$F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu} = A_{\nu,\mu} - A_{\mu,\nu}.$$

(b) Establish that the first pair of Maxwell's equations in a curved spacetime, viz.

$$F_{\mu\nu;\lambda} + F_{\lambda\mu;\nu} + F_{\nu\lambda;\mu} = 0,$$

actually reduce to 4 marks

$$F_{\mu\nu,\lambda} + F_{\lambda\mu,\nu} + F_{\nu\lambda,\mu} = 0.$$

(c) Show that the second pair of Maxwell's equations in a curved spacetime, viz.

$$F^{\mu\nu}{}_{;\nu} = \frac{4\,\pi}{c}\, j^{\mu},$$

can be written as $\boxed{\text{4 marks}}$

$$\frac{1}{\sqrt{-g}}\, \partial_\nu \left(\sqrt{-g}\, F^{\mu\nu}\right) = \frac{4\,\pi}{c}\, j^{\mu}.$$

9. <u>*Geodesics in a FLRW universe:*</u> The spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe is described by the line-element

$$\mathrm{d}s^2 = c^2\, \mathrm{d}t^2 - a^2(t)\, \left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right),$$

where the function $a(t)$ is referred to as the scale factor which describes the time evolution of the universe.

(a) Determine all the Christoffel symbols associated with the metric. $\boxed{\text{3 marks}}$

(b) Explicitly write down the geodesic equations governing massive particles in the FLRW universe. $\boxed{\text{3 marks}}$

(c) Solve the geodesic equations suitably to show that the magnitude of the three momentum of the particle decreases as inversely proportional to the scale factor with the expansion of the universe. $\boxed{\text{4 marks}}$

Note: The magnitude of the three momentum $|\boldsymbol{p}|$ is defined through the relation $|\boldsymbol{p}|^2 = -g_{ij}\, p^i\, p^j$, where, as usual, $(i, j) = (1, 2, 3)$.

10. <u>*Spacetime metric in the Newtonian limit:*</u> Recall that the action governing a relativistic particle in a curved spacetime described by the metric tensor $g_{\mu\nu}$ is given by

$$S[\tilde{x}(s)] = -m\, c \int \mathrm{d}s,$$

where $\mathrm{d}s^2 = g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu$.

(a) Obtain the non-relativistic limit of the action describing a free particle in *Minkowski spacetime.* $\boxed{\text{1 marks}}$

(b) Modify the action suitably to describe a non-relativistic particle that is moving under the influence of the gravitational potential $\phi(\boldsymbol{x})$. $\boxed{\text{2 marks}}$

Note: A constant term added to the standard Lagrangian describing the non-relativistic particle does not change the equation of motion.

(c) Compare the non-relativistic action you have obtained with the above exact relativistic action and arrive at the relation between $\mathrm{d}s$ and $\mathrm{d}t$ in the presence of the gravitational potential $\phi(\boldsymbol{x})$. $\boxed{\text{3 marks}}$

(d) Using the relation you have obtained between $\mathrm{d}s$ and $\mathrm{d}t$, and recognizing that $\mathrm{d}\boldsymbol{x} = \boldsymbol{v}\, \mathrm{d}t$, show that, upto order c^0, the line-element describing the spacetime in the presence of the potential $\phi(\boldsymbol{x})$ is given by $\boxed{\text{4 marks}}$

$$\mathrm{d}s^2 = c^2 \left[1 - \frac{2\,\phi(\boldsymbol{x})}{c^2}\right] \mathrm{d}t^2 - \mathrm{d}\boldsymbol{x}^2.$$

Note: This line-element is supposed to describe a weak gravitational field wherein $\phi/c^2 \ll 1$.