PH1010

PHYSICS I

July–November 2017

Mini tests, quizzes, end-of-semester exam and grading

- The grading will be based on two scheduled quizzes, five mini tests and an end-of-semester exam.
- The two will quizzes carry 20% weight each and the five mini tests will carry 10% weight in total.
- The course is divided into five units. The mini tests will be conducted at the end of each unit. These tests will involve a single question discussed in the tutorials associated with the unit. Each mini test will carry 2 marks. They will be held during the first 15 minutes of the class on the following Mondays: August 21, September 4, September 25, October 16 and November 6.
- The two quizzes will be on Monday, September 11 and Tuesday, October 24. The quizzes will be held during 8:00–8:50 AM on these days. (Note that on Monday, September 11, the usual times of the A and D slots will be interchanged. Similarly, on Tuesday, October 24, the times of the B and D slots will be interchanged.)
- The tentative format of the quizzes will be as follows. Each quiz will contain 3 true/false questions (for 1 mark each), 3 multiple choice questions with one correct option (for 1 mark each), 4 fill in the blanks, two questions involving detailed calculations (for 3 marks each) and one question involving some plotting (for 4 marks), adding to a total of 20 marks.
- The end-of-semester exam will be held during 9:00 AM–12:00 NOON on Friday, November 17, and the exam will carry 50% weight.
- The tentative format of the end-of-semester exam will be as follows. The exam will contain 4 true/false questions (for 1 mark each), 4 multiple choice questions with one or more correct options (for 2 marks each), 8 fill in the blanks (for 1 mark each), five questions involving moderate calculations (for 3 marks each) and three question involving detailed calculations or plotting (for 5 marks each), adding to a total of 50 marks.

Prepared by L. Sriramkumar and J. F. Libby

Last updated on August 4, 2017

Syllabus and structure

(Instructor's version)

Physics I

Unit I

1. **Revisiting Newton's laws of motion** [$\sim$ 6 lectures (July 31–August 8) + 2 tutorials (August 10 and August 17) + mini test 1 (August 21)]

   (a) Space and time – Notion of a point particle [JRT, Secs. 1.1, 1.2]

   (b) Concept of vectors – Choice of coordinates [JRT, Sec. 1.2]

   (c) Properties of vectors – Vector addition and scalar multiplication – Scalar and vector products – Examples [JRT, Sec. 1.2; KK, Secs. 1.2–1.7]

WEEK I (d) Differentiation of vectors – Velocity and acceleration [JRT, Sec. 1.2; KK, Secs. 1.8, 1.9]

   (e) Mass and force – Newton's first and second laws – Reference frames – Inertial frames [JRT, Secs. 1.3, 1.4, 1.6; KK, Secs. 2.1–2.5]

   (f) Newton's third law – Multi-particle systems – Conservation of momentum – Elastic collisions [JRT, Sec. 1.5, 3.1, 4.9; KK, Sec. 2.6, 4.5]

   (g) Kinetic energy and work – Potential energy – Conservative forces – Force as a gradient of potential energy – Gravitational and electrostatic examples [JRT, Secs. 4.1–4.4; KK, Secs. 5.1, 5.3, 5.5, 5.6, Note 5.2]

WEEK II (h) Conservation of energy – Energy for linear one-dimensional systems – Turning points [JRT, Secs. 4.6–4.7; KK, Secs. 5.2, 5.4–5.7, 5.10]

Illustrative examples 1 and 2

Exercise sheets 1 and 2

Unit II

2. **Equilibrium and oscillations in physical systems** [$\sim$ 10 lectures (August 9–September 4) + 3 tutorials (August 24, August 31 and September 7) + mini test 2 (September 4)]

   (a) Hooke's law – Simple harmonic motion – Solutions – Examples – Complex exponentials [JRT, Secs. 5.1, 5.2, 2.6; KK, Secs. 11.1, 11.2]

WEEK III (b) Damped oscillations – Driven damped oscilations – Resonance – LCR circuits [JRT, Secs. 5.4, 5.5, 5.6; KK, Secs. 11.3–11.6, Notes 11.1–11.3]

WEEK IV (c) The inverted oscillator – Solutions to the equation of motion [TM, Sec. 4.3]

   (d) Motion of a particle in a potential – Equilibrium, stability and small oscillations [TM, Sec. 4.2]

WEEK V (e) Phase space – Trajectories in phase space of the oscillator and inverted oscillator – Trajectories in a generic one-dimensional potential [TM, Sec. 3.4, 4.3, 4.4]

Illustrative examples 3 and 4

Exercise sheets 3, 4 and 5

Quiz I (September 11)

Unit III

3. **Motion in a central force field** [$\sim$ 9 lectures (September 5–25) + 2 tutorials (September 21 and October 5) + mini test 3 (September 25, covering parts of Units II and III)]

WEEK VI (a) Angular momentum of a single particle – Conservation of angular momentum – Kepler's second law [JRT, Sec. 3.4; KK, Secs. 7.1–7.2]

(b) Angular momentum for several particles – Moment of inertia [JRT, Sec. 3.5]

(c) Central force – Conservation of angular momentum [JRT, Sec. 4.8]

WEEK VII (d) Continuous symmetries – Conservation of linear momentum, angular momentum and energy

(e) Discrete symmetries – Time reversal symmetry – Reflection and inversion – Polar and axial vectors – Linear and angular momentum as examples

(f) Two dimensional polar coordinates [JRT, Sec. 1.7]

WEEK VIII (g) Centre of mass – Relative coordinates – Reduced mass [JRT, Secs. 8.1, 8.2]

(h) Conservation of energy – The equivalent one-dimensional problem – The concept of the effective potential [JRT, Sec. 8.4; KK, Secs. 10.2–10.4]

WEEK IX (i) The orbit equation – Planetary motion – Kepler's laws [JRT, Secs. 8.5, 8.6; KK, Secs. 10.5, 10.6, Notes 10.1, 10.2]

(j) The unbounded Kepler orbits [JRT, Sec. 8.7]

(k) Satellites – Changes of orbits [JRT, Sec. 8.8]

Illustrative examples 5 and 6

Exercise sheets 6 and 7

Unit IV

4. **The equation of continuity** [$\sim$ 9 lectures (September 26–October 17) + 2 tutorials (October 12 and 19) + mini test 4 (October 16, covering parts of Units III and IV)]

WEEK X (a) Vector calculus – Gradient and divergence [PM, Secs. 2.3, 2.4, 2.8; HMS, pp. 137–140]

(b) Curvilinear coordinates – Examples of electrostatic and gravitational fields

WEEK XI (c) Integral calculus – Line, surface and volume integrals [HMS, pp. 17–29, 63–70]

(d) Flux and divergence – Gauss's divergence theorem [PM, Secs. 1.9, 2.8–2.10; HMS, pp. 29–32, 36–41, 44–49]

WEEK XII (e) Determining symmetric electrostatic fields from Gauss's law [PM, Secs. 1.10–1.13]

(f) Equation of continuity – Conservation of mass and charge [JRT, Sec. 16.12; HMS, pp. 50–52]

Illustrative examples 7 and 8

Exercise sheets 8 and 9

Quiz II (October 24)

Unit V

5. **Circulation of vector fields** [$\sim$ 9 lectures (October 23–November 8) + 2 tutorials (November 2 and 9) + mini test 5 (November 6, covering parts of Units IV and V)]

WEEK XIII (a) Concepts of circulation and curl of a vector field [PM, Subsec. 2.14, 2.16, 2.17]

(b) Stokes' theorem [PM, Subsec. 2.15; HMS, pp. 92–100]

WEEK XIV (c) Irrotational vector fields – Vorticity

(d) Magentostatic fields

WEEK XV (e) Fluid flow – Bernoulli's principle [JRT, Sec. 16.12]

Illustrative examples 9

Exercise sheet 10

End-of-semester exam (November 17)

Last updated on October 31, 2017

Basic textbooks

1. J. R. Taylor, *Classical Mechanics* (University Science Books, Mill Valley, California, 2004). [JRT]
2. D. Kleppner and R. Kolenkow, *An Introduction to Mechanics*, Second Edition (Cambridge University Press, Cambridge, England, 2014). [KK]
3. D. Morin, *Introduction to Classical Mechanics: With Problems and Solutions* (Cambridge University Press, Cambridge, England, 2008). [DM]
4. E. M. Purcell and D. J. Morin, *Electricity and Magnetism*, Berkeley Physics Course, Volume II, Third Edition (Cambridge University Press, Cambridge, England, 2013). [PM]
5. H. M. Schey, *Div, Grad, Curl and All That*, Third Edition (W. W. Norton Company, New York, 1997). [HMS]

Additional references

1. S. T. Thornton and J. B. Marion, *Classical Dynamics of Particles and Systems*, (Cengage Learning, Singapore, 2004). [TM]
2. C. Kittel, W. D. Knight, M. A. Ruderman, C. A. Helmholz and B. J. Moyer, *Mechanics*, Berkeley Physics Course, Volume I, Second Edition (Tata McGraw-Hill, New Delhi, 2007). [KKRHM]
3. R. P. Feynman, R. B. Leighton and M. Sands, *The Feynman Lectures on Physics* Volume I (Narosa, New Delhi, 2008). [FLS]
4. D. J. Griffiths, *Introduction to Electrodynamics*, Fourth Edition (Prentice-Hall of India, New Delhi, 2012). [DJG]

Last updated on August 4, 2017

Syllabus and structure

(Student's version)

Physics I

Unit I

1. **Revisiting Newton's laws of motion** [$\sim$ 6 lectures + 2 tutorials]

   (a) Space and time – Notion of a point particle [JRT, Secs. 1.1, 1.2]

   (b) Concept of vectors – Choice of coordinates [JRT, Sec. 1.2]

   (c) Properties of vectors – Vector addition and scalar multiplication – Scalar and vector products – Examples [JRT, Sec. 1.2; KK, Secs. 1.2–1.7]

   (d) Differentiation of vectors – Velocity and acceleration [JRT, Sec. 1.2; KK, Secs. 1.8, 1.9]

   (e) Mass and force – Newton's first and second laws – Reference frames – Inertial frames [JRT, Secs. 1.3, 1.4, 1.6; KK, Secs. 2.1–2.5]

   (f) Newton's third law – Multi-particle systems – Conservation of momentum – Elastic collisions [JRT, Sec. 1.5, 3.1, 4.9; KK, Sec. 2.6, 4.5]

   (g) Kinetic energy and work – Potential energy – Conservative forces – Force as a gradient of potential energy – Gravitational and electrostatic examples [JRT, Secs. 4.1–4.4; KK, Secs. 5.1, 5.3, 5.5, 5.6, Note 5.2]

   (h) Conservation of energy – Energy for linear one-dimensional systems – Turning points [JRT, Secs. 4.6–4.7; KK, Secs. 5.2, 5.4–5.7, 5.10]

Illustrative examples 1 and 2

Exercise sheets 1 and 2

Unit II

2. **Equilibrium and oscillations in physical systems** [$\sim$ 10 lectures + 3 tutorials]

   (a) Hooke's law – Simple harmonic motion – Solutions – Examples – Complex exponentials [JRT, Secs. 5.1, 5.2, 2.6; KK, Secs. 11.1, 11.2]

   (b) Damped oscillations – Driven damped oscilations – Resonance – LCR circuits [JRT, Secs. 5.4, 5.5, 5.6; KK, Secs. 11.3–11.6, Notes 11.1–11.3]

   (c) The inverted oscillator – Solutions to the equation of motion [TM, Sec. 4.3]

   (d) Motion of a particle in a potential – Equilibrium, stability and small oscillations [TM, Sec. 4.2]

   (e) Phase space – Trajectories in phase space of the oscillator and inverted oscillator – Trajectories in a generic one-dimensional potential [TM, Sec. 3.4, 4.3, 4.4]

Illustrative examples 3 and 4

Exercise sheets 3, 4 and 5

Quiz I (September 11)

Unit III

3. **Motion in a central force field** [$\sim$ 9 lectures + 2 tutorials]

   (a) Angular momentum of a single particle – Conservation of angular momentum – Kepler's second law [JRT, Sec. 3.4; KK, Secs. 7.1–7.2]

   (b) Angular momentum for several particles – Moment of inertia [JRT, Sec. 3.5]

(c) Central force – Conservation of angular momentum [JRT, Sec. 4.8]

(d) Continuous symmetries – Conservation of linear momentum, angular momentum and energy

(e) Discrete symmetries – Time reversal symmetry – Reflection and inversion – Polar and axial vectors – Linear and angular momentum as examples

(f) Two dimensional polar coordinates [JRT, Sec. 1.7]

(g) Centre of mass – Relative coordinates – Reduced mass [JRT, Secs. 8.1, 8.2]

(h) Conservation of energy – The equivalent one-dimensional problem – The concept of the effective potential [JRT, Sec. 8.4; KK, Secs. 10.2–10.4]

(i) The orbit equation – Planetary motion – Kepler's laws [JRT, Secs. 8.5, 8.6; KK, Secs. 10.5, 10.6, Notes 10.1, 10.2]

(j) The unbounded Kepler orbits [JRT, Sec. 8.7]

(k) Satellites – Changes of orbits [JRT, Sec. 8.8]

Illustrative examples 5 and 6

Exercise sheets 6 and 7

Unit IV

4. **The equation of continuity** [$\sim$ 9 lectures + 2 tutorials]

(a) Vector calculus – Gradient and divergence [PM, Secs. 2.3, 2.4, 2.8; HMS, pp. 137–140]

(b) Curvilinear coordinates – Examples of electrostatic and gravitational fields

(c) Integral calculus – Line, surface and volume integrals [HMS, pp. 17–29, 63–70]

(d) Flux and divergence – Gauss's divergence theorem [PM, Secs. 1.9, 2.8–2.10; HMS, pp. 29–32, 36–41, 44–49]

(e) Determining symmetric electrostatic fields from Gauss's law [PM, Secs. 1.10–1.13]

(f) Equation of continuity – Conservation of mass and charge [JRT, Sec. 16.12; HMS, pp. 50–52]

Illustrative examples 7 and 8

Exercise sheets 8 and 9

Quiz II (October 24)

Unit V

5. **Circulation of vector fields** [$\sim$ 9 lectures + 2 tutorials]

(a) Concepts of circulation and curl of a vector field [PM, Subsec. 2.14, 2.16, 2.17]

(b) Stokes' theorem [PM, Subsec. 2.15; HMS, pp. 92–100]

(c) Irrotational vector fields – Vorticity

(d) Magentostatic fields

(e) Fluid flow – Bernoulli's principle [JRT, Sec. 16.12]

Illustrative examples 9

Exercise sheet 10

End-of-semester exam (November 17)

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3. D. Morin, *Introduction to Classical Mechanics: With Problems and Solutions* (Cambridge University Press, Cambridge, England, 2008). [DM]
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4. D. J. Griffiths, *Introduction to Electrodynamics*, Fourth Edition (Prentice-Hall of India, New Delhi, 2012). [DJG]

Last updated on August 11, 2017

Conventions and notations

We shall list here the conventions and notations that we shall adopt. We shall keep adding to this list as we make progress.

Coordinates

Depending on the problem at hand, we shall choose to work with one of the following three coordinate systems: Cartesian, cylindrical polar or spherical polar coordinates.

- Cartesian coordinates: (x, y, z). The range of the coordinates are $-\infty < (x, t, z) < \infty$.
- Cylindrical polar coordinates: (ρ, ϕ, z). The range of the coordinates are as follows: $0 \leq \rho < \infty$, $0 \leq \phi < 2\,\pi$ and $-\infty < z < \infty$.

  Note: The angle ϕ is measured with respect to the x-axis in the x-y-plane.
- Spherical polar coordinates: (r, θ, ϕ). The range of the coordinates are as follows: $0 \leq r < \infty$, $0 \leq \theta < \pi$ and $0 \leq \phi < 2\,\pi$.

  Note: As in the case of the cylindrical polar coordinates, the angle ϕ is measured with respect to the x-axis in the x-y-plane. The angle θ is measured with respect to the z-axis.

Vectors

- We shall denote the vectors with a boldface. For instance, the position, velocity, momentum, acceleration, angular momentum and force will typically be represented as $\boldsymbol{r}$, $\boldsymbol{v}$, $\boldsymbol{p}$, $\boldsymbol{a}$, $\boldsymbol{L}$ and $\boldsymbol{F}$, respectively. The corresponding amplitudes will be denoted as r, v, p, a, L and F.
- The unit vectors will be denoted in boldface, along with a hat. For instance, the unit vector associated with the vector $\boldsymbol{A}$ will be denoted as $\hat{\boldsymbol{A}}$.
- The unit vectors in the Cartesian coordinates will be often represented as $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$. These unit vectors satisfy the relations

  $$\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{x}} = \hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{y}} = \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{z}} = 1, \quad \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{y}} = \hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{z}} = \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{x}} = 0,$$

  and

  $$\hat{\boldsymbol{x}} \times \hat{\boldsymbol{y}} = \hat{\boldsymbol{z}}, \quad \hat{\boldsymbol{y}} \times \hat{\boldsymbol{z}} = \hat{\boldsymbol{x}}, \quad \hat{\boldsymbol{z}} \times \hat{\boldsymbol{x}} = \hat{\boldsymbol{y}}.$$

  We should mention that some textbooks (and instructors!) may also represent the unit vectors $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$ as $(\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}})$ or $(\hat{\boldsymbol{e}}_x, \hat{\boldsymbol{e}}_y, \hat{\boldsymbol{e}}_z)$.
- The unit vectors in the cylindrical polar coordinates will be represented as $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\phi}}$ and $\hat{\boldsymbol{z}}$. They satisfy the relations

  $$\hat{\boldsymbol{\rho}} \cdot \hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{z}} = 1, \quad \hat{\boldsymbol{\rho}} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{z}} = \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{\rho}} = 0,$$

  and

  $$\hat{\boldsymbol{\rho}} \times \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{z}}, \quad \hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{z}} = \hat{\boldsymbol{\rho}}, \quad \hat{\boldsymbol{z}} \times \hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\phi}}.$$

  The unit vectors $(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{z}})$ are also represented as $(\hat{\boldsymbol{e}}_\rho, \hat{\boldsymbol{e}}_\phi, \hat{\boldsymbol{e}}_z)$.
- The unit vectors in the spherical polar coordinates will be denoted as $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$. The satisfy the relations

  $$\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = 1, \quad \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{r}} = 0,$$

  and

  $$\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}, \quad \hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{r}}, \quad \hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{r}} = \hat{\boldsymbol{\theta}}.$$

  The unit vectors $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}})$ are also denoted as $(\hat{\boldsymbol{e}}_r, \hat{\boldsymbol{e}}_\theta, \hat{\boldsymbol{e}}_\phi)$.

Illustrative examples 1

Vector operations

1. *Area of a triangle and the law of sines:* Let the three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ form the three edges of a triangle in a cyclic fashion. Let α, β and γ be the three angles of the triangle at the vertices opposite to the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. [JRT, Problem 1.18]

   (a) Prove that the area, say, A, of the triangle is given by

   $$A = \frac{1}{2}\,|\boldsymbol{a}\times\boldsymbol{b}| = \frac{1}{2}\,|\boldsymbol{b}\times\boldsymbol{c}| = \frac{1}{2}\,|\boldsymbol{c}\times\boldsymbol{a}|.$$

   (b) Use this equality to establish that

   $$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

2. *Vector proof of trigonometric identities:* Let $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{b}}$ be two unit vectors in the x-y plane making angles θ and ϕ with the x-axis, respectively. [KK, Problem 1.8; JRT, Problem 1.22]

   (a) Show that

   $$\begin{aligned}\hat{\boldsymbol{a}} &= \cos\theta\,\hat{\boldsymbol{x}} + \sin\theta\,\hat{\boldsymbol{y}},\\ \hat{\boldsymbol{b}} &= \cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}}.\end{aligned}$$

   (b) Using vector algebra prove that

   $$\begin{aligned}\cos(\phi-\theta) &= \cos\phi\,\cos\theta + \sin\phi\,\sin\theta,\\ \sin(\phi-\theta) &= \sin\phi\,\cos\theta - \cos\phi\,\sin\theta.\end{aligned}$$

3. *Trajectory and velocity of a particle:* The position of a particle is given by [KK, Example 1.7]

   $$\boldsymbol{r}(t) = A\,\left(\mathrm{e}^{\alpha t}\,\hat{\boldsymbol{x}} + \mathrm{e}^{-\alpha t}\,\hat{\boldsymbol{y}}\right),$$

   where A and α are positive constants. Determine the velocity of the particle. Sketch its trajectory and indicate the direction and magnitude of the velocity vector at different points on the trajectory.

4. *Particle on a circular orbit:* A particle is moving in the x-y-plane on a circular trajectory with the origin as its centre and radius R. It is moving in a counter-clockwise direction with angular frequency ω. Let the particle start from the position $(x, y) = (R, 0)$ at $t = 0$. [KK, Example 1.8; JRT, Problem 1.10]

   (a) Show that the trajectory of the particle is described by the position vector

   $$\boldsymbol{r}(t) = R\,[\cos(\omega\,t)\,\hat{\boldsymbol{x}} + \sin(\omega\,t)\,\hat{\boldsymbol{y}}]\,.$$

   (b) Determine the particle's velocity and acceleration.

   (c) By taking the scalar product of the particle's position vector and velocity, show that the direction of velocity is perpendicular to the direction of the position vector.

   (d) What are the magnitude and direction of acceleration?

   Note: The acceleration that is directed radially inwards is known as the centripetal acceleration.

5. *Derivative of a triple product:* Let $\boldsymbol{r}$, $\boldsymbol{v}$ and $\boldsymbol{a}$ denote the position, velocity and acceleration of a particle. Prove that [JRT, Problem 1.19]

   $$\frac{\mathrm{d}}{\mathrm{d}t}\,[\boldsymbol{a}\cdot(\boldsymbol{v}\times\boldsymbol{r})] = \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\cdot(\boldsymbol{v}\times\boldsymbol{r})\,.$$

Illustrative examples 1 with solutions

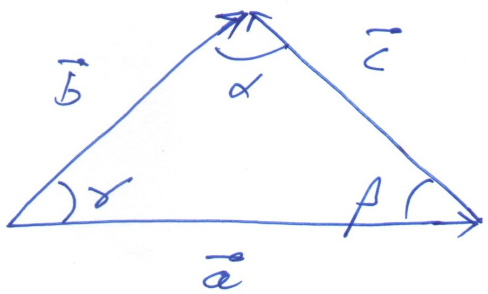
Vector operations

1. <u>*Area of a triangle and the law of sines:*</u> Let the three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ form the three edges of a triangle in a cyclic fashion. Let α, β and γ be the three angles of the triangle at the vertices opposite to the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. [JRT, Problem 1.18]

   (a) Prove that the area, say, A, of the triangle is given by

   $$A = \frac{1}{2}\,|\boldsymbol{a}\times\boldsymbol{b}| = \frac{1}{2}\,|\boldsymbol{b}\times\boldsymbol{c}| = \frac{1}{2}\,|\boldsymbol{c}\times\boldsymbol{a}|.$$

   <u>*Solution:*</u> Consider the triangle as shown in the figure below.



   Note that a is the base of the triangle and the height of the triangle is $h = b\sin\gamma$. Therefore, the area of the triangle is given by

   $$A = \frac{1}{2}\,a\,(b\sin\gamma).$$

   These arguments can be applied by considering the other sides of the triangle to be bases, to arrive at the required relation.

   (b) Use this equality to establish that

   $$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

   <u>*Solution:*</u> By expressing the area of the triangle in terms of the three sides as bases, we obtain that

   $$a\,b\sin\gamma = b\,c\sin\alpha = c\,a\sin\beta,$$

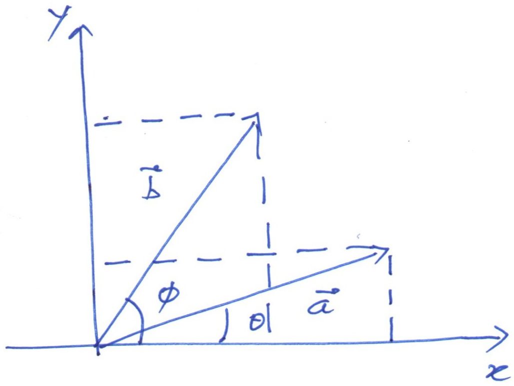
   which, upon diving by $a\,b\,c$, leads to the required result.

2. <u>*Vector proof of trigonometric identities:*</u> Let $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{b}}$ be two unit vectors in the x-y plane making angles θ and ϕ with the x-axis, respectively. [KK, Problem 1.8; JRT, Problem 1.22]

   (a) Show that

   $$\begin{aligned}\hat{\boldsymbol{a}} &= \cos\theta\,\hat{\boldsymbol{x}} + \sin\theta\,\hat{\boldsymbol{y}},\\ \hat{\boldsymbol{b}} &= \cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}}.\end{aligned}$$

   <u>*Solution:*</u> Consider the figure below.



It is evident from the geometry that

$$(a_x = a\cos\theta, a_y = a\sin\theta), \quad (b_x = b\cos\phi, b_y = b\sin\phi),$$

which immediately leads to the required result if we set $a = 1$ and $b = 1$.

(b) Using vector algebra prove that

$$\begin{aligned}\cos(\phi - \theta) &= \cos\phi\,\cos\theta + \sin\phi\,\sin\theta,\\ \sin(\phi - \theta) &= \sin\phi\,\cos\theta - \cos\phi\,\sin\theta.\end{aligned}$$

Solution: Note that $\boldsymbol{a} = a\,\hat{\boldsymbol{a}}$ and $\boldsymbol{b} = b\,\hat{\boldsymbol{b}}$. Hence, we have (when $a = b = 1$)

$$\hat{\boldsymbol{a}} \cdot \hat{\boldsymbol{b}} = \cos(\phi - \theta) = \cos\phi\,\cos\theta + \sin\phi\,\sin\theta$$

and

$$\hat{\boldsymbol{a}} \times \hat{\boldsymbol{b}} = \sin(\phi - \theta)\,\hat{\boldsymbol{z}} = (\sin\phi\,\cos\theta - \cos\phi\,\sin\theta)\;\hat{\boldsymbol{z}},$$

which are the required results.

3. *Trajectory and velocity of a particle:* The position of a particle is given by [KK, Example 1.7]

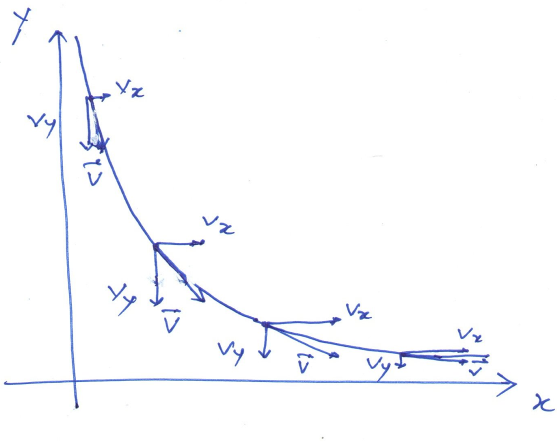
$$\boldsymbol{r}(t) = A\,\left(\mathrm{e}^{\alpha\,t}\,\hat{\boldsymbol{x}} + \mathrm{e}^{-\alpha\,t}\,\hat{\boldsymbol{y}}\right),$$

where A and α are positive constants. Determine the velocity of the particle. Sketch its trajectory and indicate the direction and magnitude of the velocity vector at different points on the trajectory.

Solution: We have

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = A\,\alpha\,\left(\mathrm{e}^{\alpha\,t}\,\hat{\boldsymbol{x}} - \mathrm{e}^{-\alpha\,t}\,\hat{\boldsymbol{y}}\right),$$

which implies that, as the velocity along the x-direction increases exponentially, the velocity along the (negative) y-direction decreases exponentially. Note that the trajectory of the curve in the x-y-plane is described by the function $y = A^2/x$, while the velocity components are related as $v_y = -(A\,\alpha)^2/v_x$. It is useful to notice that the range of time is $-\infty < t < \infty$. The y-component of the velocity (which is always pointing along the negative direction) is very large as $t \to -\infty$, and goes to zero as $t \to -\infty$, In contrast, the x-component of the velocity is very small at very early times (i.e. as $t \to -\infty$), and grows to be large at very late times (i.e. as $t \to \infty$). The velocity vector is described by the tangent to the curve in the x-y plane, as illustrated in the figure below.



4. *Particle on a circular orbit:* A particle is moving in the x-y-plane on a circular trajectory with the origin as its centre and radius R. It is moving in a counter-clockwise direction with angular frequency ω. Let the particle start from the position $(x, y) = (R, 0)$ at $t = 0$.
[KK, Example 1.8; JRT, Problem 1.10]

(a) Show that the trajectory of the particle is described by the position vector

$$\boldsymbol{r}(t) = R\,[\cos(\omega\,t)\,\hat{\boldsymbol{x}} + \sin(\omega\,t)\,\hat{\boldsymbol{y}}]\,.$$

Solution: Clearly, corresponding to the initial condition mentioned above, at any given time, we have

$$x(t) = R\cos(\omega\,t), \quad y(t) = R\sin(\omega\,t),$$

which leads to the required expression. Note that $x^2 + y^2 = R^2$ at all times, which confirms that the orbit is a circle.

(b) Determine the particle's velocity and acceleration.
Solution: We have

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \omega\,R\,[-\sin(\omega\,t)\,\hat{\boldsymbol{x}} + \cos(\omega\,t)\,\hat{\boldsymbol{y}}]\,,$$
$$\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\omega^2\,R\,[\cos(\omega\,t)\,\hat{\boldsymbol{x}} + \sin(\omega\,t)\,\hat{\boldsymbol{y}}]\,.$$

(c) By taking the scalar product of the particle's position vector and velocity, show that the direction of velocity is perpendicular to the direction of the position vector.
Solution: We have

$$\boldsymbol{r}\cdot\boldsymbol{v} = \omega\,R^2\,[-\cos(\omega\,t)\sin(\omega\,t) + \sin(\omega\,t)\cos(\omega\,t)] = 0,$$

which implies that the direction of velocity is perpendicular to the direction of the position vector. Since the particle's position vector is along the radius, the direction of the velocity is then along the tangent to the circle.

(d) What are the magnitude and direction of acceleration?
Solution: We have

$$\boldsymbol{a} = -\omega^2\,\boldsymbol{r},$$

i.e. the acceleration is directed radially inwards and it has the magnitude $a = -\omega^2\,r$.

Note: The acceleration that is directed radially inwards is known as the centripetal acceleration.

5. *Derivative of a triple product:* Let $\boldsymbol{r}$, $\boldsymbol{v}$ and $\boldsymbol{a}$ denote the position, velocity and acceleration of a particle. Prove that [JRT, Problem 1.19]

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\boldsymbol{a}\cdot(\boldsymbol{v}\times\boldsymbol{r})\right] = \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\cdot(\boldsymbol{v}\times\boldsymbol{r})\,.$$

Solution: The dot and the cross products obey the Liebniz rule so that we have

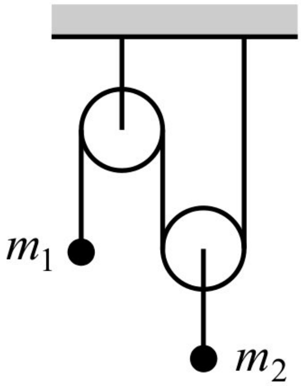
$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[\boldsymbol{a}\cdot(\boldsymbol{v}\times\boldsymbol{r})\right] &= \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\cdot(\boldsymbol{v}\times\boldsymbol{r}) + \boldsymbol{a}\cdot\left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\times\boldsymbol{r}\right) + \boldsymbol{a}\cdot\left(\boldsymbol{v}\times\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)\\
&= \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\cdot(\boldsymbol{v}\times\boldsymbol{r}) + \boldsymbol{a}\cdot(\boldsymbol{a}\times\boldsymbol{r}) + \boldsymbol{a}\cdot(\boldsymbol{v}\times\boldsymbol{v})\\
&= \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\cdot(\boldsymbol{v}\times\boldsymbol{r}) + \boldsymbol{r}\cdot(\boldsymbol{a}\times\boldsymbol{a})\\
&= \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\cdot(\boldsymbol{v}\times\boldsymbol{r})\,,
\end{aligned}$$

where, in the third equality, we have made use of the cyclic property of the scalar triple product.

Illustrative examples 2

Newton's laws of motion and conserved quantities

1. An Atwood's machine: Consider a pulley system as shown in the figure below. [DM, pages 58–59]



Assuming the pulleys and string to be massless, determine the accelerations of the two masses and also find the tension in the string.

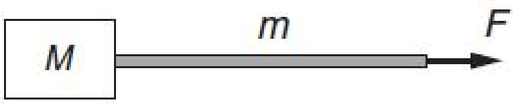
2. Projectile motion: Consider a particle that is moving freely under the influence of gravity so that it has a constant downward acceleration g. Without any loss of generality, assume that the particle is moving in the x-y-plane, where the y axis is vertically upward. [JRT, Problem 1.40]

   (a) Neglecting air resistance, integrate the second order differential equation governing the system to obtain the trajectory of the particle. Show that the particle describes a parabola in the x-y-plane.

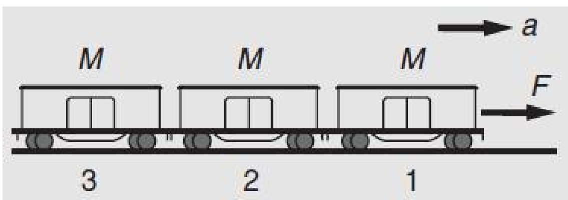
   (b) The mouth of a cannon which is located at the origin in the x-y-plane shoots a ball at an angle θ with respect to the x-axis. Let $r(t)$ denote the ball's distance from the cannon. What is the largest possible value of θ if $r(t)$ is to increase throughout the ball's flight?

   Hint: Using your solution to first part, you can write down r^2 as $x^2 + y^2$, and then find the condition that r^2 is always increasing.

3. (a) Pulling a block: A string of mass m attached to a block of mass M is pulled with a force F as shown in the figure below. Neglecting gravity, determine the force that the string transmits to the block. [KK, Example 3.2]



   (b) Forces on a freight train: Three freight cars of mass M each are pulled with a force F by a locomotive. Assuming friction to be negligible, find the forces on each car. [KK, Example 2.4]



4. Conservation of angular momentum: Consider a particle that is moving in a force field which is directed radially from a centre, i.e. $\boldsymbol{F} \propto \boldsymbol{r}$. Let $\boldsymbol{r}$ and $\boldsymbol{p}$ denote the position vector of the particle

and its momentum with respect to the centre. Show that the angular momentum of the particle $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ is conserved. [JRT, Section 3.4]

Note: We shall make use of this result later when we study the motion of the planets around the Sun.

5. *Shifting the equilibrium position:* Consider the force exerted by a one-dimensional spring, which is fixed at one end. The force can be expressed as $F = -k\,x$, where x is the displacement of the free end from its equilibrium position. [JRT, Problem 4.9]

   (a) Assuming that this force is conservative, show that the corresponding potential energy is $U = k\,x^2/2$, if we choose U to be zero at the equilibrium position.

   (b) Suppose that this spring is hung vertically from the ceiling with a mass m suspended from the other end and constrained to move in the vertical direction only. Find the extension x_0 of the new equilibrium position with the suspended mass. Show that the total potential energy (spring plus gravity) has the same form, viz. $k\,y^2/2$, if we use the coordinate y equal to the displacement measured from the new equilibrium position at $x = x_0$ (and redefine our reference point so that $U = 0$ at $y = 0$).

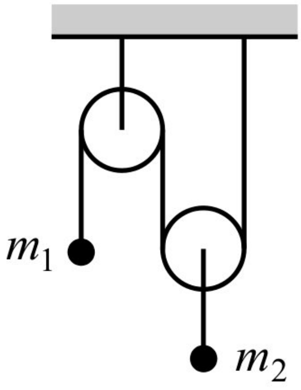
   (c) Plot the original and the modified potentials as a function of x. What are the allowed ranges of energy?

Last updated on August 4, 2017

Illustrative examples 2 with solutions

Newton's laws of motion and conserved quantities

1. *An Atwood's machine:* Consider a pulley system as shown in the figure below. [DM, pages 58–59]

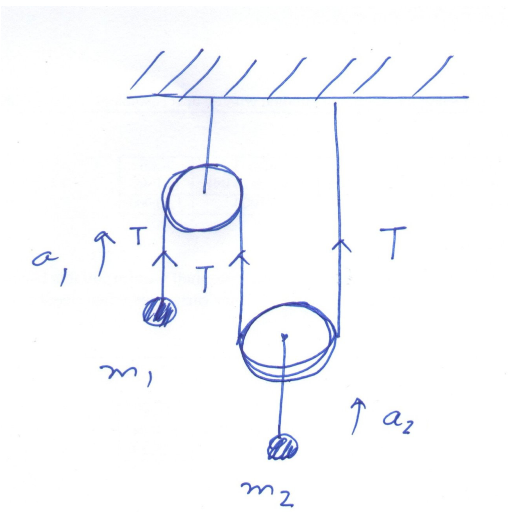


Assuming the pulleys and string to be massless, determine the accelerations of the two masses and also find the tension in the string.

Solution: Let T be tension throughout the *massless* string. In such a case, the masses m_1 and m_2 satisfy the equations

$$T - m_1\, g = m_1\, a_1, \quad 2\,T - m_2\, g = m_2\, a_2,$$

where a_1 and a_2 are the accelerations of the masses m_1 and m_2, and g, of course, is the acceleration due to gravity (see figure below).



Now, note that, when m_2 moves up or down by a given length, the mass m_1 moves by twice as much, which implies that $a_1 = -2\, a_2$. Note that the relative minus sign arises due to the fact that the mass m_1 goes down when m_2 goes up and vice-versa. Therefore, we have

$$T - m_1\, g = -2\, m_1\, a_2, \quad 2\,T - m_2\, g = m_2\, a_2$$

or, equivalently,

$$2\,T - 2\, m_1\, g = -4\, m_1\, a_2, \quad 2\,T - m_2\, g = m_2\, a_2.$$

Upon subtracting these two equations, we obtain that

$$a_2 = \frac{2\,m_1 - m_2}{4\,m_1 + m_2}\,g,$$

which then leads to

$$a_1 = -2\,a_2 = \frac{2\,m_2 - 4\,m_1}{4\,m_1 + m_2}\,g.$$

One of these results for a_1 and a_2 can be used in the above equations to determine T to be

$$T = \frac{3\,m_1\,m_2}{4\,m_1 + m_2}\,g.$$

2. <u>*Projectile motion:*</u> Consider a particle that is moving freely under the influence of gravity so that it has a constant downward acceleration g. Without any loss of generality, assume that the particle is moving in the x-y-plane, where the y axis is vertically upward. [JRT, Problem 1.40]

   (a) Neglecting air resistance, integrate the second order differential equation governing the system to obtain the trajectory of the particle. Show that the particle describes a parabola in the x-y-plane.

   <u>*Solution:*</u> The equation of motion governing the particle is given by

   $$m\,\ddot{\boldsymbol{r}} = -m\,g\,\hat{\boldsymbol{y}},$$

   where $\boldsymbol{r} = x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}}$ and g is the acceleration due to gravity. Clearly, the mass m of the particle cancels out in the above equation, which implies that the trajectory is actually independent of mass. The equations along the x and y-directions are given by

   $$\ddot{x} = 0, \quad \ddot{y} = -g,$$

   which can be immediately integrated to arrive at

   $$x(t) = x_0 + v_x^0\,t, \qquad y(t) = y_0 + v_y^0\,t - \frac{1}{2}\,g\,t^2,$$

   where (x_0, y_0) and (v_x^0, v_y^0) are the initial positions and the velocities along the two directions at $t = 0$. Let us set $x_0 = y_0 = 0$ without any loss of generality. In such a case, we have $t = x/v_x^0$ so that we can write

   $$y = \frac{v_y^0}{v_x^0}\,x - \frac{g}{2\,v_x^{0\,2}}\,x^2,$$

   which evidently describes a parabola in the x-y-plane.

