PH5870

INTRODUCTION TO GENERAL RELATIVITY

July–November 2026

Lecture schedule and meeting hours

- The course will consist of about 42 lectures, including about 10 tutorial sessions. However, note that there will be no separate tutorial sessions, and they will be integrated with the lectures.
- The duration of each lecture will be 50 minutes. We will be meeting in HSB 209.
- The first lecture will be on Tuesday, July 28, and the last one will be on Friday, November 6.
- We will meet three or four times a week. We shall meet during the following hours: 11:00–11:50 AM on Wednesdays, 9:00–9:50 AM on Thursdays, and 8:00–8:50 AM on Fridays.
- We may also meet during 5:00–5:50 PM on Tuesdays for either the quizzes or to make up for lectures that I may have to miss due to other unavoidable commitments. Changes in schedule, if any, will be notified sufficiently in advance.
- If you would like to discuss with me about the course outside the lecture hours, you are welcome to meet me in my office (in HSB 202) during 12:00 Noon–1:00 PM on Wednesdays. In case you are unable to find me in my office on more than occasion, please send me an e-mail at `sriram@physics.iitm.ac.in`.

Information about the course

- All the information regarding the course such as the schedule of the lectures, the structure and the syllabus of the course, suitable textbooks and additional references will be available on the course's page on Moodle at the following URL:

  `https://courses.iitm.ac.in/`

- The exercise sheets and other additional material will also be made available on Moodle.
- A PDF file containing these information as well as completed quizzes will also be available at the link on this course at the following URL:

  `http://physics.iitm.ac.in/~sriram/professional/teaching/teaching.html`

  I will keep updating this file and the course's page on Moodle as we make progress.

Quizzes, end-of-semester exam and grading

- The grading will be based on three scheduled quizzes and an end-of-semester exam.
- I will consider the best two quizzes for grading, and the two will carry 25% weight each.
- The three quizzes will be held on August 25, September 29 and October 27. All these three dates are Tuesdays, and the quizzes will be held during 5:00–6:30 PM.
- The end-of-semester exam will be held during 9:00 AM–12:00 Noon on Wednesday, November 18, and the exam will carry 50% weight.

Syllabus and structure

1. **Special theory of relativity** [$\sim$ 10 lectures]
   (a) The Michelson-Morley interferometric experiment – Postulates of special relativity
   (b) Lorentz transformations – The relativity of simultaneity – Length contraction and time dilation
   (c) Transformation of velocities and acceleration – Uniform acceleration – Doppler effect
   (d) Metric tensor – The light cone
   (e) Contravariant and covariant vectors – Action for the relativistic free particle – Charges in an electromagnetic field and the Lorentz force law
   (f) Conservation of relativistic energy and momentum
   (g) Infinitesimal generators of translations, rotations and boosts – Algebra of the generators – The Lorenz and the Poincare groups

Exercise sheets 1, 2, 3 and 4

2. **Theory of a real scalar field** [$\sim$ 5 lectures]
   (a) An illustrative example – Action formulation for a string
   (b) Action describing a real, canonical, scalar field – The Euler-Lagrange field equation
   (c) The conjugate momentum – Hamiltonian density – The stress-energy tensor – Physical interpretation

Exercise sheet 5

Quiz I

3. **The case of the complex scalar field** [$\sim$ 3 lectures]
   (a) Action governing the complex scalar field – Equations of motion
   (b) Global gauge invariance
   (c) Local gauge invariance and the need for the electromagnetic field

Exercise sheet 6

Additional exercises I

4. **The theory of the electromagnetic field** [$\sim$ 5 lectures]
   (a) The electromagnetic field tensor – The first pair of Maxwell's equations
   (b) The four current vector – The continuity equation – Charge conservation
   (c) Action governing the free electromagnetic field – Interaction of the electromagnetic field with charges and currents – The second pair of Maxwell's equations
   (d) Gauge invariance of the electromagnetic field – The Lorentz and the Coulomb gauges
   (e) Equations governing the free field – Electromagnetic waves – Polarization
   (f) Energy density – Poynting vector – The stress-energy tensor of the electromagnetic field
   (g) Lorentz transformation properties of the electric and the magnetic fields
   (h) The retarded Green's function – The Lienard-Wiechart potentials – Radiation from moving charges

Exercise sheets 7 and 8

Quiz II

5. **Symmetries and conservation laws** [∼ 4 lectures]
   (a) Noether's theorem
   (b) External symmetries – Symmetry under translations, rotations and Lorentz transformations – Conserved quantities
   (c) Dilatations – The conformal stress-energy tensor
   (d) Internal symmetries and gauge transformations