   (b) The mouth of a cannon which is located at the origin in the x-y-plane shoots a ball at an angle θ with respect to the x-axis. Let $r(t)$ denote the ball's distance from the cannon. What is the largest possible value of θ if $r(t)$ is to increase throughout the ball's flight?

   Hint: Using your solution to first part, you can write down r^2 as $x^2 + y^2$, and then find the condition that r^2 is always increasing.

   <u>*Solution:*</u> Note that $v_x^0 = v_0 \cos\theta$ and $v_y^0 = v_0 \sin\theta$, where v_0 is the magnitude of the velocity of the ball as it emerges from the mouth of the canon. In such a case, if we make use of the above result, we obtain that

   $$r^2(t) = x^2(t) + y^2(t) = v_0^2\,t^2 + \frac{g^2}{4}\,t^4 - g\,v_0 \sin\theta\,t^3.$$

   If $r(t)$ is to increase throughout the ball's flight, then we require $\mathrm{d}r^2/\mathrm{d}t > 0$, which leads to

   $$\frac{\mathrm{d}r^2}{\mathrm{d}t} = 2\,v_0^2\,t + g^2\,t^3 - 3\,g\,v_0 \sin\theta\,t^2 > 0$$

or, equivalently (since we are assuming that $t > 0$)

$$g^2\, t^2 - 3\, g\, v_0 \sin\theta\, t + 2\, v_0^2 > 0$$

Let us set this inequality to be an equality, in which case it becomes a quadratic equation in t. The solutions for t are

$$t = \frac{1}{2\, g^2} \left(3\, g\, v_0 \sin\theta \pm 3\, g\, v_0 \sqrt{\sin^2\theta - \frac{8}{9}}\right),$$

which will be real only when

$$\sin^2\theta \geq \frac{8}{9}.$$

The equality corresponds to $\theta \simeq 70.5°$. Now, note that, when $\theta = 0$, y is zero and $x(= r)$ is always an increasing function of t. In contrast, when $\theta = 90°$, $x = 0$, while $y(= r)$ increases initially, reaches a maximum before decreasing and eventually reducing to zero. These arguments suggest that $r^2(t)$ is an increasing function of t for $\theta < 70.5°$.

Actually, the above argument is convoluted and there is a simpler way of arriving at the above result. Note that a quadratic function is positive if its discriminant is negative (and, vice versa). This is easy to establish. Consider the general inequality

$$a\, x^2 + b\, x + c > 0.$$

This can be written as (when $a > 0$, as in our case of interest)

$$\left(\sqrt{a}\, x + \frac{b}{2\sqrt{a}}\right)^2 - \left(\frac{b^2}{4\, a} - c\right) > 0,$$

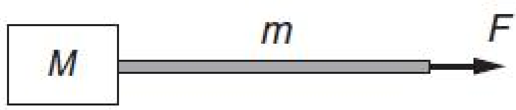
an inequality which will clearly be satisfied if $\left(b^2 - 4\, a\, c\right) < 0$ or $b^2 < 4\, a\, c$. Therefore, in our case, $\mathrm{d}r^2/\mathrm{d}t$ will be positive when

$$(3\, g\, v_0 \sin\theta)^2 < 8\, g^2\, v_0^2$$

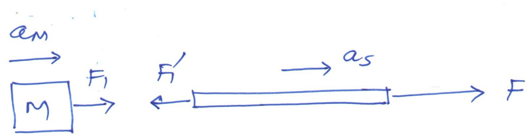
or, equivalently, when

$$\sin^2\theta < \frac{8}{9}.$$

3. (a) *Pulling a block:* A string of mass m attached to a block of mass M is pulled with a force F as shown in the figure below. Neglecting gravity, determine the force that the string transmits to the block. [KK, Example 3.2]



Solution: Let a_{s} and a_{M} be the acceleration of the string and the block, respectively.



From the figure above, it is clear that the equations of motion of the block and the string are

$$F_1 = M\, a_{\rm M}, \quad F - F_1' = m\, a_{\rm s},$$

where m is the mass of the string. We shall assume that the string does not stretch. In such a case, we have $a_{\rm s} = a_{\rm M} = a$. Also, according to Newton's third law $F_1 = F_1'$, so that we obtain

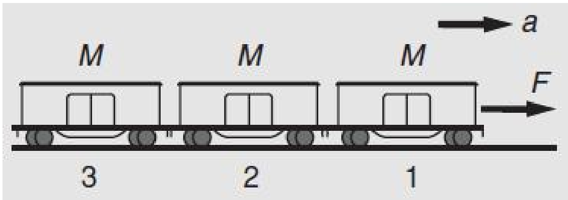
$$a = \frac{F}{M+m}.$$

Therefore,

$$F_1 = F_1' = \frac{M}{M+m}\, F,$$

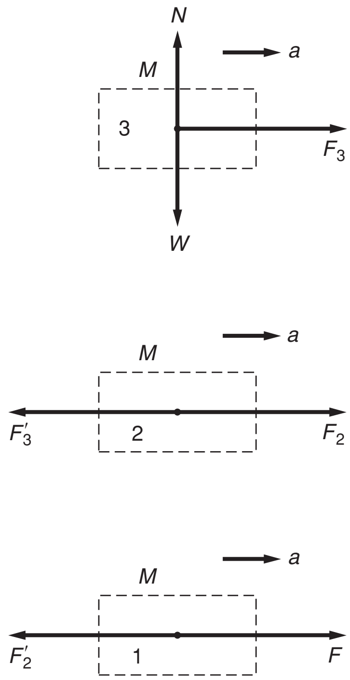
i.e. the force on the block is less than F. In other words, the string does not transmit the full applied force. Note that, when $m \ll M$, $F_1 \simeq F$ and the string transmits most of the force. At the other extreme, when $m \gg M$, $F_1 \simeq 0$, as there is practically no load for the string to pull.

(b) *Forces on a freight train:* Three freight cars of mass M each are pulled with a force F by a locomotive. Assuming friction to be negligible, find the forces on each car. [KK, Example 2.4]



Solution: As the cars are joined, they move with the same acceleration and hence

$$a = \frac{F}{3\,M}.$$



Let us first consider car 3. From the above figure, it is clear that $N = W$ (as there is no upward or downward motion) and the force F_3 on car 3 is

$$F_3 = M\,a = M\,\frac{F}{3\,M} = \frac{F}{3}.$$

As far as the middle car is concerned, we have

$$F_2 - F_3' = M\,a$$

and, as $F_3' = F_3$, we obtain that

$$F_2 = F_3 + M\,a = \frac{F}{3} + \frac{F}{3} = \frac{2\,F}{3}.$$

Let us now turn to car 1. We have

$$F - F_2' = M\,a,$$

so that, since $F_2 = F_2'$,

$$F = F_2 + M\,a = \frac{2\,F}{3} + \frac{F}{3} = F,$$

which is the last of the required results. In summary, each car is pulled by the force $F/3$ to the right.

4. *Conservation of angular momentum:* Consider a particle that is moving in a force field which is directed radially from a centre, i.e. $\boldsymbol{F} \propto \boldsymbol{r}$. Let $\boldsymbol{r}$ and $\boldsymbol{p}$ denote the position vector of the particle and its momentum with to respect the centre. Show that the angular momentum of the particle $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ is conserved. [JRT, Section 3.4]

   Note: We shall make use of this result later when we study the motion of the planets around the Sun.

   Solution: Since $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ and $\boldsymbol{p} = m\,\boldsymbol{v}$, we have

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \times \boldsymbol{p} + \boldsymbol{r} \times \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \boldsymbol{v} \times (m\,\boldsymbol{v}) + \boldsymbol{r} \times \boldsymbol{F}.$$

   The first term in the final equality evidently vanishes. As we have assumed that $\boldsymbol{F} \propto \boldsymbol{r}$, the second term vanishes as well implying that $\mathrm{d}\boldsymbol{L}/\mathrm{d}t = 0$ or, equivalently, $\boldsymbol{L}$ is conserved.

   This result implies that angular momentum is conserved in central force fields.

5. *Shifting the equilibrium position:* Consider the force exerted by a one-dimensional spring, which is fixed at one end. The force can be expressed as $F = -k\,x$, where x is the displacement of the free end from its equilibrium position. [JRT, Problem 4.9]

   (a) Assuming that this force is conservative, show that the corresponding potential energy is $U = k\,x^2/2$, if we choose U to be zero at the equilibrium position.

   Solution: In one dimension, we have $U(x) = -\int \mathrm{d}x\,F(x)$, so that the potential associated with the spring is

$$U_{\mathrm{s}}(x) = -\int \mathrm{d}x\,F(x) = k\int \mathrm{d}x\,x = \frac{k\,x^2}{2},$$

   and we have chosen the constant of integration such that $U = 0$ at $x = 0$.

   (b) Suppose that this spring is hung vertically from the ceiling with a mass m suspended from the other end and constrained to move in the vertical direction only. Find the extension x_0 of the new equilibrium position with the suspended mass. Show that the total potential energy (spring plus gravity) has the same form, viz. $k\,y^2/2$, if we use the coordinate y equal to the displacement measured from the new equilibrium position at $x = x_0$ (and redefine our reference point so that $U = 0$ at $y = 0$).

Solution: When the spring is hung vertically from the ceiling, the potential due to the gravitational field is $U_{\mathrm{g}}(x) = -m\,g\,x$, where g is the acceleration due to gravity. Therefore, the total potential energy associated with the mass is given by

$$U(x) = U_{\mathrm{s}}(x) + U_{\mathrm{g}}(x) = \frac{k\,x^2}{2} - m\,g\,x.$$

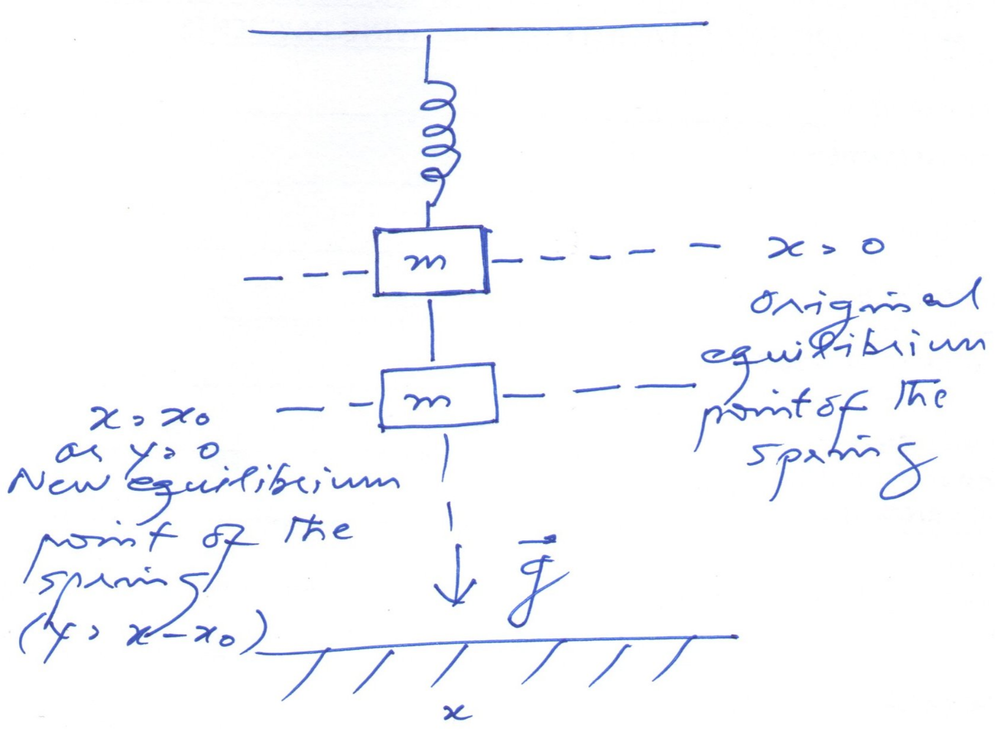
Recall that the location where the potential vanishes is always chosen as per convenience. Here, we have chosen such that the gravitational potential too vanishes at $x = 0$. The complete potential can be rewritten as

$$U(x) = \frac{k}{2}\left(x - \frac{m\,g}{k}\right)^2 - \frac{m^2\,g^2}{2\,k}$$

or, equivalently,

$$U(y) = \frac{k\,y^2}{2} - U_0,$$

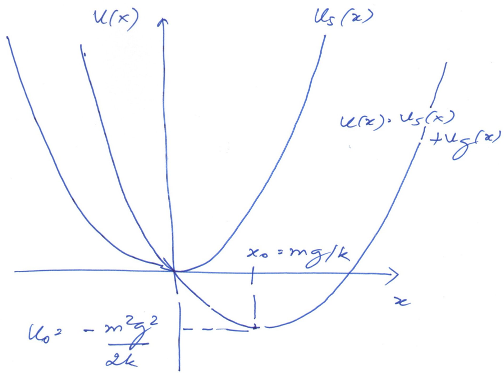
where $y = x - m\,g/k$ and $U_0 = m^2\,g^2/(2\,k)$. Note that $y = 0$ is the new point of equilibrium. We can redefine our potential energy such that $U = 0$ at $y = 0$, if needed. In terms of x, the new point of equilibrium will be $x_0 = m\,g/k$. This is illustrated in the figure below, with the positive x axis pointing vertically downwards.



In other words, gravity shifts the equilibrium point of the spring downwards.

(c) Plot the original and the modified potentials as functions of x. What are the allowed ranges of energy?

Solution: The potentials in the two cases will be as illustrated in the figure below.



If we work with the x coordinates, in the original case when gravity is absent, the allowed range of energy is $0 < E < \infty$. When acceleration due to gravity is included, the allowed range of energy is $U_0 < E < \infty$. It should be borne in mind that the 'zero' of the potential energy can be shifted, as is convenient. For that reason, the allowed range of energy will actually depend on the choice of the 'zero' of the potential energy.

Exercise sheet 1

Vector operations and Newton's laws of motion

1. (a) *Direction cosines:* The direction cosines of a vector are the cosines of the angles it makes with the coordinate axes. The cosines of the angles between the vector and the x, y, and z axes are usually denoted by α, β, and γ, respectively. Using vector algebra, prove that $\alpha^2+\beta^2+\gamma^2=1$. [KK, Problem 1.4]

   (b) *Constructing an unknown vector:* The unknown vector $\boldsymbol{v}$ satisfies the conditions $\boldsymbol{b}\cdot\boldsymbol{v}=\lambda$ and $\boldsymbol{b}\times\boldsymbol{v}=\boldsymbol{c}$, where λ, $\boldsymbol{b}$ and $\boldsymbol{c}$ are fixed and known. Express $\boldsymbol{v}$ in terms of λ, $\boldsymbol{b}$ and $\boldsymbol{c}$. [JRT, Problem 1.23]

   Hint: It is useful to note that

   $$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})=\boldsymbol{B}\,(\boldsymbol{A}\cdot\boldsymbol{C})-\boldsymbol{C}\,(\boldsymbol{A}\cdot\boldsymbol{B})\,.$$

   (c) *Trajectory of a particle:* The position of a moving particle is given as a function of time t to be [JRT, Problem 1.12]

   $$\boldsymbol{r}(t)=b\cos(\omega\,t)\,\hat{\boldsymbol{x}}+c\sin(\omega\,t)\,\hat{\boldsymbol{y}}+v_0\,t\,\hat{\boldsymbol{z}},$$

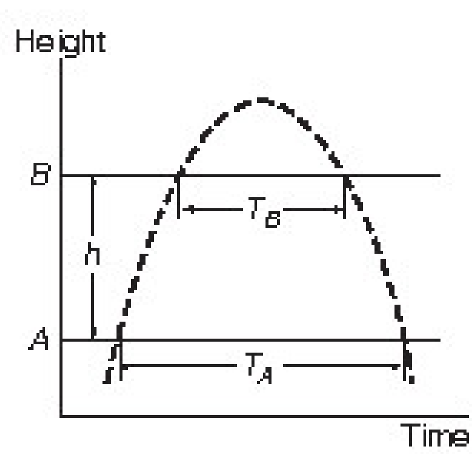
   where b, c, v_0 and ω are constants. Describe the particle's orbit. Find the particle's velocity and acceleration and interpret your results for the special case $b=c$.

2. *Charges in the ionosphere:* The ionosphere is a region of electrically neutral gas, composed of positively charged ions and negatively charged electrons, which surrounds the Earth at a height of approximately 200 km. If a radio wave passes through the ionosphere, its electric field accelerates the charged particles. Because the electric field oscillates in time, the charged particles tend to oscillate back and forth. Determine the motion of an electron of charge $-e$ and mass m that is initially at rest, and which is suddenly subjected to the electric field $E=E_0\sin(\omega\,t)$, where ω is the frequency of oscillation (in radians/second). [KK, Example 1.11]

3. *Measuring acceleration due to gravity:* The acceleration due to gravity g can be measured by projecting a body upward and measuring the time that it takes to pass two given points in both directions. Show that if the time the body takes to pass a horizontal line A in both directions is T_A, and the time to go by a second line B in both directions is T_B, then, assuming that the acceleration is constant, its magnitude is given by [KK, Problem 1.16]

   $$g=\frac{8\,h}{T_A^2-T_B^2},$$

   where h is the height of line B above line A (see figure below).



4. *Rain drops and terminal velocity:* Consider a raindrop of mass m that is falling with a given initial velocity under acceleration due to gravity on Earth. The air in the Earth's atmosphere exerts a viscous force $-\alpha\, v$ on the raindrop, where α is a positive constant and v is the velocity of the body. The quantity α is given by $\alpha = 6\,\pi\,\eta\,r$, where η is called the dynamic viscosity and r is the radius of the raindrop. [KK, Examples 3.8 and 3.9]

   (a) The viscous drag ensures that the velocity of the raindrop does not increase indefinitely, but leads to a maximum velocity called the terminal velocity. Determine the terminal velocity in terms of the coefficient α, the acceleration due to gravity g and the mass m of the raindrop.

   (b) Given that the density of water is $\rho_{\mathrm{W}} = 1000\,\mathrm{kg/m^3}$ and the dynamic viscosity of air at $20^\circ\,\mathrm{C}$ is $1.8 \times 10^5\,\mathrm{kg/(m\,s)}$, determine the terminal velocity of a raindrop with radius $1\,\mathrm{mm}$. Will a larger or a smaller raindrop have a higher terminal velocity?

   (c) Solve the equation of motion to determine the velocity of the raindrop as a function of time.

   (d) How long will it take for the raindrop (for values of the various quantities mentioned above) to reach the terminal velocity?

5. *Validity of the third law:* Consider two charges one moving along the positive x-direction and another moving along the positive y-direction. Determine the directions of the forces due to the magnetic field of one charge on the other and show that the forces do not obey Newton's third law. Can you identify a possible reason for the behavior? [JRT, Pages 21–23]

Last updated on August 11, 2017

Exercise sheet 1 with solutions

Vector operations and Newton's laws of motion

1. (a) *Direction cosines:* The direction cosines of a vector are the cosines of the angles it makes with the coordinate axes. The cosines of the angles between the vector and the x, y, and z axes are usually denoted by α, β, and γ, respectively. Using vector algebra, prove that $\alpha^2+\beta^2+\gamma^2 = 1$. [KK, Problem 1.4]

   Solution: Consider a vector $\boldsymbol{A}$, which makes angles a, b, and c with respect to the x, y, and z axes. In such a case, we have $\boldsymbol{A}\cdot\hat{\boldsymbol{x}} = A\cos a = A\,\alpha = A_x$, $\boldsymbol{A}\cdot\hat{\boldsymbol{y}} = A\cos b = A\,\beta = A_y$ and $\boldsymbol{A}\cdot\hat{\boldsymbol{z}} = A\cos c = A\,\gamma = A_z$, where $A = |\boldsymbol{A}|$. Therefore, we have

$$\alpha^2+\beta^2+\gamma^2 = \cos^2 a + \cos^2 b + \cos^2 c = \frac{A_x^2 + A_y^2 + A_z^2}{A^2} = 1,$$

   as required.

   (b) *Constructing an unknown vector:* The unknown vector $\boldsymbol{v}$ satisfies the conditions $\boldsymbol{b}\cdot\boldsymbol{v} = \lambda$ and $\boldsymbol{b}\times\boldsymbol{v} = \boldsymbol{c}$, where λ, $\boldsymbol{b}$ and $\boldsymbol{c}$ are fixed and known. Express $\boldsymbol{v}$ in terms of λ, $\boldsymbol{b}$ and $\boldsymbol{c}$. [JRT, Problem 1.23]

   Hint: It is useful to note that

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}\,(\boldsymbol{A}\cdot\boldsymbol{C}) - \boldsymbol{C}\,(\boldsymbol{A}\cdot\boldsymbol{B}).$$

   Solution: Since $\boldsymbol{b}\cdot\boldsymbol{v} = \lambda$ and $\boldsymbol{b}\times\boldsymbol{v} = \boldsymbol{c}$, upon taking a cross-product with $\boldsymbol{b}$ on both sides of the second equation, we obtain

$$\boldsymbol{b}\times(\boldsymbol{b}\times\boldsymbol{v}) = \boldsymbol{b}\times\boldsymbol{c},$$

   which leads to (upon using the above expression for the vector triple product)

$$\boldsymbol{b}\,(\boldsymbol{b}\cdot\boldsymbol{v}) - \boldsymbol{v}\,(\boldsymbol{b}\cdot\boldsymbol{b}) = \lambda\,\boldsymbol{b} - b^2\,\boldsymbol{v} = \boldsymbol{b}\times\boldsymbol{c},$$

   so that we arrive at

$$\boldsymbol{v} = \frac{\lambda\,\boldsymbol{b} - (\boldsymbol{b}\times\boldsymbol{c})}{b^2}.$$

   (c) *Trajectory of a particle:* The position of a moving particle is given as a function of time t to be [JRT, Problem 1.12]

$$\boldsymbol{r}(t) = b\cos(\omega\,t)\,\hat{\boldsymbol{x}} + c\sin(\omega\,t)\,\hat{\boldsymbol{y}} + v_0\,t\,\hat{\boldsymbol{z}},$$

   where b, c, v_0 and ω are constants. Describe the particle's orbit. Find the particle's velocity and acceleration and interpret your results for the special case $b = c$.

   Solution: Note that on the x-y-plane, the motion of the particle is governed by the equation

$$\frac{x^2}{b^2} + \frac{y^2}{c^2} = 1,$$

   which describes an ellipse. Apart from this, the particle also undergoes translational motion along the z-direction. Therefore, the resultant orbit of the particle is an 'elliptical helix'. For the special case wherein $b = c$, the orbit of the particle simplifies to a circular helix.

   The particle's velocity is given by

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = -b\,\omega\sin(\omega\,t)\,\hat{\boldsymbol{x}} + c\,\omega\cos(\omega\,t)\,\hat{\boldsymbol{y}} + v_0\,\hat{\boldsymbol{z}},$$

   while its acceleration is given by

$$\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -b\,\omega^2\cos(\omega\,t)\,\hat{\boldsymbol{x}} - c\,\omega^2\sin(\omega\,t)\,\hat{\boldsymbol{y}}.$$

2. *Charges in the ionosphere:* The ionosphere is a region of electrically neutral gas, composed of positively charged ions and negatively charged electrons, which surrounds the Earth at a height of approximately 200 km. If a radio wave passes through the ionosphere, its electric field accelerates the charged particles. Because the electric field oscillates in time, the charged particles tend to oscillate back and forth. Determine the motion of an electron of charge $-e$ and mass m that is initially at rest, and which is suddenly subjected to the electric field $E = E_0 \sin(\omega t)$, where ω is the frequency of oscillation (in radians/second). [KK, Example 1.11]

   Solution: The force on the electron due to the electric field $\boldsymbol{E}$ is given by

$$\boldsymbol{F} = -e\,\boldsymbol{E} = -e\,\boldsymbol{E_0}\,\sin(\omega\,t),$$

   where e is the charge of the electron. The acceleration produced due to this force is

$$\boldsymbol{a} = \frac{\boldsymbol{F}}{m} = -\frac{e\,\boldsymbol{E_0}}{m}\,\sin(\omega\,t),$$

   where m is the mass of the electron. As the acceleration is produced only in the direction of $\boldsymbol{E_0}$, let us consider this component alone, so that, we have

$$a(t) = -\frac{e\,E_0}{m}\,\sin(\omega\,t).$$

   Upon integrating this equation, we obtain the velocity of the electron to be

$$v(t) = v_0 - \frac{eE_0}{m}\int_0^t \mathrm{d}t\,\sin(\omega\,t) = v_0 + \frac{e\,E_0}{m\,\omega}\,[\cos(\omega\,t) - 1]\,,$$

   where the constant of integration v_0 is the velocity of the electron at time $t = 0$. Since the electron is assumed to be initially at rest, we need to set $v_0 = 0$. Let us choose a coordinate system so that $\boldsymbol{E_0}$ points in the x-direction. Upon integrating the above expression for velocity, we then arrive at

$$x(t) = x_0 + \frac{e\,E_0}{m\,\omega^2}\,\sin(\omega\,t) - \frac{e\,E_0}{m\,\omega}\,t,$$

   where, evidently, x_0 is a constant of integration which corresponds to $x(t)$ at $t = 0$. If we set $x_0 = 0$, then the trajectory simplifies to

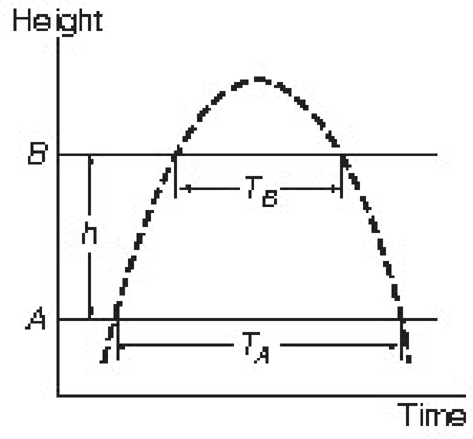
$$x(t) = \frac{e\,E_0}{m\,\omega^2}\,\sin(\omega\,t) - \frac{e\,E_0}{m\,\omega}\,t.$$

   Therefore, apart from the oscillatory motion due to the oscillations in the electric field, the electron also undergoes translational motion.

3. *Measuring acceleration due to gravity:* The acceleration due to gravity g can be measured by projecting a body upward and measuring the time that it takes to pass two given points in both directions. Show that if the time the body takes to pass a horizontal line A in both directions is T_A, and the time to go by a second line B in both directions is T_B, then, assuming that the acceleration is constant, its magnitude is given by [KK, Problem 1.16]

$$g = \frac{8\,h}{T_A^2 - T_B^2},$$

   where h is the height of line B above line A (see figure below).



<u>*Solution:*</u> Let the body be moving in the x-y-plane, with the y-axis chosen to be vertically upward. Let us choose our time suitably such that at $T = 0$, the vertical height of the body is $y = y_A$ during its upward motion. Also, let its vertical velocity at this time be $v = v_A$. Then, at any subsequent time T, the vertical displacement of the body from its initial position is given by

$$y_{\mathrm{u}}(T) = y_A + v_A\, T - \frac{g}{2}\, T^2.$$

Moreover, its velocity during this period is given by

$$v_u = v_A - g\, T.$$

From the symmetry of the figure, it is evident that the maximum height is reached at $T = T_A/2$. At this time, as the velocity of the body is zero, we have $v_A = g\, T_A/2$. Further, the body reaches B on its way upward journey at $T = (T_A - T_B)/2$. Therefore, we have

$$y_B = y_{\mathrm{u}}[(T_A - T_B)/2] = y_A + v_A\, \left(\frac{T_A - T_B}{2}\right) - \frac{g}{2}\, \left(\frac{T_A - T_B}{2}\right)^2$$

and, since $v_A = g\, T_A/2$, we obtain that

$$h = y_B - y_A = \frac{g}{8}\, \left(T_A^2 - T_B^2\right)$$

or, equivalently,

$$g = \frac{8\, h}{T_A^2 - T_B^2}.$$

It is useful to note that the maximum height reached by the body is

$$y_{\mathrm{max}} = y_{\mathrm{u}}(T_A/2) = y_A + \frac{g\, T_A^2}{8}.$$

Note that on the downward journey, the trajectory of the particle is given by

$$y_{\mathrm{d}}(T) = y_{\mathrm{max}} - \frac{g}{2}\, \left(T - \frac{T_A}{2}\right)^2 = y_A + \frac{g\, T_A^2}{8} - \frac{g}{2}\, \left(T - \frac{T_A}{2}\right)^2,$$

since the velocity is zero at the maximum height. The particle reaches y_B on its way down at time $T = (T_A + T_B)/2$. At this time, upon using the above expression $y_{\mathrm{d}}(T)$, we obtain that

$$y_B = y_{\mathrm{d}}[(T_A + T_B)/2] = y_A + \frac{g}{8}\, \left(T_A^2 - T_B^2\right)$$

or

$$h = y_B - y_A = \frac{g}{8}\, \left(T_A^2 - T_B^2\right),$$

exactly as have obtained earlier. Actually, the above y_B can also be arrived at from the original $y_{\mathrm{u}}(T)$ when evaluated at the time $T = (T_A + T_B)/2$.

4. *Rain drops and terminal velocity:* Consider a raindrop of mass m that is falling with a given initial velocity under acceleration due to gravity on Earth. The air in the Earth's atmosphere exerts a viscous force $-\alpha\, v$ on the raindrop, where α is a positive constant and v is the velocity of the body. The quantity α is given by $\alpha = 6\,\pi\,\eta\,r$, where η is called the dynamic viscosity and r is the radius of the raindrop. [KK, Examples 3.8 and 3.9]

   (a) The viscous drag ensures that the velocity of the raindrop does not increase indefinitely, but leads to a maximum velocity called the terminal velocity. Determine the terminal velocity in terms of the coefficient α, the acceleration due to gravity g and the mass m of the raindrop.

   Solution: Let v denote the velocity of the raindrop along the vertically downward direction. Upon taking into account the viscous and the gravitational forces, the equation of motion governing the vertical motion of the raindrop is given by

$$m\,\frac{\mathrm{d}v}{\mathrm{d}t} = -\alpha\, v + m\, g.$$

   When the velocity reaches a maximum value, say, v_{ter}, we have $\mathrm{d}v/\mathrm{d}t = 0$, a condition which leads to

$$v_{\mathrm{ter}} = \frac{m\,g}{\alpha}.$$

   (b) Given that the density of water is $\rho_{\mathrm{W}} = 1000\,\mathrm{kg/m^3}$ and the dynamic viscosity of air at 20° C is $\eta = 1.8 \times 10^{-5}\,\mathrm{kg/(m\,s)}$, determine the terminal velocity of a raindrop with radius 1 mm. Will a larger or a smaller raindrop have a higher terminal velocity?

   Solution: Recall that the quantity α is given by $\alpha = 6\,\pi\,\eta\,r$. Also, the mass of a spherical raindrop of radius r is given by

$$m = \frac{4\,\pi}{3}\,r^3\,\rho_{\mathrm{W}}.$$

   Therefore, the terminal velocity is given by

$$v_{\mathrm{ter}} = \frac{(4/3)\,\pi\,r^3\,\rho_{\mathrm{W}} g}{6\,\pi\,\eta\,r} = \frac{2\,r^2\,\rho_{\mathrm{W}}\,g}{9\,\eta} \simeq 120\,\mathrm{m/s},$$

   where, in arriving at the final equality, we have substituted the given values for ρ_{W}, r, and η. From the above expression for the terminal velocity, it is clear that larger raindrops will have higher terminal velocities. It should be stressed here that the value obtained above is a rather '*large*' velocity. (It will make getting drenched in the rain unpleasant and unbearable!) Typical raindrops are of the order of a few microns (i.e. 10^{-6} m), which have velocities of the order of 10^{-3} m/s.

   (c) Solve the equation of motion to determine the velocity of the raindrop as a function of time.

   Solution: In order to solve the equation of motion, let us substitute $v = u + v_{\mathrm{ter}}$. In terms of u, the equation governing the velocity of the raindrop can be rewritten as

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{1}{\tau}\,(u + v_{\mathrm{ter}}) + g = -\frac{u}{\tau},$$

   where τ is the characteristic timescale given by $\tau = m/\alpha$. This equation can be immediately integrated to yield

$$u = u_0\,\mathrm{e}^{-t/\tau}.$$

   or

$$v = u_0\,\mathrm{e}^{-t/\tau} + v_{\mathrm{ter}}.$$

   If we choose the initial velocity of the raindrop to be zero, i.e. $v = 0$ at $t = 0$, then we obtain that

$$v(t) = v_{\mathrm{ter}}\left(1 - \mathrm{e}^{-t/\tau}\right).$$

(d) How long will it take for the raindrop (for values of the various quantities mentioned above) to reach the terminal velocity?

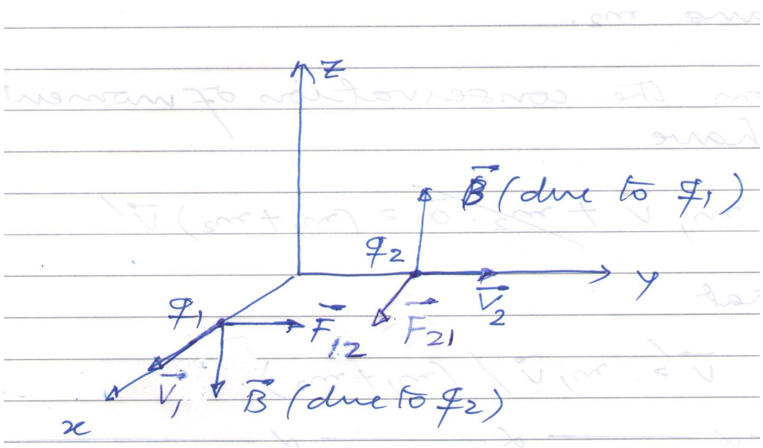
Solution: In order for the raindrop to reach its terminal velocity, mathematically, it would require infinite time. However, from the above solution we have obtained describing the velocity of the raindrop, it is clear that in a time equal to the characteristic time τ, the velocity of the raindrop reaches a value equal to $(1 - \mathrm{e}^{-1}) = 0.63$ times its terminal velocity. In a time of about about $5\,\tau$, the velocity reaches 99% of the terminal velocity. Therefore, for all purposes, when $t \gg \tau$, the raindrop falls at a velocity which is very close to its asymptotic terminal velocity.

For the values of the various quantities mentioned above, the characteristic time for the raindrop can be estimated to be

$$\tau = \frac{m}{\alpha} = \frac{(4/3)\,\pi\,r^3\,\rho_{\mathrm{W}}}{6\,\pi\,\eta\,r} = \frac{2\,r^2\,\rho_{\mathrm{W}}}{9\,\eta} \simeq 12\,\mathrm{s}.$$

5. *Validity of the third law:* Consider two charges one moving along the positive x-direction and another moving along the positive y-direction. Determine the directions of the forces due to the magnetic field of one charge on the other and show that the forces do not obey Newton's third law. Can you identify a possible reason for the behavior? [JRT, Pages 21–23]

Solution: Consider a charge q_1 that is moving along the positive x-direction with a velocity $\boldsymbol{v}_1$ and another charge q_2 which is moving along the positive y-direction with a velocity $\boldsymbol{v}_2$, as shown in the figure below.



Recall that, according to the Biot-Savart's law, given a current I flowing through an element $\mathbf{d}\boldsymbol{l}$, the magnetic field $\boldsymbol{B}$ generated by the current is given by

$$\boldsymbol{B} = \frac{\mu_0}{4\,\pi}\,\frac{I\,\mathrm{d}\boldsymbol{l} \times \boldsymbol{r}}{r^3},$$

where μ_0 is the vacuum permeability of free space and $\boldsymbol{r}$ is the radial vector running from the current carrying element to the point of observation. This implies that the direction of the magnetic field generated due to a charge moving with a velocity $\boldsymbol{v}$ can be expressed as $\boldsymbol{B} \propto \boldsymbol{v} \times \boldsymbol{r}$. Hence, the magnetic field, say, $\boldsymbol{B}_1$, due to the charge q_1 at the position of the charge q_2 is in the positive z-direction. Similarly, the direction of the magnetic field $\boldsymbol{B}_2$ due to the charge q_2 at the position of the charge q_1 is in the negative z-direction.

The force $\boldsymbol{F}$ on a charge q due to a magnetic field $\boldsymbol{B}$ is given by $\boldsymbol{F} \propto q\,(\boldsymbol{v} \times \boldsymbol{B})$. Hence, the force $\boldsymbol{F}_{21}$ on charge q_2 due to the magnetic field $\boldsymbol{B}_1$ is along the positive x-direction. In a similar manner, the force $\boldsymbol{F}_{12}$ on charge q_1 due to the magnetic field $\boldsymbol{B}_2$ is along the positive y-direction. Evidently, these forces are *not* equal and opposite to each other, i.e. $\boldsymbol{F}_{12} \neq -\boldsymbol{F}_{21}$.

Therefore, the charges violate Newton's third law. The fact that Newton's third law is not valid in the above example implies that the total momentum of the two charges is not conserved. This is because that the mechanical momentum of the particles is not the only kind of momentum. Electromagnetic fields too carry momentum. The loss in mechanical momentum contributes to the electromagnetic momentum of the system. It should be mentioned that this loss in momentum is not significant at non-relativistic velocities, i.e. when $v \ll c$, where c denotes the velocity of light.

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A simple proof of Taylor expansion

Let us say that we need to understand the behavior of a function $f(x)$ near a point x_0. Let us write $x = x_0 + h$ and express the function $f(x)$ as a series in the following fashion:

$$f(x) = f(x_0 + h) \simeq f_0 + f_1\, h + f_2\, h^2 + f_3\, h^3 + \cdots,$$

where the quantities f_0, f_1, f_2, f_3, ... are constants. If we can determine these constants, then, evidently, we would have arrived at a series expansion for the function $f(x)$ near x_0.

Note that, $f(x_0) = f_0$ and, since $\mathrm{d}h/\mathrm{d}x = 1$,

$$\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)_{x=x_0} = \left(\frac{\mathrm{d}f}{\mathrm{d}h}\right)_{h=0} = f_1.$$

Similarly, we have

$$\left(\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right)_{x=x_0} = \left(\frac{\mathrm{d}^2 f}{\mathrm{d}h^2}\right)_{h=0} = 2\, f_2$$

and

$$\left(\frac{\mathrm{d}^3 f}{\mathrm{d}x^3}\right)_{x=x_0} = \left(\frac{\mathrm{d}^3 f}{\mathrm{d}h^3}\right)_{h=0} = 6\, f_3 = 3!\, f_3$$

so that we can write

$$f(x) = f(x_0 + h) \simeq f(x_0) + \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)_{x=x_0} h + \frac{1}{2!}\left(\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right)_{x=x_0} h^2 + \frac{1}{3!}\left(\frac{\mathrm{d}^3 f}{\mathrm{d}x^3}\right)_{x=x_0} h^3,$$

Such an expansion is known as the Taylor series.

In particular, when x_0 is chosen to be zero, the expansion is often referred to as the Maclaurin series. The Maclaurin series for some of the standard functions are as follows:

$$\begin{aligned}
\mathrm{e}^x &\simeq 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \\
\sin x &\simeq x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots, \\
\cos x &\simeq 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots, \\
\log(1+x) &\simeq x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots, \\
(1+x)^n &\simeq 1 + n\,x + \frac{n\,(n-1)}{2!}\,x^2 + \cdots.
\end{aligned}$$

Note that the last result can also be obtained from a binomial expansion. We shall make use of some these results in due course.

Exercise sheet 2

Newton's laws of motion and conserved quantities

1. *Projectile motion in the presence of viscous drag:* Recall that the motion of a projectile under the influence of the uniform gravitational field near the surface of the Earth is confined to, say, the x-y-plane, with the y-axis pointing upward. Let, as usual, g denote the acceleration due to gravity pointing along the negative y-direction. Consider a situation wherein the air in the Earth's atmosphere exerts the viscous force $-\alpha\,\mathbf{v}$ on the projectile. [JRT, Section 2.3, Problem 2.15]

   (a) Solve the resulting equations of motion to arrive at the trajectory of the particle as a function of time, i.e. obtain $x(t)$ and $y(t)$.

   (b) Assuming $x = y = 0$ at $t = 0$, eliminate time t from the two equations to obtain the following trajectory of the particle in the x-y-plane:

   $$y(x) = \frac{v_{y0} + v_{\rm ter}}{v_{x0}}\, x + v_{\rm ter}\,\tau\,\ln\left(1 - \frac{x}{v_{x0}\,\tau}\right),$$

   where v_{x0} and v_{y0} are initial velocities along the x and y directions, respectively, $v_{\rm ter} = g\,\tau$ is the terminal velocity, with $\tau = m/\alpha$ being the characteristic time scale associated with the dissipative force.

   (c) Utilize the above expression to show that (by expanding the logarithmic term) the range of the projectile *in the absence of the viscous drag* is given by

   $$R_{\rm vac} = \frac{2\,v_{x0}\,v_{y0}}{g}.$$

   (d) Assuming that the effect of air resistance is negligible so that the ratio $R/(v_{x0}\,\tau)$ is very small, where R is new range of the projectile, expand the logarithmic term in the above expression to the next order to obtain that

   $$R \simeq R_{\rm vac}\left(1 - \frac{4\,v_{y0}}{3\,v_{\rm ter}}\right).$$

2. *Motion in a uniform magnetic field:* Consider a charged particle that is moving in a uniform and constant magnetic field pointed along the positive z-direction, i.e. $\boldsymbol{B} = B_0\,\hat{\boldsymbol{k}}$, where B_0 is a constant. The force $\boldsymbol{F}$ on the particle is given by the following Lorentz force law

   $$\boldsymbol{F} = q\,(\boldsymbol{v} \times \boldsymbol{B}),$$

   where q is the charge of the particle and $\boldsymbol{v}$ is its velocity.

   [JRT, Sections 2.5 and 2.7, and Problem 2.54]

   (a) Show that the magnetic field does not do any work on the particle.

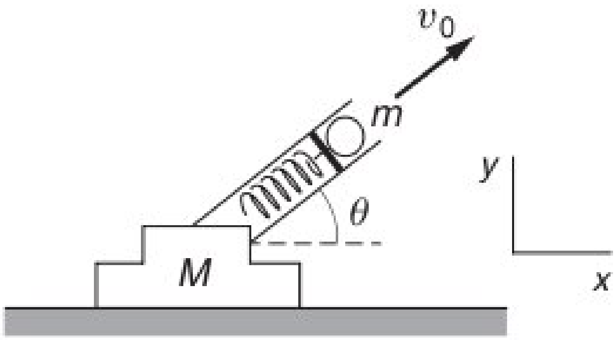
   (b) If the initial velocity of the particle along the direction of the magnetic field is zero, show that the motion of the particle is confined to the x-y-plane.

   (c) Solve the equations of motion to show that the charge moves on a circle (in the x-y-plane) with angular frequency $\omega = q\,B/m$.
   Note: The quantity $\omega = q\,B_0/m$ is known as the cyclotron frequency.

   (d) What is the radius of the circle? Does it depend on the initial velocity of the charge in the x-y-plane?

3. *Recoil of a spring gun:* A loaded spring gun, initially at rest on a horizontal frictionless surface, fires a marble at angle of elevation θ as shown in the figure below. The mass of the gun is M, the mass of the marble is m, and the muzzle velocity of the marble (the speed with which the marble is ejected, relative to the muzzle) is v_0 (see accompanying figure). Use the conservation of linear momentum to determine the final speed of the gun, when the marble has left the barrel.

   [KK, Example 4.8, JRT, Problem 3.1]



4. *Rockets:* Rockets are propelled by ejecting spent fuel at an exhaust speed, say, $v_{\rm ex}$, relative to the rocket. If m is the mass of the system at a given instant, using the conservation of momentum, show that the velocity of the rocket can be obtained to be

$$v = v_0 + v_{\rm ex} \ln(m_0/m),$$

where v_0 and m_0 are the initial velocity and the initial mass of the rocket. [JRT, Section 3.2]

5. *Period and energy:* Recall that the equation of motion governing the trajectory of a particle is a second order differential equation in time. Second order differential equations can, in general, be difficult to solve, whereas it is often straightforward to solve a first order differential equation. In the case of motion in one dimension, the conservation of energy allows us to reduce the second order differential equation to one of first order.

   (a) Consider a particle that is described by the potential energy $U(x)$ in one dimension. Write down the total energy of the system.

   (b) Differentiate the energy with respect to time and using Newton's second law, show that energy is conserved *if the potential is independent of time.*

   (c) Consider the case of a particle connected to a spring that we had discussed earlier, which is described by the potential $U(x) = k\,x^2/2$, where k is called the spring constant. Given an energy E, determine the turning points of the system.

   (d) Recall that the time period of a bounded system is the time it takes for the system to return to the same point. Using the conservation of energy, given an energy, say, E, determine the time period of the spring to return to the original turning point. Does the time period depend on E?

   (e) Now, consider a system that is described by the potential $U(x) = \alpha\,x^4$, where $\alpha > 0$. Does the system exhibit bounded motion? If it does, what are the turning points?

   (f) Evaluate the time period associated with the system and determine if the time period depends on the energy E.

   (g) Can you comment on the situation wherein $U(x) = \beta\,x^{2\,n}$, where $\beta > 0$ and $n > 0$?

Exercise sheet 2 with solutions

Newton's laws of motion and conserved quantities

1. *Projectile motion in the presence of viscous drag:* Recall that the motion of a projectile under the influence of the uniform gravitational field near the surface of the Earth is confined to, say, the x-y-plane, with the y-axis pointing upward. Let, as usual, g denote the acceleration due to gravity pointing along the negative y-direction. Consider a situation wherein the air in the Earth's atmosphere exerts the viscous force $-\alpha\,\mathbf{v}$ on the projectile. [JRT, Section 2.3, Problem 2.15]

   (a) Solve the resulting equations of motion to arrive at the trajectory of the particle as a function of time, i.e. obtain $x(t)$ and $y(t)$.

   Solution: The equation governing the motion of the projectile is given by

$$m\,\dot{\boldsymbol{v}} = -m\,g\,\hat{\boldsymbol{y}} - \alpha\,\boldsymbol{v}.$$

   The x and y components of the above equation can be written as

$$\dot{v}_x = -\frac{v_x}{\tau}, \quad \dot{v}_y = -g - \frac{v_y}{\tau},$$

   where $\tau = m/\alpha$ is the characteristic timescale associated with the system. The first equation can be solved to obtain

$$v_x(t) = v_{x0}\,\mathrm{e}^{-t/\tau},$$

   where, evidently, v_{x0} is the initial velocity along the x-direction specified at time $t=0$. On integrating this equation further, we obtain that

$$x(t) = x_0 + \int_0^t \mathrm{d}t'\,v_{x0}\,\mathrm{e}^{-t'/\tau} = x_0 + v_{x0}\,\tau\,\left(1 - \mathrm{e}^{-t/\tau}\right).$$

   Let us now consider the equation of motion of the projectile in the y-direction. We can rewrite the equation governing v_y as

$$\int_{v_{y0}}^{v} \frac{\mathrm{d}v_y}{v_{\mathrm{ter}} + v_y} = -\int \frac{\mathrm{d}t}{\tau},$$

   where $v_{\mathrm{ter}} = g\,\tau$ denotes the terminal velocity and v_{y0} is the velocity v_y at time $t=0$. We can integrate the above equation to obtain that

$$\ln\left(\frac{v_y + v_{\mathrm{ter}}}{v_{y0} + v_{\mathrm{ter}}}\right) = -\frac{t}{\tau},$$

   which can be expressed as

$$v_y(t) = v_{y0}\,\mathrm{e}^{-t/\tau} - v_{\mathrm{ter}}\,\left(1 - \mathrm{e}^{-t/\tau}\right).$$

   This can further integrated to arrive at

$$y(t) = \int_0^t \mathrm{d}t'\,v_y(t') = y_0 + (v_{y0} + v_{\mathrm{ter}})\,\tau\,\left(1 - \mathrm{e}^{-t/\tau}\right) - v_{\mathrm{ter}}\,t,$$

   where we have set $y = y_0$ at $t = 0$.

   (b) Assuming $x = y = 0$ at $t = 0$, eliminate time t from the two equations to obtain the following trajectory of the particle in the x-y-plane:

$$y(x) = \frac{v_{y0} + v_{\mathrm{ter}}}{v_{x0}}\,x + v_{\mathrm{ter}}\,\tau\,\ln\left(1 - \frac{x}{v_{x0}\,\tau}\right),$$

where v_{x0} and v_{y0} are initial velocities along the x and y directions, respectively, $v_{\rm ter} = g\,\tau$ is the terminal velocity, with $\tau = m/\alpha$ being the characteristic time scale associated with the dissipative force.

<u>*Solution:*</u> Using the solution we have obtained for $x(t)$ above, when $x_0 = 0$, we can write

$$\left(1 - {\rm e}^{-t/\tau}\right) = \frac{x}{v_{x0}\,\tau}$$

and

$$t = -\tau \ln\left(1 - \frac{x}{v_{x0}\,\tau}\right).$$

Therefore, if we set $y_0 = 0$, we can express the $y(t)$ we had obtained above as follows:

$$y(x) = \left(\frac{v_{y0} + v_{\rm ter}}{v_{x0}}\right) x + v_{\rm ter}\,\tau \ln\left(1 - \frac{x}{v_{x0}\,\tau}\right),$$

which is the required result.

(c) Utilize the above expression to show that (by expanding the logarithmic term) the range of the projectile *in the absence of the viscous drag* is given by

$$R_{\rm vac} = \frac{2\,v_{x0}\,v_{y0}}{g}.$$

<u>*Solution:*</u> In order to find the range R of the projectile, we need to solve the equation $y(x) = 0$. The non-zero root for x will correspond to the range, say, R. For the expression of $y(x)$ we obtained above, this equation can be written as

$$\left(\frac{v_{y0} + v_{\rm ter}}{v_{x0}}\right) R + v_{\rm ter}\,\tau \ln\left(1 - \frac{R}{v_{x0}\,\tau}\right) = 0.$$

This is a transcendental equation which cannot be solved analytically in terms of elementary functions.

In the absence of viscous drag (i.e. when $\alpha = 0$), $\tau \to \infty$. Therefore, the second term in the argument of the logarithmic function can be considered to be very small. Note that, using the Taylor series, we can write

$$\ln(1 - x) \simeq -x - \frac{x^2}{2} - \frac{x^3}{3} + \cdots.$$

On expanding the logarithmic function in the above equation in a Taylor series up to the third order, we obtain that

$$\left(\frac{v_{y0} + v_{\rm ter}}{v_{x0}}\right) R - v_{\rm ter}\,\tau \left[\frac{R}{v_{x0}\,\tau} + \frac{1}{2}\left(\frac{R}{v_{x0}\,\tau}\right)^2 + \frac{1}{3}\left(\frac{R}{v_{x0}\,\tau}\right)^3\right] \simeq 0$$

or, equivalently,

$$\frac{v_{y0}}{v_{x0}}\,R - v_{\rm ter}\,\tau \left[\frac{1}{2}\left(\frac{R}{v_{x0}\,\tau}\right)^2 + \frac{1}{3}\left(\frac{R}{v_{x0}\,\tau}\right)^3\right] \simeq 0.$$

Since $v_{\rm ter} = g\,\tau$, we obtain

$$\frac{v_{y0}}{v_{x0}}\,R - g\left(\frac{R^2}{2\,v_{x0}^2} + \frac{R^3}{3\,v_{x0}^3\,\tau}\right) \simeq 0.$$

The last term in the above equation can be neglected when $\tau \to \infty$. Also, we are interested in non-trivial solutions wherein $R \neq 0$. Hence, in the absence of viscous drag, from the above equation, we obtain the range of the projectile to be

$$R_{\rm vac} = \frac{2\,v_{x0}\,v_{y0}}{g},$$

which is the well known result for the range of the projectile.

(d) Assuming that the effect of air resistance is negligible so that the ratio $R/(v_{x0}\,\tau)$ is very small, where R is new range of the projectile, expand the logarithmic term in the above expression to the next order to obtain that

$$R \simeq R_{\rm vac}\,\left(1 - \frac{4\,v_{y0}}{3\,v_{\rm ter}}\right).$$

<u>*Solution:*</u> Previously, we had obtained that

$$\frac{v_{y0}}{v_{x0}}\,R - g\,\left(\frac{R^2}{2\,v_{x0}^2} + \frac{R^3}{3\,v_{x0}^3\,\tau}\right) \simeq 0.$$

and, since we are interested in the case wherein $R \neq 0$, this equation reduces to

$$R \simeq \frac{2\,v_{x0}\,v_{y0}}{g} - \frac{2\,R^2}{3\,v_{x0}\,\tau} \simeq R_{\rm vac} - \frac{2\,R^2}{3\,v_{x0}\,\tau}.$$

The second term in this expression is a small correction term. We can therefore replace R in this term with the approximate value $R \simeq R_{\rm vac}$ to obtain

$$R \simeq R_{\rm vac} - \frac{2\,R_{\rm vac}^2}{3\,v_{x0}\,\tau} \simeq R_{\rm vac}\,\left(1 - \frac{4\,v_{y0}}{3\,v_{\rm ter}}\right),$$

which is the required result. As expected, the viscous drag reduces the range of the projectile.

2. <u>*Motion in a uniform magnetic field:*</u> Consider a charged particle that is moving in a uniform and constant magnetic field pointed along the positive z-direction, i.e. $\boldsymbol{B} = B_0\,\hat{\boldsymbol{z}}$, where B_0 is a constant. The force $\boldsymbol{F}$ on the particle is given by the following Lorentz force law

$$\boldsymbol{F} = q\,\left(\boldsymbol{v} \times \boldsymbol{B}\right),$$

where q is the charge of the particle and $\boldsymbol{v}$ is its velocity.

[JRT, Sections 2.5 and 2.7, and Problem 2.54]

(a) Show that the magnetic field does not do any work on the particle.

<u>*Solution:*</u> Since the infinitesimal work done is given by

$$\mathrm{d}W = \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \boldsymbol{F}\cdot\boldsymbol{v}\;\mathrm{d}t,$$

we have, in our case,

$$\mathrm{d}W = q\,\left(\boldsymbol{v} \times \boldsymbol{B}\right)\cdot\boldsymbol{v}\;\mathrm{d}t$$

which vanishes implying that the magnetic field does not do any work on the particle.

(b) If the initial velocity of the particle along the direction of the magnetic field is zero, show that the motion of the particle is confined to the x-y-plane.

<u>*Solution:*</u> Since

$$m\,\ddot{\boldsymbol{r}} = m\,\dot{\boldsymbol{v}} = \boldsymbol{F} = q\,\left(\boldsymbol{v} \times \boldsymbol{B}\right)$$

and $\boldsymbol{B} = B_0\,\hat{\boldsymbol{z}}$, we obtain that

$$m\,\dot{v}_x = q\,B_0\,v_y, \quad m\,\dot{v}_y = -q\,B_0\,v_x, \quad m\,\dot{v}_z = 0.$$

Note that there is no force exerted along the direction of the magnetic field. Therefore, if the initial velocity of the particle along the z-direction is zero, its final velocity is also zero and hence it does not undergo any motion in the z-direction. The motion of the particle is restricted to the x-y plane.

(c) Solve the equations of motion to show that the charge moves on a circle (in the x-y-plane) with angular frequency $\omega = q\,B_0/m$.

Note: The quantity $\omega = q\,B_0/m$ is known as the cyclotron frequency.

Solution: Note that the equations governing the velocities v_x and v_y in the x-y-plane are given by

$$\dot{v}_x = \omega\,v_y, \quad \dot{v}_y = -\omega\,v_x,$$

where $\omega = q\,B_0/m$. Upon differentiating the first of these two equations with respect to time and using the second equation and, similarly, differentiating the second and using the first, we obtain that

$$\ddot{v}_x = -\omega^2\,v_x, \quad \ddot{v}_y = -\omega^2\,v_y.$$

The general solution to these differential equations can be written as

$$v_x(t) = \mathcal{A}\sin(\omega\,t) + \mathcal{B}\cos(\omega\,t), \quad v_y(t) = \mathcal{C}\sin(\omega\,t) + \mathcal{D}\cos(\omega\,t)$$

and the above equations relating v_x and v_y imply that $\mathcal{C} = -\mathcal{B}$ and $\mathcal{D} = \mathcal{A}$ so that

$$v_y(t) = -\mathcal{B}\sin(\omega\,t) + \mathcal{A}\cos(\omega\,t).$$

Let $v_x(t=0) = \mathcal{B} = v_{x0}$ and $v_y(t=0) = \mathcal{A} = v_{y0}$. Note that $v_x^2 + v_y^2 = v_{x0}^2 + v_{y0}^2$, which implies that the speed of the particle in the x-y-plane is a constant. The above solutions for v_x and v_y can be further integrated to arrive at

$$x(t) = x_0 + \frac{v_{x0}}{\omega}\sin(\omega\,t) - \frac{v_{y0}}{\omega}\cos(\omega\,t), \quad y(t) = y_0 + \frac{v_{y0}}{\omega}\sin(\omega\,t) + \frac{v_{x0}}{\omega}\cos(\omega\,t),$$

where x_0 and y_0 are constants of integration. We have, from these solutions,

$$(x - x_0)^2 + (y - y_0)^2 = \frac{v_{x0}^2 + v_{y0}^2}{\omega^2},$$

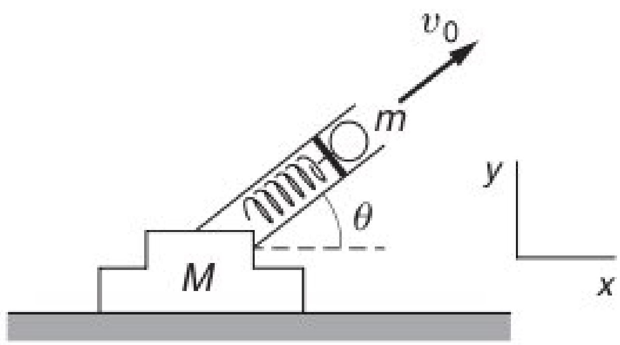
which describes a circle in the x-y-plane.

(d) What is the radius of the circle? Does it depend on the initial velocity of the charge in the x-y-plane?

Solution: It is clear from the above solution that the radius of the circle is given by v_0/ω, where $v_0 = \sqrt{v_{x0}^2 + v_{y0}^2}$. Obviously, the radius of the circle depends on the initial velocity of the charge. The larger the initial velocity, the larger is the radius of the circle.

3. *Recoil of a spring gun:* A loaded spring gun, initially at rest on a horizontal frictionless surface, fires a marble at angle of elevation θ as shown in the figure below. The mass of the gun is M, the mass of the marble is m, and the muzzle velocity of the marble (the speed with which the marble is ejected, relative to the muzzle) is v_0 (see accompanying figure). Use the conservation of linear momentum to determine the final speed of the gun, when the marble has left the barrel.

[KK, Example 4.8, JRT, Problem 3.1]



Solution: The forces acting on the system are the force due to gravity and the normal force. These two forces are in the vertical direction (i.e. along the y-axis) and they balance each other.

Note that there are no horizontal forces (i.e. along the x-direction) and hence the momentum of the complete system must be conserved. Since the complete system is initially at rest, $p_{x,\mathrm{i}} = 0$, where p_x represents the total momentum of the system along the x-direction and the subscript i denotes the fact that it is the initial momentum, before the gun is fired.

When the gun has been fired and the marble has left the muzzle/mouth of the gun, the gun recoils (i.e. moves backward), so that its horizontal momentum is $-M\,V_\mathrm{f}$, where V_f is the speed of the gun. Since the marble leaves the mouth of the gun with a velocity v_0 *with respect to the muzzle*, its velocity with respect to the table or the laboratory frame is actually $v_0\cos\theta - V_\mathrm{f}$. Hence, the momentum of the marble is $m\,(v_0\cos\theta - V_\mathrm{f})$ so that the final momentum of the system is $p_{x,\mathrm{f}} = m\,(v_0\cos\theta - V_\mathrm{f}) - M\,V_\mathrm{f}$.

Therefore, according to the conservation of linear momentum (recall that there are no forces acting in the horizontal direction), we have $p_{x,\mathrm{i}} = p_{x,\mathrm{f}}$ so that we obtain

$$m\,(v_0\cos\theta - V_\mathrm{f}) - M\,V_\mathrm{f} = 0$$

or

$$V_\mathrm{f} = \frac{m\,v_0\cos\theta}{M+m}.$$

The recoil velocity of the gun is smaller when the mass of the gun is larger (i.e. when $M \gg m$), as one would expect.

4. *Rockets:* Rockets are propelled by ejecting spent fuel at an exhaust speed, say, v_ex, relative to the rocket. If m is the mass of the system at a given instant, using the conservation of momentum, show that the velocity of the rocket can be obtained to be

$$v = v_0 + v_\mathrm{ex}\ln(m_0/m),$$

where v_0 and m_0 are the initial velocity and the initial mass of the rocket. [JRT, Section 3.2]

Solution: Let the momentum of the rocket at a given time t be $p(t) = m\,v$. At the time $t + \mathrm{d}t$, the momentum of the rocket will be: $(m + \mathrm{d}m)\,(v + \mathrm{d}v)$, where $\mathrm{d}m$ is the mass of the fuel ejected by the rocket in time $\mathrm{d}t$ and, it should be emphasized that $\mathrm{d}m$ is a negative quantity. Let this fuel be ejected with the exhaust speed v_ex so that the speed of the ejected fuel with respect to the ground is $v - v_\mathrm{ex}$. Then the total momentum of the system at the time $t + \mathrm{d}t$ is given by

$$p(t+\mathrm{d}t) = (m+\mathrm{d}m)\,(v+\mathrm{d}v) - \mathrm{d}m\,(v - v_\mathrm{ex}) \simeq m\,v + m\,\mathrm{d}v + v_\mathrm{ex}\,\mathrm{d}m = p(t) + m\,\mathrm{d}v + v_\mathrm{ex}\,\mathrm{d}m$$

up to the first order in the differentials involved.

In the absence of external forces, the total momentum is conserved. Therefore, $p(t+\mathrm{d}t) = p(t)$ and hence

$$\mathrm{d}v = -v_\mathrm{ex}\,\frac{\mathrm{d}m}{m}.$$

Considering the exhaust speed v_ex to be constant, we can integrate this equation to obtain

$$v - v_0 = -v_\mathrm{ex}\int_{m_0}^{m}\frac{\mathrm{d}m}{m} = v_\mathrm{ex}\ln(m_0/m)$$

or

$$v = v_0 + v_\mathrm{ex}\ln(m_0/m),$$

which is the required result.

5. *Period and energy:* Recall that the equation of motion governing the trajectory of a particle is a second order differential equation in time. Second order differential equations can, in general, be difficult to solve, whereas it is often straightforward to solve a first order differential equation. In the case of motion in one dimension, the conservation of energy allows us to reduce the second order differential equation to one of first order.

   (a) Consider a particle that is described by the potential energy $U(x)$ in one dimension. Write down the total energy of the system.

   Solution: The total mechanical energy of the system is the sum of its kinetic and potential energies, and is given by
$$E = \mathcal{T} + U = \frac{m}{2}\,\dot{x}^2 + U(x).$$
   Note that, in order to avoid 'degeneracy' with time period T, which we will encounter soon, we have used $\mathcal{T}$ to denote the kinetic energy.

   (b) Differentiate the energy with respect to time and using Newton's second law, show that energy is conserved *if the potential is independent of time.*

   Solution: We have
$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}\mathcal{T}}{\mathrm{d}t} + \frac{\mathrm{d}U}{\mathrm{d}t} = m\,\dot{x}\,\ddot{x} + \frac{\partial U}{\partial t} + \frac{\partial U}{\partial x}\,\dot{x}$$
   and, if the potential is not explicitly dependent on time, i.e. when $\partial U/\partial t = 0$, the above equation simplifies to
$$\frac{\mathrm{d}E}{\mathrm{d}t} = \dot{x}\,\left(m\,\ddot{x} + \frac{\partial U}{\partial x}\right).$$
   Now, according to Newton's second law,
$$m\,\ddot{x} = -\frac{\partial U}{\partial x},$$
   so that we have
$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0.$$
   In other words, the total mechanical energy of a system is conserved if the potential energy of the system is independent of time. This is easy to understand physically since the energy of a closed physical system is expected to be conserved. Such a system will be described by a potential energy that is independent of time.

   Note that a closed system is one that is *not* subject to external forces. As an example, consider the bob of a pendulum that is hung by a string through, say, the roof of a laboratory. If the length of the string is constant, then the system can be said to be closed and the mechanical energy of the pendulum will be conserved. On the other hand, if someone is tugging at the string from above the roof, then the system ceases to be closed as, evidently, it is subject to external influences which leads to modifications in the length of the string. This will be reflected as a potential energy that is dependent on time (since the length of the string is changing). It is not surprising that the energy of the pendulum will *not* be conserved in such a case.

   (c) Consider the case of a particle connected to a spring that we had discussed earlier, which is described by the potential $U(x) = k\,x^2/2$, where k is called the spring constant. Given an energy E, determine the turning points of the system.

   Solution: The turning points correspond to positions where the kinetic energy of a system vanishes (since the velocity is zero) so that the total energy of the system is given by the potential energy. In the case of the potential $U(x) = k\,x^2/2$, given that the energy of the particle is E, the turning points, say, $x_\pm$, are given by
$$\frac{k}{2}\,x_\pm^2 = E$$

or

$$x_\pm = \pm\sqrt{\frac{2\,E}{k}}.$$

(d) Recall that the time period of a bounded system is the time it takes for the system to return to the same point. Using the conservation of energy, given an energy, say, E, determine the time period of the spring to return to the original turning point. Does the time period depend on E?

<u>*Solution:*</u> Since

$$E = \mathcal{T} + U = \frac{m}{2}\,\dot{x}^2 + U(x) = \frac{m}{2}\,\dot{x}^2 + \frac{k}{2}\,x^2,$$

we can write

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{\frac{2}{m}\,\left(E - \frac{k}{2}\,x^2\right)}$$

or

$$\mathrm{d}t = \frac{\sqrt{m/2}\,\mathrm{d}x}{\sqrt{E - (k\,x^2/2)}}.$$

The period, say, T, of the system is the time taken by the particle to go from x_- to x_+ and return to x_-. Note that the time taken by the particle to go from x_- to x_+ will be the same as the time taken by it to return to x_- from x_+. Therefore, the above equation can be integrated to yield the time period to be

$$T = 2\int_{x_-}^{x_+} \mathrm{d}x\,\frac{\sqrt{m/2}}{\sqrt{E - (k\,x^2/2)}}.$$

Moreover, since the potential is symmetric in x, the time taken by the particle to go from x_- to the origin is the same the as the time taken from the origin to x_+. Hence, we obtain that

$$\begin{aligned} T &= 4\int_0^{x_+} \mathrm{d}x\,\frac{\sqrt{m/2}}{\sqrt{E - (k\,x^2/2)}} = 4\sqrt{\frac{m}{2\,E}}\int_0^{x_+} \mathrm{d}x\,\frac{1}{\sqrt{1 - k\,x^2/(2\,E)}} \\ &= 4\sqrt{\frac{m}{2\,E}}\,x_+\int_0^1 \mathrm{d}z\frac{1}{\sqrt{1-z^2}} = \frac{4}{\omega}\int_0^1 \mathrm{d}z\frac{1}{\sqrt{1-z^2}} = \frac{4}{\omega}\,\frac{\pi}{2} = \frac{2\,\pi}{\omega}, \end{aligned}$$

where we have set $z = x/x_+$ and have used the fact that $\omega = \sqrt{k/m}$. Note that the integral

$$I = \int_0^1 \mathrm{d}z\frac{1}{\sqrt{1-z^2}}$$

can be easily evaluated by setting $z = \sin\theta$ as follows:

$$I = \int_0^{\pi/2} \mathrm{d}\theta\,\frac{\cos\theta}{\sqrt{1 - \sin^2\theta}} = \int_0^{\pi/2} \mathrm{d}\theta\, = \frac{\pi}{2}.$$

(e) Now, consider a system that is described by the potential $U(x) = \alpha\,x^4$, where $\alpha > 0$. Does the system exhibit bounded motion? If it does, what are the turning points?

<u>*Solution:*</u> Yes, the system exhibits bounded motion in a fashion somewhat similar to that of the particle in a quadratic potential we considered above. Note that the total energy of the system is conserved since the potential energy is not explicitly dependent on time. As the turning points correspond to positions wherein the kinetic energy vanishes, they are given by the real roots of the equation

$$\alpha\,x^4 = E$$

or

$$x_\pm = \pm\left(\frac{E}{\alpha}\right)^{1/4},$$

where, evidently, E is the energy of the particle.

(f) Evaluate the time period associated with the system and determine if the time period depends on the energy E.

<u>Solution:</u> As in the case of the quadratic potential, the quartic potential is symmetric about the origin (i.e. $x = 0$). Therefore, the time period of the system can be written as

$$\begin{aligned} T &= 2\int_{x_-}^{x_+} \mathrm{d}x\, \frac{\sqrt{m/2}}{\sqrt{E - \alpha\, x^4}} = 4\int_0^{x_+} \mathrm{d}x\, \frac{\sqrt{m/2}}{\sqrt{E - \alpha\, x^4}} \\ &= 4\sqrt{\frac{m}{2\,E}}\int_0^{x_+} \mathrm{d}x\, \frac{1}{\sqrt{1 - \alpha\, x^4/E}} = 4\sqrt{\frac{m}{2\,E}}\, x_+ \int_0^1 \mathrm{d}z \frac{1}{\sqrt{1 - z^4}} \\ &= \frac{2\sqrt{2\,m}}{\alpha^{1/4}}\, E^{1/4-1/2} \int_0^1 \mathrm{d}z \frac{1}{\sqrt{1 - z^4}} \end{aligned}$$

where we have set $z = x/x_+$. Note that the time period depends on the energy E of the particle as $T \propto E^{-1/4}$, i.e. the larger the energy the smaller is the time period of the system.

(g) Can you comment on the situation wherein $U(x) = \beta\, x^{2\,n}$, where $\beta > 0$ and $n > 0$?

<u>Solution:</u> In such a case, the turning points are given by

$$x_\pm = \pm \left(\frac{E}{\beta}\right)^{1/(2\,n)}$$

and the time period of the system can be expressed as

$$\begin{aligned} T &= 2\int_{x_-}^{x_+} \mathrm{d}x\, \frac{\sqrt{m/2}}{\sqrt{E - \beta\, x^{2\,n}}} = 4\int_0^{x_+} \mathrm{d}x\, \frac{\sqrt{m/2}}{\sqrt{E - \beta\, x^{2\,n}}} \\ &= 4\sqrt{\frac{m}{2\,E}}\int_0^{x_+} \mathrm{d}x\, \frac{1}{\sqrt{1 - \beta\, x^{2\,n}/E}} = 4\sqrt{\frac{m}{2\,E}}\, x_+ \int_0^1 \mathrm{d}z \frac{1}{\sqrt{1 - z^{2\,n}}} \\ &= \frac{2\sqrt{2\,m}}{\beta^{1/(2\,n)}}\, E^{1/(2\,n)-1/2} \int_0^1 \mathrm{d}z \frac{1}{\sqrt{1 - z^{2\,n}}}, \end{aligned}$$

where, as earlier, we have set $z = x/x_+$. Note that $T \propto E^{1/(2\,n)-1/2}$ and it is only in the case wherein $n = 1$ (which corresponds to the quadratic potential we discussed above) that the time period is independent of energy. For this reason, the simple harmonic oscillator holds a special place in physics. It is important to also note that we have arrived at the energy dependence of the time period without actually having to explicitly evaluate the integral!

Department of Physics
Indian Institute of Technology Madras

Mock quiz

From Newton's laws to simple harmonic motion

Date: September 11, 2017 Time: 08:00 – 08:50 AM

Name:

Roll No.:

Instructor:

Batch:

Instructions

1. Begin by completing the information requested above. Please write your complete name, your roll number, the name of your instructor, and your batch number (out of I–XII). **The answer sheet will not be evaluated unless both your name and roll number are written.**
2. This question paper cum answer sheet booklet contains **six** pages. Please check right away that all the pages are present.
3. As we had announced earlier, this quiz consists of 3 true/false questions (for 1 mark each), 3 multiple choice questions with one correct option (for 1 mark each), 4 fill in the blanks, two questions involving detailed calculations (for 3 marks each) and one question involving some plotting (for 4 marks), adding to a total of 20 marks.
4. You are expected to answer **all** the questions. There are no negative marks.
5. The answers have to be written in the boxes provided. Answers written elsewhere in the booklet will **not** be evaluated.
6. Kindly write the answers, including sketches, with a blue or black pen. Note that answers written with pencils or pens of other colors will **not** be evaluated.
7. You can use the empty reverse sides for rough work. No extra sheets will be provided.
8. You are not allowed to use a calculator or any other electronic device during the quiz. Please note that you will not be permitted to continue with the quiz if you are found with any such device.
9. Make sure that you return question paper cum answer sheet booklet before you leave the examination hall.

For use by examiners
(Do not write in this space)

Q1–Q10	Q11	Q12	Q13	Total

✦ **True or false** (1 mark each, write True (T)/False (F) in the box provided)

1. The angular momentum of a particle moving in a central force field is conserved.
2. The kinetic and potential energies of an oscillator averaged over one period are equal.
3. A particle moving in the potential $U(x) = \alpha\, x^2$, where $\alpha < 0$, exhibits bounded motion.

✦ **Multiple choice questions** (1 mark each, write the one correct option in the box provided)

4. A particle is moving along the trajectory $\boldsymbol{r}(t) = A\, t\, \hat{\boldsymbol{x}} + B\, t^2\, \hat{\boldsymbol{y}}$. The trajectory of the particle in the x-y-plane is described by

   [**A**] $y \propto x$ [**B**] $y \propto x^2$ [**C**] $y \propto x^{1/2}$ [**D**] None of the above

5. A particle is exerted upon by the force $\mathbf{F} = -\beta\, x^{-2}\, \hat{\boldsymbol{x}}$. The potential energy $U(x)$ associated with the force is

   [**A**] $U(x) = \beta/x$ [**B**] $U(x) = -\beta/x$ [**C**] $U(x) = \beta/x^2$ [**D**] $U(x) = -\beta/x^2$

6. The position of a mass exhibiting simple harmonic motion in one dimension is described by the function: $x(t) = \sqrt{3}\cos(\omega\, t) + \sin(\omega\, t)$ for $t \geq 0$. The particle crosses the origin at the times (with $n = 0, 1, 2, \ldots$)

   [**A**] $t = (3\, n + 2)\, \pi/(3\, \omega)$ [**B**] $t = (3\, n + 1)\, \pi/(6\, \omega)$

   [**C**] $t = n\, \pi/(2\, \omega)$ [**D**] $t = n\, \pi/\omega$

✦ **Fill in the blanks** (1 mark each, write the answer in the box provided)

7. A vector $\boldsymbol{a}$ that depends on time has a constant magnitude. In such a case, $\boldsymbol{a} \cdot \mathrm{d}\boldsymbol{a}/\mathrm{d}t$ equals
8. A particle that is falling under the influence of gravity is exerted by the drag force $\boldsymbol{F} = -\beta\, v^2\, \hat{\boldsymbol{v}}$, where v is the speed of the particle and $\hat{\boldsymbol{v}}$ denotes the unit vector along the direction of velocity of the particle. In such a case, the terminal velocity v_{ter} of the particle is given by
9. A charge q is moving in a constant electric field $\boldsymbol{E}$. If $\boldsymbol{r}$ and $\boldsymbol{v}$ denote the position and velocity of the charge, then the rate of change of work done by the electric field on the charge is given by
10. At critical damping, the complete solution governing a damped simple harmonic oscillator is given by

✦ Questions with detailed answers

11. *Kicking a puck up an inclined plane:* A puck is kicked up a frictionless inclined plane with an initial velocity v_0. Let the angle of incline of the inclined plane be θ. (a) Write down the Newton's equation governing the motion of the puck. (b) Solve the equation and determine the distance that the puck will travel on the inclined plane. $\boxed{\text{1+2 marks}}$

12. *Bead on a wire:* A bead, under the influence of gravity, slides along a frictionless wire whose height is given by the function $y(x)$. (a) Express the kinetic and the potential energies of the particle in terms of $\dot{x}$, y and $\mathrm{d}y/\mathrm{d}x$. (b) Obtain the equation of motion governing the system. $\boxed{\text{1+2 marks}}$

13. *Motion in one dimension:* A particle moves in the one-dimensional potential $U(x) = \alpha\,\left(x^2 - x_0^2\right)^2$, where $\alpha > 0$. (a) Plot the potential. (b) Identify the range of energies for which the particle has four turning points. (c) Determine the frequency of small oscillations near the minima. 2+1+1 marks

Illustrative examples 3

Simple harmonic motion

1. *Linearity and superposition:* Let $x_1(t)$ and $x_2(t)$ be solutions to the linear differential equation $\ddot{x} = \alpha\, x$, where α is a constant. Show that the linear superposition $a\, x_1(t) + b\, x_2(t)$, where a and b are constants, is also a solution to the differential equation. In contrast, show that, a linear superposition of solutions to the non-linear equation $\ddot{x}^2 = \beta\, x$ is *not* a solution to the differential equation. [DM, Problem 4.1]

2. *An oscillating mass:* Consider a mass on the end of a spring which is oscillating with angular frequency ω. [JRT, Problems 5.6, 5.7, 5.9]

   (a) At $t = 0$, its position is $x_0 > 0$ and it is given a kick so that it moves back toward the origin and executes simple harmonic motion with amplitude $2\, x_0$. Find its position as a function of time in the form $x(t) = A \cos(\omega\, t - \delta)$.

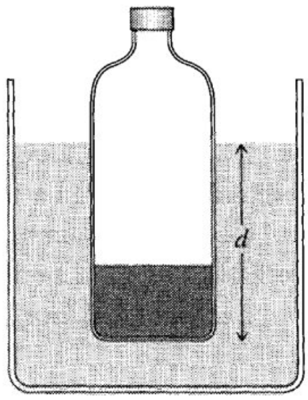
   (b) Solve for the coefficients B_1 and B_2 of the form $x(t) = B_1 \cos(\omega\, t) + B_2 \sin(\omega\, t)$ in terms of the initial position x_0 and velocity v_0 at $t = 0$.

   (c) If the oscillator's mass is $m = 0.5\,\mathrm{kg}$ and the force constant is $k = 50\,\mathrm{N/m}$, what is the angular frequency ω?

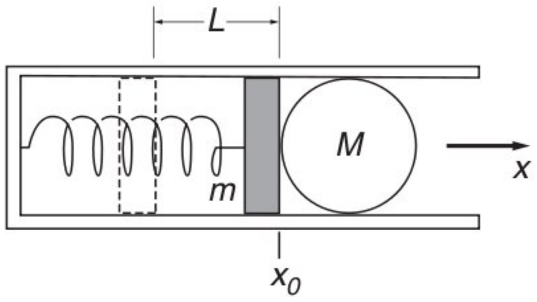
   (d) If $x_0 = 3.0\,\mathrm{m}$ and $v_0 = 40\,\mathrm{m/s}$, what are B_1 and B_2? Sketch $x(t)$ for a couple of cycles. What are the earliest times at which $x = 0$ and at which $\dot{x} = 0$?

   (e) If the maximum displacement of the mass is $0.2\,\mathrm{m}$, and its maximum speed is $1.2\,\mathrm{m/s}$, what is the period of its oscillations?