Exercise sheet 9

6. **Tensor algebra and tensor calculus** [∼ 10 lectures]
   (a) Manifolds and coordinates – Curves and surfaces
   (b) Transformation of coordinates – Contravariant, covariant and mixed tensors – Elementary operations with tensors
   (c) The partial derivative of a tensor – Covariant differentiation and the affine connection
   (d) The metric – Geodesics
   (e) Isometries – The Killing equation and conserved quantities
   (f) The Riemann tensor – The equation of geodesic deviation
   (g) The curvature and the Weyl tensors

Exercise sheets 10, 11, 12, 13 and 14

Additional exercises II

Quiz III

7. **Field equations of general relativity** [∼ 5 lectures]
   (a) The equivalence principle – The principle of general covariance – The principle of minimal gravitational coupling
   (b) The vacuum Einstein equations
   (c) Derivation of vacuum Einstein equations from the action – The Bianchi identities
   (d) The stress-energy tensor – The cases of perfect fluid, scalar and electromagnetic fields
   (e) Non-canonical scalar fields – Relation to relativistic fluids
   (f) The structure of the Einstein equations

Exercise sheet 15

End-of-semester exam

Advanced problems

Basic textbooks

1. L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Course of Theoretical Physics, Volume 2), Fourth Edition (Pergamon Press, New York, 1975).
2. B. F. Schutz, *A First Course in General Relativity* (Cambridge University Press, Cambridge, 1990).
3. R. d'Inverno, *Introducing Einstein's Relativity* (Oxford University Press, Oxford, 1992).
4. B. Felsager, *Geometry, Particles and Fields* (Springer, Heidelberg, 1998).
5. J. B. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (Pearson Education, Delhi, 2003).
6. S. Carroll, *Spacetime and Geometry* (Addison Wesley, New York, 2004).
7. M. P. Hobson, G. P. Efstathiou and A. N. Lasenby, *General Relativity: An Introduction for Physicists* (Cambridge University Press, Cambridge, 2006).
8. F. Scheck, *Classical Field Theory* (Springer, Heidelberg, 2012).

Additional references

1. S. Weinberg, *Gravitation and Cosmology* (John Wiley, New York, 1972).
2. A. P. Lightman, W. H. Press, R. H. Price and S. A. Teukolsky, *Problem Book in Relativity and Gravitation* (Princeton University Press, New Jersey, 1975).
3. L. H. Ryder, *Quantum Field Theory*, Second Edition (Cambridge University Press, Cambridge, England, 1996).
4. W. Rindler, *Relativity: Special, General and Cosmological* (Oxford University Press, Oxford, 2006).
5. T. Padmanabhan, *Gravitation: Foundation and Frontiers* (Cambridge University Press, Cambridge, 2010).
6. A. Zee, *Einstein Gravity in a Nutshell* (Princeton University Press, Princeton, New Jersey, 2013).

Advanced texts

1. S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge University Press, Cambridge, 1973).
2. C. W. Misner, K. S. Thorne and J. W. Wheeler, *Gravitation* (W. H. Freeman and Company, San Francisco, 1973).
3. R. M. Wald, *General Relativity* (The University of Chicago Press, Chicago, 1984).
4. E. Poisson, *A Relativist's Toolkit* (Cambridge University Press, Cambridge, 2004).

Exercise sheet 1

Lorentz transformations and some consequences

1. *Superluminal motion:* Consider a blob of plasma that is moving at a speed v along a direction that makes an angle θ with respect to the line of sight. Show that the *apparent* transverse speed of the source, projected on the sky, will be related to the actual speed v by the relation

$$v_{\rm app} = \frac{v\,\sin\theta}{1-(v/c)\cos\theta}.$$

From this expression conclude that the apparent speed $v_{\rm app}$ can exceed the speed of light.

2. *Aberration of light:* Consider two inertial frames S and S', with the frame S' moving along the x-axis with a velocity v with respect to the frame S. Let the velocity of a particle in the frames S and S' be $\mathbf{u}$ and $\mathbf{u}'$, and let θ and θ' be the angles subtended by the velocity vectors with respect to the common x-axis, respectively.

   (a) Show that

$$\tan\theta = \frac{u'\sin\theta'}{\gamma\,[u'\cos\theta' + v]},$$

   where $\gamma = \left[1-(v/c)^2\right]^{-1/2}$.