3. *Bottle in a bucket:* A bottle is floating upright in a large bucket of water as shown in the figure below. In equilibrium it is submerged to a depth d_0 below the surface of the water. Show that if it is pushed down to a depth d and released, it will execute harmonic motion, and find the frequency of its oscillations. If $d_0 = 20\,\mathrm{cm}$, what is the period of the oscillations? [JRT, Example 5.2]



4. *Spring gun and initial conditions:* A toy spring gun fires a marble of mass M by means of a spring and piston in a barrel, as shown in the figure below. [KK, Example 3.11]



The piston has mass m and is attached to the end of a spring having spring constant k. The piston and marble are pulled back a distance L from equilibrium and released. Ignoring the effects of gravity and friction, determine the speed of the marble just as it loses contact with the piston.

5. *Virial theorem:* Consider an undamped, harmonic oscillator with period T. Let $\langle f\rangle$ denote the average value of any variable $f(t)$, averaged over one complete cycle as follows:

$$\langle f\rangle = \frac{1}{T}\int_0^T \mathrm{d}t\, f(t).$$

Prove that the average values of the kinetic and the potential energies $\mathcal{T}$ and U are given by

$$\langle \mathcal{T}\rangle = \langle U\rangle = E/2,$$

where E is the total energy of the oscillator. [JRT, Problem 5.12]

Hint: Start by proving the more general and, extremely useful, results that $\langle \sin^2(\omega\, t-\delta)\rangle = \langle \cos^2(\omega\, t-\delta)\rangle = 1/2$.

Note: The above relation between $\langle \mathcal{T}\rangle$ and $\langle U\rangle$ is a specific example of a general result governing bounded systems known as the virial theorem. (We are using $\mathcal{T}$ here to denote the kinetic energy in order to distinguish it from the time period T!)

Illustrative examples 3 with solutions

Simple harmonic motion

1. Linearity and superposition: Let $x_1(t)$ and $x_2(t)$ be solutions to the linear differential equation $\ddot{x} = \alpha\, x$, where α is a constant. Show that the linear superposition $a\, x_1(t) + b\, x_2(t)$, where a and b are constants, is also a solution to the differential equation. In contrast, show that, a linear superposition of solutions to the non-linear equation $\ddot{x}^2 = \beta\, x$ is *not* a solution to the differential equation. [DM, Problem 4.1]

   Solution: Since $x_1(t)$ and $x_2(t)$ are solutions to the linear differential equation $\ddot{x} = \alpha\, x$, we have

   $$\ddot{x}_1 = \alpha\, x_1, \quad \ddot{x}_2 = \alpha\, x_2,$$

   so that when

   $$x(t) = a\, x_1(t) + b\, x_2(t),$$

   we have

   $$\ddot{x}(t) = a\, \ddot{x}_1(t) + b\, \ddot{x}_2(t) = \alpha\, [a\, x_1(t) + b\, x_2(t)] = \alpha\, x(t).$$

   In other words, the linear superposition $x(t) = a\, x_1(t) + b\, x_2(t)$ is also a solution to the differential equation.

   Let us now consider the non-linear equation $\ddot{x}^2 = \beta\, x$. If $x_1(t)$ and $x_2(t)$ are solutions to the differential equation, then the linear superposition $x(t) = a\, x_1(t) + b\, x_2(t)$ satisfies the equation

   $$\ddot{x}^2 = [a\, \ddot{x}_1(t) + b\, \ddot{x}_2(t)]^2 = a^2\, \ddot{x}_1^2 + b^2\, \ddot{x}_2^2 + 2\, a\, b\, \ddot{x}_1\, \ddot{x}_2 = \beta\, \left(a^2\, x_1 + b^2\, x_2\right) + 2\, a\, b\, \ddot{x}_1\, \ddot{x}_2,$$

   which, evidently, cannot be written in the same form as the original equation. The non-linear nature of the differential equation does not permit a linear superposition of the solutions to be a solution.

2. An oscillating mass: Consider a mass on the end of a spring which is oscillating with angular frequency ω. [JRT, Problems 5.6, 5.7, 5.9]

   (a) At $t = 0$, its position is $x_0 > 0$ and it is given a kick so that it moves back toward the origin and executes simple harmonic motion with amplitude $2\, x_0$. Find its position as a function of time in the form $x(t) = A \cos(\omega\, t - \delta)$.

   Solution: Since

   $$x(t) = A \cos(\omega\, t - \delta),$$

   at $t = 0$, we have

   $$x(t = 0) = x_0 = A \cos(-\delta)$$

   or $A = x_0/\cos(-\delta)$. Also, the velocity is given by

   $$v(t) = \dot{x}(t) = -A\, \omega \sin(\omega\, t - \delta).$$

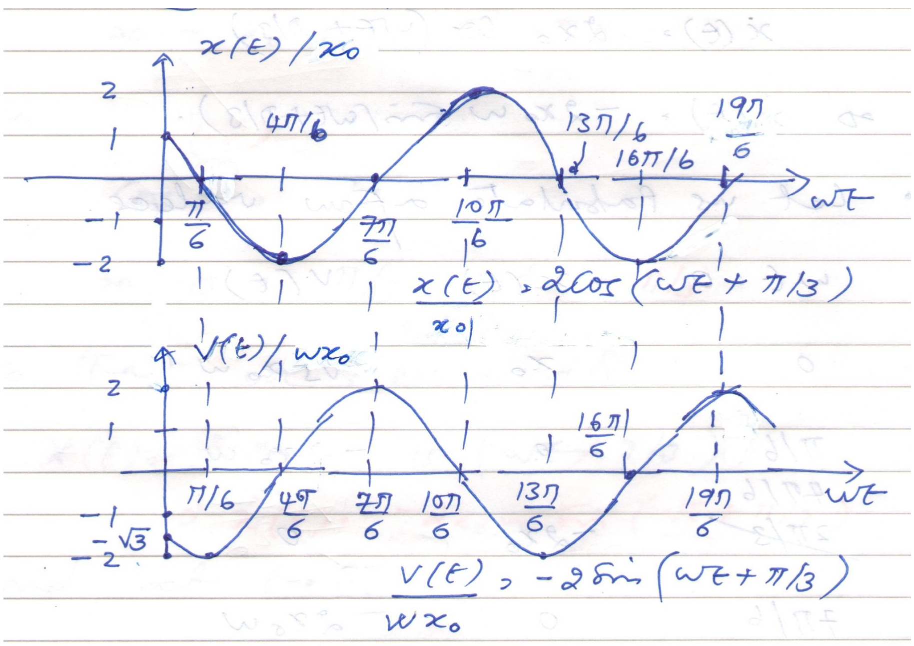
   Note that $x(t)$ reaches the first maximum, say, at $t_{\rm max}$, when $v(t)$ crosses zero for the first time. It is clear that the first zero of $v(t)$ occurs at $\omega\, t_{\rm max} = \delta + \pi$. Then, at $t = t_{\rm max}$

   $$x(t_{\rm max}) = \frac{x_0}{\cos(-\delta)} \cos(\omega\, t_{\rm max} - \delta) = \frac{x_0}{\cos(-\delta)} \cos\pi = -2\, x_0,$$

   which implies that $\cos(-\delta) = 1/2$ or $\delta = -\pi/3$. Therefore, we can write

   $$x(t) = 2\, x_0 \cos\left(\omega\, t + \frac{\pi}{3}\right), \quad v(t) = -2\, x_0 \sin\left(\omega\, t + \frac{\pi}{3}\right).$$

   Note that the particle was to start at x_0 at $t = 0$ and head towards $-2\, x_0$, which implies that its initial velocity is negative. These points are indeed satisfied by the $x(t)$ and $v(t)$ we have arrived at above. We have plotted these results in the figure below.



(b) Solve for the coefficients B_1 and B_2 of the form $x(t) = B_1 \cos(\omega t) + B_2 \sin(\omega t)$ in terms of the initial position x_0 and velocity v_0 at $t = 0$.

<u>*Solution:*</u> Since

$$x(t) = B_1 \cos(\omega t) + B_2 \sin(\omega t),$$

we have

$$v(t) = \omega \left[-B_1 \sin(\omega t) + B_2 \cos(\omega t)\right].$$

Clearly, $B_1 = x_0$ and $B_2 = v_0/\omega$.

(c) If the oscillator's mass is $m = 0.5\,\mathrm{kg}$ and the force constant is $k = 50\,\mathrm{N/m}$, what is the angular frequency ω?

<u>*Solution:*</u> As

$$\omega = \sqrt{\frac{k}{m}},$$

for the above values of m and k, we find that $\omega = 10\,\mathrm{s}^{-1}$.

(d) If $x_0 = 3.0\,\mathrm{m}$ and $v_0 = 40\,\mathrm{m/s}$, what are B_1 and B_2? Sketch $x(t)$ for a couple of cycles. What are the earliest times at which $x = 0$ and at which $\dot{x} = 0$?

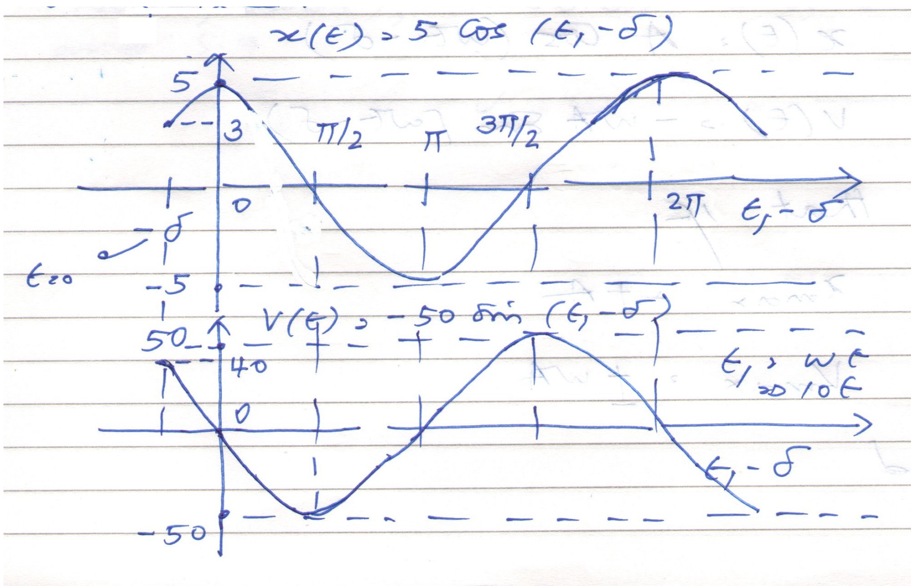
<u>*Solution:*</u> For $\omega = 10\,\mathrm{s}^{-1}$ and the above initial conditions, we have

$$x(t) = 3\cos(\omega t) + 4\sin(\omega t), \quad v(t) = -30\sin(\omega t) + 40\cos(\omega t),$$

in units of m and m/s, respectively. These equations can be expressed as

$$x(t) = 5\cos(\omega t - \delta), \quad v(t) = -50\sin(\omega t - \delta),$$

where $\cos\delta = 3/5$, which corresponds to $\delta \simeq 0.93$ radians. We have plotted these results in the figure below.



Note that the x-axis in the figure is $t_1 - \delta = \omega\, t - \delta$, which starts at $-\delta$ corresponding to $t = 0$. It should be clear from the figure that the earliest time at which $x = 0$ is when

$$t_1 - \delta = \omega\, t - \delta = \pi/2,$$

which corresponds to $t = [(\pi/2) + \delta]/\omega \simeq 0.25\,\mathrm{s}$, for $\omega = 10\,\mathrm{s}^{-1}$. Similarly, the earliest time at which $v = 0$ is when

$$t_1 - \delta = \omega\, t - \delta = 0,$$

i.e. at $t = \delta/\omega \simeq 0.09\,\mathrm{s}$, for $\omega = 10\,\mathrm{s}^{-1}$.

(e) If the maximum displacement of the mass is $0.2\,\mathrm{m}$, and its maximum speed is $1.2\,\mathrm{m/s}$, what is the period of its oscillations?

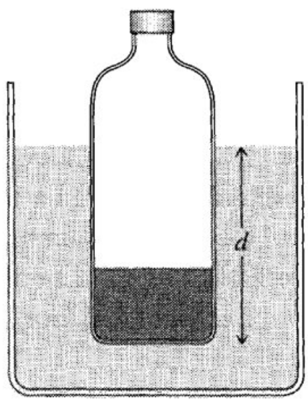
<u>*Solution:*</u> Since we can write

$$x(t) = A\cos(\omega\, t - \delta), \quad v(t) = -\omega\, A\sin(\omega\, t - \delta),$$

evidently, the maximum values of position and velocity are $x_{\mathrm{max}} = A$ and $v_{\mathrm{max}} = \omega\, A$, so that $\omega = v_{\mathrm{max}}/x_{\mathrm{max}}$. So, for the given values of maximum displacement and velocity, we have $\omega = 6\,\mathrm{s}^{-1}$ and the corresponding time period is

$$T = \frac{2\,\pi}{\omega} = \frac{2\,\pi}{6}\,\mathrm{s} \simeq 1.05\,\mathrm{s}.$$

3. <u>*Bottle in a bucket:*</u> A bottle is floating upright in a large bucket of water as shown in the figure below. In equilibrium it is submerged to a depth d_0 below the surface of the water. Show that if it is pushed down to a depth d and released, it will execute harmonic motion, and find the frequency of its oscillations. If $d_0 = 20\,\mathrm{cm}$, what is the period of the oscillations? [JRT, Example 5.2]



Solution: The two forces on the bottle are its weight $m\,g$ which acts downward and the upward buoyant force $\rho\,A\,d\,g$ (which is equal to the weight of the water displaced), where ρ is the density of water, A is the cross-sectional area of the bottle and d is the depth to which the bottle is immersed in the water. The equilibrium depth of the bottle is determined by the condition

$$m\,g = \rho\,A\,d_0\,g.$$

If the bottle is at a depth, say, $d = d_0 + x$, where x is, evidently, the displacement from the equilibrium position, then, according to Newton's second law, we have

$$m\,\ddot{x} = m\,g - \rho\,A\,g\,(d_0 + x).$$

But, as $m\,g = \rho\,A\,d_0\,g$, this equation reduces to

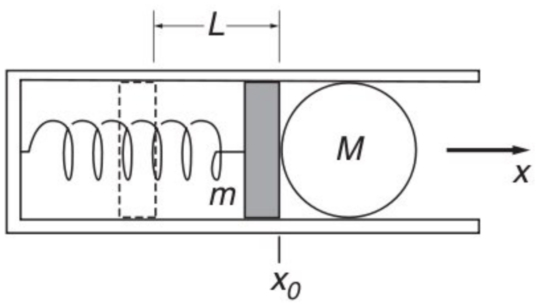
$$\ddot{x} = -\frac{\rho\,A\,g}{m}\,x = -\frac{g}{d_0}\,x,$$

which describes simple harmonic motion. The bottle oscillates about the equilibrium position with frequency $\omega = \sqrt{g/d_0}$ so that the time period of the oscillations is given by

$$T = 2\,\pi\,\sqrt{\frac{d_0}{g}}.$$

For $d_0 = 20\,\mathrm{cm}$, the time period amounts to about $T \simeq 0.9\,s$.

4. *Spring gun and initial conditions:* A toy spring gun fires a marble of mass M by means of a spring and piston in a barrel, as shown in the figure below. [KK, Example 3.11]



The piston has mass m and is attached to the end of a spring having spring constant k. The piston and marble are pulled back a distance L from equilibrium and released. Ignoring the effects of gravity and friction, determine the speed of the marble just as it loses contact with the piston.

Solution: The piston pushes the total mass $M + m$. Therefore, the position of the piston obeys the following equation of motion:

$$(M + m)\,\ddot{x} + k\,x = 0$$

or, equivalently,

$$\ddot{x} + \frac{k}{M+m}\,x = 0,$$

which has the general solution

$$x(t) = A\sin(\omega\,t) + B\cos(\omega\,t),$$

where $\omega = \sqrt{k/(M+m)}$ while A and B are constants. The corresponding velocity of the piston is given by

$$v(t) = \dot{x}(t) = \omega\,\left[A\cos(\omega\,t) - B\sin(\omega\,t)\right].$$

At $t = 0$, the piston is pulled by a length L and released from rest. In other words, at $t = 0$, $x = -L$ and $v = 0$. Upon using the above expressions for $x(t)$ and $v(t)$, these conditions lead to $B = -L$ and $A = 0$, respectively. Hence, we have

$$x(t) = -L\cos(\omega\, t), \quad v(t) = \omega\, L \sin(\omega\, t)$$

and this solution holds until the marble loses contact with the piston (and is, eventually, ejected out of the toy gun). The velocity of the piston reaches the maximum value at $t_{\rm max} = \pi/(2\,\omega)$, i.e. when $x(t_{\rm max}) = 0$. The marble loses contact as the spring passes the equilibrium point. The final speed of the marble is

$$v(t_{\rm max}) = \omega\, L = \sqrt{\frac{k}{M+m}}\, L.$$

To generate a high speed, k and L should be large, while $(M + m)$ should be small.

This result could have also been easily arrived at using the conservation of energy. Note that, when the piston and marble are pulled back by a distance L and the system is at rest, the total energy, say, E, of the system is given completely by its potential energy, i.e.

$$E = \frac{k}{2}\, L^2.$$

At the equilibrium point when the marble is released, the total energy is purely kinetic and it is given by

$$E = \frac{1}{2}\,(M+m)\, v^2(t_{\rm max}).$$

From the conservation of energy, we immediately have

$$\frac{1}{2}\,(M+m)\, v^2(t_{\rm max}) = \frac{k}{2}\, L^2$$

so that

$$v(t_{\rm max}) = \sqrt{\frac{k}{M+m}}\, L,$$

which is the result we had obtained above.

5. _Virial theorem:_ Consider an undamped, harmonic oscillator with period T. Let $\langle f\rangle$ denote the average value of any variable $f(t)$, averaged over one complete cycle as follows:

$$\langle f\rangle = \frac{1}{T}\int_0^T {\rm d}t\, f(t).$$

Prove that the average values of the kinetic and the potential energies $\mathcal{T}$ and U are given by

$$\langle \mathcal{T}\rangle = \langle U\rangle = E/2,$$

where E is the total energy of the oscillator. [JRT, Problem 5.12]

Hint: Start by proving the more general and, extremely useful, results that $\langle \sin^2(\omega\, t - \delta)\rangle = \langle \cos^2(\omega\, t - \delta)\rangle = 1/2$.

Note: The above relation between $\langle \mathcal{T}\rangle$ and $\langle U\rangle$ is a specific example of a general result governing bounded systems known as the virial theorem. (We are using $\mathcal{T}$ here to denote the kinetic energy in order to distinguish it from the time period T!)

Solution: Recall that, we can write the generic solution to equation of motion describing the simple harmonic oscillator as

$$x(t) = A\cos(\omega\, t - \delta)\,.$$

The kinetic energy of the oscillator is then given by

$$\mathcal{T} = \frac{m}{2}\,\dot{x}^2 = \frac{m}{2}\,\omega^2\,A^2\,\sin^2(\omega\,t-\delta) = \frac{k}{2}\,A^2\,\sin^2(\omega\,t-\delta)\,,$$

where we have made use of the fact that $\omega = \sqrt{k/m}$, with k being the spring constant. The potential energy of the oscillator is given by

$$U = \frac{k}{2}\,x^2 = \frac{k}{2}\,A^2\cos^2(\omega\,t-\delta)\,.$$

Now, note that

$$\begin{aligned}\langle\sin^2(\omega\,t-\delta)\rangle &= \frac{1}{T}\int_0^T \mathrm{d}t\,\sin^2(\omega\,t-\delta) = \frac{1}{2\,T}\int_0^T \mathrm{d}t\,\Big\{1-\cos\left[2\,(\omega\,t-\delta)\right]\Big\}\\ &= \frac{1}{2\,T}\left\{t-\frac{\sin\left[2\,(\omega\,t-\delta)\right]}{2\,\omega}\right\}_0^T = \frac{1}{2}-\frac{1}{4\,\omega\,T}\Big\{\sin\left[2\,(\omega\,T-\delta)\right]+\sin\left(2\,\delta\right)\Big\}\end{aligned}$$

and, since, $T = 2\,\pi/\omega$, this expression reduces to

$$\langle\sin^2(\omega\,t-\delta)\rangle = \frac{1}{2}-\frac{1}{8\,\pi}\,\left[\sin\left(4\,\pi-2\,\delta\right)+\sin\left(2\,\delta\right)\right] = \frac{1}{2}.$$

Similarly, we also have

$$\begin{aligned}\langle\cos^2(\omega\,t-\delta)\rangle &= \frac{1}{T}\int_0^T \mathrm{d}t\,\cos^2(\omega\,t-\delta) = \frac{1}{2\,T}\int_0^T \mathrm{d}t\,\Big\{1+\cos\left[2\,(\omega\,t-\delta)\right]\Big\}\\ &= \frac{1}{2\,T}\left\{t+\frac{\sin\left[2\,(\omega\,t-\delta)\right]}{2\,\omega}\right\}_0^T = \frac{1}{2}+\frac{1}{4\,\omega\,T}\Big\{\sin\left[2\,(\omega\,T-\delta)\right]+\sin\left(2\,\delta\right)\Big\}\\ &= \frac{1}{2}+\frac{1}{8\,\pi}\,\left[\sin\left(4\,\pi-2\,\delta\right)+\sin\left(2\,\delta\right)\right] = \frac{1}{2}.\end{aligned}$$

Upon using these expressions, it is easy to establish the required result that, in the case of the simple harmonic oscillator

$$\langle\mathcal{T}\rangle = \langle U\rangle = \frac{k}{4}\,A^2 = \frac{E}{2},$$

where E is the total energy of the oscillator.

Illustrative examples 4

Damped and forced simple harmonic motion

1. *Decaying energy:* A damped oscillator satisfies the equation [JRT, Problem 5.23]

$$m\,\ddot{x} + b\,\dot{x} + k\,x = 0,$$

where $F_{\rm damp} = -b\,\dot{x}$ is the damping force. Determine the rate of change of the energy $E = (m/2)\,\dot{x}^2 + (k/2)\,x^2$ (by straightforward differentiation of the solution) and, with the help of the above equation of motion, show that $\mathrm{d}E/\mathrm{d}t$ is (minus) the rate at which energy is dissipated by $F_{\rm damp}$.

2. *The second solution at critical damping:* When we had discussed the case of critical damping (wherein $\beta = \omega_0$), we had only provided plausible arguments to arrive at the second solution, viz. $x_2(t) = t\,\mathrm{e}^{-\beta\,t}$. The solution can be obtained in a systematic way by starting with the solutions for $\beta < \omega_0$ and suitably taking the limit $\beta \to \omega_0$ as described below. [JRT, Problem 5.24]

   (a) Recall that, for $\beta < \omega_0$, we can write the two solutions as $x_1(t) = \mathrm{e}^{-\beta\,t}\cos(\omega_1\,t)$ and $x_2(t) = \mathrm{e}^{-\beta\,t}\sin(\omega_1\,t)$, where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$, with $\omega_0 = \sqrt{k/m}$ and $\beta = b/(2\,m)$. Show that, as $\beta \to \omega_0$, the first term approaches the first solution for critical damping, viz. $x_1(t) = \mathrm{e}^{-\beta\,t}$.

   (b) Note that, as $\beta \to \omega_0$, the second solution seems to vanish. However, as long as $\beta < \omega_0$, one can divide $x_2(t)$ by ω_1, which will also be a perfectly good second solution. Show that, as $\beta \to \omega_0$, this new second solution behaves as $x_2(t) = t\,\mathrm{e}^{-\beta\,t}$.

3. *Period of an underdamped oscillator:* Consider a damped oscillator with $\beta < \omega_0$. There arises a difficulty in defining the period T_1 of the oscillator, since the motion, say, $x_1(t) = \mathrm{e}^{-\beta\,t}\cos(\omega_1\,t)$, is not periodic. However, a possible definition for the period of the damped oscillator would be the time T_1 between successive maxima of $x_1(t)$. [JRT, Problem 5.25]

   (a) Plot the behavior of $x_1(t)$ against t and indicate the time corresponding to the above definition of T_1 on the graph. Show that $T_1 = 2\,\pi/\omega_1$.

   (b) Show that an equivalent definition is that T_1 is twice the time between successive zeros of $x(t)$. Also indicate this on the plot.

   (c) If $\beta = \omega_0/2$, determine the factor by which the amplitude shrinks in one period.

4. *The Q of an oscillator:* The degree of damping of an oscillator is often characterized by a dimensionless parameter Q, known as the quality factor, defined by [KK, Section 11.3.2]

$$Q = \frac{\text{average energy stored in the oscillator}}{\text{average energy dissipated during 1 radian of motion}}.$$

Show that the Q of a lightly damped oscillator is given by $Q = \omega_0/(2\,\beta)$.

5. *Full width at half maximum:* Consider a damped and driven oscillator, with a natural frequency ω_0 and damping constant β, as defined earlier. For an underdamped oscillator such that $\beta \ll \omega_0$, as we have discussed, if the driving frequency ω is varied, the maximum amplitude $A_{\rm max}$ occurs at $\omega \simeq \omega_0$. Show that $A_{\rm max}^2$ is equal to half its maximum value when $\omega \simeq \omega_0 \pm \beta$, so that the full width at half maximum is just $2\,\beta$. [JRT, Problem 5.41]

Hint: To arrive at the result, one has to be careful with the approximations involved. For instance, while it fine to choose $\omega + \omega_0 = 2\,\omega_0$, one cannot set $\omega - \omega_0 = 0$.

Illustrative examples 4 with solutions

Damped and forced simple harmonic motion

1. *Decaying energy:* A damped oscillator satisfies the equation [JRT, Problem 5.23]

$$m\,\ddot{x} + b\,\dot{x} + k\,x = 0,$$

where $F_{\mathrm{damp}} = -b\,\dot{x}$ is the damping force. Determine the rate of change of the energy $E = (m/2)\,\dot{x}^2 + (k/2)\,x^2$ (by straightforward differentiation of the solution) and, with the help of the above equation of motion, show that $\mathrm{d}E/\mathrm{d}t$ is (minus) the rate at which energy is dissipated by F_{damp}.

Solution: Recall that, according to the work-energy theorem, the change in the energy, say, ΔE, of a system can be expressed as

$$\Delta E = \Delta\mathcal{T} + \Delta U = W_{\mathrm{nc}},$$

where $\Delta\mathcal{T}$ and ΔU represent the changes in the kinetic and potential energies, respectively, while W_{nc} is the work done by the non-conservative forces. For instance, if $\boldsymbol{F}_{\mathrm{damp}}$ denotes the damping force, then the above relation can be expressed as

$$\Delta E = \Delta\mathcal{T} + \Delta U = \boldsymbol{F}_{\mathrm{damp}}\cdot \mathrm{d}\boldsymbol{r},$$

where $\mathrm{d}\boldsymbol{r}$ denotes the displacement of the system. In the one-dimensional case of our interest, this reduces to

$$\Delta E = \Delta\mathcal{T} + \Delta U = F_{\mathrm{damp}}\,\mathrm{d}x$$

or, equivalently,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}\mathcal{T}}{\mathrm{d}t} + \frac{\mathrm{d}U}{\mathrm{d}t} = F_{\mathrm{damp}}\,\dot{x} = -b\,\dot{x}^2,$$

in the case of the damped oscillator.

The solution for the damped oscillator is given by

$$x(t) = A\,\mathrm{e}^{-\beta\,t}\cos\left(\omega_1\,t - \delta\right),$$

where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$ and we have assumed weak damping, i.e. $\beta < \omega_0$. Also, we have $b = 2\,m\,\beta$. In such a case,

$$\dot{x}(t) = -\beta\,A\,\mathrm{e}^{-\beta\,t}\cos\left(\omega_1\,t - \delta\right) - \omega_1\,A\,\mathrm{e}^{-\beta\,t}\sin\left(\omega_1\,t - \delta\right)$$

so that we obtain

$$\begin{aligned} E &= \frac{m}{2}\,\dot{x}^2 + \frac{k}{2}\,x^2 \\ &= \frac{1}{2}\left(A\,\mathrm{e}^{-\beta\,t}\right)^2\left\{m\left[\beta\cos\left(\omega_1\,t-\delta\right) + \omega_1\sin\left(\omega_1\,t-\delta\right)\right]^2 + k\cos^2\left(\omega_1\,t-\delta\right)\right\}. \end{aligned}$$

Using this result, the fact that $\omega_1^2 = \omega_0^2 - \beta^2$, $\omega_0^2 = k/m$ and quite a bit of algebra, one can show that

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} &= -\beta\left(A\,\mathrm{e}^{-\beta\,t}\right)^2\left\{m\left[\beta\cos\left(\omega_1\,t-\delta\right) + \omega_1\sin\left(\omega_1\,t-\delta\right)\right]^2 + k\cos^2\left(\omega_1\,t-\delta\right)\right\} \\ &\quad + \omega_1\left(A\,\mathrm{e}^{-\beta\,t}\right)^2\left\{m\left[\beta\cos\left(\omega_1\,t-\delta\right) + \omega_1\sin\left(\omega_1\,t-\delta\right)\right]\right. \\ &\quad \left.\times\left[-\beta\sin\left(\omega_1\,t-\delta\right) + \omega_1\cos\left(\omega_1\,t-\delta\right)\right] - k\cos\left(\omega_1\,t-\delta\right)\sin\left(\omega_1\,t-\delta\right)\right\} \\ &= -2\,m\,\beta\left(A\,\mathrm{e}^{-\beta\,t}\right)^2\left[\beta\cos\left(\omega_1\,t-\delta\right) + \omega_1\sin\left(\omega_1\,t-\delta\right)\right]^2 = -b\,\dot{x}^2, \end{aligned}$$

as required. It should be mentioned here that a couple steps between the penultimate and final expressions have been omitted. They are lengthy but straightforward manipulations involving trigonometric functions.

2. *The second solution at critical damping:* When we had discussed the case of critical damping (wherein $\beta = \omega_0$), we had only provided plausible arguments to arrive at the second solution, viz. $x_2(t) = t\, \mathrm{e}^{-\beta t}$. The solution can be obtained in a systematic way by starting with the solutions for $\beta < \omega_0$ and suitably taking the limit $\beta \to \omega_0$ as described below. [JRT, Problem 5.24]

   (a) Recall that, for $\beta < \omega_0$, we can write the two solutions as $x_1(t) = \mathrm{e}^{-\beta t} \cos(\omega_1 t)$ and $x_2(t) = \mathrm{e}^{-\beta t} \sin(\omega_1 t)$, where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$, with $\omega_0 = \sqrt{k/m}$ and $\beta = b/(2\,m)$. Show that, as $\beta \to \omega_0$, the first term approaches the first solution for critical damping, viz. $x_1(t) = \mathrm{e}^{-\beta t}$.
   Solution: For $\beta < \omega_0$, the two solutions are

   $$x_1(t) = \mathrm{e}^{-\beta t} \cos(\omega_1 t), \quad x_2(t) = \mathrm{e}^{-\beta t} \sin(\omega_1 t),$$

   where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$. As $\beta \to \omega_0$, $\omega_1 \to 0$, and it is straightforward to see that, in this limit,

   $$x_1(t) = \mathrm{e}^{-\beta t},$$

   since the cosine term reduces to unity.

   (b) Note that, as $\beta \to \omega_0$, the second solution seems to vanish. However, as long as $\beta < \omega_0$, one can divide $x_2(t)$ by ω_1, which will also be a perfectly good second solution. Show that, as $\beta \to \omega_0$, this new second solution behaves as $x_2(t) = t\, \mathrm{e}^{-\beta t}$.
   Solution: To avoid the fact that the second solution $x_2(t)$ seems to vanish as $\beta \to \omega_0$, let us express the solution as follows:

   $$x_2(t) = \frac{1}{\omega_1}\, \mathrm{e}^{-\beta t} \sin(\omega_1 t),$$

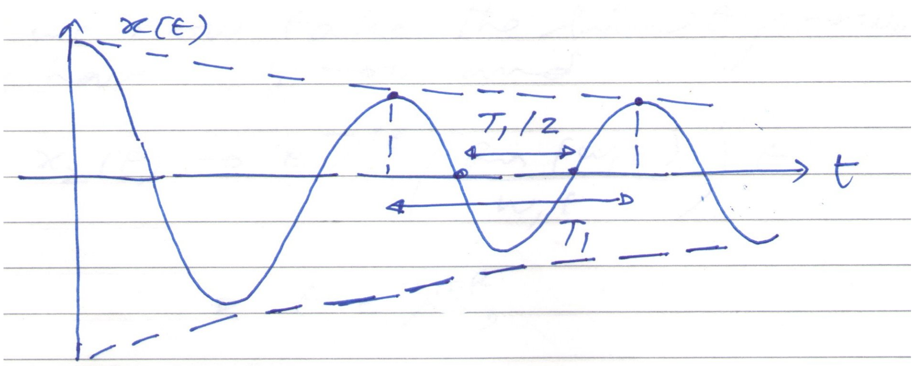
   since an overall constant will not make a difference. If we now take the limit $\omega_1 \to 0$, we obtain that

   $$x_2(t) = \lim_{\omega_1 \to 0} \frac{\sin(\omega_1 t)}{\omega_1 t}\, t\, \mathrm{e}^{-\beta t} = t\, \mathrm{e}^{-\beta t},$$

   as required.

3. *Period of an underdamped oscillator:* Consider a damped oscillator with $\beta < \omega_0$. There arises a difficulty in defining the period T_1 of the oscillator, since the motion, say, $x_1(t) = \mathrm{e}^{-\beta t} \cos(\omega_1 t)$, is not periodic. However, a possible definition for the period of the damped oscillator would be the time T_1 between successive maxima of $x_1(t)$. [JRT, Problem 5.25]

   (a) Plot the behavior of $x_1(t)$ against t and indicate the time corresponding to the above definition of T_1 on the graph. Show that $T_1 = 2\,\pi/\omega_1$.
   Solution: The behavior of $x_1(t)$ is plotted in the figure below.



   Since $\mathrm{e}^{-\beta t}$ is a monotonically decreasing function, the period of the solution

   $$x_1(t) = \mathrm{e}^{-\beta t} \cos(\omega_1 t)$$

is determined by the cosine term. Evidently, the period in such a case is $T_1 = 2\,\pi/\omega_1$. This period can also be arrived at by determining the time interval between the points where the velocity vanishes [corresponding to maxima and minima of $x_1(t)$], which occurs whenever $\tan(\omega_1\,t) = -\beta/\omega_1$.

(b) Show that an equivalent definition is that T_1 is twice the time between successive zeros of $x_1(t)$. Also indicate this on the plot.

<u>Solution:</u> Similarly, the zeros of the $x(t)$ occur when the cosine term vanishes. Clearly, the time between successive zeros is

$$\frac{T_1}{2} = \frac{\pi}{\omega_1}$$

or $T_1 = 2\,\pi/\omega_1$, as we had arrived at above.

(c) If $\beta = \omega_0/2$, determine the factor by which the amplitude shrinks in one period.

<u>Solution:</u> The solution $x_1(t)$ exhibits maxima when

$$\omega\,t_n = 2\,n\,\pi$$

or $t_n = 2\,n\,\pi/\omega_1$, where $n = 0, 1, 2, 3 \ldots$. At $t_n = 2\,n\,\pi/\omega_1$, we have

$$x_1(t_n) = \mathrm{e}^{-\beta\,t_n}\cos(\omega_1\,t_n) = \mathrm{e}^{-2\,n\,\pi\,\beta/\omega_1}$$

and, at $t_{n+1} = 2\,(n+1)\,\pi/\omega_1$, we have

$$x_1(t_{n+1}) = \mathrm{e}^{-\beta\,t_{n+1}}\cos(\omega_1\,t_n) = \mathrm{e}^{-2\,(n+1)\,\pi\,\beta/\omega_1},$$

since $\cos(\omega\,t_n) = \cos(\omega\,t_{n+1}) = 1$. Therefore,

$$\frac{x_1(t_{n+1})}{x_1(t_n)} = \frac{\mathrm{e}^{-2\,(n+1)\,\pi\,\beta/\omega_1}}{\mathrm{e}^{-2\,n\,\pi\,\beta/\omega_1}} = \mathrm{e}^{-2\,\pi\,\beta/\omega_1},$$

which is the amount by which the amplitude of the oscillation shrinks by one period. When $\beta = \omega_0/2$, we have

$$\omega_1 = \sqrt{\omega_0^2 - \beta^2} = \sqrt{\omega_0^2 - (\omega_0/2)^2} = \sqrt{3}\,\omega_0/2.$$

In such a case, amplitude of the oscillation shrinks by the amount

$$\frac{x_1(t_{n+1})}{x_1(t_n)} = \mathrm{e}^{-2\,\pi\,(\omega_0/2)/(\sqrt{3}\,\omega_0/2)} = \mathrm{e}^{-2\,\pi/\sqrt{3}} \simeq 0.026.$$

4. <u>*The Q of an oscillator:*</u> The degree of damping of an oscillator is often characterized by a dimensionless parameter Q, known as the quality factor, defined by [KK, Section 11.3.2]

$$Q = \frac{\text{average energy stored in the oscillator}}{\text{average energy dissipated during 1 radian of motion}}.$$

Show that the Q of a lightly damped oscillator is given by $Q = \omega_0/(2\,\beta)$.

<u>Solution:</u> Recall that, in the case of a damped oscillator, we have

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -b\,\dot{x}^2,$$

where $b = 2\,m\,\beta$. Therefore, for a lightly damped oscillator wherein $\beta \ll \omega_0$, we can write

$$\Delta E = -2\,m\,\beta\,\langle\dot{x}^2\rangle\,\Delta t = -4\,\beta\,\langle\mathcal{T}\rangle\,\Delta t = -4\,\beta\,E\,\Delta t/2 = -2\,\beta\,E\,\Delta t,$$

where we have made use of the fact that $\langle \mathcal{T} \rangle = m\,\langle \dot{x}^2 \rangle/2 = E/2$. If we choose $\Delta t \simeq \omega_0^{-1}$, then

$$\Delta E \simeq 2\,\beta\,E/\omega_0$$

so that the quality factor of the oscillator is given by

$$Q = \frac{E}{\Delta E} = \frac{E}{2\,\beta\,E/\omega_0} = \frac{\omega_0}{2\,\beta},$$

as required.

Note that a lightly damped oscillator has $Q \gg 1$, while a heavily damped oscillator has a low Q.

5. *Full width at half maximum:* Consider a damped and driven oscillator, with a natural frequency ω_0 and damping constant β, as defined earlier. For an underdamped oscillator such that $\beta \ll \omega_0$, as we have discussed, if the driving frequency ω is varied, the maximum amplitude A_{max} occurs at $\omega \simeq \omega_0$. Show that A_{max}^2 is equal to half its maximum value when $\omega \simeq \omega_0 \pm \beta$, so that the full width at half maximum is just $2\,\beta$. [JRT, Problem 5.41]

   Hint: To arrive at the result, one has to be careful with the approximations involved. For instance, while it fine to choose $\omega + \omega_0 = 2\,\omega_0$, one cannot set $\omega - \omega_0 = 0$.

   Solution: Recall that, the amplitude of the oscillations in such a situation is given by

$$A^2 = \frac{f_0^2}{\left(\omega^2 - \omega_0^2\right)^2 + 4\,\beta^2\,\omega^2},$$

where f_0 is the amplitude of the forcing term. Note that

$$\frac{\mathrm{d}A^2}{\mathrm{d}\omega} = -f_0^2\,\frac{4\,\left(\omega^2 - \omega_0^2\right)\,\omega + 8\,\beta^2\,\omega}{\left[\left(\omega^2 - \omega_0^2\right)^2 + 4\,\beta^2\,\omega^2\right]^2} = 0$$

occurs at

$$\omega_2 = \sqrt{\omega_0^2 - 2\,\beta^2}.$$

At $\omega = \omega_2$,

$$A_{\mathrm{max}}^2 = \frac{f_0^2}{4\,\beta^2\,\left(\omega_0^2 - \beta^2\right)} \simeq \frac{f_0^2}{4\,\beta^2\,\omega_0^2},$$

for $\beta \ll \omega_0$. Now, let us set

$$\frac{A_{\mathrm{max}}^2}{2} = \frac{f_0^2}{8\,\beta^2\,\omega_0^2} = \frac{f_0^2}{\left(\omega^2 - \omega_0^2\right)^2 + 4\,\beta^2\,\omega^2},$$

so that

$$\left(\omega^2 - \omega_0^2\right)^2 + 4\,\beta^2\,\omega^2 = 8\,\beta^2\,\omega_0^2.$$

This can be written as

$$(\omega + \omega_0)^2\,(\omega - \omega_0)^2 = 4\,\beta^2\,\left(2\,\omega_0^2 - \omega^2\right)$$

or

$$(2\,\omega_0)^2\,(\omega - \omega_0)^2 \simeq 4\,\beta^2\,\left(2\,\omega_0^2 - \omega^2\right) \simeq 4\,\beta^2\,\omega_0^2,$$

which implies that

$$(\omega - \omega_0)^2 \simeq \beta^2$$

or

$$\omega \simeq \omega_0 \pm \beta.$$

Hence, the full width at half maximum is

$$\Delta\omega = (\omega_0 + \beta) - (\omega_0 - \beta) = 2\,\beta.$$

Hyperbolic functions

Recall that, we can Taylor expand the sine and cosine functions as follows:

$$\begin{aligned}\sin x &\simeq x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\,,\\ \cos x &\simeq 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\,.\end{aligned}$$

Also, recall that, we can express the exponential function as

$$\mathrm{e}^x \simeq 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\,.$$

Using this result, we can write

$$\begin{aligned}\mathrm{e}^{i\,x} &= 1 + i\,x - \frac{x^2}{2!} - i\,\frac{x^3}{3!} + \frac{x^4}{4!} + i\,\frac{x^5}{5!} + \ldots\\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + i\,\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right) = \cos x + i\sin x,\end{aligned}$$

a famous result known as Euler's formula. We can invert this result to write

$$\cos x = \frac{1}{2}\,\left(\mathrm{e}^{i\,x} + \mathrm{e}^{-i\,x}\right), \quad \sin x = \frac{1}{2\,i}\,\left(\mathrm{e}^{i\,x} - \mathrm{e}^{-i\,x}\right).$$

The cosine and sine hyperbolic functions are defined as follows:

$$\cosh x = \frac{1}{2}\,\left(\mathrm{e}^{x} + \mathrm{e}^{-x}\right), \quad \sinh x = \frac{1}{2}\,\left(\mathrm{e}^{x} - \mathrm{e}^{-x}\right).$$

It is straightforward to establish that

$$\cosh^2 x - \sinh^2 x = 1,$$

which is equivalent to the following well known result involving the trigonometric functions:

$$\cos^2 x + \sin^2 x = 1.$$

Note that

$$\cos\,(i\,x) = \cosh x, \quad \sin\,(i\,x) = i\sinh x$$

and, hence, it can be shown that

$$\begin{aligned}\cosh\,(x+y) &= \cosh x\cosh y + \sinh x\sinh y, \quad \cosh\,(x-y) = \cosh x\cosh y - \sinh x\sinh y,\\ \sinh\,(x+y) &= \sinh x\cosh y + \cosh x\sinh y, \quad \sinh\,(x-y) = \sinh x\cosh y - \cosh x\sinh y.\end{aligned}$$

As in the case of the standard trigonometric functions, we can define

$$\tanh x = \frac{\sinh x}{\cosh x}.$$

To understand their behavior, it will be useful to plot the three hyperbolic functions. Also, establish that

$$1 - \tanh^2 x = \mathrm{sech}^2 x, \quad \coth^2 x - 1 = \mathrm{cosech}^2 x,$$

where, as in the trigonometric case, we have defined

$$\sec x = \frac{1}{\cosh x}, \quad \mathrm{cosec}\,x = \frac{1}{\sinh x}.$$

Exercise sheet 3

Undamped and damped simple harmonic motion

1. *The simple pendulum and the harmonic approximation:* Consider a pendulum of mass m and length l that is moving under the influence of gravity. Let the motion of the pendulum be confined to a plane and let the angle of deviation from the vertical be ϕ. [JRT, Problem 5.3]

   (a) Express the potential energy in terms of the angle ϕ.

   (b) Show that for sufficiently small angles the pendulum behaves as a harmonic oscillator.

   (c) Arrive at the well known result for the frequency and time period of the pendulum for small displacements about the mean value.

2. *Periodic and bounded motion:* Consider a particle exerted by the force $F = -F_0 \sinh{(\alpha\, x)}$, where F_0 and α are positive constants. [JRT, Problem 5.10]

   (a) What is the potential energy of the particle?

   (b) Does the particle exhibit harmonic motion for small oscillations? What is the frequency of the oscillations?

   (c) Does the particle exhibit bounded motion for larger energies? Can you determine the period for an arbitrary energy?

3. *Leading corrections to the standard period of simple pendulum:* We had seen earlier that a simple pendulum behaves as a harmonic oscillator for small displacements from the equilibrium value and had also determined the standard time period, say, T, of the pendulum. [DM, Problem 4.23]

   (a) Obtain an integral expression that describes the time period of the pendulum for an arbitrary initial displacement, say, ϕ_0, from the equilibrium value (i.e. $\phi = 0$).

   (b) Expanding the expression suitably in ϕ_0, find the corrections to the time period T at the order ϕ_0^2.

4. *Crossing the origin:* Show that an over damped or critically damped oscillator can cross the origin at most once. [DM, Problem 4.24; JRT, Problem 5.27]

5. *Maximum initial speeds:* A damped oscillator with natural frequency ω_0 and damping coefficient β starts out at position $x_0 > 0$. What is the maximum initial speed (directed toward the origin) the oscillator can have and not cross the origin, if the oscillator is (i) critically damped and (ii) over damped? [DM, Problems 4.26 and 4.27]

Last updated on August 19, 2017

Exercise sheet 3 with solutions

Undamped and damped simple harmonic motion

1. *The simple pendulum and the harmonic approximation:* Consider a pendulum of mass m and length l that is moving under the influence of gravity. Let the motion of the pendulum be confined to a plane and let the angle of deviation from the vertical be ϕ. [JRT, Problem 5.3]

   (a) Express the potential energy in terms of the angle ϕ.
   Solution: Let ϕ denote the angle of the pendulum from the vertical. Let us set the potential energy to be zero at the location of the mass when it is *not* displaced from the vertical, i.e. when $\phi = 0$. In such a case, the potential energy is given by

   $$U(\phi) = m\, g\, l\, (1 - \cos\phi)\,.$$

   (b) Show that for sufficiently small angles the pendulum behaves as a harmonic oscillator.
   Solution: For small ϕ, we can write

   $$\cos\phi \simeq 1 - \frac{\phi^2}{2}$$

   so that the potential energy reduces to

   $$U(\phi) \simeq \frac{m}{2}\, g\, l\, \phi^2 \simeq \frac{m}{2}\, \frac{g}{l}\, (l\, \phi)^2 \simeq \frac{m}{2}\, \frac{g}{l}\, x^2,$$

   where $x = l\, \phi$ is the linear displacement of the mass near $\phi = 0$, which holds true if the angle ϕ remains small. Evidently, the above potential corresponds to that of a simple harmonic oscillator with frequency $\omega^2 = g/l$.

   (c) Arrive at the well known result for the frequency and time period of the pendulum for small displacements about the mean value.
   Solution: Under the small angle approximation, we have seen above that $\omega = \sqrt{g/l}$. Therefore, the corresponding time period T is given by

   $$T = \frac{2\,\pi}{\omega} = 2\,\pi\,\sqrt{\frac{l}{g}}.$$

2. *Periodic and bounded motion:* Consider a particle exerted by the force $F = -F_0 \sinh\,(\alpha\, x)$, where F_0 and α are positive constants. [JRT, Problem 5.10]

   (a) What is the potential energy of the particle?
   Solution: The potential energy of the particle is given by

   $$U(x) = -\int \mathrm{d}x\, F(x) = F_0 \int \mathrm{d}x \sinh\,(\alpha\, x) = \frac{F_0}{\alpha} \cosh\,(\alpha\, x)\,.$$

   (b) Does the particle exhibit harmonic motion for small oscillations? What is the frequency of the oscillations?
   Solution: For small oscillations, the above potential can be expressed as

   $$U(x) \simeq \frac{F_0}{\alpha}\, \left(1 + \frac{\alpha^2\, x^2}{2}\right).$$

   Since $x = 0$ is a stable minimum, the particle exhibits harmonic motion for small oscillations about this point. The frequency of these small oscillations is determined by the relation

   $$\frac{m}{2}\, \omega^2 = \frac{F_0\, \alpha}{2},$$

   so that we have

   $$\omega = \sqrt{\frac{F_0\, \alpha}{m}}.$$

(c) Does the particle exhibit bounded motion for larger energies? Can you determine the period for an arbitrary energy?

Solution: Yes, the particle exhibits bounded motion for larger energies. Since the potential is independent of time, the energy of the system is conserved and the period of the system can be expressed as

$$T = 2\int_{x_-}^{x_+} \mathrm{d}x\, \frac{\sqrt{m/2}}{\sqrt{E - U(x)}},$$

where x_- and x_+ are the left and the right turning points of the system. The turning points are given by

$$x_\pm = \pm\frac{1}{\alpha}\cosh^{-1}\left(\frac{\alpha\, E}{F_0}\right).$$

Moreover, as the potential is symmetric in x, the period can be written as

$$T = 4\int_{0}^{x_+} \mathrm{d}x\, \frac{\sqrt{m/2}}{\sqrt{E - (F_0/\alpha)\cosh(\alpha\, x)}}.$$

However, it turns out to be impossible to express this integral in terms of simple functions. It should be appreciated that the integral can be evaluated easily numerically, with the aid of the simplest of methods, if needed.

3. _Leading corrections to the standard period of simple pendulum:_ We had seen earlier that a simple pendulum behaves as a harmonic oscillator for small displacements from the equilibrium value and had also determined the standard time period, say, T, of the pendulum. [DM, Problem 4.23]

(a) Obtain an integral expression that describes the time period of the pendulum for an arbitrary initial displacement, say, ϕ_0, from the equilibrium value (i.e. $\phi = 0$).

Solution: Recall that the potential describing the system is given by

$$U(\phi) = m\, g\, l\, (1 - \cos\phi).$$

Since the potential is time independent, the total energy of the system is conserved and is given by

$$E = \frac{m}{2}\left(l\,\dot{\phi}\right)^2 + U(\phi),$$

so that we can write

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \sqrt{2/(m\, l^2)}\,\sqrt{E - U(\phi)}$$

or

$$\mathrm{d}t = \frac{\mathrm{d}\phi\,\sqrt{m\, l^2/2}}{\sqrt{E - U(\phi)}}.$$

Since the potential is symmetric in ϕ, the time period of the system can be expressed as

$$T = 4\int_0^{\phi_+} \frac{\mathrm{d}\phi\,\sqrt{m\, l^2/2}}{\sqrt{E - m\, g\, l\, (1 - \cos\phi)}}.$$

As the particle starts with zero velocity at ϕ_0, clearly, $\phi_+ = \phi_0$ and we also have

$$E = m\, g\, l\, (1 - \cos\phi_0).$$

Upon using the relation $\cos\phi = 1 - 2\sin^2(\phi/2)$, we obtain

$$T(\phi_0) = 4\int_0^{\phi_0} \frac{\mathrm{d}\phi\,\sqrt{l/(2\, g)}}{\sqrt{\cos\phi - \cos\phi_0}} = 4\int_0^{\phi_0} \frac{\mathrm{d}\phi\,\sqrt{l/(4\, g)}}{\sqrt{\sin^2(\phi_0/2) - \sin^2(\phi/2)}}.$$

(b) Expanding the expression suitably in ϕ_0, find the corrections to the time period T at the order ϕ_0^2.

Solution: Let us first consider the leading order term. For small ϕ_0 and hence ϕ, we have $\sin\phi \simeq \phi$ and $\sin\phi_0 \simeq \phi_0$, so that the above integral for the time period reduces to

$$T_0 \simeq 4\sqrt{\frac{l}{g}}\int_0^{\phi_0}\frac{\mathrm{d}\phi}{\sqrt{\phi_0^2-\phi^2}}.$$

Upon substituting

$$\phi = \phi_0 \sin\theta,$$

this integral can be easily evaluated to arrive at

$$T_0 = 4\sqrt{\frac{l}{g}}\,\frac{\pi}{2} = 2\,\pi\sqrt{\frac{l}{g}},$$

which is the standard result.

Let us now turn to evaluate the leading corrections to this result. We can write

$$\frac{T(\phi_0)}{T_0} = \frac{1}{\pi\sin(\phi_0/2)}\int_0^{\phi_0}\frac{\mathrm{d}\phi}{\sqrt{1-\sin^2(\phi/2)/\sin^2(\phi_0/2)}}$$

and let us set

$$\sin\theta = \frac{\sin(\phi/2)}{\sin(\phi_0/2)},$$

so that

$$\cos\theta\,\mathrm{d}\theta = \frac{1}{2}\,\frac{\cos(\phi/2)}{\sin(\phi_0/2)}\,\mathrm{d}\phi$$

and hence

$$\frac{T(\phi_0)}{T_0} = \frac{2}{\pi}\int_0^{\pi/2}\frac{\mathrm{d}\theta\,\cancel{\cos\theta}}{\cancel{\sqrt{1-\sin^2\theta}}}\,\frac{1}{\cos(\phi/2)} = \frac{2}{\pi}\int_0^{\pi/2}\frac{\mathrm{d}\theta}{\sqrt{1-\sin^2(\phi_0/2)\sin^2\theta}}.$$

Let us expand the denominator in a binomial expansion. In such a case, we have

$$\begin{aligned}\frac{T(\phi_0)}{T_0} &\simeq \frac{2}{\pi}\int_0^{\pi/2}\mathrm{d}\theta\,\left[1+\frac{1}{2}\sin^2\left(\frac{\phi_0}{2}\right)\sin^2\theta+\frac{3}{8}\sin^4\left(\frac{\phi_0}{2}\right)\sin^4\theta+\cdots\right]\\ &\simeq 1+\frac{1}{\pi}\sin^2\left(\frac{\phi_0}{2}\right)\int_0^{\pi/2}\mathrm{d}\theta\sin^2\theta+\frac{3}{4\,\pi}\sin^4\left(\frac{\phi_0}{2}\right)\int_0^{\pi/2}\mathrm{d}\theta\sin^4\theta+\cdots.\end{aligned}$$

Now,

$$\int_0^{\pi/2}\mathrm{d}\theta\sin^2\theta = \frac{1}{2}\int_0^{\pi/2}\mathrm{d}\theta\,\left[1-\cos(2\,\phi)\right] = \frac{1}{2}\left\{\frac{\pi}{2}-\left[\frac{\sin(2\,\phi)}{2}\right]_0^{\pi/2}\right\} = \frac{\pi}{4}$$

and

$$\begin{aligned}\int_0^{\pi/2}\mathrm{d}\theta\sin^4\theta &= \frac{1}{4}\int_0^{\pi/2}\mathrm{d}\theta\,\left[1-\cos(2\,\phi)\right]^2 = \frac{1}{4}\int_0^{\pi/2}\mathrm{d}\theta\,\left[1+\cos^2(2\,\phi)-2\cos(2\phi)\right]\\ &= \frac{1}{4}\int_0^{\pi/2}\mathrm{d}\theta\,\left[1-2\cos(2\,\phi)\right]+\frac{1}{8}\int_0^{\pi/2}\mathrm{d}\theta\,\left[1+\cos(4\,\phi)\right] = \frac{\pi}{8}+\frac{\pi}{16} = \frac{3\,\pi}{16},\end{aligned}$$

so that we finally obtain

$$\begin{aligned}\frac{T(\phi_0)}{T_0} &\simeq 1+\frac{1}{4}\sin^2\left(\frac{\phi_0}{2}\right)+\frac{9}{64}\sin^4\left(\frac{\phi_0}{2}\right)+\cdots\\ &\simeq 1+\frac{\phi_0^2}{16}+\frac{9\,\phi_0^4}{1024}+\cdots,\end{aligned}$$

where, in arriving at the final expression, we have assumed that ϕ_0 is small enough to permit writing $\sin\phi_0$ as ϕ_0.

Let us consider the correction due to the quadratic term in ϕ_0 and define the correction in percentage C as follows:

$$C = \frac{T - T_0}{T_0} \times 100\%.$$

We find that the correction C is about 0.002% for $\phi_0 = 1°$, 0.05% for $\phi_0 = 5°$, 0.2% for $\phi_0 = 10°$ and 2% for $\phi_0 = 30°$.

4. *Crossing the origin:* Show that an over damped or critically damped oscillator can cross the origin at most once. [DM, Problem 4.24; JRT, Problem 5.27]

Solution: Let us first consider the critically damped case wherein $\beta = \omega_0$. In such a case, recall that the general solution is given by

$$x(t) = C_1\, \mathrm{e}^{-\beta\, t} + C_2\, t\, \mathrm{e}^{-\beta\, t}.$$

If we assume that $x = x_0$ at $t = 0$, then, evidently, $C_1 = x_0$. Also, we have

$$v(t) = \dot{x}(t) = -C_1\, \beta\, \mathrm{e}^{-\beta\, t} + C_2\, (1 - \beta\, t)\, \mathrm{e}^{-\beta\, t}$$

so that, if $v = -v_0$ at $t = t_0$, we have $v_0 = -\beta\, C_1 + C_2$ or $C_2 = v_0 + \beta\, x_0$. Therefore, the complete solution can be written in terms of the initial conditions as follows:

$$x(t) = x_0\, \mathrm{e}^{-\beta\, t} + (v_0 + \beta\, x_0)\; t\, \mathrm{e}^{-\beta\, t}.$$

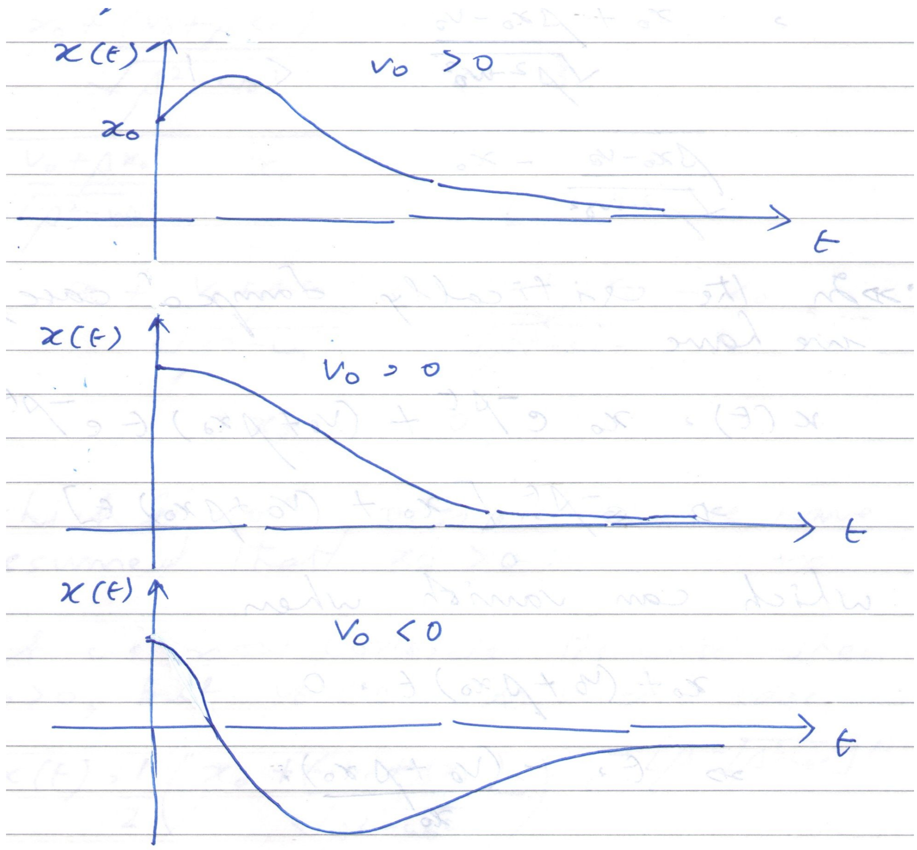
Now, if $x(t_1) = 0$, where t_1 is the time when the particle crosses the origin, we obtain that

$$x_0 + (v_0 + \beta\, x_0)\; t_1 = 0$$

or, equivalently,

$$t_1 = -\frac{x_0}{v_0 + \beta\, x_0}.$$

This implies that the origin is crossed at most by the critically damped oscillator. Note that, we require $t_1 > 0$, which is possible when v_0 is negative, i.e. when the oscillator is set in motion towards the origin. The different possibilities corresponding to different initial velocities are illustrated in the figure below.



Let us now turn to the over damped case wherein $\beta > \omega_0$. In such a situation, the solution is given by

$$x(t) = \mathrm{e}^{-\beta\, t}\, \left(C_1 \mathrm{e}^{\omega_1\, t} + C_2\, \mathrm{e}^{-\omega_1\, t}\right),$$

where $\omega_1 = \sqrt{\beta^2 - \omega_0^2}$. If we assume that $x = x_0$ at $t = 0$, then we have $x_0 = C_1 + C_2$. Also, we have

$$v(t) = \dot{x}(t) = -\mathrm{e}^{-\beta\, t}\, \left[(\beta - \omega_1)\, C_1\, \mathrm{e}^{\omega_1\, t} + (\beta + \omega_1)\, C_2\, \mathrm{e}^{-\omega_1\, t}\right]$$

so that, if $v = v_0$ at $t = 0$, we obtain that

$$v_0 = -\beta\, (C_1 + C_2) + \omega_1\, (C_1 - C_2)\,.$$

Therefore, the general solution can be expressed in terms of the initial conditions as follows:

$$x(t) = \frac{\mathrm{e}^{-\beta\, t}}{2\, \omega_1}\, \left\{[v_0 + (\beta + \omega_1)\, x_0]\, \mathrm{e}^{\omega_1\, t} - [v_0 + (\beta - \omega_1)\, x_0]\, \mathrm{e}^{-\omega_1\, t}\right\}.$$

If $x(t_1) = 0$, then we evidently require that

$$\mathrm{e}^{-2\, \omega_1\, t_1} = \frac{v_0 + (\beta + \omega_1)\, x_0}{v_0 + (\beta - \omega_1)\, x_0}$$

or, equivalently,

$$t_1 = -\frac{1}{2\, \omega_1}\, \ln\left[\frac{v_0 + (\beta + \omega_1)\, x_0}{v_0 + (\beta - \omega_1)\, x_0}\right].$$

Recall that we are interested in the domain wherein $t > 0$. Hence, t_1 has to be a positive quantity. This implies that we require

$$\frac{v_0 + (\beta + \omega_1)\, x_0}{v_0 + (\beta - \omega_1)\, x_0} < 1.$$

As $x_0 > 0$, this condition cannot be satisfied when $v_0 > 0$ or when $v_0 = 0$.

When v_0 is negative such that $-|v_0| + (\beta + \omega_1)\, x_0 < 0$, the above condition can be written as

$$-|v_0| + (\beta + \omega_1)\, x_0 > -|v_0| + (\beta - \omega_1)\, x_0,$$

which can indeed be satisfied. Clearly, the over damped oscillator too crosses the origin at most once. The behavior of $x(t)$ in the three cases (i.e. when $v_0 > 0$, $v_0 = 0$ and $v_0 < 0$) is broadly similar to the behavior in the critically damped case. *However, it should be emphasized that, actually, the critically damped oscillator reaches the origin faster than the over damped case.*

5. *Maximum initial speeds:* A damped oscillator with natural frequency ω_0 and damping coefficient β starts out at position $x_0 > 0$. What is the maximum initial speed (directed toward the origin) the oscillator can have and not cross the origin, if the oscillator is (i) critically damped and (ii) over damped? [DM, Problems 4.26 and 4.27]

   Solution: We had seen above that, in the critically damped case, the oscillator crosses the origin at the time

$$t_1 = -\frac{x_0}{v_0 + \beta\, x_0} = \frac{x_0}{|v_0| - \beta\, x_0},$$

   when $v_0 = -|v_0|$, i.e. when the oscillator is set in motion towards the origin. Since we are interested in the domain $t_1 > 0$, this condition implies that we require $|v_0| > \beta\, x_0$. In other words, $v_0 = -\beta\, x_0$ is the maximum initial speed that the critically damped oscillator can have and not cross the origin.

   In the over damped case, we had seen above that the oscillator will cross the origin when the following condition is satisfied: $-|v_0| + (\beta + \omega_1)\, x_0 < 0$. Therefore, in this case, the maximum initial speed that oscillator can have and not cross the origin is $v_0 = -\,(\beta + \omega_1)\, x_0$.

Exercise sheet 4

Damped and forced simple harmonic motion

1. *Logarithmic decrement:* The logarithmic decrement δ of a free damped oscillator is defined to be the natural logarithm of the ratio of successive maximum displacements in the same direction. Show that $\delta = \pi/Q$, where Q is the quality factor we have discussed earlier. [KK, Problem 11.5]

2. *Velocity and driving force in phase:* Find the driving frequency for which the velocity of a driven and damped oscillator is exactly in phase with the driving force. [KK, Problem 11.9]

3. *Energy lost is energy gained:* Consider a damped oscillator (with natural frequency ω_0 and damping constant β, both fixed) that is driven by the force $F(t) = F_0 \cos(\omega\, t)$. [JRT, Problem 5.45]

   (a) Find the rate $P(t)$ at which $F(t)$ does work and show that the average rate $\langle P \rangle$ over any number of cycles is $m\,\beta\,\omega^2\,A^2$, where m is the mass of the oscillator and A is the amplitude.

   (b) Verify that the energy gained by the oscillator from the driving force is the same as the average rate at which energy is lost to the resistive force.

   (c) Show that, as ω is varied, $\langle P \rangle$ is maximum when $\omega = \omega_0$, i.e. the resonance of the power occurs exactly at the natural frequency.

4. *Grandfather clock:* The pendulum of a grandfather clock activates an escapement mechanism every time it passes through the vertical. The escapement is under tension (provided by a hanging weight) and gives the pendulum a small impulse a distance l from the pivot. The energy transferred by this impulse compensates for the energy dissipated by friction, so that the pendulum swings with a constant amplitude. [KK, Problem 11.10]

   (a) What is the impulse needed to sustain the motion of a pendulum of length L and mass m, with an amplitude of swing ϕ_0 and quality factor Q?

   (b) Why is it desirable for the pendulum to engage the escapement as it passes the vertical rather than at some other point of the cycle?

5. *Cuckoo clock:* A small cuckoo clock has a pendulum 25 cm long with a mass of 10 g and a period of 1 s. The clock is powered by a 200 gm weight which falls 2 m between the daily windings. The amplitude of the swing is 0.2 rad. [KK, Problem 11.12]

   (a) What is the Q of the clock?

   (b) How long would the clock run if it were powered by a battery with 1 J capacity?

Last updated on August 15, 2017

Exercise sheet 4 with solutions

Damped and forced simple harmonic motion

1. Logarithmic decrement: The logarithmic decrement δ of a free damped oscillator is defined to be the natural logarithm of the ratio of successive maximum displacements in the same direction. Show that $\delta = \pi/Q$, where Q is the quality factor we have discussed earlier. [KK, Problem 11.5]

   Solution: Recall that, we had shown that $Q = \omega_0/(2\,\beta)$. In a damped oscillator, we had also seen that, over a period, the amplitude shrinks by the amount

   $$\frac{x(t_{n+1})}{x(t_n)} = \mathrm{e}^{-2\,\pi\,\beta/\omega_1} \simeq \mathrm{e}^{-2\,\pi\,\beta/\omega_0},$$

   where, in arriving at the final equality, we have assumed that the degree of damping is small, i.e. $\beta \ll \omega_0$, so that $\omega_1 \simeq \omega_0$.

   Therefore, we have

   $$\delta = \ln\left[\frac{x(t_n)}{x(t_{n+1})}\right] = \frac{2\,\pi\,\beta}{\omega_0} = \frac{\pi}{Q},$$

   which is the required result.

2. Velocity and driving force in phase: Find the driving frequency for which the velocity of a driven and damped oscillator is exactly in phase with the driving force. [KK, Problem 11.9]

   Solution: Recall that a damped and driven oscillator is governed by the differential equation

   $$\ddot{x} + 2\,\beta\,\dot{x} + \omega_0^2\,x = F(t)/m,$$

   where $F(t)$ is the driving force. For a driving of the form $F(t) = F_0\cos(\omega\,t)$, the solution to the above equation at late times (i.e. when $\beta\,t \gg 1$) can be expressed as

   $$x(t) = A\cos(\omega\,t - \delta),$$

   where $f_0 = F_0/m$ (with m being the mass of the oscillator), the amplitude A is given by

   $$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\,\beta^2\,\omega_0^2}$$

   and the phase δ is defined through the relation

   $$\tan\delta = \frac{2\,\beta\,\omega}{\omega_0^2 - \omega^2}.$$

   The velocity of the oscillator is given by

   $$v(t) = \dot{x}(t) = -\omega\,A\sin(\omega\,t - \delta)$$

   and if we choose $\delta = \pi/2$, we obtain that

   $$v(t) = \omega\,A\cos(\omega\,t),$$

   which is exactly in phase with the forcing term.

   Now, for $\delta = \pi/2$, $\tan\delta$ is infinite, which can occur (according to the definition above) only when $\omega = \omega_0$, which is the required result.

3. Energy lost is energy gained: Consider a damped oscillator (with natural frequency ω_0 and damping constant β, both fixed) that is driven by the force $F(t) = F_0\cos(\omega\,t)$. [JRT, Problem 5.45]

(a) Find the rate $P(t)$ at which $F(t)$ does work and show that the average rate $\langle P\rangle$ over any number of cycles is $m\,\beta\,\omega^2\,A^2$, where m is the mass of the oscillator and A is the amplitude.

<u>*Solution:*</u> The rate at which work is done by the driving force is given by

$$P(t) = \frac{\mathrm{d}W}{\mathrm{d}t} = F(t)\,v(t).$$

For the case of the damped, driven oscillator, at late times, we have

$$x(t) = A\cos(\omega\,t-\delta), \quad v(t) = -\omega\,A\sin(\omega\,t-\delta),$$

with A and δ as defined in the previous exercise. Therefore, the average $\langle P\rangle$ over one period, say, T, is given by

$$\begin{aligned}\langle P\rangle &= \frac{-\omega\,A\,F_0}{T}\int_0^T \mathrm{d}t\cos(\omega\,t)\sin(\omega\,t-\delta)\\ &= \frac{-\omega\,A\,F_0}{T}\int_0^T \mathrm{d}t\cos(\omega\,t)\,\left[\sin(\omega\,t)\cos\delta-\cos(\omega\,t)\sin\delta\right]\\ &= -\omega\,A\,F_0\,\left[\cos\delta\,\langle\cos(\omega\,t)\sin(\omega\,t)\rangle-\sin\delta\,\langle\cos^2(\omega\,t)\rangle\right] = \frac{\omega\,A\,F_0}{2}\sin\delta,\end{aligned}$$

because of the fact that $\langle\cos^2(\omega\,t)\rangle = 1/2$ (as we have seen earlier) and $\langle\cos(\omega\,t)\sin(\omega\,t)\rangle = 0$ (prove this). Since

$$\tan\delta = \frac{2\,\beta\,\omega}{\omega_0^2-\omega^2}$$

and

$$\sin\delta = \frac{\tan\delta}{\sqrt{1+\tan^2\delta}} = \frac{2\,\beta\,\omega}{\left[\left(\omega_0^2-\omega^2\right)^2+4\,\beta^2\,\omega^2\right]^{1/2}},$$

we have

$$\langle P\rangle = \frac{\omega\,A\,F_0}{2}\sin\delta = \frac{m\,\beta\,\omega^2\,A\,f_0}{\left[\left(\omega_0^2-\omega^2\right)^2+4\,\beta^2\,\omega^2\right]^{1/2}} = m\,\beta\,\omega^2\,A^2,$$

which is the required result.

(b) Verify that the energy gained by the oscillator from the driving force is the same as the average rate at which energy is lost to the resistive force.

<u>*Solution:*</u> The damping force is given by $F_{\mathrm{damp}} = 2\,m\,\beta\,v$ so that the corresponding power is given by

$$P(t) = 2\,m\,\beta\,v^2(t).$$

Upon using the above expression for $v(t)$ and averaging over one cycle, we obtain that

$$\langle P\rangle = 2\,m\,\beta\,\langle v^2(t)\rangle = 2\,m\,\beta\,\omega^2\,A^2\,\langle\sin^2(\omega\,t-\delta)\rangle = m\,\beta\,\omega^2\,A^2,$$

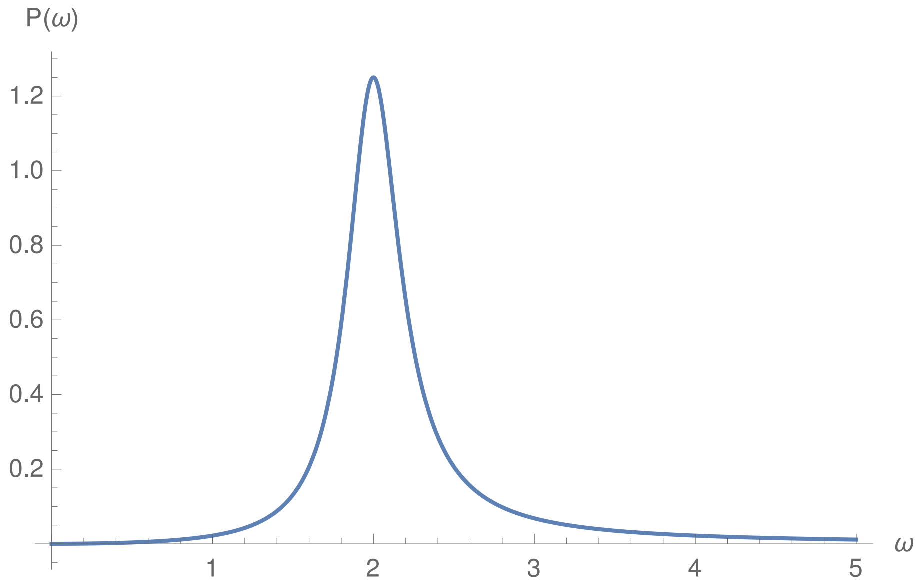
since $\langle\sin^2(\omega\,t-\delta)\rangle = 1/2$ (prove this). The above average power is the same as the power gained by the system.

(c) Show that, as ω is varied, $\langle P\rangle$ is maximum when $\omega = \omega_0$, i.e. the resonance of the power occurs exactly at the natural frequency.

<u>*Solution:*</u> If we use the expression for A, we find that

$$\langle P(\omega)\rangle = \frac{m\,\beta\,\omega^2\,f_0^2}{\left(\omega_0^2-\omega^2\right)^2+4\,\beta^2\,\omega^2}.$$

This average power has been plotted as a function of ω in the figure below for the following values of the parameters involved: $m = 1$, $\omega_0 = 2$, $\beta = 0.2$ and $f_0 = 1$, in suitable units.



Clearly, $\langle P(\omega)\rangle$ peaks at $\omega = \omega_0$.
Also, note that, if we set

$$\frac{\mathrm{d}\langle P(\omega)\rangle}{\mathrm{d}\omega} = \frac{2\,m\,\beta\,\omega\,f_0^2}{\left(\omega_0^2-\omega^2\right)^2+4\,\beta^2\,\omega^2} - \frac{m\,\beta\,\omega^2\,f_0^2}{\left[\left(\omega_0^2-\omega^2\right)^2+4\,\beta^2\,\omega^2\right]^2}\,\left[2\,\left(\omega_0^2-\omega^2\right)\,(-2\,\omega)+8\,\beta^2\,\omega\right] = 0,$$

we obtain that

$$2\,\omega^2\,\left(\omega_0^2-\omega^2\right)-4\,\beta^2\,\omega^2+\left(\omega_0^2-\omega^2\right)^2+4\,\beta^2\,\omega^2=0$$

or

$$\left(\omega_0^2-\omega^2\right)\,\left(2\,\omega^2+\omega_0^2-\omega^2\right)=\left(\omega_0^2-\omega^2\right)\,\left(\omega_0^2+\omega^2\right)=0,$$

which is clearly satisfied when $\omega = \omega_0$.

4. *Grandfather clock:* The pendulum of a grandfather clock activates an escapement mechanism every time it passes through the vertical. The escapement is under tension (provided by a hanging weight) and gives the pendulum a small impulse a distance l from the pivot. The energy transferred by this impulse compensates for the energy dissipated by friction, so that the pendulum swings with a constant amplitude. [KK, Problem 11.10]

   (a) What is the impulse needed to sustain the motion of a pendulum of length L and mass m, with an amplitude of swing ϕ_0 and quality factor Q?

   Solution: To begin with, it is important to appreciate the fact that we are considered weakly damped motion. Recall that the quality factor Q is defined as

$$Q = \frac{E}{\Delta E},$$

   where E is the energy of the oscillator and ΔE is the energy lost due to damping during one radian of motion. If A is the amplitude of the oscillator, then we have

$$E = \frac{k}{2}\,A^2 = \frac{m}{2}\,\omega^2\,A^2,$$

   where

$$\omega = \frac{2\,\pi}{T} = \frac{\cancel{2\pi}}{\cancel{2\pi}}\sqrt{\frac{g}{L}}$$

so that

$$E = \frac{m\,g}{2\,L}\,A^2 = \frac{m\,g}{2\,\cancel{L}}\,L^{\cancel{2}}\,\phi_0^2 = \frac{m\,g\,L}{2}\,\phi_0^2,$$

where we have made use of the fact that $A = L\,\phi_0$.

The impulse I given to the pendulum is the change in the momentum so that

$$I = \Delta p = F\,\Delta t = \frac{\Delta E}{L\,\phi_0}\,\frac{T}{2} = \frac{\Delta E}{L\,\phi_0}\,\pi\,\sqrt{\frac{L}{g}}.$$

As $\Delta E = E/Q$, we can write

$$I = \frac{\pi\,\Delta E}{\sqrt{g\,L}\,\phi_0} = \frac{\pi\,E}{Q\,\sqrt{g\,L}\,\phi_0} = \frac{\pi}{Q\,\sqrt{g\,L}\,\phi_0}\,\frac{m\,g\,L}{2}\,\phi_0^2 = \frac{\pi\,m\,\sqrt{g\,L}\,\phi_0}{2\,Q}.$$

There is another useful way of calculating this impulse. Let v_0 be the velocity of the pendulum as it crosses the vertical at a given instant, and let v_1 be the velocity of the pendulum at the same point half a cycle later. Then, we can write the energy lost by the pendulum during the half cycle as

$$\Delta E = \frac{m}{2}\,\left(v_0^2 - v_1^2\right)$$

so that the quality factor is given by

$$\begin{aligned} Q &= \frac{\text{Energy of the oscillator}}{\text{Energy dissipated per radian}} = \frac{m\,v_0^2/2}{\Delta E/\pi} = \frac{\pi\,m\,v_0^2/2}{(m/2)\,\left(v_0^2 - v_1^2\right)} = \frac{\pi\,v_0^2}{\left(v_0^2 - v_1^2\right)} \\ &= \frac{\pi\,v_0^2}{(v_0 + v_1)\,(v_0 - v_1)} \simeq \frac{\pi\,v_0}{2\,\Delta v}, \end{aligned}$$

where $\Delta v = v_0 - v_1$. Therefore, we can write

$$\Delta v = \frac{\pi\,v_0}{2\,Q}$$

and hence the impulse is given by

$$I = m\,\Delta v = \frac{\pi\,m\,v_0}{2\,Q}.$$

As $v_0 = L\,\dot{\phi} \simeq L\,\omega\,\phi_0 \simeq \sqrt{g\,L}\,\phi_0$, we obtain that

$$I = \frac{\pi\,m\,\sqrt{g\,L}\,\phi_0}{2\,Q},$$

which is the result we had arrived at earlier.