   (b) For $u = u' = c$, show that

$$\cos\theta = \frac{\cos\theta' + (v/c)}{1+(v/c)\cos\theta'}$$

   and

$$\sin\theta = \frac{\sin\theta'}{\gamma\,[1+(v/c)\cos\theta']}.$$

   (c) For $(v/c) \ll 1$, show that

$$\Delta\theta = (v/c)\,\sin\theta',$$

   where $\Delta\theta = (\theta' - \theta)$.

3. *Decaying muons:* Muons are unstable and decay according to the radioactive decay law $N = N_0\,\exp -(0.693\,t/t_{_{1/2}})$, where N_0 and N are the number of muons at times $t=0$ and t, respectively, while $t_{_{1/2}}$ is the half life. The half life of the muons in their own rest frame is 1.52×10^{-6} s. Consider a detector on top of a $2,000$ m mountain which counts the number of muons traveling at the speed of $v = 0.98\,c$. Over a given period of time, the detector counts 10^3 muons. When the relativistic effects are taken into account, how many muons can be expected to reach the sea level?

4. *Binding energy:* As you may know, the deuteron which is the nucleus of deuterium, an isotope of hydrogen, consists of one proton and one neutron. Given that the mass of a proton and a neutron are $m_{_{\rm p}} = 1.673\times10^{-27}$ kg and $m_{_{\rm n}} = 1.675\times10^{-27}$ kg, while the mass of the deuteron is $m_{_{\rm d}} = 3.344\times10^{-27}$ kg, show that the binding energy of the deuteron in about 2.225 MeV.

   Note: MeV refers to Million electron Volts, and an electron Volt is 1.602×10^{-19} J.

5. *Form invariance of the Minkowski line element:* Show that the following Minkowski line element is invariant under the Lorentz transformations:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}\boldsymbol{x}^2.$$

Exercise sheet 2

Working in terms of four vectors

1. *Compton effect using four vectors:* Consider the scattering between a photon of frequency ω and a relativistic electron with velocity $\mathbf{v}$ leading to a photon of frequency ω' and electron with velocity $\mathbf{v}'$. Such a scattering is known as Compton scattering. Let α be the angle between the incident and the scattered photon. Also, let θ and θ' be the angles subtended by the directions of propagation of the incident and the scattered photon with the velocity vector of the electron before the collision.

   (a) Using the conservation of four momentum, show that

   $$\frac{\omega'}{\omega} = \frac{1 - (v/c)\cos\theta}{1 - (v/c)\cos\theta' + (\hbar\,\omega/\gamma\, m_{\mathrm{e}}\, c^2)\,(1 - \cos\alpha)},$$

   where $\gamma = \left[1 - (v/c)^2\right]^{-1/2}$ and m_{e} is the mass of the electron.

   (b) When $\hbar\,\omega \ll \gamma\, m_{\mathrm{e}}\, c^2$, show that the frequency shift of the photon can be written as

   $$\frac{\Delta\omega}{\omega} = \frac{(v/c)\,\left(\cos\theta - \cos\theta'\right)}{1 - (v/c)\cos\theta'},$$

   where $\Delta\omega = (\omega' - \omega)$.

2. *Creation of electron-positron pairs:* A purely relativistic process corresponds to the production of electron-positron pairs in a collision of two high energy gamma ray photons. If the energies of the photons are ϵ_1 and ϵ_2 and the relative angle between their directions of propagation is θ, then, by using the conservation of energy and momentum, show that the process can occur only if

   $$\epsilon_1\,\epsilon_2 > \frac{2\,m_{\mathrm{e}}^2\,c^4}{1 - \cos\theta},$$

   where m_{e} is the mass of the electron.

3. *Transforming four vectors and invariance under Lorentz transformations:* Consider two inertial frames K and K', with K' moving with respect to K, say, along the common x-axis with a certain velocity.

   (a) Given a four vector A^μ in the K frame, construct the corresponding contravariant and covariant four vectors, say, $A^{\mu\prime}$ and A'_μ, in the K' frame.

   (b) Explicitly illustrate that the scalar product $A_\mu\, A^\mu$ is a Lorentz invariant quantity, i.e. show that $A_\mu\, A^\mu = A'_\mu\, A^{\mu\prime}$.

4. *Lorentz invariance of the wave equation:* Show that the following wave equation:

   $$\frac{1}{c^2}\,\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi = 0$$

   satisfied by, say, light, is invariant under the Lorentz transformations.

5. *Mirrors in motion:* A mirror moves with the velocity v in a direction perpendicular its plane. A ray of light of frequency ν_1 is incident on the mirror at an angle of incidence θ, and is reflected at an angle of reflection ϕ and frequency ν_2.

   (a) Show that

   $$\frac{\tan(\theta/2)}{\tan(\phi/2)} = \frac{c+v}{c-v} \qquad \text{and} \qquad \frac{\nu_2}{\nu_1} = \frac{c + v\cos\theta}{c - v\cos\phi}.$$

   (b) What happens if the mirror was moving parallel to its plane?

Exercise sheet 3

Tensors and transformations

1. *Four velocity and four acceleration:* The four acceleration of a relativistic particle is defined as $a^\mu = \mathrm{d}u^\mu/\mathrm{d}s$, where $u^\mu = \mathrm{d}x^\mu/\mathrm{d}s$ is the four velocity of the particle.

   (a) Express a^μ in terms of the three velocity $\mathbf{v}$ and the three acceleration $\mathbf{a} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t$ of the particle.

   (b) Evaluate $a^\mu\, u_\mu$ and $a^\mu\, a_\mu$ in terms of $\boldsymbol{v}$ and $\boldsymbol{a}$.

2. *The Lorentz force:* The action for a relativistic particle that is interacting with the electromagnetic field is given by

$$S[x^\mu(s)] = -m\,c \int \mathrm{d}s - \frac{e}{c} \int \mathrm{d}x_\mu\, A^\mu,$$

   where m is the mass of the particle, while e is its electric charge. The quantity $A^\mu = (\phi, \boldsymbol{A})$ is the four vector potential that describes the electromagnetic field, with, evidently, ϕ and $\mathbf{A}$ being the conventional scalar and three vector potentials.

   (a) Vary the above action with respect to x^μ to arrive at the following Lorentz force law:

$$m\,c\,\frac{\mathrm{d}u^\mu}{\mathrm{d}s} = \frac{e}{c}\, F^{\mu\nu}\, u_\nu,$$

   where u^μ is the four velocity of the particle and the electromagnetic field tensor $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} = \partial_\mu\, A_\nu - \partial_\nu\, A_\mu$$

   with $\partial_\mu \equiv \partial/\partial x^\mu$.

   (b) Show that the components of the field tensor $F_{\mu\nu}$ are given by

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix},$$

   where (E_x, E_y, E_z) and (B_x, B_y, B_z) are the components of the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ which are related to the components of the four vector potential by the following standard expressions:

$$\boldsymbol{E} = -\frac{1}{c}\,\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi \qquad \text{and} \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A}.$$

   (c) Express the above equation governing the motion of the charge in the more familiar three vector notation. What does the zeroth component of the equation describe?

3. *Reducing tensors to vectors:* If $X^\lambda{}_{\mu\nu}$ is a mixed tensor of rank $(1,2)$, show that the contracted quantity $Y_\mu = X^\nu{}_{\mu\nu}$ is a covariant vector.

4. *Transformation of electric and magnetic fields:* Consider two inertial frames, say, K and K', with the frame K' moving with a velocity v with respect to the frame K along the common x-axes.

   (a) Given the components of the electric and the magnetic fields, say, $\mathbf{E}$ and $\mathbf{B}$, in the frame K, using the transformation properties of the electromagnetic field tensor $F_{\mu\nu}$, construct the corresponding components in the frame K'.

   (b) Show that $|\boldsymbol{E}|^2 - |\boldsymbol{B}|^2$ is invariant under the Lorentz transformations.

   (c) Express the quantity $|\boldsymbol{E}|^2 - |\boldsymbol{B}|^2$ explicitly as a scalar in terms of the field tensor $F_{\mu\nu}$.

5. *The Lorentz invariant four volume:* Show that the differential spacetime volume $\mathrm{d}^4x = c\;\mathrm{d}t\,\mathrm{d}^3\boldsymbol{x}$ is a Lorentz invariant quantity.

Exercise sheet 4

Spacetime diagrams

1. <u>*Axes of inertial frames I:*</u> Consider two inertial frames K and K', with K' moving at the velocity v along their common, positive x-direction. Let the two frames coincide at $t = t' = 0$, and let us ignore the y and the z-directions for simplicity.

   (a) In the plane of the spacetime coordinates $(c\,t, x)$, determine the $c\,t'$ axis.
   Hint: This is essentially given by the trajectory of an observer located at, say, $x' = 0$, in the K' frame.

   (b) Consider a beam of light emitted by a source located $x = 0$ and is being reflected by a mirror at, say, $x = a$. Evidently, if the source emits the beam of light at $c\,t = -a$, it will return to the source, after being reflected by the mirror, at $c\,t = a$. Using this method and the fact that the velocity of light is the same in all inertial frames of reference, determine the x' axis in the $(c\,t, x)$ plane.