(b) Why is it desirable for the pendulum to engage the escapement as it passes the vertical rather than at some other point of the cycle?

<u>*Solution:*</u> Note that, we can write

$$\Delta E = \frac{m}{2}\,\left[(v + \Delta v)^2 - v^2\right] = m\,v\,\Delta v + \frac{m}{2}\,\Delta v^2 = I\,v + \frac{I^2}{2\,m}.$$

Due to mechanical imperfections, the point where the impulse (by the external agent) acts can vary. To minimize these variations, it would be best if the impulse can be applied when v does not vary to the first order in ϕ. Recall that, in the absence of damping,

$$E = \frac{m}{2}\,v^2 + m\,g\,L\,\left(1 - \cos\phi\right).$$

and, for a constant E, we have

$$m\,v\,\frac{\mathrm{d}v}{\mathrm{d}\phi} = -m\,g\,L\sin\phi.$$

Since $\mathrm{d}v/\mathrm{d}\phi = 0$ when $\phi = 0$, the best time to provide a specific impulse to the pendulum is when it crosses the vertical point.

5. *Cuckoo clock:* A small cuckoo clock has a pendulum 25 cm long with a mass of 10 g and a period of 1 s. The clock is powered by a 200 gm weight which falls 2 m between the daily windings. The amplitude of the swing is 0.2 rad. [KK, Problem 11.12]

   (a) What is the Q of the clock?
   Solution: We have, as in the earlier problem,

   $$E = \frac{k}{2}\,A^2 = \frac{m}{2}\,\omega^2\,A^2,$$

   where A is the amplitude of the oscillations. Also, since $\omega = \sqrt{g/L}$ and $A = L\,\phi_0$, we obtain that

   $$E = \frac{m\,g\,L}{2}\,\phi_0^2.$$

   Note that the clock is driven by the falling mass, say, M, which weighs 200 gm and falls by the length $\ell = 2\,\mathrm{m}$ in a day. Therefore, the energy lost by the pendulum per radian is given by

   $$\Delta E = \frac{M\,g\,\ell}{2\,\pi}\,\left(\frac{T}{1\,\mathrm{day}}\right),$$

   where $T = 1\,\mathrm{s}$ is the period of the pendulum.
   Given $m = 10\,\mathrm{gm}$, $L = 25\,\mathrm{cm}$, $\phi_0 = 0.2\,\mathrm{radian}$, $M = 200\,\mathrm{gm}$ and $\ell = 2\,\mathrm{m}$, we find that

   $$Q = \frac{E}{\Delta E} = \left(\frac{m\,g\,L\,\phi_0^2}{2}\right)\,\left(\frac{2\,\pi}{M\,g\,\ell}\right)\,\left(\frac{1\,\mathrm{day}}{1\,\mathrm{s}}\right) \simeq 68.$$

   (b) How long would the clock run if it were powered by a battery with 1 J capacity?
   Solution: Note that for the values of $M = 200\,\mathrm{gm}$ and $\ell = 2\,\mathrm{m}$

   $$M\,g\,\ell \simeq 3.9\,\mathrm{J}.$$

   This amount of energy runs the clock for a day. Therefore, a battery with 1 J capacity can keep the clock running for a little over 6 hours.

The inverted oscillator

As we have discussed, given an arbitrary one-dimensional potential, the behavior of the system can broadly be classified into the behavior near maxima and minima. We have seen that, for energies just above the minima (which are not points of inflection), the system will behave like an oscillator. We have now studied such regular oscillators extensively. However, near a maximum (and when the maximum is also not a point of inflection), the system behaves like an 'inverted' oscillator with a potential that can be expressed as

$$U(x) \simeq -m\,\varpi^2\,x^2,$$

where $\varpi^2 > 0$. It should be emphasized that x here denotes the displacement from the point of maximum. Also, for simplifying the discussion, we have chosen the potential energy conveniently such that it vanishes at a maximum. Note that, near maxima, the potential behaves as a hilltop, rather that a valley as it does near minima.

Recall that the equation of motion is given by

$$m\,\ddot{x} = -\frac{\mathrm{d}U}{\mathrm{d}x}$$

and, near a maximum described by the above potential, the equation reduces to

$$\ddot{x} = \varpi^2\,x$$

or

$$\ddot{x} - \varpi^2\,x = 0.$$

A general solution to this differential equation is given by

$$x(t) = \mathcal{A}\,\mathrm{e}^{\varpi\,t} + \mathcal{B}\,\mathrm{e}^{-\varpi\,t},$$

so that

$$v(t) = \dot{x}(t) = \varpi\,\left(\mathcal{A}\,\mathrm{e}^{\varpi\,t} - \mathcal{B}\,\mathrm{e}^{-\varpi\,t}\right).$$

If it is given that $x(t=0) = x_0$ and $v(t=0) = v_0$, then we have $x_0 = \mathcal{A} + \mathcal{B}$ and $v_0 = \omega\,(\mathcal{A} - \mathcal{B})/\varpi$. The constants $\mathcal{A}$ and $\mathcal{B}$ can be determined to be

$$\mathcal{A} = \frac{1}{2}\,\left(x_0 + \frac{v_0}{\varpi}\right), \quad \mathcal{B} = \frac{1}{2}\,\left(x_0 - \frac{v_0}{\varpi}\right),$$

so that the solution can be expressed as

$$x(t) = \frac{1}{2}\,\left(x_0 + \frac{v_0}{\varpi}\right)\,\mathrm{e}^{\varpi\,t} + \frac{1}{2}\,\left(x_0 - \frac{v_0}{\varpi}\right)\,\mathrm{e}^{-\varpi\,t}.$$

This solution can be easily expressed in terms of the hyperbolic functions as follows:

$$x(t) = x_0\cosh\left(\varpi\,t\right) + \frac{v_0}{\varpi}\sinh\left(\varpi\,t\right).$$

Note that, since $\sinh x = 0$ and $\cosh x = 1$ for $x = 0$, it is clear that $x(t=0) = x_0$, as required. Also, we have

$$v(t) = \varpi\,x_0\sinh\left(\varpi\,t\right) + v_0\cosh\left(\varpi\,t\right),$$

which reduces to v_0 at $t = 0$, as required.

Phase space and phase portraits

Until now, we have been studying the motion of particles in the coordinate space. It proves to be insightful to understand the motion of a particle in the so-called phase space, which is space of coordinates and momenta. One can define a momentum associated with each of the coordinates and, hence, if a particle has N degrees of freedom, the corresponding phase space has $2\,N$-dimensions. In Classical Mechanics, while it is not necessary to study the motion of particles in phase space, as we mentioned, it provides a complementary view to understand the dynamics involved. In Quantum and Statistical Mechanics, the concept of phase space proves to be essential and one finds that, in particular, the idea is inevitable in Statistical Mechanics.

Time-independent potentials and constancy of energy

We shall restrict ourselves to studying the motion of a particle moving in one dimension, say, the x-direction, under the influence of a given potential $U(x)$. As we have seen, if the potential energy of the particle is time-independent, the total energy of the system, viz.

$$E = T + U = \frac{m\,\dot{x}^2}{2} + U(x),$$

is conserved. In other words, it is a constant of motion. The constancy of energy immediately allows us to understand the behavior of a particle on phase space.

The case of the simple harmonic oscillator

For instance, consider the case of the simple harmonic oscillator whose total energy is given by

$$E = \frac{m\,\dot{x}^2}{2} + \frac{m}{2}\,\omega^2\,x^2.$$

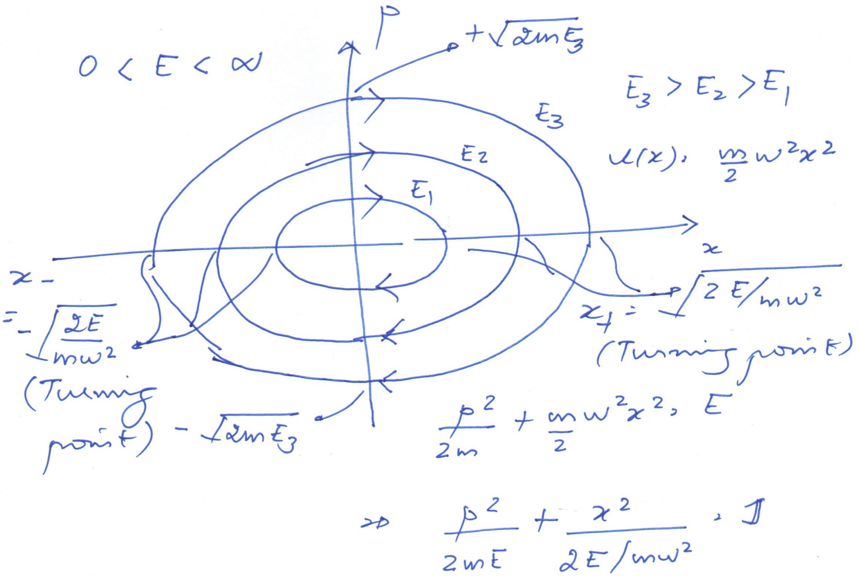
Recall that the momentum of the particle given by $p = m\,\dot{x}$ and, hence, we can express the total energy as

$$E = \frac{p^2}{2\,m} + \frac{m}{2}\,\omega^2\,x^2.$$

Since the total energy E is a constant, this equation can expressed as

$$\frac{p^2}{2\,m\,E} + \frac{m}{2\,E}\,\omega^2\,x^2 = 1,$$

which, evidently, describes an ellipse which intersects the x and p axes at $x = \pm\sqrt{2\,E/(m\,\omega^2)}$ and $p = \pm\sqrt{2\,m\,E}$. Each such ellipse in the phase space is referred to as a *phase trajectory*. A complete collection of such trajectories for all the allowed values of the energy is referred to as the *phase portrait*. In the case of harmonic oscillator, the allowed range of energy is $0 < E < \infty$. Clearly, as we increase or decrease the energy of the oscillator, the size of the ellipse correspondingly increases or decreases. Such a collection of ellipses, as shown in the figure below, form the phase portrait for the system.



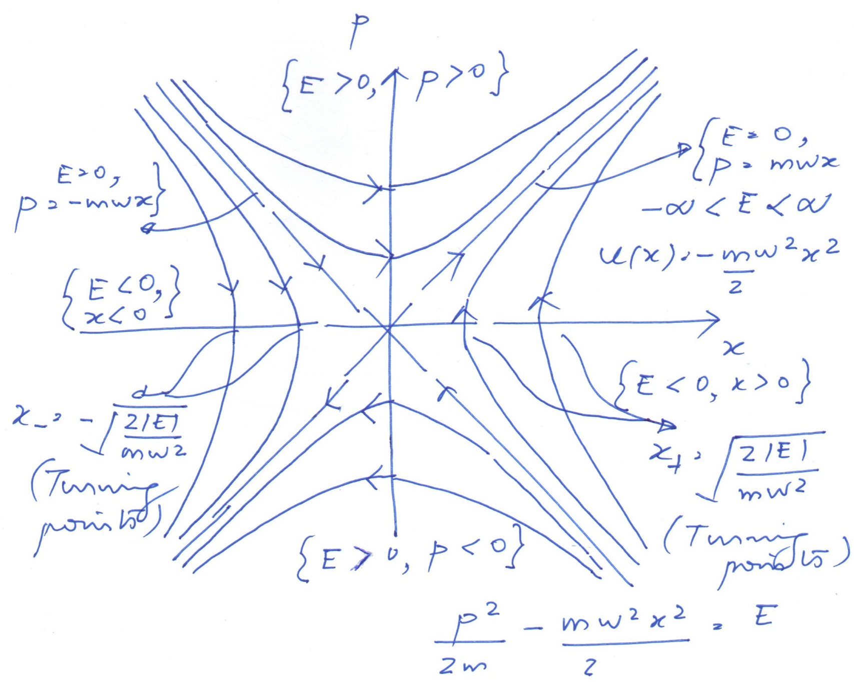
Note that the points where the curves cross the x-axis, i.e. at $\pm\sqrt{2\,E/(m\,\omega^2)}$, are the turning points of the system. The phase trajectories also have a sense of direction about them, which can be indicated with the aid of suitable arrows. For example, consider a specific ellipse. A particle starting at a positive turning point with a negative momentum will go 'down' the ellipse (i.e. below the x-axis) to eventually reach the other turning point, before going 'up' the ellipse (i.e. above the x-axis). This behavior sets the direction for the 'flow' in phase space.

The case of the inverted oscillator

The other case that is important to consider is the so-called inverted oscillator that is described by the potential $U(x) = -m\,\omega^2\,x^2/2$. In contrast to the conventional oscillator, note that the potential energy is negative in this case. Also, the potential energy is unbounded from below, i.e. it does not have a minimum so that the allowed range of energy is $-\infty < E < \infty$. Since the potential energy is independent of time, the total energy is a constant and is given by

$$E = \frac{p^2}{2\,m} - \frac{m}{2}\,\omega^2\,x^2.$$

This equation describes hyperbolae. The $E = 0$ case leads to the straight lines $p = \pm m\,\omega\,x$, which form the asymptotes to the hyperbolae, creating four wedges in the phase space, as shown in the figure below.



The hyperbolae corresponding to $E < 0$ and $E > 0$ fall in the right or left wedges and the top or the bottom wedges, respectively, depending on their initial position and momentum. For instance, a particle with $E < 0$ will follow a hyperbola in the right or the left wedge depending on whether its initial position is positive or negative. Similarly, a particle with $E > 0$ will follow a hyperbola in the top or the bottom wedge depending on whether its initial momentum is positive or negative. Consider a particle with $E < 0$ coming in from left infinity. This particle has large negative values for x and hence large positive values of momentum (so that the total energy is finite and negative). So, in the phase space, it starts at the left top corner and approaches the x-axis. Upon hitting the x-axis at the turning point, its momentum changes direction (i.e. sign) and starts following a hyperbola heading to the bottom left corner. Such behavior lead to corresponding directions for the phase space trajectories.

The generic case

Consider a generic potential $U(x)$. It should be clear that the potential will behave as a simple harmonic oscillator near the minima and as an inverted oscillator near its maxima. This property can be utilized to construct the phase portrait for a generic one-dimensional system. We would urge you to construct the phase portrait for the following simple systems:

1. A particle confined to a box
2. A harmonic oscillator wherein a hard wall has been introduced at the origin
3. A charge in a uniform and time-independent electric field
4. A charged harmonic oscillator that is in a uniform and time-independent electric field

Exercise sheet 5

Small oscillations and trajectories in phase space

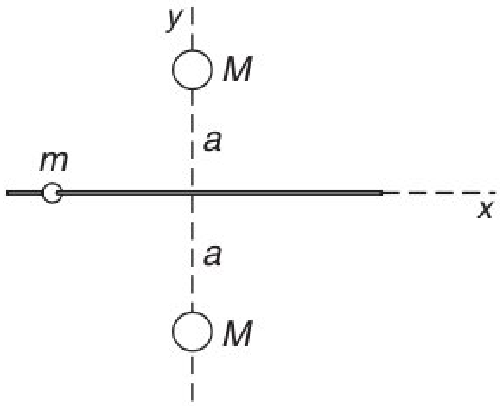
1. *Particle acted upon by two forces:* A particle of mass m moves in one dimension, say, along the positive x-axis. It is acted on by a constant force directed towards the origin with magnitude B, and an inverse-square law repulsive force with magnitude A/x^2. [KK, Problem 5.14]

   (a) What is the potential energy $U(x)$ describing the system? Plot the function $U(x)$ over the domain $0 < x < \infty$.

   (b) Determine the equilibrium position of the particle.

   (c) Does the particle exhibit small oscillations about the equilibrium point?

2. *Oscillation of a bead between gravitating masses:* A bead of mass m slides without friction on a smooth rod along the x axis, as shown in the figure below. The rod is equidistant between two spheres of mass M which are located at $(x, y) = (0, a)$ and $(0, -a)$, and attract the bead gravitationally. Find the frequency of small oscillations of the bead about the origin. [KK, Problem 5.13]



3. *Potential characterizing a molecule:* The Lennard-Jones potential given by

$$U(r) = \epsilon \left[\left(\frac{r_0}{r}\right)^{12} - 2\left(\frac{r_0}{r}\right)^{6} \right],$$

   where $0 < r < \infty$ and $\epsilon > 0$, is a commonly used potential energy function to describe the interactions between two atoms. The term $(r_0/r)^{12}$ rises steeply for $r < r_0$ which models the strong hard sphere repulsion between the two atoms at close separation. The term $(r_0/r)^6$ decreases slowly for $r > r_0$ to model the strong attractive tail between the two atoms at larger separations. The two terms together produce a potential capable of binding atoms. [KK, Problem 5.12; KK, Example 6.2]

   (a) Sketch the potential and determine the equilibrium point of the potential.

   (b) Calculate the frequency of small oscillations about the equilibrium point.

   (c) Estimate the equilibrium point and the frequency of small oscillations for the chlorine diatomic molecule Cl_2, wherein $r_0 = 2.98\,\text{Å} = 2.98 \times 10^{-10}\,\text{m}$ and $\epsilon = 2.48\,\text{eV} = 3.97 \times 10^{-19}\,\text{J}$.

4. *Motion in a potential I:* Consider a particle that is moving in the one-dimensional potential

$$U(x) = \alpha\, x^2 - \beta\, x^3,$$

   where α and β are positive constants. For simplicity, let us assume that $\alpha = 1$ and $\beta = 1$ in suitable units.

(a) Sketch the above potential.

(b) What is the allowed range of energy of the particle?

(c) Determine the values of energy and the domain in x when the particle exhibits bounded motion.

(d) Draw the complete phase portrait of the system.

5. *Motion in a potential II:* Consider a particle that is moving in the following so-called Morse potential:

$$U(x) = \alpha \left(\mathrm{e}^{-2\,\beta\,x} - 2\,\mathrm{e}^{-\beta\,x}\right),$$

where α and β are positive constants. Let us assume that $\alpha = 1$ and $\beta = 1$ in suitable units, for convenience.

(a) Plot the potential.

(b) What is the allowed range of energy?

(c) Determine the domain in energy that leads to bounded motion. Evaluate the period associated with the bounded motion.

(d) Draw the complete phase portrait of the system.

Exercise sheet 5 with solutions

Small oscillations and trajectories in phase space

1. <u>*Particle acted upon by two forces:*</u> A particle of mass m moves in one dimension, say, along the positive x-axis. It is acted on by a constant force directed towards the origin with magnitude B, and an inverse-square law repulsive force with magnitude A/x^2. [KK, Problem 5.14]

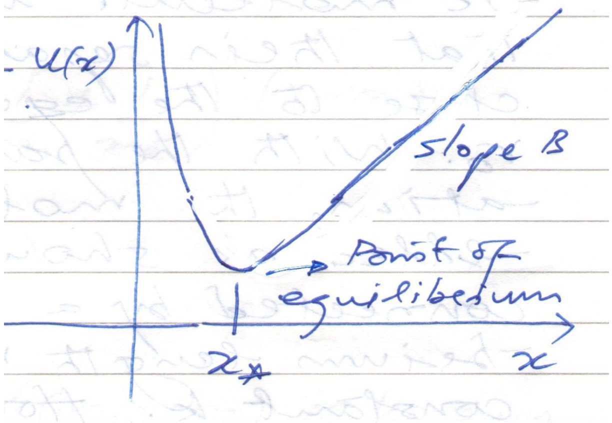
   (a) What is the potential energy $U(x)$ describing the system? Plot the function $U(x)$ over the domain $0 < x < \infty$.
   <u>*Solution:*</u> Note that the force can be written as

   $$\boldsymbol{F} = F(x)\,\hat{\boldsymbol{x}} = \left(-B + \frac{A}{x^2}\right)\hat{\boldsymbol{x}},$$

   so that the potential energy is given by

   $$U(x) = -\int \mathrm{d}\boldsymbol{r}\cdot\boldsymbol{F} = -\int \mathrm{d}x\, F(x) = -\int \mathrm{d}x\,\left(-B + \frac{A}{x^2}\right) = B\,x + \frac{A}{x},$$

   where we have chosen the constant of integration to be zero. It should be stressed that (as we have done earlier), this is a choice which can be made as per convenience. Note that, for our choice, the potential does not vanish for any x. The potential broadly behaves as in the figure below.



   (b) Determine the equilibrium position of the particle.
   <u>*Solution:*</u> The equilibrium position, say, x_*, is the point where the first derivative of the potential vanishes or, equivalently, where the force is zero. In our case, we have $F(x_*) = 0$, which implies that

   $$-B + \frac{A}{x_*^2} = 0$$

   or $x_* = \sqrt{A/B}$.

   (c) Does the particle exhibit small oscillations about the equilibrium point?
   <u>*Solution:*</u> We have, at the equilibrium point,

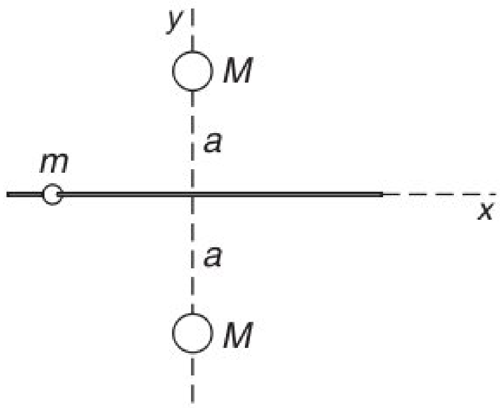
   $$k = \left(\frac{\mathrm{d}^2 U}{\mathrm{d}x^2}\right)_{x=x_*} = \frac{2\,A}{x_*^3} = \frac{2\,B^{3/2}}{A^{1/2}}$$

   and hence the corresponding frequency of oscillations is given by

   $$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2\,B^{3/2}}{m\,A^{1/2}}}.$$

   Since k is positive (as A and B are positive quantities), the particle indeed exhibits small oscillations about the equilibrium point.

2. *Oscillation of a bead between gravitating masses:* A bead of mass m slides without friction on a smooth rod along the x axis, as shown in the figure below. The rod is equidistant between two spheres of mass M which are located at $(x, y) = (0, a)$ and $(0, -a)$, and attract the bead gravitationally. Find the frequency of small oscillations of the bead about the origin. [KK, Problem 5.13]



Solution: Recall that the gravitational potential at a given point is the sum of the gravitational potentials due to different masses. Therefore, the potential energy of the small mass m in the presence of the two large masses M is given by

$$U(x) = U_1(x) + U_2(x),$$

where $U_1(x)$ and $U_2(x)$ are the potential energies due to the individual masses. Since

$$U_1(x) = U_2(x) = -\frac{G\,M\,m}{(a^2 + x^2)^{1/2}},$$

where G is Newton's gravitational constant, we have

$$U(x) = -\frac{2\,G\,M\,m}{(a^2 + x^2)^{1/2}},$$

so that the force along the x-direction is given by

$$F_x = -\frac{\partial U}{\partial x} = -\frac{2\,G\,M\,m\,x}{(a^2 + x^2)^{3/2}}.$$

The point of equilibrium is wherein $F_x = 0$ which, evidently, occurs at $x = 0$.

Also, we have

$$\frac{\mathrm{d}^2 U}{\mathrm{d}x^2} = \frac{2\,G\,M\,m}{(a^2 + x^2)^{3/2}} - \frac{6\,G\,M\,m\,x}{(a^2 + x^2)^{5/2}},$$

which, at $x = 0$ reduces to

$$k = \left(\frac{\mathrm{d}^2 U}{\mathrm{d}x^2}\right)_{x=0} = \frac{2\,G\,M\,m}{a^3}.$$

Since this is a positive definite quantity, the small mass m exhibits oscillations about the equilibrium point $x = 0$ with frequency $\omega = \sqrt{k/m} = \sqrt{2\,G\,M/a^3}$.

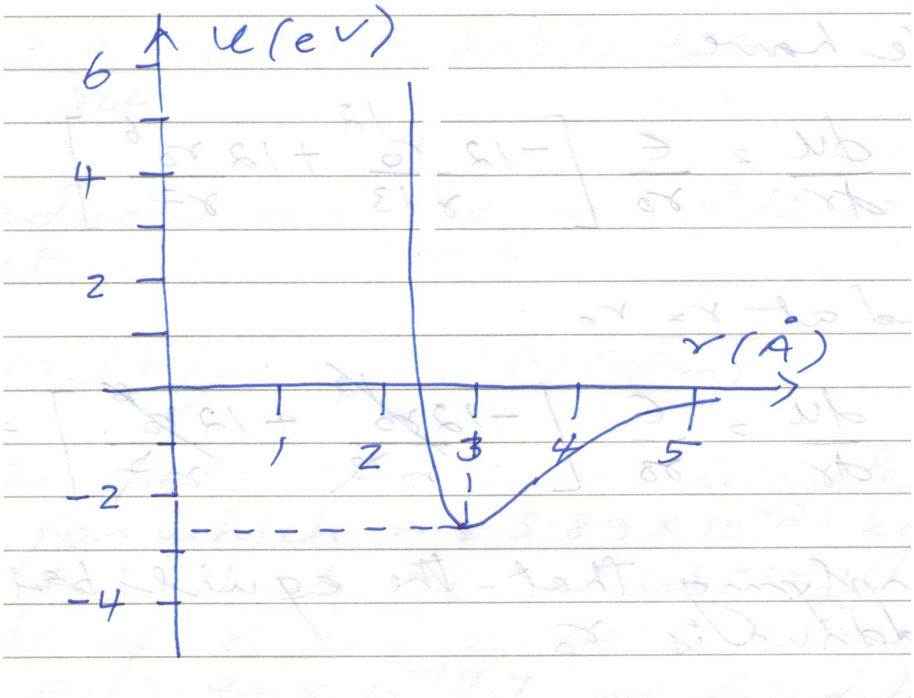
3. *Potential characterizing a molecule:* The Lennard-Jones potential given by

$$U(r) = \epsilon \left[\left(\frac{r_0}{r}\right)^{12} - 2\left(\frac{r_0}{r}\right)^{6}\right],$$

where $0 < r < \infty$ and $\epsilon > 0$, is a commonly used potential energy function to describe the interactions between two atoms. The term $(r_0/r)^{12}$ rises steeply for $r < r_0$ which models the strong hard sphere repulsion between the two atoms at close separation. The term $(r_0/r)^6$ decreases slowly for $r > r_0$ to model the strong attractive tail between the two atoms at larger separations. The two terms together produce a potential capable of binding atoms. [KK, Problem 5.12; KK, Example 6.2]

(a) Sketch the potential and determine the equilibrium point of the potential.

<u>*Solution:*</u> It should be clear that the first term rises sharply (and goes to infinity) as one approaches the origin (actually, for $r \ll r_0$), while the second term is dominant at large distances (i.e. when $r \gg r_0$). Also, note that the potential is positive when the first term dominates, whereas it is negative when the second term is the dominant one. The potential behaves as shown in the accompanying figure. The numbers in the sketch correspond to values for the chlorine molecule mentioned below.



(b) Calculate the frequency of small oscillations about the equilibrium point.

<u>*Solution:*</u> We have

$$\frac{\mathrm{d}U}{\mathrm{d}r} = \epsilon\left(-12\,\frac{r_0^{12}}{r^{13}} + 12\,\frac{r_0^6}{r^7}\right),$$

which vanishes at $r = r_0$, indicating that it is the point of equilibrium. Also, we have

$$\frac{\mathrm{d}^2U}{\mathrm{d}r^2} = 12\,\epsilon\left(13\,\frac{r_0^{12}}{r^{14}} - 7\,\frac{r_0^6}{r^8}\right)$$

so that at $r = r_0$

$$k = \left(\frac{\mathrm{d}^2U}{\mathrm{d}r^2}\right)_{r=r_0} = 12\,\epsilon\left(\frac{13}{r_0^2} - \frac{7}{r_0^2}\right) = \frac{72\,\epsilon}{r_0^2},$$

which is a positive definite quantity. Therefore, the frequency of small oscillations about the equilibrium point is given by

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{72\,\epsilon}{m\,r_0^2}},$$

where m is the mass of the molecule.

Note: Actually, instead of the mass m of the chlorine molecule, we should be using the so-called reduced mass μ, which in this case corresponds to $m/2$. We will learn about the concept of the reduced mass in next unit, when we discuss central forces.

(c) Estimate the equilibrium point and the frequency of small oscillations for the chlorine diatomic molecule Cl_2, wherein $r_0 = 2.98\,\text{Å} = 2.98 \times 10^{-10}\,\text{m}$ and $\epsilon = 2.48\,\text{eV} = 3.97 \times 10^{-19}\,\text{J}$.

<u>Solution:</u> For these values, one finds that

$$\omega \simeq 10^{14}\,\text{s}^{-1} \simeq 10^{14}\,\text{Hz},$$

which is found to be in good agreement with the frequencies measured in molecular spectroscopy.

Note: The mass of the chlorine molecule is about 35 amu, with $1\,\text{amu} = 1.67 \times 10^{-27}\,\text{kg}$.

4. <u>*Motion in a potential I:*</u> Consider a particle that is moving in the one-dimensional potential

$$U(x) = \alpha\, x^2 \,-\, \beta\, x^3,$$

where α and β are positive constants. For simplicity, let us assume that $\alpha = 1$ and $\beta = 1$ in suitable units.

(a) Sketch the above potential.

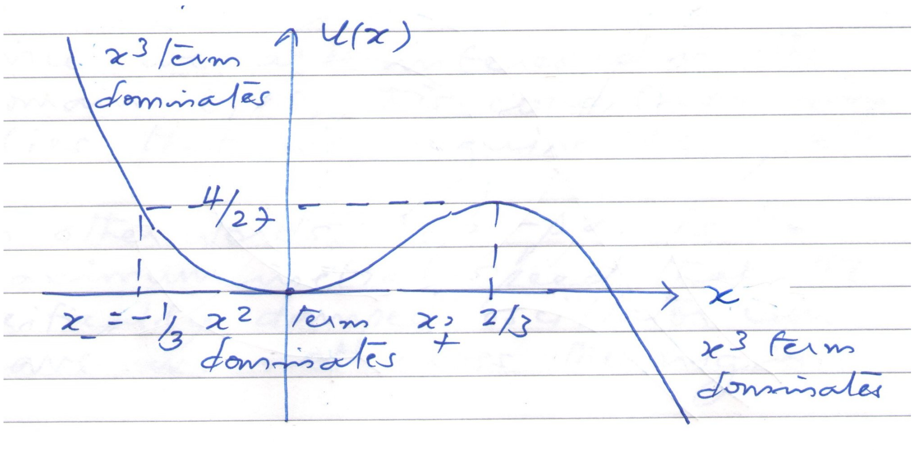
<u>Solution:</u> We have (since $\alpha = \beta = 1$),

$$\frac{\mathrm{d}U}{\mathrm{d}x} = 2\,x - 3\,x^2$$

and $\mathrm{d}U/\mathrm{d}x$ vanishes at $x = 0$ and $= 2/3$. Also,

$$\frac{\mathrm{d}^2U}{\mathrm{d}x^2} = 2 - 6\,x,$$

so that $\mathrm{d}^2U/\mathrm{d}x^2 = 2$ at $x = 0$ and $\mathrm{d}^2U/\mathrm{d}x^2 = -2$ at $x = 2/3$, which implies that $x = 0$ is a minimum, while $x = 2/3$ is a maximum. The complete potential is sketched below.



(b) What is the allowed range of energy of the particle?

<u>Solution:</u> Since the potential energy goes to large negative values, the range of allowed energy of the particle is $-\infty < E < \infty$.

(c) Determine the values of energy and the domain in x when the particle exhibits bounded motion.

<u>Solution:</u> Note that the potential energy corresponding to the maximum in the potential is $U = 4/27$. It is useful to note here that the turning turning points corresponding to $E = 4/27$

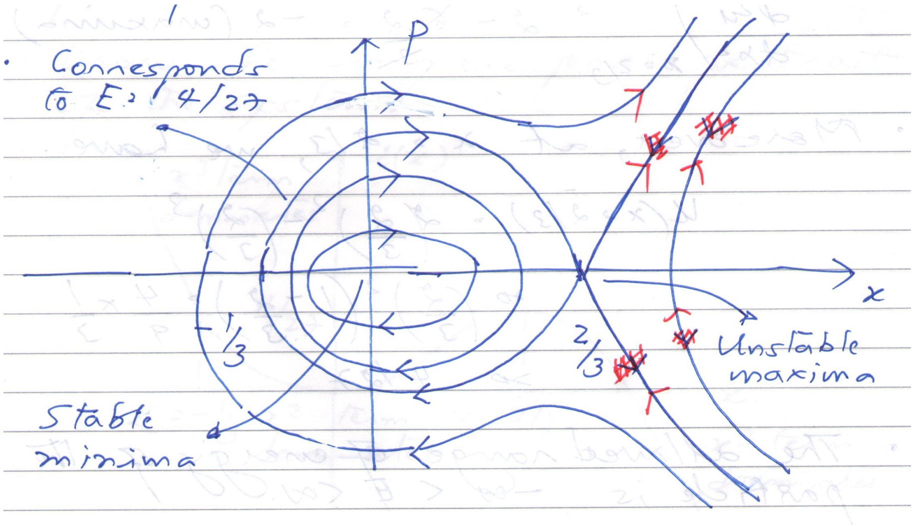
are located at $x_- = -1/3$ and $x_+ = 2/3$. At the turning points, $U(x_\pm) = E$, so that for $E = 4/27$ we have

$$x_\pm^2 - \beta\, x_\pm^3 = x_\pm^2\,(1 - x_\pm) = \frac{4}{27}.$$

Evidently, $x_+ = 2/3$, as it is the maxima of the potential at this point that leads to $E = 4/27$. It is straightforward to inspect that $x_- = -1/3$. The particle will exhibit bounded motion if $0 < E \leq 4/27$ and $x_- < x < x_+$.

(d) Draw the complete phase portrait of the system.

Solution: The complete phase portrait of the particle is illustrated in the figure below.



Note that the trajectories are broadly elliptic in shape around the minimum, while they have a hyperbolic form near the maxima. The trajectory corresponding to $E = 4/27$ separates these two types of behavior. The arrows indicate the direction of the trajectories.

5. *Motion in a potential II:* Consider a particle that is moving in the following so-called Morse potential:

$$U(x) = \alpha\left(\mathrm{e}^{-2\,\beta\,x} - 2\,\mathrm{e}^{-\beta\,x}\right),$$

where α and β are positive constants. Let us assume that $\alpha = 1$ and $\beta = 1$ in suitable units, for convenience.

(a) Plot the potential.

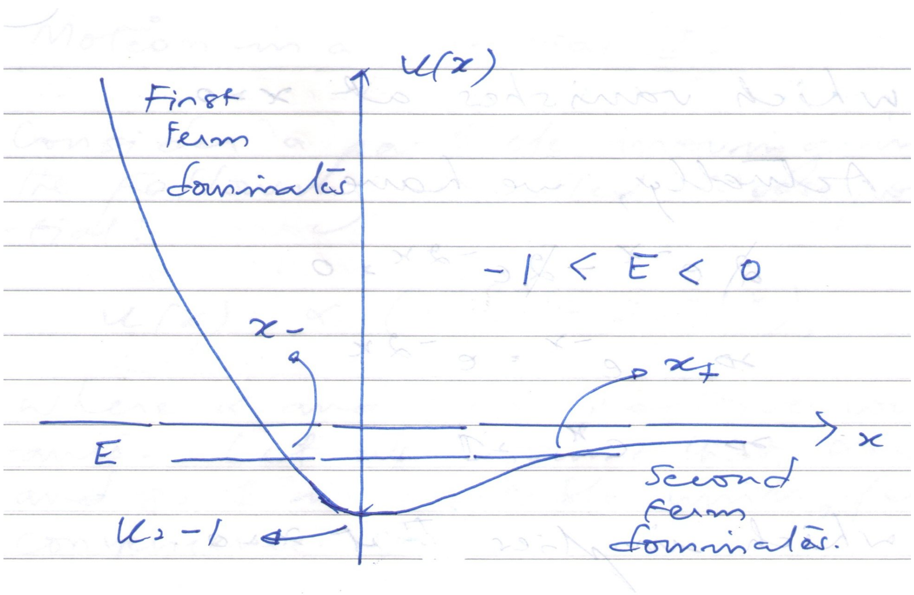
Solution: We have (for $\alpha = \beta = 1$)

$$\frac{\mathrm{d}U}{\mathrm{d}x} = -2\,\mathrm{e}^{-2\,x} + 2\,\mathrm{e}^{-x},$$

which vanishes when $x = 0$. Also, at $x = 0$

$$\left(\frac{\mathrm{d}^2U}{\mathrm{d}x^2}\right)_{x=0} = \left(4\,\mathrm{e}^{-2\,x} - 2\,\mathrm{e}^{-x}\right)_{x=-0} = 2,$$

implying that the point is a minimum. Moreover, the minimum value of the potential is $U = -1$. Further, the first term dominates at large negative values of x, indicating that the potential goes to infinity in this domain. Whereas, at large positive values of x, the second term dominates. This term is negative and it vanishes as $x \to \infty$. The resulting potential is illustrated in the figure below.



As in the case of the Lennard-Jones potential, the Morse potential is also used to describe inter-atomic interactions in molecules.

(b) What is the allowed range of energy?

<u>*Solution:*</u> Evidently, the allowed range of energy is $-1 \le E < \infty$.

(c) Determine the domain in energy that leads to bounded motion. Evaluate the period associated with the bounded motion.

<u>*Solution:*</u> The particle exhibits bounded motion for $-1 < E < 0$. The time period of the system is given by

$$T = 2\sqrt{\frac{m}{2}}\int_{x_-}^{x_+}\frac{\mathrm{d}x}{[E - U(x)]^{1/2}} = 2\sqrt{\frac{m}{2}}\int_{x_-}^{x_+}\frac{\mathrm{d}x}{[E - \mathrm{e}^{-2\,x} + 2\,\mathrm{e}^{-x}]^{1/2}},$$

where x_- and x_+ denote the left and the right turning points. If we set $y = \mathrm{e}^{-x}$, then the turning points are determined by the condition

$$E = y^2 - 2\,y,$$

which can be immediately solved to yield that

$$y_\mp = 1 \pm \sqrt{1 + E}$$

or, equivalently,

$$x_\pm = -\ln\left(1 \mp \sqrt{1 - E}\right).$$

It is useful to note that as $E \to 0$, $x_+ \to \infty$, as expected.