2. <u>*Axes of inertial frames II:*</u> In the previous exercise, you had determined $c\,t'$ and x' axes in the $(c\,t, x)$ plane.

   (a) Determine the angles between the $c\,t$ and $c\,t'$ axes as well as the x and x' axes.

   (b) Draw the $c\,t$ and x axes in the plane of the spacetime coordinates $(c\,t', x')$ and determine the angles involved.

3. <u>*Invariant hyperbolae:*</u> Consider two Lorentz frames as discussed in the previous two exercises.

   (a) Draw hyperbolae corresponding to different possible values of the Lorentz invariant quantity $c^2\,t^2 - x^2$ in the $(c\,t, x)$ plane.

   (b) Using the invariant hyperbolae, calibrate the $c\,t'$ and x' axes with the respect to the values on the $c\,t$ and x axes.

   (c) A line of simultaneity at a given point, say, P, on the $c\,t$ axis in the spacetime diagram is, evidently, described by the tangent to the hyperbola passing through that point. Draw a line of simultaneity in the $(c\,t, x)$ plane and illustrate the same line of simultaneity (along with the point P and the original hyperbola passing through it) in the $(c\,t', x')$ plane.

   (d) Show that the event P can be shifted anywhere on the hyperbola by working in a suitable Lorentz frame.

   (e) Argue that the tangent to the hyperbola at any event P is the line of simultaneity of the Lorentz frame whose time axis joins P to the origin of the spacetime diagram.

4. <u>*Lorentz contraction:*</u> Consider a rod of a given length at rest in the K' frame. Let one end of the rod be at $x = 0$ at $t = 0$.

   (a) As you have done earlier, draw the $c\,t'$ and x' axes in the $(c\,t, x)$ plane. Also, draw the trajectory of the two ends of the rod in the $(c\,t, x)$ plane.

   (b) From the geometry, express the length of the rod at time $t = 0$ in the K frame, in terms of the actual length of the rod in the K' frame.
   Note: You will find that the length of the rod measured at a given time in the K frame is smaller than its original length. This is Lorentz contraction.

5. <u>*Global view of Minkowski spacetime:*</u> Consider the following Minkowski line-element in $(1 + 1)$-spacetime dimensions:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}x^2.$$

(a) Show that, in terms of the null coordinates

$$u = c\,t - x, \quad v = c\,t + x,$$

the Minkowski line-element reduces to

$$\mathrm{d}s^2 = \mathrm{d}u\,\mathrm{d}v.$$

(b) Show that, if we perform the following coordinate transformation:

$$u' = 2\tan^{-1} u, \qquad v' = 2\tan^{-1} v,$$

where $-\pi \leq u' \leq \pi$ and $-\pi < v' \leq \pi$, then the Minkowski line-element can be expressed as

$$\mathrm{d}s^2 = \frac{1}{4}\sec^2(u'/2)\sec^2(v'/2)\,\mathrm{d}\bar{s}^2,$$

with $\mathrm{d}\bar{s}^2$ being given by

$$\mathrm{d}\bar{s}^2 = \mathrm{d}u'\,\mathrm{d}v'.$$

Note: The line-element $\mathrm{d}\bar{s}^2$ has the same structure as the original Minkowski line-element $\mathrm{d}s^2$. The above relation between $\mathrm{d}s^2$ and $\mathrm{d}\bar{s}^2$ is referred to as a conformal transformation. It is important to appreciate that it is *not* a coordinate transformation.

(c) Working with the line-element $\mathrm{d}\bar{s}^2$, identify the following points of the Minkowski coordinates in the u'-v' plane: (i) past and future time-like infinities ($t \to \mp\infty$, often referred to as i^- and i^+), (ii) space-like infinities ($x \to \mp\infty$ denoted as i^0), and (iii) past and future null-like infinities [$(u, v) \to -\infty$ and $(u, v) \to \infty$, denoted as $\mathcal{I}^-$ and $\mathcal{I}^+$].

Note: The conformal transformation, even as it preserves the form of the Minkowski metric, brings in the infinities of the original Minkowski coordinates to finite values. This helps us create an image of the originally infinite Minkowski domain over a compact region. Such a diagram is known as a Penrose diagram.

(d) Do the the light cones in the Penrose diagram have the same shape as in the original Minkowski spacetime?

(e) Indicate how time-like trajectories behave in the diagram.

(f) Also, draw space-like surfaces corresponding to constant values for the time coordinate t.