In order to evaluate the integral, let us set $z = e^x$ so that we have

$$T = \sqrt{2\,m}\int_{z_-}^{z_+}\frac{\mathrm{d}z}{(-|E|\,z^2 - 1 + 2\,z)^{1/2}},$$

where we have set $E = -|E|$. Upon completing the squares in the denominator, we find that we can write

$$T = \sqrt{2\,m\,|E|}\int_{z_-}^{z_+}\frac{\mathrm{d}z}{\left[(-|E| + 1) - (|E|\,z - 1)^2\right]^{1/2}} = \sqrt{\frac{2\,m}{|E|}}\int_{v_-}^{v_+}\frac{\mathrm{d}v}{(1 - v^2)^{1/2}},$$

where we have set

$$b^2 = 1 - |E|, \quad b\,v = |E|\,z - 1.$$

The integral over v can be easily carried out by setting $v = \sin\theta$, which leads to

$$T = \sqrt{\frac{2\,m}{|E|}}\,(\theta_+ - \theta_-)\,.$$

We find that

$$z_+ = \mathrm{e}^{x_+} = \exp - \left[\ln\left(1 - \sqrt{1-E}\right)\right] = \frac{b\,v_+ + 1}{|E|} = \frac{b\sin\theta_+ + 1}{|E|} = \frac{\sqrt{1-|E|}\,\sin\theta_+ + 1}{|E|}$$

so that

$$\theta_+ = \sin^{-1}\left(\frac{|E| - 1 + \sqrt{1-|E|}}{|E| - 1 + \sqrt{1-|E|}}\right) = \sin^{-1}(1) = \frac{\pi}{2}$$

and, similarly,

$$z_- = \mathrm{e}^{x_-} = \exp - \left[\ln\left(1 + \sqrt{1-E}\right)\right] = \frac{b\,v_- + 1}{|E|} = \frac{b\sin\theta_- + 1}{|E|} = \frac{\sqrt{1-|E|}\,\sin\theta_- + 1}{|E|}$$

so that

$$\theta_+ = \sin^{-1}\left[\frac{|E| - 1 - \sqrt{1-|E|}}{-\left(|E| - 1 - \sqrt{1-|E|}\right)}\right] = \sin^{-1}(-1) = -\frac{\pi}{2},$$

leading to the amazingly simple result

$$T = \pi\sqrt{\frac{2\,m}{|E|}}.$$

Note that $T \to \infty$, as $E \to 0$, as expected.
For the system of interest, we can write

$$t - t_0 = \sqrt{\frac{m}{2}}\int \frac{\mathrm{d}x}{\left[E - \mathrm{e}^{-2\,x} + 2\,\mathrm{e}^{-x}\right]^{1/2}}$$

and, for $E = -|E|$, if we set, $z = \mathrm{e}^x$, we obtain that

$$\begin{aligned} t - t_0 &= \sqrt{\frac{m}{2}}\int \frac{\mathrm{d}z}{\left[-|E|\,z^2 - 1 + 2\,z\right]^{1/2}} = \sqrt{\frac{m\,|E|}{2}}\int \frac{\mathrm{d}z}{\left[(-|E| + 1) - (|E|\,z - 1)^2\right]^{1/2}} \\ &= \sqrt{\frac{m}{2\,|E|}}\int \frac{\mathrm{d}v}{(1 - v^2)^{1/2}}, \end{aligned}$$

where we have set

$$b^2 = 1 - |E|, \quad b\,v = |E|\,z - 1.$$

The integral over v can be easily carried out by setting $v = \sin\theta$ to arrive at

$$t - t_0 = \sqrt{\frac{m}{2\,|E|}}\,\sin^{-1}v$$

or

$$v = \frac{|E|\,z - 1}{\sqrt{1 - |E|}} = \sin\left[\sqrt{\frac{2\,|E|}{m}}\,(t - t_0)\right]$$

so that we can write

$$z = \mathrm{e}^x = \frac{1}{|E|}\left\{1 + \sqrt{1-|E|}\,\sin\left[\sqrt{\frac{2\,|E|}{m}}\,(t-t_0)\right]\right\}.$$

Because of the sinusoidal term, it is clear that the system exhibits periodic motion with frequency

$$\omega = \sqrt{\frac{2\,|E|}{m}}$$

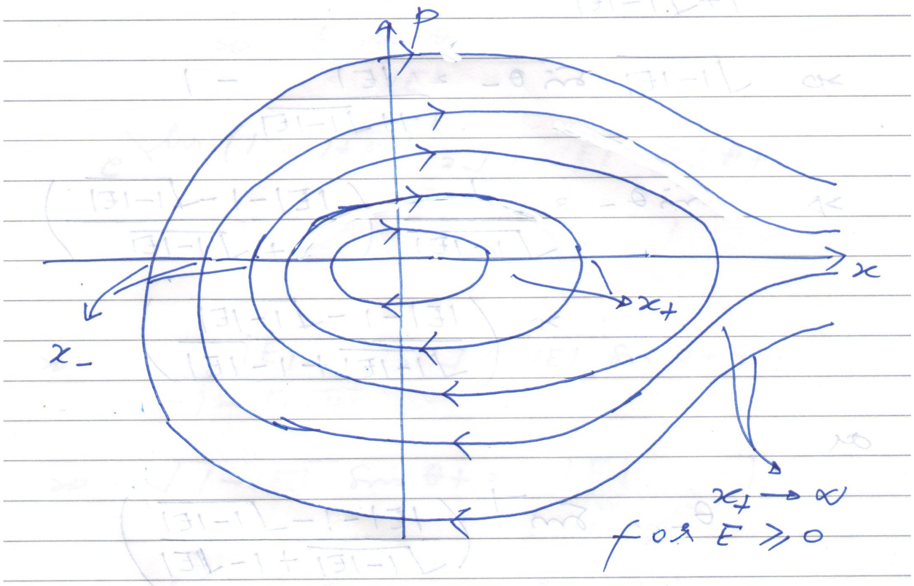
or, equivalently, the time period

$$T = \frac{2\,\pi}{\omega} = \pi\,\sqrt{\frac{2\,m}{|E|}},$$

exactly as we have arrived at above.

(d) Draw the complete phase portrait of the system.

Solution: As we discussed, the particle exhibits bounded motion for $-1 < E < 0$ and unbounded motion (on one side) for $E > 0$. The complete phase portrait of the system is illustrated in the figure below.



Department of Physics
Indian Institute of Technology Madras

Quiz I

From Newton's laws to oscillations

Date: September 11, 2017 Time: 08:00 – 08:50 AM

Name:

Roll No.:

Instructor:

Batch:

Instructions

1. Begin by completing the information requested above. Please write your complete name, your roll number, the name of your instructor, and your batch number (out of I–XII). **The answer sheet will not be evaluated unless both your name and roll number are written.**
2. This question paper cum answer sheet booklet contains **six** single-sided pages. Please check right away that all the pages are present.
3. As we had announced earlier, this quiz consists of 3 true/false questions (for 1 mark each), 3 multiple choice questions with one correct option (for 1 mark each), 4 fill in the blanks, two questions involving detailed calculations (for 3 marks each) and one question involving some plotting (for 4 marks), adding to a total of 20 marks.
4. You are expected to answer **all** the questions. There are no negative marks.
5. The answers have to be written in the boxes provided. Answers written elsewhere in the booklet will **not** be evaluated.
6. Kindly write the answers, including sketches, with a blue or black pen. Note that answers written with pencils or pens of other colors will **not** be evaluated.
7. You can use the empty reverse sides for rough work. No extra sheets will be provided.
8. You are not allowed to use a calculator or any other electronic device during the quiz. Please note that you will not be permitted to continue with the quiz if you are found with any such device.
9. Make sure that you return question paper cum answer sheet booklet before you leave the examination hall.

For use by examiners
(Do not write in this space)

Q1–Q10	Q11	Q12	Q13	Total

✦ **True or false** (1 mark each, write True (T)/False (F) in the box provided)

1. The total energy of a closed system is always conserved.

2. The acceleration of an underdamped oscillator vanishes when its position is zero.

3. A particle moving in the potential $U(x) = -\alpha\, x - \beta\, x^2$, where α and β are positive constants, never exhibits bounded motion.

✦ **Multiple choice questions** (1 mark each, write the one correct option in the box provided)

4. The unit vector perpendicular to the vectors $\vec{\boldsymbol{a}} = \hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} - \hat{\boldsymbol{z}}$ and $\vec{\boldsymbol{b}} = 2\,\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} - 3\,\hat{\boldsymbol{z}}$ is

   [**A**] $(-2\,\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} - \hat{\boldsymbol{z}})\,/\sqrt{6}$ [**B**] $(2\,\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}})\,/\sqrt{6}$
   [**C**] $(-2\,\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}})\,/\sqrt{6}$ [**D**] $(2\,\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} - \hat{\boldsymbol{z}})\,/\sqrt{6}$

5. For a weakly damped oscillator, $x(t+T)$ is found to be 94% of $x(t)$, where T is the time period of the system. The quantity β/ω_1 (where β is the damping constant and ω_1 is the actual frequency) for the system is approximately

   [**A**] 10^{-5} [**B**] 10^{-4} [**C**] 10^{-3} [**D**] 10^{-2}

6. Consider a particle of mass m which is falling under the influence of gravity and subject to the drag force $\vec{\boldsymbol{F}} = -\alpha\,\vec{\boldsymbol{v}}$. The terminal velocity of the particle is given by $v_{\rm term} = g\,\tau$, where $\tau = m/\alpha$ denotes the characteristic time scale. If the particle falls from rest, it will reach 95% of $v_{\rm term}$ in time

   [**A**] $t \simeq \tau$ [**B**] $t \simeq 3\,\tau$ [**C**] $t \simeq 5\,\tau$ [**D**] $t \simeq 10\,\tau$

✦ **Fill in the blanks** (1 mark each, write the answer in the box provided)

7. A point mass has the trajectory $\vec{\boldsymbol{r}}(t) = A\cos\,(\omega\,t)\;\hat{\boldsymbol{x}} + \sin\,(\omega\,t)\;\hat{\boldsymbol{y}}$. If the angle between the velocity $\vec{\boldsymbol{v}}$ and the acceleration $\vec{\boldsymbol{a}}$ at time $t = \pi/(4\,\omega)$ is 60°, what is the value of A?

8. Consider a particle moving in the one-dimensional potential $U(x) = -U_0\,\exp\left[-(x-\mu)^2/(2\,\sigma^2)\right]$, where U_0, μ and σ are positive quantities of suitable dimensions. Determine the frequency of small oscillations about the minimum of the potential.

9. Let x_1 and x_2 be two different positions of an undamped oscillator at times t_1 and t_2. Let v_1 and v_2 be the corresponding velocities. Express the angular frequency of the oscillations in terms of x_1, x_2, v_1 and v_2.

10. Given the function $f(r) = 1/r$, where $r = \sqrt{x^2 + y^2 + z^2}$, determine $\vec{\boldsymbol{\nabla}} f$.

✦ **Questions with detailed answers** (write the calculations and answers within the boxes provided)

11. *Dropping from a plane:* A plane flying horizontally at a constant speed v_0 and at a height h above the sea, drops a bundle on a small static raft. (a) Neglecting air resistance, write down Newton's second law for the bundle as it falls from the plane, and solve the equations to obtain the bundle's position as a function of time t. (b) If $v_0 = 50\,\mathrm{m/s}$, $h = 125\,\mathrm{m}$ and $g \simeq 10\,\mathrm{m/s^2}$, what is the horizontal distance before the raft that the pilot must drop the bundle to hit the raft? (c) Determine the interval of time ($\pm\Delta t$) within which the pilot must drop the bundle if it is to land within $\pm 10\,\mathrm{m}$ of the raft. 1+1+1 marks

12. <u>*Decaying force:*</u> An undamped oscillator with natural frequency ω_0 is subject to the external force $F(t) = F_0\, \mathrm{e}^{-\alpha\, t}$, where F_0 and α are positive quantities. (a) Determine the homogeneous and inhomogeneous (i.e. particular) solutions. (b) Which of the two solutions dominates at late times, i.e. when $\alpha\, t \gg 1$? $\boxed{\text{2+1 marks}}$

13. *Motion in one dimension:* Consider the one-dimensional potential

$$U(x) = \frac{U_0\, x^2}{x^2 + a^2},$$

where U_0 and a are positive constants. (a) Plot the quantity $U(x)/U_0$ against x/a. (b) Identify the stable and/or unstable point(s) of equilibrium. (c) Find the turning points for $E = U_0/4$. 2+1+1 marks

Solutions to Quiz I

From Newton's laws to oscillations

✦ True or false

1. The total energy of a closed system is always conserved.

   Solution: True. Since the system is closed, there will be no transfer of energy (to or from the system) and hence the energy of the system will remain a constant.

2. The acceleration of an underdamped oscillator vanishes when its position is zero.

   Solution: False. For an underdamped oscillator, we have

$$x(t) = A\,\mathrm{e}^{-\beta\,t}\cos\left(\omega_1\,t - \delta\right),$$

   where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$. Therefore, we have

$$v(t) = \dot{x}(t) = -A\,\mathrm{e}^{-\beta\,t}\left[\beta\cos\left(\omega_1\,t-\delta\right) + \omega_1\sin\left(\omega_1\,t-\delta\right)\right]$$

   and

$$\begin{aligned} a(t) = \dot{v}(t) &= \beta\,A\,\mathrm{e}^{-\beta\,t}\left[\beta\cos\left(\omega_1\,t-\delta\right) + \omega_1\sin\left(\omega_1\,t-\delta\right)\right] \\ &\quad - A\,\mathrm{e}^{-\beta\,t}\left[-\beta\,\omega_1\sin\left(\omega_1\,t-\delta\right) + \omega_1^2\cos\left(\omega_1\,t-\delta\right)\right] \\ &= -2\,\beta\,\dot{x} - \omega_0^2\,x, \end{aligned}$$

   where we have set $\omega_1^2 = \omega_0^2 - \beta^2$. Note that $x(t)$ vanishes when the cosine term vanishes, i.e. when $\omega_1\,t - \delta = n\,\pi/2$, where $n = 1, 2, 3, \ldots$. However, since the velocity also contains a sine term, it will *not* vanish at these times. Hence, the acceleration will not vanish when the position is zero.

3. A particle moving in the potential $U(x) = -\alpha\,x - \beta\,x^2$, where $\alpha > 0$ and $\beta > 0$, never exhibits bounded motion.

   Solution: True. The potential

$$U(x) = -\alpha\,x - \beta\,x^2,$$

   where $\alpha > 0$ and $\beta > 0$, has a maximum at $x = -\alpha/(2\,\beta)$. The potential vanishes at $x = 0$ and goes to negative infinity at large positive as well as negative values of x. So, the allowed range of energy is $-\infty < E < \infty$. Since there arises no minima, the system does not exhibit bounded motion for any value of the energy.

✦ Multiple choice questions

4. The unit vector perpendicular to the vectors $\vec{a} = \hat{x} + \hat{y} - \hat{z}$ and $\vec{b} = 2\,\hat{x} + \hat{y} - 3\,\hat{z}$ is

   [A] $(-2\,\hat{x} + \hat{y} - \hat{z})/\sqrt{6}$ **[B]** $(2\,\hat{x} + \hat{y} + \hat{z})/\sqrt{6}$

   [C] $(-2\,\hat{x} + \hat{y} + \hat{z})/\sqrt{6}$ **[D]** $(2\,\hat{x} + \hat{y} - \hat{z})/\sqrt{6}$

   Solution: A. We have, say,

$$\vec{c} = \vec{a} \times \vec{b} = (-2\,\hat{x} + \hat{y} - \hat{z}),$$

   so that

$$\hat{c} = \frac{\vec{c}}{|\boldsymbol{c}|} = (-2\,\hat{x} + \hat{y} - \hat{z})/\sqrt{6}.$$

5. For a weakly damped oscillator, $x(t+T)$ is found to be 94% of $x(t)$, where T is the time period of the system. The quantity β/ω_1 (where β is the damping constant and ω_1 is the actual frequency) for the system is approximately

   [A] 10^{-5} **[B]** 10^{-4} **[C]** 10^{-3} **[D]** 10^{-2}

Solution: D. Recall that, for a underdamped oscillator,

$$\frac{x(t+T)}{x(t)} = \mathrm{e}^{-2\,\pi\,\beta/\omega_1}$$

so that, in our case, we have

$$0.94 = \mathrm{e}^{-2\,\pi\,\beta/\omega_1} \simeq 1 - \frac{2\,\pi\,\beta}{\omega_1},$$

or

$$\frac{\beta}{\omega_1} = \frac{\sqrt{\omega_0^2 - \omega_1^2}}{\omega_1} = \frac{0.06}{2\,\pi} \simeq 0.01 = 10^{-2}.$$

6. Consider a particle of mass m which is falling under the influence of gravity and subject to the drag force $\vec{\boldsymbol{F}} = -\alpha\,\vec{\boldsymbol{v}}$. The terminal velocity of the particle is given by $v_{\rm term} = g\,\tau$, where $\tau = m/\alpha$ denotes the characteristic time scale. If the particle falls from rest, it will reach 95% of $v_{\rm term}$ in time

   [**A**] $t \simeq \tau$ [**B**] $t \simeq 3\,\tau$ [**C**] $t \simeq 5\,\tau$ [**D**] $t \simeq 10\,\tau$

   Solution: B. Recall that, in the case of a particle falling from rest, we have

$$v_y(t) = v_{\rm term}\,\left(1 - \mathrm{e}^{-t/\tau}\right),$$

   so that, when $t = 3\,\tau$,

$$\frac{v_y}{v_{\rm term}} = 1 - \frac{1}{\mathrm{e}^3} \simeq 1 - \frac{1}{2.72^3} \simeq 1 - \frac{1}{25} \simeq 1 - 0.04 \simeq 0.96.$$

✦ Fill in the blanks

7. A point mass has the trajectory $\vec{\boldsymbol{r}}(t) = A\cos\,(\omega\,t)\;\hat{\boldsymbol{x}} + \sin\,(\omega\,t)\;\hat{\boldsymbol{y}}$. If the angle between the velocity $\vec{\boldsymbol{v}}$ and the acceleration $\vec{\boldsymbol{a}}$ at time $t = \pi/(4\,\omega)$ is 60°, what is the value of A?

   Solution: Since

$$\vec{\boldsymbol{r}}(t) = A\cos\,(\omega\,t)\;\hat{\boldsymbol{x}} + \sin\,(\omega\,t)\;\hat{\boldsymbol{y}},$$

   we have

$$\vec{\boldsymbol{v}}(t) = -A\,\omega\sin\,(\omega\,t)\;\hat{\boldsymbol{x}} + \omega\cos\,(\omega\,t)\;\hat{\boldsymbol{y}}, \quad \vec{\boldsymbol{a}}(t) = -A\,\omega^2\cos\,(\omega\,t)\;\hat{\boldsymbol{x}} - \omega^2\sin\,(\omega\,t)\;\hat{\boldsymbol{y}},$$

   so that

$$\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{a}} = \omega^3\,\left(A^2-1\right)\,\sin\,(\omega\,t)\,\cos\,(\omega\,t) = (\omega^3/2)\,\left(A^2-1\right)\,\sin\,(2\,\omega\,t)\,.$$

   At $\omega\,t = \pi/4$, we have

$$\vec{\boldsymbol{v}} = -\frac{A\,\omega}{\sqrt{2}}\,\hat{\boldsymbol{x}} + \frac{\omega}{\sqrt{2}}\,\hat{\boldsymbol{y}}, \quad \vec{\boldsymbol{a}}(t) = -\frac{A\,\omega^2}{\sqrt{2}}\,\hat{\boldsymbol{x}} - \frac{\omega^2}{\sqrt{2}}\,\hat{\boldsymbol{y}}$$

   and, at this time,

$$\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{a}} = |\vec{\boldsymbol{v}}|\,|\vec{\boldsymbol{a}}|\cos 60^\circ = \frac{1}{2}\,\frac{\omega}{\sqrt{2}}\,\sqrt{A^2+1}\,\frac{\omega^2}{\sqrt{2}}\,\sqrt{A^2+1} = \frac{\omega^3}{2}\,\left(A^2-1\right),$$

   which leads to

$$\frac{A^2-1}{A^2+1} = \frac{1}{2},$$

   or $A = \pm\sqrt{3}$.

8. Consider a particle moving in the one-dimensional potential $U(x) = -U_0 \exp\left[-(x-\mu)^2/(2\,\sigma^2)\right]$, where U_0, μ and σ are positive quantities of suitable dimensions. Determine the frequency of small oscillations about the minimum of the potential.

<u>*Solution*</u>: Since

$$\frac{\mathrm{d}U}{\mathrm{d}x} = \frac{U_0\,(x-\mu)}{\sigma^2}\,\mathrm{e}^{-(x-\mu)^2/(2\,\sigma^2)}$$

and

$$\frac{\mathrm{d}^2U}{\mathrm{d}x^2} = \frac{U_0}{\sigma^2}\,\mathrm{e}^{-(x-\mu)^2/(2\,\sigma^2)} - \frac{U_0\,(x-\mu)^2}{\sigma^4}\,\mathrm{e}^{-(x-\mu)^2/(2\,\sigma^2)},$$

evidently, the minimum is located at $x = \mu$. Therefore, we have

$$k = \left(\frac{\mathrm{d}^2U}{\mathrm{d}x^2}\right)_{x=\mu} = \frac{U_0}{\sigma^2}$$

and, hence, the frequency of small oscillations about the minimum is given by

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{U_0}{m\,\sigma^2}}.$$

9. Let x_1 and x_2 be two different positions of an undamped oscillator at times t_1 and t_2. Let v_1 and v_2 be the corresponding velocities. Express the angular frequency of the oscillations in terms of x_1, x_2, v_1 and v_2.

<u>*Solution*</u>: We can write

$$x(t) = A\cos\left(\omega\,t - \delta\right),$$

where ω is the frequency of the oscillator. Then, we have

$$v(t) = -\omega\,A\sin\left(\omega\,t - \delta\right),$$

so that

$$\begin{aligned} x_1 = x(t_1) &= A\cos\left(\omega\,t_1 - \delta\right), \quad v_1 = v(t_1) = -\omega\,A\sin\left(\omega\,t_1 - \delta\right),\\ x_2 = x(t_2) &= A\cos\left(\omega\,t_2 - \delta\right), \quad v_2 = v(t_2) = -\omega\,A\sin\left(\omega\,t_2 - \delta\right), \end{aligned}$$

and, hence,

$$x_1^2 + \frac{v_1^2}{\omega^2} = x_2^2 + \frac{v_2^2}{\omega^2} = A^2$$

or

$$\omega = \sqrt{\frac{v_1^2 - v_2^2}{x_2^2 - x_1^2}},$$

which is the required result.

10. Given the function $f(r) = 1/r$, where $r = \sqrt{x^2+y^2+z^2}$, determine $\vec{\boldsymbol{\nabla}} f$.

<u>*Solution*</u>: We have

$$\vec{\boldsymbol{\nabla}} f = \frac{\partial f}{\partial x}\,\hat{\boldsymbol{x}} + \frac{\partial f}{\partial y}\,\hat{\boldsymbol{y}} + \frac{\partial f}{\partial z}\,\hat{\boldsymbol{z}},$$

where

$$\frac{\partial f}{\partial x} = \frac{\mathrm{d}f}{\mathrm{d}r}\,\frac{\partial r}{\partial x}, \quad \frac{\partial f}{\partial y} = \frac{\mathrm{d}f}{\mathrm{d}r}\,\frac{\partial r}{\partial y}, \quad \frac{\partial f}{\partial z} = \frac{\mathrm{d}f}{\mathrm{d}r}\,\frac{\partial r}{\partial z}.$$

Since $r = \sqrt{x^2+y^2+z^2}$, we have

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}, \quad \frac{\partial r}{\partial x} = \frac{y}{\sqrt{x^2+y^2+z^2}} = \frac{y}{r}, \quad \frac{\partial r}{\partial x} = \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{z}{r},$$

so that

$$\vec{\boldsymbol{\nabla}} f = \frac{\mathrm{d}f}{\mathrm{d}r}\,\frac{x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}} + z\,\hat{\boldsymbol{z}}}{r} = \frac{\mathrm{d}f}{\mathrm{d}r}\,\frac{\vec{\boldsymbol{r}}}{r} = \frac{\mathrm{d}f}{\mathrm{d}r}\,\hat{\boldsymbol{r}}$$

and, in the case wherein $f(r) = 1/r$, we obtain that

$$\vec{\boldsymbol{\nabla}} f = -\frac{1}{r^2}\,\hat{\boldsymbol{r}}.$$

✦ Questions with detailed answers

11. *Dropping from a plane:* A plane flying horizontally at a constant speed v_0 and at a height h above the sea, drops a bundle on a small static raft. (a) Neglecting air resistance, write down Newton's second law for the bundle as it falls from the plane, and solve the equations to obtain the bundle's position as a function of time t. (b) If $v_0 = 50\,\mathrm{m/s}$, $h = 125\,\mathrm{m}$ and $g \simeq 10\,\mathrm{m/s^2}$, what is the horizontal distance before the raft that the pilot must drop the bundle to hit the raft? (c) Determine the interval of time ($\pm\Delta t$) within which the pilot must drop the bundle if it is to land within $\pm 10\,\mathrm{m}$ of the raft.

    Solution: The equations of motion along the horizontal (x, pointing rightward) and vertical (y, pointing upward) directions are given by

    $$\ddot{x} = 0, \qquad \ddot{y} = -g,$$

    which can be integrated to yield

    $$x = v_0\,t, \quad y = h - \frac{g}{2}\,t^2.$$

    The time taken for the bundle to drop on the raft is (i.e. when $y = 0$)

    $$t_{\mathrm{fall}} = \sqrt{2\,h/g}$$

    and, over this time, the horizontal distance traveled by the bundle is

    $$x = v_0\,t_{\mathrm{fall}} = v_0\,\sqrt{2\,h/g}.$$

    For $v_0 = 50\,\mathrm{m/s}$, $h = 100\,\mathrm{m}$, and $g \simeq 10\,\mathrm{m/s^2}$,

    $$x = 50\,\sqrt{2 \times 125/10}\,\mathrm{m} = 250\,\mathrm{m}.$$

    Note that, if the drop is delayed by a time Δt, then the bundle will overshoot the raft by $\Delta x = v_0\,\Delta t$. Therefore, we have

    $$\Delta t = \frac{\Delta x}{v_0} = \frac{10\,\mathrm{m}}{50\,\mathrm{m/s}} = 0.2\,\mathrm{s},$$

    so that $\Delta t = \pm 0.2\,\mathrm{s}$.

12. *Decaying force:* An undamped oscillator with natural frequency ω_0 is subject to the external force $F(t) = F_0\,\mathrm{e}^{-\alpha\,t}$, where F_0 and α are positive quantities. (a) Determine the homogeneous and inhomogeneous (i.e. particular) solutions. (b) Which of the two solutions dominates at late times, i.e. when $\alpha\,t \gg 1$?

    Solution: We have

    $$\ddot{x} + \omega_0^2\,x = \frac{F_0}{m}\,\mathrm{e}^{-\alpha\,t},$$

    and the homogeneous solution is evidently given by

    $$x_{\mathrm{h}}(t) = A\cos\left(\omega_0\,t - \delta\right),$$

    where A and δ are constants determined by the initial conditions.

Let us propose the inhomogeneous solution to be $x_{\rm ih}(t) = C\,{\rm e}^{-\alpha\,t}$. Upon substituting this solution in the above differential equation, we obtain that

$$\left(\alpha^2+\omega_0^2\right)\,C\,{\rm e}^{-\alpha\,t} = \frac{F_0}{m}\,{\rm e}^{-\alpha\,t}$$

or, equivalently,

$$C = \frac{F_0/m}{\alpha^2+\omega_0^2},$$

so that we have

$$x_{\rm ih}(t) = \frac{F_0/m}{\alpha^2+\omega_0^2}\,{\rm e}^{-\alpha\,t}.$$

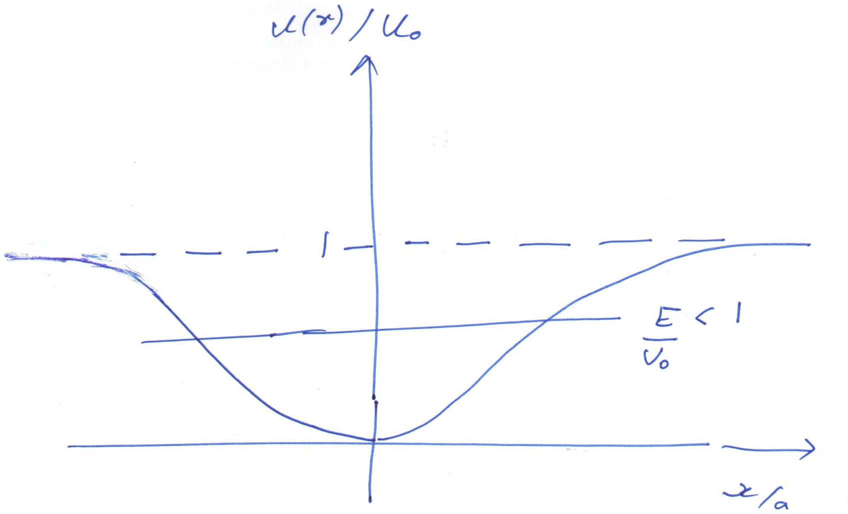
Clearly, at late times such that $\alpha\,t \gg 1$, $x_{\rm ih}(t)$ dies away. Therefore, at such times, it is the homogeneous solution $x_{\rm h}(t)$ that will dominate.

13. *Motion in one dimension:* Consider the one-dimensional potential

$$U(x) = \frac{U_0\,x^2}{x^2+a^2},$$

where U_0 and a are positive constants. (a) Plot the quantity $U(x)/U_0$ against x/a. (b) Identify the stable and/or unstable point(s) of equilibrium. (c) Find the turning points for $E = U_0/4$.

Solution: Note that the potential is symmetric in x. Also, $U(x)=0$ at $x=0$ and $U(x)\to U_0$ as $x\to\pm\infty$.



Moreover, as

$$\frac{{\rm d}U}{{\rm d}x} = \frac{2\,U_0\,x}{x^2+a^2} - \frac{2\,U_0\,x^3}{\left(x^2+a^2\right)^2} = \frac{2\,U_0\,a^2\,x}{\left(x^2+a^2\right)^2},$$

which vanishes only at $x=0$. Further,

$$\frac{{\rm d}^2U}{{\rm d}x^2} = \frac{2\,U_0\,a^2}{\left(x^2+a^2\right)^2} - \frac{8\,U_0\,a^2\,x^2}{\left(x^2+a^2\right)^3},$$

so that $({\rm d}^2U/{\rm d}x^2) = 2\,U_0/a^2$ at $x=0$. Clearly, the potential has one stable equilibrium point at $x=0$.

For $E = U_0/4$, we have, when $E = U(x_*)$,

$$\frac{U_0}{4} = \frac{U_0\, x_*^2}{x_*^2 + a^2}$$

so that

$$x_*^2 + a^2 = 4\, x_*^2,$$

and, hence, the turning points are given by $x_* = \pm a/\sqrt{3}$.

Department of Physics
Indian Institute of Technology Madras

Quiz I – Make up

From Newton's laws to oscillations

Date: January 11, 2018 Time: 09:00 – 09:50 AM

Name: **Roll No.:**

Instructor: **Batch:**

Instructions

1. Begin by completing the information requested above. Please write your complete name, your roll number, the name of your instructor, and your batch number (out of I–XII). **The answer sheet will not be evaluated unless both your name and roll number are written.**
2. This question paper cum answer sheet booklet contains **six** single-sided pages. Please check right away that all the pages are present.
3. As we had announced earlier, this quiz consists of 3 true/false questions (for 1 mark each), 3 multiple choice questions with one correct option (for 1 mark each), 4 fill in the blanks (for 1 mark each), two questions involving detailed calculations (for 3 marks each) and one question involving some plotting (for 4 marks), adding to a total of 20 marks.
4. You are expected to answer **all** the questions. There are no negative marks.
5. The answers have to be written in the boxes provided. Answers written elsewhere in the booklet will **not** be evaluated.
6. Kindly write the answers, including sketches, with a blue or black pen. Note that answers written with pencils or pens of other colors will **not** be evaluated.
7. You can use the empty reverse sides for rough work. No extra sheets will be provided.
8. You are not allowed to use a calculator or any other electronic device during the quiz. Please note that you will not be permitted to continue with the quiz if you are found with any such device.
9. Make sure that you return question paper cum answer sheet booklet before you leave the examination hall.

For use by examiners
(Do not write in this space)

Q1–Q10	Q11	Q12	Q13	Total

✦ **True or false** (1 mark each, write True (T)/False (F) in the box provided)

1. Consider a charged particle moving in a constant and uniform magnetic field. The linear momentum of the particle along the direction of the magnetic field is conserved.

2. The value of the product of the position and velocity of an undamped, one-dimensional oscillator, when averaged over one period, is zero.

3. A particle moving in the potential $U(x) = \alpha\, x^2 - \beta\, x^3$, where α and β are positive constants, exhibits only unbounded motion.

✦ **Multiple choice questions** (1 mark each, write the one correct option in the box provided)

4. If two vectors $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ satisfy the condition $|\vec{\boldsymbol{A}}+\vec{\boldsymbol{B}}| = |\vec{\boldsymbol{A}}-\vec{\boldsymbol{B}}|$, then the angle between the vectors $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ is

   [**A**] 45° [**B**] 60° [**C**] 90° [**D**] 180°

5. A 0.3 kg mass is attached to a spring and oscillates at 2 Hz with the quality factor $Q = 60$. The value of the spring constant k and damping constant β of the system are

   [**A**] $k = 1.2\,\mathrm{N/m},\ \beta = 1/60\,\mathrm{s}^{-1}$ [**B**] $k = 2.4\,\mathrm{N/m},\ \beta = 1/120\,\mathrm{s}^{-1}$

   [**C**] $k = 2.4\,\mathrm{N/m},\ \beta = 1/60\,\mathrm{s}^{-1}$ [**D**] $k = 1.2\,\mathrm{N/m},\ \beta = 1/120\,\mathrm{s}^{-1}$

6. A sphere (of mass $m = 0.15\,\mathrm{kg}$ and diameter $D = 0.07\,\mathrm{m}$) that is falling under the influence of gravity is subject to the *quadratic* drag force $\vec{\boldsymbol{F}} = -\alpha\, v^2\, \hat{\boldsymbol{v}}$. If $\alpha = \gamma\, D^2$, where $\gamma = 0.25\,\mathrm{N\,s^2/m^4}$, the terminal velocity of the sphere is

   [**A**] 140.0 m/s [**B**] 17.5 m/s [**C**] 35.0 m/s [**D**] 70.0 m/s

✦ **Fill in the blanks** (1 mark each, write the answer in the box provided)

7. The unknown vector $\vec{\boldsymbol{v}}$ satisfies the conditions $\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{b}} = \lambda$ and $\vec{\boldsymbol{v}}\times\vec{\boldsymbol{b}} = \vec{\boldsymbol{c}}$, where λ, $\vec{\boldsymbol{b}}$ and $\vec{\boldsymbol{c}}$ are known. Express the vector $\vec{\boldsymbol{v}}$ in terms of λ, $\vec{\boldsymbol{b}}$ and $\vec{\boldsymbol{c}}$.

8. Consider a particle moving in the one-dimensional potential $U(x) = U_0\,(x/d)^2 \exp[-(x/d)]$, where U_0 and d are positive quantities of suitable dimensions. Determine the time period of small oscillations about the minimum of the potential.

9. Express the solution to the critically damped, one-dimensional oscillator in terms of the initial position x_0 and initial velocity v_0.

10. Given $f(x, y, z) = (x\,y + y\,z + z\,x)$, express $\vec{\boldsymbol{r}}\cdot\vec{\boldsymbol{\nabla}} f$ in terms of f.

✦ **Questions with detailed answers** (write the calculations and answers within the boxes provided)

11. <u>*Bead on a wire:*</u> A bead, under the influence of gravity, slides along a frictionless wire whose height is given by the function $y(x)$. (a) Express the kinetic and the potential energies of the particle in terms of $\dot{x}$, y and $\mathrm{d}y/\mathrm{d}x$. (b) Obtain the equation of motion governing the system. $\boxed{\text{1+2 marks}}$

12. *Oscillator subject to a constant force:* An undamped, one-dimensional oscillator of mass m and angular frequency ω is subject to an additional, constant force F_0. (a) What is the equation of motion governing the system? (b) What is the most general solution to the inhomogeneous equation? $\boxed{\text{1+2 marks}}$

13. *Motion in one dimension:* Consider the one-dimensional potential

$$U(x) = U_0 \left(\frac{x^2}{d^2} + \frac{d^2}{x^2}\right),$$

where U_0 and d are positive constants. (a) Plot the quantity $U(x)/U_0$ against x/d. (b) Determine the stable and/or unstable point(s) of equilibrium. (c) Find the turning points for $E = 17\,U_0/4$. $\boxed{\text{2+1+1 marks}}$

Symmetries and conservation laws

Systems can possess certain physical symmetries and these symmetries, if they exist, play a crucial in governing the dynamics of the system. Symmetries can be broadly classified into two types, continuous and discrete symmetries. A continuous transformation is one which can be constructed out of repeated operations of an infinitesimal version of the transformation. Whereas a discrete transformation is one which *cannot* be constructed in such a fashion. Continuous and discrete symmetries are those under which the system remains invariant (the same) under such transformations. As we shall see, there is a close relationship between continuous symmetries and quantities that remain conserved as the system evolves. In fact, we have already encountered most of them, viz. conservation of linear momentum, angular momentum and energy. The aims of this note are twofold: to emphasize the points again and also present them in a broader perspective.

Time translational invariance and conservation of energy

We had earlier seen that energy of a system is conserved if the potential energy is independent of time. For simplicity, let us consider the case of a system in one-dimension, say, along the x-axis. In such a case, the equation of motion governing a particle of mass that is moving under the influence of the force F is given by

$$m\,\ddot{x} = F.$$

If the force is conservative, we know that it can be expressed in terms of the potential $U(x)$ as $F = -\mathrm{d}U/\mathrm{d}x$. Let us write the energy of system as

$$E = T + U = \frac{m\,\dot{x}^2}{2} + U(x),$$

where T is the kinetic energy of the particle. In such a case, we have

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{m\,\dot{x}^2}{2} + U(x)\right] = m\,\dot{x}\,\ddot{x} + \frac{\partial U}{\partial t} + \frac{\partial U}{\partial x}\,\dot{x}$$

and, if the potential does not explicitly depend on time (i.e. $\partial U/\partial t = 0$), then the above expression reduces to

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \dot{x}\,\left(m\,\ddot{x} + \frac{\mathrm{d}U}{\mathrm{d}x}\right) = \dot{x}\,(m\,\ddot{x} - F) = 0,$$

where, in order to arrive at the final equality, we have made use of the equation of motion. If the potential is time-independent it implies that it is time-translational invariant (it matter matter whether you study the system today or sometime later!), which, in turn, leads to the conservation of energy. This result is often quoted as follows: *the energy of a system possessing time-translational invariance is conserved.*

We should add that these arguments can be easily extended to a system in three dimensions. In such a case, while the equation of motion will be given by

$$m\,\ddot{\boldsymbol{x}} = \boldsymbol{F},$$

the time derivative of the energy can be determined to be

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{m\,\dot{\boldsymbol{x}}^2}{2} + U(\boldsymbol{x})\right] = m\,\dot{\boldsymbol{x}}\cdot\ddot{\boldsymbol{x}} + \frac{\partial U}{\partial t} + \boldsymbol{\nabla}U\cdot\dot{\boldsymbol{x}}.$$

In the case wherein $\partial U/\partial t = 0$, we have

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \dot{\boldsymbol{x}}\cdot(m\,\ddot{\boldsymbol{x}} + \boldsymbol{\nabla}U) = \dot{\boldsymbol{x}}\cdot(m\,\ddot{\boldsymbol{x}} - \boldsymbol{F}) = 0,$$

where again we have made use of the equation of motion to arrive at the final equality.

These arguments can also be extended to a system consisting of many particles. In the case of a system of many particles, it is trivial to extend the above arguments to a situation wherein the particles are interacting only with an external potential. With some care, the arguments can also extended to systems wherein the particles are interacting with each other as well. In the latter situation, the energy of the complete system of particles will also include the contributions due to the self interactions. In such a case, the equation of motion governing the α-th particle will be given by

$$m_\alpha\, \ddot{\boldsymbol{x}}_\alpha = \boldsymbol{F}_\alpha + \sum_{\beta\neq\alpha} \boldsymbol{F}_{\alpha\beta},$$

where $\boldsymbol{F}_\alpha = -\boldsymbol{\nabla}_\alpha U$ is the force on the particle due to the external potential U, while $\boldsymbol{F}_{\alpha\beta} = -\boldsymbol{\nabla}_\alpha U_\beta$ is the force on the α-th particle due to its interaction with the β-th particle which is generated by the potential U_β. Note that the quantity $\boldsymbol{\nabla}_\alpha$ implies that the derivatives are taken at the location of the α-th particle. Also, self-interactions, i.e. the effects of a particle on itself, are assumed to be absent. The total energy of the system can be written as

$$E = \sum_\alpha \frac{m_\alpha\, \dot{\boldsymbol{x}}_\alpha^2}{2} + \sum_\alpha U_\alpha(\boldsymbol{x}) + U(\boldsymbol{x}),$$

where, as we have already described, U is the external potential, while U_α is the potential due to α-th particle.

Therefore, we have

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_\alpha \frac{m_\alpha\, \dot{\boldsymbol{x}}_\alpha^2}{2} + \sum_\alpha U_\alpha(\boldsymbol{x}) + U(\boldsymbol{x})\right]\\ &= \sum_\alpha m_\alpha\, \dot{\boldsymbol{x}}_\alpha \cdot \ddot{\boldsymbol{x}}_\alpha + \sum_\alpha \frac{\partial U_\alpha}{\partial t} + \sum_\alpha \sum_{\beta\neq\alpha} \dot{\boldsymbol{x}}_\alpha \cdot \boldsymbol{\nabla}_\alpha U_\beta + \dot{\boldsymbol{x}}_\alpha \cdot \sum_\alpha \boldsymbol{\nabla}_\alpha U + \frac{\partial U}{\partial t}\end{aligned}$$

and, if the potentials do not explicitly depend on time, this expression reduces to

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \sum_\alpha \dot{\boldsymbol{x}}_\alpha \cdot \left(m_\alpha\, \ddot{\boldsymbol{x}}_\alpha + \sum_{\beta\neq\alpha} \boldsymbol{\nabla}_\alpha U_\beta + \boldsymbol{\nabla}_\alpha U\right) = \sum_\alpha \dot{\boldsymbol{x}}_\alpha \cdot \left(m_\alpha\, \ddot{\boldsymbol{x}}_\alpha - \sum_{\beta\neq\alpha} \boldsymbol{F}_{\alpha\beta} - \boldsymbol{F}_\alpha\right) = 0,$$

where we have made use of the equation of motion to arrive at the final equality.

Translational invariance and conservation of linear momentum

Let us now turn to a system possessing translational invariance. Again, let us first consider the case of a single particle and then extend the arguments to the case of a system of many particles.

Consider motion in one dimension. If the potential is translational invariant, independent of the extent of translation, then it is clear that the potential is a constant, which, in turn, implies that the corresponding force on the particle is zero. Clearly, this implies that $p = m\,\dot{x}$ is a constant.

If we have a collection of many particles, the total momentum of the system is given by

$$\boldsymbol{p} = \sum_\alpha m_\alpha\, \dot{\boldsymbol{x}}_\alpha,$$

so that

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \sum_\alpha m_\alpha\, \ddot{\boldsymbol{x}}_\alpha$$

and, upon using the equation of motion for the system (including forces due to mutual interactions between the particles, we obtain

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \sum_\alpha \left(\boldsymbol{F}_\alpha + \sum_{\beta\neq\alpha} \boldsymbol{F}_{\alpha\beta}\right).$$

If the external potential U possesses translational invariance, then the forces $\boldsymbol{F}_\alpha$ will be zero, in which case we have

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \sum_\alpha \sum_{\beta\neq\alpha} \boldsymbol{F}_{\alpha\beta}.$$

According to Newton's third law, the forces due to mutual interactions satisfy the relation $\boldsymbol{F}_{\alpha\beta} = -\boldsymbol{F}_{\beta\alpha}$, as a result of which the above sum vanishes, implying that the total momentum of the system is conserved.

Rotational invariance and conservation of angular momentum

Recall that, in the case of a single particle, the angular momentum is defined as

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}.$$

Hence, we have

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \times \boldsymbol{p} + \boldsymbol{r} \times \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \boldsymbol{v} \times \boldsymbol{p} + \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{r} \times \boldsymbol{F},$$

where we have made use of the fact that $\boldsymbol{v} \times \boldsymbol{p} = 0$ and $\mathrm{d}\boldsymbol{p}/\mathrm{d}t = \boldsymbol{F}$.

Now, if the force is spherically symmetric so that $\boldsymbol{F} \propto \boldsymbol{r}$, then $\boldsymbol{r} \times \boldsymbol{F} = 0$ implying that angular momentum is conserved. In other words, the spherical symmetry if responsible for the conservation of angular momentum.

Let us now turn to the case of a multi-particle system. In such a case, the angular momentum of the system is given by

$$\boldsymbol{L} = \sum_\alpha (\boldsymbol{r}_\alpha \times \boldsymbol{p}_\alpha)$$

so that

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \sum_\alpha \left(\frac{\mathrm{d}\boldsymbol{r}_\alpha}{\mathrm{d}t} \times \boldsymbol{p}_\alpha + \boldsymbol{r}_\alpha \times \frac{\mathrm{d}\boldsymbol{p}_\alpha}{\mathrm{d}t}\right) = \sum_\alpha \left[\boldsymbol{r}_\alpha \times \left(\boldsymbol{F}_\alpha + \sum_{\beta\neq\alpha} \boldsymbol{F}_{\alpha\beta}\right)\right],$$

where, as we have discussed earlier, $\boldsymbol{F}_\alpha$ and $\boldsymbol{F}_{\beta\alpha}$ represent the forces on the α-th particle due to an external field and the mutual interaction with the β-th particle, respectively. We can rewrite the above expression as

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \sum_\alpha \boldsymbol{r}_\alpha \times \boldsymbol{F}_\alpha + \sum_\alpha \sum_{\beta>\alpha} (\boldsymbol{r}_\alpha \times \boldsymbol{F}_{\alpha\beta} + \boldsymbol{r}_\beta \times \boldsymbol{F}_{\beta\alpha}) = \sum_\alpha \boldsymbol{r}_\alpha \times \boldsymbol{F}_\alpha + \sum_\alpha \sum_{\beta>\alpha} (\boldsymbol{r}_\alpha - \boldsymbol{r}_\beta) \times \boldsymbol{F}_{\alpha\beta},$$

where we have made use of the fact that $\boldsymbol{F}_{\beta\alpha} = -\boldsymbol{F}_{\alpha\beta}$. Since $\boldsymbol{F}_{\alpha\beta}$ acts along the vector connecting α and β, the last term vanishes leading to

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \sum_\alpha \boldsymbol{r}_\alpha \times \boldsymbol{F}_\alpha,$$

i.e. the change in the angular momentum arises due to the total external torque on the system.

Discrete symmetries

Some systems may possess discrete symmetries such as reflection about an axis or inversion through the origin. For instance, the transformation $(x, y, z) \to (-x, y, z)$ implies reflection about the y-z plane, while inversion about the origin is described by the transformation $(x, y, z) \to (-x, -y, -z)$. A potential describing a system may be symmetric under such transformations. However, note that these transformations are discrete transformations, i.e. they cannot be constructed out of successive transformations of their infinitesimal versions. In fact, no such infinitesimal transformation is possible at all. It is found that, in contrast to continuous symmetries, there are no conserved quantities associated with discrete symmetries.

Axial and polar vectors

Note that the inversion of coordinates can be represented as $\boldsymbol{r} \to -\boldsymbol{r}$. Under such a transformation, the velocity and acceleration behave as $\boldsymbol{v} \to -\boldsymbol{v}$ and $\boldsymbol{a} \to -\boldsymbol{a}$, whereas angular momentum behaves as $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} \to -\boldsymbol{r} \times (-\boldsymbol{p}) \to \boldsymbol{L}$. In other words, some vectors change sign under inversion whereas others do not. The former are called polar vectors, while the latter are referred to as axial or pseudo-vectors. Apart from velocity and acceleration, standard examples are conservative forces (that can be derived from a potential) and electric fields which we will encounter later. Classic examples of axial or pseudo vectors include (apart from angular momentum that we mentioned above), vorticity and magnetic fields, both of which we will study later.

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Illustrative examples 5

Many body systems, non-Cartesian coordinates and central forces

1. *Center of mass of two extended bodies:* Consider a system of two extended bodies, which have masses M_1 and M_2 and centers of mass $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$. Prove that the center of mass of the entire system can be expressed as follows: [JRT, Problem 3.20]

$$\mathrm{R} = \frac{M_1\, \boldsymbol{R}_1 + M_2\, \boldsymbol{R}_2}{M_1 + M_2}.$$

Note: This implies that, when finding the centre of mass of a complicated system, one can treat its components just as one would have treated point masses located at their separate centers of masses, even when the components themselves are extended bodies.

2. *Working with the polar coordinates in two dimensions:* On a plane, it is evident that, in terms of the polar coordinates, the radial unit vector $\hat{\boldsymbol{\rho}}$ can be written as [JRT, Section 1.7, Problem 1.43]

$$\hat{\boldsymbol{\rho}} = \cos\phi\, \hat{\boldsymbol{x}} + \sin\phi\, \hat{\boldsymbol{y}}.$$

   (a) In the polar coordinates, the second unit vector is denoted as $\hat{\boldsymbol{\phi}}$. Obtain a similar expression relating the unit vector $\hat{\boldsymbol{\phi}}$ to the Cartesian unit vectors $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$. Is the unit vector $\hat{\boldsymbol{\phi}}$ normal to the unit vector $\hat{\boldsymbol{\rho}}$?

   (b) Consider a particle moving along the following trajectory on the plane:

$$\boldsymbol{r} = \rho\, \hat{\boldsymbol{\rho}},$$

   where ρ is a function of time. Express the corresponding velocity $\boldsymbol{v}$ and acceleration $\boldsymbol{a}$ in terms of the unit vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$.

   (c) Consider a particle that is moving on a plane along a circle of radius R at an angular velocity ω, a problem we had discussed earlier. Determine the position $\boldsymbol{r}$, velocity $\boldsymbol{v}$ and acceleration $\boldsymbol{a}$ of the particle in terms of the unit vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$.

3. *The cylindrical and spherical polar coordinates in three dimensions:* Let us first consider the case of cylindrical polar coordinates in three dimensions. It is clear that that the third unit vector, apart from the unit vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$, is given by the Cartesian unit vector $\hat{\boldsymbol{z}}$.

   (a) In a similar fashion, starting from the radial unit vector $\hat{\boldsymbol{r}}$, obtain the complete orthonormal set of unit vectors, say, $(\hat{\boldsymbol{r}}, \hat{\theta}, \hat{\boldsymbol{\phi}})$, in the spherical polar coordinates in three dimensions. [JRT, pp. 134–136, Problem 4.40]

   (b) Working in the two coordinates systems, express the displacement (with respect to the origin), velocity and acceleration in terms of the corresponding unit vectors.

   (c) Determine the kinetic energy of a particle in three dimensions in the Cartesian, cylindrical polar and the spherical polar coordinate systems.

4. *Angular momentum of a rotating rigid body:* Consider a rigid body rotating with angular velocity ω about the z-axis. Let us use the cylindrical polar coordinates $(\rho_\alpha, \phi_\alpha, z_\alpha)$ to denote the positions of the particles $\alpha = 1, 2, 3, \ldots, N$ that constitute the rigid body. [JRT, Problem 3.30]

   (a) Show that the velocity of the particle α is $\rho_\alpha\, \omega$ along the $\hat{\boldsymbol{\phi}}$ direction.

   (b) Using the above result, show that the z-component of angular momentum ℓ_α of the particle α is $m_\alpha\, \rho_\alpha^2\, \omega$.

   (c) Show that the z-component of the total angular momentum of the system can be written as $L_z = I\, \omega$, where I is the moment of inertia of the system about the z-axis given by

$$I = \sum_{\alpha=1}^{N} m_\alpha\, \rho_\alpha^2.$$

5. *Virial theorem for a circular orbit in a central potential:* A mass m moves in a circular orbit (centered on the origin) in the field of an attractive central force with potential energy $U(r) = k\,r^n$. Prove the virial theorem that $T = n\,U/2$, where T denotes the kinetic energy of the particle. [JRT, Problem 4.41]

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Illustrative examples 5 with solutions

Many body systems, non-Cartesian coordinates and central forces

1. *Center of mass of two extended bodies:* Consider a system of two extended bodies, which have masses M_1 and M_2 and centers of mass $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$. Prove that the center of mass of the entire system can be expressed as follows: [JRT, Problem 3.20]

$$\mathrm{R} = \frac{M_1\,\boldsymbol{R}_1 + M_2\,\boldsymbol{R}_2}{M_1 + M_2}.$$

Note: This implies that, when finding the centre of mass of a complicated system, one can treat its components just as one would have treated point masses located at their separate centers of masses, even when the components themselves are extended bodies.

Solution: Given a collection of N point masses, each with mass m_i and position vector $\boldsymbol{r}_i$, the center of mass of the system is given by

$$\boldsymbol{R} = \frac{\sum_i m_i\,\boldsymbol{r}_i}{\sum_i m_i}.$$

Let N_1 particles constitute the mass M_1 and let N_2 particles form the mass M_2 so that

$$\boldsymbol{R}_1 = \frac{\sum_{i=1}^{N_1} m_i\,\boldsymbol{r}_i}{\sum_{i=1}^{N_1} m_i}, \quad \boldsymbol{R}_2 = \frac{\sum_{i=1}^{N_2} m_i\,\boldsymbol{r}_i}{\sum_{i=1}^{N_2} m_i}.$$

Hence, we can write, for $N = N_1 + N_2$,

$$\begin{aligned}\boldsymbol{R} &= \frac{\sum_{i=1}^{N} m_i\,\boldsymbol{r}_i}{\sum_{i=1}^{N} m_i} = \frac{\sum_{i=1}^{N_1} m_i\,\boldsymbol{r}_i + \sum_{i=1}^{N_2} m_i\,\boldsymbol{r}_i}{\sum_{i=1}^{N} m_i}\\ &= \frac{\left(\sum_{i=1}^{N_1} m_i\right)\boldsymbol{R}_1 + \left(\sum_{i=1}^{N_2} m_i\right)\boldsymbol{R}_2}{\sum_{i=1}^{N_1} m_i + \sum_{i=1}^{N_2} m_i} = \frac{M_1\,\boldsymbol{R}_1 + M_2\,\boldsymbol{R}_2}{M_1 + M_2},\end{aligned}$$

as required.

2. *Working with the polar coordinates in two dimensions:* On a plane, it is evident that, in terms of the polar coordinates, the radial unit vector $\hat{\boldsymbol{\rho}}$ can be written as [JRT, Section 1.7, Problem 1.43]

$$\hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}}.$$

(a) In the polar coordinates, the second unit vector is denoted as $\hat{\boldsymbol{\phi}}$. Obtain a similar expression relating the unit vector $\hat{\boldsymbol{\phi}}$ to the Cartesian unit vectors $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$. Is the unit vector $\hat{\boldsymbol{\phi}}$ normal to the unit vector $\hat{\boldsymbol{\rho}}$?

Solution: On the x-y-plane, the radial vector vector is given by

$$\boldsymbol{\rho} = x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}},$$

so that the corresponding unit vector can be written as

$$\hat{\boldsymbol{\rho}} = \frac{\boldsymbol{\rho}}{\rho} = \frac{x}{\rho}\,\hat{\boldsymbol{x}} + \frac{y}{\rho}\,\hat{\boldsymbol{y}}$$

and, since, $x = \rho \cos\phi$ and $y = \rho \sin\phi$, we obtain that

$$\hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}}.$$

It should be recognized that the unit vectors point to directions wherein a specific coordinate is always increasing, when all the other coordinates are kept fixed. Note that the Cartesian unit vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$ indeed possess this feature. Therefore, the unit vector $\hat{\boldsymbol{\phi}}$ should point to a direction which corresponds to increasing values of ϕ. This implies that we can define

$$\hat{\boldsymbol{\phi}} = \frac{\mathrm{d}\boldsymbol{\rho}/\mathrm{d}\phi}{|\mathrm{d}\boldsymbol{\rho}/\mathrm{d}\phi|},$$

with r kept fixed. Since

$$\boldsymbol{\rho} = \rho\cos\phi\,\hat{\boldsymbol{x}} + \rho\sin\phi\,\hat{\boldsymbol{y}},$$

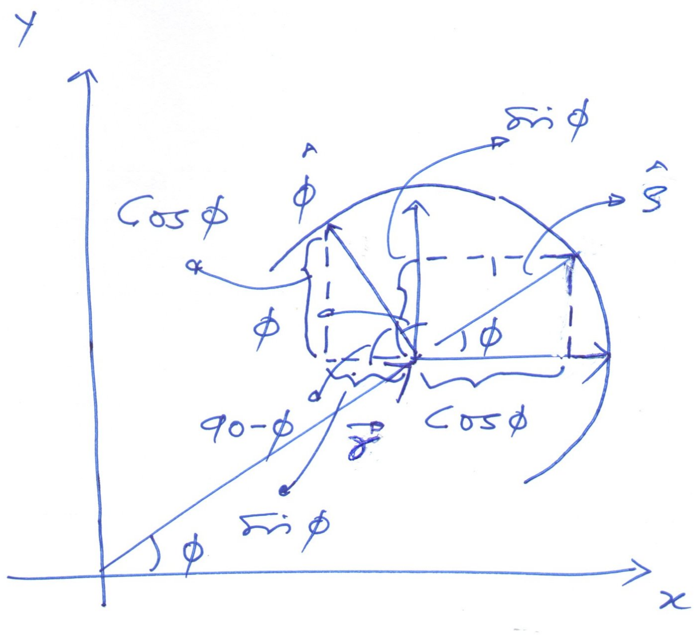
we then obtain that

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}}$$

and it is easy to see that $\hat{\boldsymbol{\rho}} \cdot \hat{\boldsymbol{\phi}} = 0$, as required.
There is another way of arriving at the result. Since, we have $\hat{\boldsymbol{\rho}} \cdot \hat{\boldsymbol{\rho}} = 1$, we have

$$\frac{\mathrm{d}\,(\hat{\boldsymbol{\rho}} \cdot \hat{\boldsymbol{\rho}})}{\mathrm{d}\phi} = 2\,\hat{\boldsymbol{\rho}} \cdot \frac{\mathrm{d}\hat{\boldsymbol{\rho}}}{\mathrm{d}\phi} = 0,$$

which implies that $\mathrm{d}\hat{\boldsymbol{\rho}}/\mathrm{d}\phi$ is perpendicular to $\hat{\boldsymbol{\rho}}$. It is easy to see that $\hat{\boldsymbol{\phi}}$ is indeed $\mathrm{d}\hat{\boldsymbol{\rho}}/\mathrm{d}\phi$.
Yet another way of obtaining the above expression for $\hat{\boldsymbol{\phi}}$ is to arrive at it geometrically.



In the figure above, the circle corresponds has a unit radius. Note that $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ have the same magnitude and directions at all positions. It should be evident from the above figure that, we can write

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}}.$$

which is clearly perpendicular to $\hat{\boldsymbol{\rho}}$ since $\hat{\boldsymbol{\rho}} \cdot \hat{\boldsymbol{\phi}} = 0$. It should be emphasized that, in contrast to the Cartesian unit vectors $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ which are independent of position, the unit vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ depend on position, specifically on the angular coordinate ϕ.

(b) Consider a particle moving along the following trajectory on the plane:

$$\boldsymbol{r} = \rho\,\hat{\boldsymbol{\rho}},$$

where ρ is a function of time. Express the corresponding velocity $\boldsymbol{v}$ and acceleration $\boldsymbol{a}$ in terms of the unit vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$.

Solution: We have

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \frac{\mathrm{d}\rho}{\mathrm{d}t}\,\hat{\boldsymbol{\rho}} + \rho\,\frac{\mathrm{d}\hat{\boldsymbol{\rho}}}{\mathrm{d}t} = \dot{\rho}\,\hat{\boldsymbol{\rho}} + \rho\,\frac{\mathrm{d}\hat{\boldsymbol{\rho}}}{\mathrm{d}t}$$

and as

$$\frac{\mathrm{d}\hat{\boldsymbol{\rho}}}{\mathrm{d}t} = (-\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}})\,\frac{\mathrm{d}\phi}{\mathrm{d}t} = \dot{\phi}\,\hat{\boldsymbol{\phi}},$$

we can write

$$\boldsymbol{v} = \dot{\rho}\,\hat{\boldsymbol{\rho}} + \rho\,\dot{\phi}\,\hat{\boldsymbol{\phi}}.$$

Therefore, we also have

$$\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \ddot{\rho}\,\hat{\boldsymbol{\rho}} + \dot{\rho}\,\frac{\mathrm{d}\hat{\boldsymbol{\rho}}}{\mathrm{d}t} + \dot{\rho}\,\dot{\phi}\,\hat{\boldsymbol{\phi}} + \rho\,\ddot{\phi}\,\hat{\boldsymbol{\phi}} + \rho\,\dot{\phi}\,\frac{\mathrm{d}\hat{\boldsymbol{\phi}}}{\mathrm{d}t}$$

and as

$$\frac{\mathrm{d}\hat{\boldsymbol{\phi}}}{\mathrm{d}t} = -\left(\cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}}\right)\,\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\dot{\phi}\,\hat{\boldsymbol{\rho}},$$

we can write

$$\boldsymbol{a} = \left(\ddot{\rho} - \rho\,\dot{\phi}^2\right)\,\hat{\boldsymbol{\rho}} + \left(2\,\dot{\rho}\,\dot{\phi} + \rho\,\ddot{\phi}\right)\,\hat{\boldsymbol{\phi}}.$$

(c) Consider a particle that is moving on a plane along a circle of radius R at an angular velocity ω, a problem we had discussed earlier. Determine the position $\boldsymbol{r}$, velocity $\boldsymbol{v}$ and acceleration $\boldsymbol{a}$ of the particle in terms of the unit vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$.

Solution: In the plane polar coordinates, the coordinates of a particle moving on a circle of radius R at angular velocity ω corresponds to $\rho = R$, $\dot{\phi} = \omega$ and $\ddot{\phi} = 0$ (as ω is a constant). In such a case, we have

$$\boldsymbol{r} = R\,\hat{\boldsymbol{\rho}},$$

so that the corresponding velocity and acceleration are given by (since $\dot{\rho} = \ddot{\rho} = 0$)

$$\boldsymbol{v} = R\,\dot{\phi}\,\hat{\boldsymbol{\phi}} = R\,\omega\,\hat{\boldsymbol{\phi}}, \quad \boldsymbol{a} = -R\,\dot{\phi}^2\,\hat{\boldsymbol{\rho}} = -R\,\omega^2\,\hat{\boldsymbol{\rho}}.$$

These expressions imply that the particle moves with a constant speed (i.e. $\omega\,R$) along the $\hat{\boldsymbol{\phi}}$ direction, while the acceleration of the particle is a constant (viz. $\omega^2\,R$) and is directed towards the centre. This corresponds to the centripetal acceleration experienced by the particle, which we have discussed earlier.

3. *The cylindrical and spherical polar coordinates in three dimensions:* Let us first consider the case of cylindrical polar coordinates in three dimensions. It is clear that that the third unit vector, apart from the unit vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$, is given by the Cartesian unit vector $\hat{\boldsymbol{z}}$.

(a) In a similar fashion, starting from the radial unit vector $\hat{\boldsymbol{r}}$, obtain the complete orthonormal set of unit vectors, say, $(\hat{\boldsymbol{r}}, \hat{\theta}, \hat{\boldsymbol{\phi}})$, in the spherical polar coordinates in three dimensions. [JRT, pp. 134–136, Problem 4.40]

Solution: Recall that the spherical polar coordinates are defined as

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta,$$

where θ is the angle made by the radial vector $\boldsymbol{r}$ with respect to the z-axis and ϕ is the angle made with respect to the x-axis by the projection of the radial vector on to the x-y-plane. Therefore, we can express the unit radial vector as

$$\hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{r} = \frac{x}{r}\,\hat{\boldsymbol{x}} + \frac{y}{r}\,\hat{\boldsymbol{x}} + \frac{z}{r}\,\hat{\boldsymbol{z}}$$

and hence

$$\hat{\boldsymbol{r}} = \sin\theta\cos\phi\,\hat{\boldsymbol{x}} + \sin\theta\sin\phi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}}.$$

It is straightforward to see that $\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{r}} = 1$, as required.

In order to span the three-dimensional space, we now require two more unit vectors, say, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$, which are perpendicular to the unit vector $\hat{\boldsymbol{r}}$ and are also perpendicular to each other. These unit vectors point to the directions in which the angles θ and ϕ are increasing. From the geometry, it should be clear that the unit vector $\hat{\boldsymbol{\phi}}$ should be parallel to the x-y-plane suggesting that it should have no z-component. Also, it should be clear that $\hat{\boldsymbol{\phi}}$ will be independent of the angle θ. Therefore, we can write

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}},$$

so that $\hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = 1$ and $\hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{r}} = 0$, as required.

We can finally write

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\boldsymbol{x}} + \cos\theta\sin\phi\,\hat{\boldsymbol{y}} - \sin\theta\,\hat{\boldsymbol{z}},$$

so that $\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} = 1$ and $\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{r}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = 0$, as required.

(b) Working in the two coordinates systems, express the displacement (with respect to the origin), velocity and acceleration in terms of the corresponding unit vectors.

Solution: In the cylindrical coordinates, we can express the displacement with respect to the origin as

$$\boldsymbol{r} = \rho\,\hat{\boldsymbol{\rho}} + z\,\hat{z},$$

so that

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \frac{\mathrm{d}\,(\rho\,\hat{\boldsymbol{\rho}})}{\mathrm{d}t} + \frac{\mathrm{d}z}{\mathrm{d}t}\,\hat{z}.$$

We have already calculated the first of these terms (in the case of the plane polar coordinates) and the second term is straightforward to evaluate (since $\hat{\boldsymbol{z}}$ is a constant), so that we can write

$$\boldsymbol{v} = \dot{\rho}\,\hat{\boldsymbol{\rho}} + \rho\,\dot{\phi}\,\hat{\boldsymbol{\phi}} + \dot{z}\,\hat{\boldsymbol{z}}.$$

Upon using the result for the acceleration in the plane polar coordinates, we can arrive at

$$\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \left(\ddot{\rho} - \rho\,\dot{\phi}^2\right)\,\hat{\boldsymbol{\rho}} + \left(2\,\dot{\rho}\,\dot{\phi} + \rho\,\ddot{\phi}\right)\,\hat{\boldsymbol{\phi}} + \ddot{z}\,\hat{\boldsymbol{z}}.$$

Let us now turn to the case of the spherical polar coordinates. In this case, since $\boldsymbol{r} = r\,\hat{\boldsymbol{r}}$, we have

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}t}\,\hat{\boldsymbol{r}} + r\,\frac{\mathrm{d}\hat{\boldsymbol{r}}}{\mathrm{d}t}$$

and as

$$\begin{aligned}
\frac{\mathrm{d}\hat{\boldsymbol{r}}}{\mathrm{d}t} &= \left(\cos\theta\cos\phi\,\dot{\theta} - \sin\theta\sin\phi\,\dot{\phi}\right)\,\hat{\boldsymbol{x}} + \left(\cos\theta\sin\phi\,\dot{\theta} + \sin\theta\cos\phi\,\dot{\phi}\right)\,\hat{\boldsymbol{y}} - \sin\theta\,\dot{\theta}\,\hat{\boldsymbol{z}}\\
&= \left(\cos\theta\cos\phi\,\hat{\boldsymbol{x}} + \cos\theta\sin\phi\,\hat{\boldsymbol{y}} - \sin\theta\,\hat{\boldsymbol{z}}\right)\,\dot{\theta} + \sin\theta\,\left(-\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}}\right)\,\dot{\phi}\\
&= \dot{\theta}\,\hat{\boldsymbol{\theta}} + \sin\theta\,\dot{\phi}\,\hat{\boldsymbol{\phi}}
\end{aligned}$$

we can write

$$\boldsymbol{v} = \dot{r}\,\hat{\boldsymbol{r}} + r\,\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\sin\theta\,\dot{\phi}\,\hat{\boldsymbol{\phi}}.$$

Now, note that

$$\begin{aligned}
\frac{\mathrm{d}\hat{\boldsymbol{\theta}}}{\mathrm{d}t} &= -\left(\sin\theta\cos\phi\,\dot{\theta} + \cos\theta\sin\phi\,\dot{\phi}\right)\,\hat{\boldsymbol{x}} - \left(\sin\theta\sin\phi\,\dot{\theta} - \cos\theta\cos\phi\,\dot{\phi}\right)\,\hat{\boldsymbol{y}} - \cos\theta\,\dot{\theta}\,\hat{\boldsymbol{z}} \\
&= -\left(\sin\theta\cos\phi\,\hat{\boldsymbol{x}}\ + \sin\theta\sin\phi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}}\right)\,\dot{\theta} + \cos\theta\,\left(-\sin\phi\,\hat{\boldsymbol{x}}\ + \cos\phi\,\hat{\boldsymbol{y}}\right)\,\dot{\phi} \\
&= -\dot{\theta}\,\hat{\boldsymbol{r}} + \cos\theta\,\dot{\phi}\,\hat{\boldsymbol{\phi}}
\end{aligned}$$

and

$$\frac{\mathrm{d}\hat{\boldsymbol{\phi}}}{\mathrm{d}t} = -\left(\cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}}\right)\,\dot{\phi} = -\dot{\phi}\,\left(\cos\theta\,\hat{\boldsymbol{\theta}} + \sin\theta\,\hat{\boldsymbol{r}}\right),$$

so that

$$\begin{aligned}
\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} &= \frac{\mathrm{d}\left(\dot{r}\,\hat{\boldsymbol{r}}\right)}{\mathrm{d}t} + \frac{\mathrm{d}\left(r\,\dot{\theta}\,\hat{\boldsymbol{\theta}}\right)}{\mathrm{d}t} + \frac{\mathrm{d}\left(r\sin\theta\,\dot{\phi}\,\hat{\boldsymbol{\phi}}\right)}{\mathrm{d}t} \\
&= \ddot{r}\,\hat{\boldsymbol{r}} + \dot{r}\,\left(\dot{\theta}\,\hat{\boldsymbol{\theta}} + \sin\theta\,\dot{\phi}\,\hat{\boldsymbol{\phi}}\right) + \left(\dot{r}\,\dot{\theta} + r\,\ddot{\theta}\right)\,\hat{\boldsymbol{\theta}} + r\,\dot{\theta}\,\left(-\dot{\theta}\,\hat{\boldsymbol{r}} + \cos\theta\,\dot{\phi}\,\hat{\boldsymbol{\phi}}\right) \\
&\quad + \left(\dot{r}\sin\theta\,\dot{\phi} + r\cos\theta\,\dot{\theta}\,\dot{\phi} + r\sin\theta\,\ddot{\phi}\right)\,\hat{\boldsymbol{\phi}} - r\sin\theta\,\dot{\phi}^2\,\left(\cos\theta\,\hat{\boldsymbol{\theta}} + \sin\theta\,\hat{\boldsymbol{r}}\right) \\
&= \left(\ddot{r} - r\,\dot{\theta}^2 - r\sin^2\!\theta\,\dot{\phi}^2\right)\,\hat{\boldsymbol{r}} + \left(2\,\dot{r}\,\dot{\theta} - r\sin\theta\cos\theta\,\dot{\phi}^2\right)\,\hat{\boldsymbol{\theta}} \\
&\quad + \left(2\,\dot{r}\sin\theta\,\dot{\phi} + 2\,r\cos\theta\,\dot{\theta}\,\dot{\phi} + r\sin\theta\,\ddot{\phi}\right)\,\hat{\boldsymbol{\phi}}.
\end{aligned}$$

Note that, when, say, $\theta = \pi/2$ (which corresponds to motion in the x-y-plane), so that $\dot{\theta} = 0$, this expression simplifies to

$$\boldsymbol{a} = \left(\ddot{r} - r\,\dot{\phi}^2\right)\,\hat{\boldsymbol{r}} + \left(2\,\dot{r}\,\dot{\phi} + r\,\ddot{\phi}\right)\,\hat{\boldsymbol{\phi}},$$

which is the expression we had obtained earlier when working in the polar coordinates.

(c) Determine the kinetic energy of a particle in three dimensions in the Cartesian, cylindrical polar and the spherical polar coordinate systems.

<u>*Solution:*</u> In the Cartesian coordinates, we have

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\,\hat{\boldsymbol{x}} + \frac{\mathrm{d}y}{\mathrm{d}t}\,\hat{\boldsymbol{y}} + \frac{\mathrm{d}z}{\mathrm{d}t}\,\hat{\boldsymbol{z}} = \dot{x}\,\hat{\boldsymbol{x}} + \dot{y}\,\hat{\boldsymbol{y}} + \dot{z}\,\hat{\boldsymbol{z}}.$$

Therefore, the corresponding kinetic energy is given by

$$T = \frac{m}{2}\,v^2 = \frac{m}{2}\,\boldsymbol{v}\cdot\boldsymbol{v} = \frac{m}{2}\,\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right).$$

In the cylindrical polar coordinates, if we use the fact that,

$$\boldsymbol{v} = \dot{\rho}\,\hat{\boldsymbol{\rho}} + \rho\,\dot{\phi}\,\hat{\boldsymbol{\phi}} + \dot{z}\,\hat{\boldsymbol{z}},$$

we have

$$T = \frac{m}{2}\,\boldsymbol{v}\cdot\boldsymbol{v} = \frac{m}{2}\,\left[\dot{\rho}^2 + \left(\rho\,\dot{\phi}\right)^2 + \dot{z}^2\right].$$

Similarly, in the spherical polar coordinates, since,

$$\boldsymbol{v} = \dot{r}\,\hat{\boldsymbol{r}} + r\,\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\sin\theta\,\dot{\phi}\,\hat{\boldsymbol{\phi}},$$

the kinetic energy is given by

$$T = \frac{m}{2}\,\boldsymbol{v}\cdot\boldsymbol{v} = \frac{m}{2}\,\left[\dot{r}^2 + \left(r\,\dot{\theta}\right)^2 + \left(r\sin\theta\,\dot{\phi}\right)^2\right].$$

4. <u>*Angular momentum of a rotating rigid body:*</u> Consider a rigid body rotating with angular velocity ω about the z-axis. Let us use the cylindrical polar coordinates $(\rho_\alpha, \phi_\alpha, z_\alpha)$ to denote the positions of the particles $\alpha = 1, 2, 3, \ldots, N$ that constitute the rigid body. [JRT, Problem 3.30]

   (a) Show that the velocity of the particle α is $\rho_\alpha\,\omega$ along the $\hat{\boldsymbol{\phi}}$ direction.

   <u>*Solution:*</u> Note that, in the cylindrical polar coordinates, we have, in general,

   $$\boldsymbol{v} = \dot{\rho}\,\hat{\boldsymbol{\rho}} + \rho\,\dot{\phi}\,\hat{\boldsymbol{\phi}} + \dot{z}\,\hat{\boldsymbol{z}}.$$

   In the case of our interest, for the individual particles as well as the rigid body, $\dot{\rho} = \dot{z} = 0$ and $\dot{\phi} = \omega$, so that

   $$\boldsymbol{v}_\alpha = \rho_\alpha\,\omega\,\hat{\boldsymbol{\phi}}.$$

   (b) Using the above result, show that the z-component of angular momentum $\boldsymbol{L}_\alpha$ of the particle α is $m_\alpha\,\rho_\alpha^2\,\omega$.

   <u>*Solution:*</u> Since the radial vector of the α-th particle is given by

   $$\boldsymbol{r}_\alpha = \rho_\alpha\,\hat{\boldsymbol{\rho}} + z_\alpha\,\hat{\boldsymbol{z}},$$

   the corresponding angular momentum can be determined to be

   $$\begin{aligned} \boldsymbol{L}_\alpha &= m_\alpha\,(\boldsymbol{r}_\alpha \times \boldsymbol{v}_\alpha) = m_\alpha\,\left([\rho_\alpha\,\hat{\boldsymbol{\rho}} + z_\alpha\,\hat{\boldsymbol{z}}] \times \rho_\alpha\,\omega\,\hat{\boldsymbol{\phi}}\right) \\ &= m_\alpha\,\rho_\alpha\,\omega\,\left(\rho_\alpha\,\hat{\boldsymbol{\rho}} \times \hat{\boldsymbol{\phi}} + z_\alpha\,\hat{\boldsymbol{z}} \times \hat{\boldsymbol{\phi}}\right) = m_\alpha\,\rho_\alpha\,\omega\,\left(-z_\alpha\,\hat{\boldsymbol{\rho}} + \rho_\alpha\,\hat{\boldsymbol{z}}\right). \end{aligned}$$

   This implies that the z-component of the angular momentum associated with the α-th particle is

   $$L_\alpha^z = m_\alpha\,\rho_\alpha^2\,\omega.$$

   (c) Show that the z-component of the total angular momentum of the system can be written as $L_z = I\,\omega$, where I is the moment of inertia of the system about the z-axis given by

   $$I = \sum_{\alpha=1}^{N} m_\alpha\,\rho_\alpha^2.$$

   <u>*Solution:*</u> The total angular momentum of the system is given by

   $$L_z = \sum_{\alpha=1}^{N} L_\alpha^z = \sum_{\alpha=1}^{N} m_\alpha\,\rho_\alpha^2\,\omega = I\,\omega,$$

   where I is the moment of inertia defined as

   $$I = \sum_{\alpha=1}^{N} m_\alpha\,\rho_\alpha^2.$$

5. <u>*Virial theorem for a circular orbit in a central potential:*</u> A mass m moves in a circular orbit (centered on the origin) in the field of an attractive central force with potential energy $U(r) = k\,r^n$. Prove the virial theorem that $T = n\,U/2$, where T denotes the kinetic energy of the particle. [JRT, Problem 4.41]

   <u>*Solution:*</u> Note that the effective potential governing the system is given by

   $$U_{\text{eff}}(r) = U(r) + \frac{L^2}{2\,m\,r^2},$$

where the second term is the angular momentum barrier, with L denoting the angular momentum of the particle. The circular trajectory will arise if there is an extremum of the effective potential $U_{\rm eff}(r)$. Therefore, for $U(r) = k\,r^n$, we have

$$\frac{{\rm d}U_{\rm eff}}{{\rm d}r} = n\,k\,r_*^{n-1} - \frac{L^2}{m\,r_*^3} = 0,$$

where r_* denotes the radius of the circular trajectory. Clearly, we have

$$L^2 = n\,m\,k\,r_*^{n+2}.$$

Recall that the motion in a central potential occurs in a plane. Along a circular trajectory $\dot{r} = 0$, so that the kinetic energy is given by

$$T = \frac{m}{2}\,r_*^2\,\dot{\phi}^2,$$

where we have set $r = r_*$. Also, as $L = m\,r^2\,\dot{\phi}$, we can write, at $r = r_ast$,

$$T = \frac{m}{2}\,r_*^2\,\left(\frac{L}{m\,r_*^2}\right)^2 = \frac{L^2}{2\,m\,r_*^2}.$$

If we use the above expression for L^2 in terms of r_*, we arrive at

$$T = \frac{n\,k\,r_*^n}{2} = \frac{n\,U(r_*)}{2},$$

which is the required result.

Illustrative examples 6

Motion in the Keplerian potential

1. *Period of a low orbit Earth satellite:* Use Kepler's third law to estimate the period of a satellite in a circular orbit a few tens of kilometers above the Earth's surface. [JRT, Example 8.5]

2. *Halley's comet:* Halley's comet is in an elliptic orbit around the Sun. The eccentricity of the orbit is 0.967 and the period is 75.3 years. The mass of the Sun is 1.99×10^{30} kg. [JRT, Example 8,4; KK, Problem 10.9]

   (a) Using these data, determine the distance of Halley's comet from the Sun at perihelion and at aphelion.

   (b) What is the speed of Halley's comet when it is closest to the Sun?

   Note: The value of the gravitational constant is $G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2/kg^2}$.

3. *Mass of the Moon:* Before landing astronauts on the Moon, the Apollo 11 space vehicle was in an orbit about the Moon. The mass of the vehicle was 9979 kg and the period of the orbit was 120 min. The maximum and minimum distances of the orbits from the center of the Moon were 1861 km and 1838 km. Assuming the Moon to be a uniform spherical body, determine the mass of the Moon according to these data. [KK, Problem 10.11]

4. *Time dependence of bound Keplerian orbits:* Recall that, we had earlier arrived at the orbital trajectory $r(\phi)$ describing a particle moving in the Keplerian central potential $U(r) = -\alpha/r$, with $\alpha > 0$. Solve for the trajectory of the particle as a function of time and show that the elliptical orbits can be expressed parametrically as

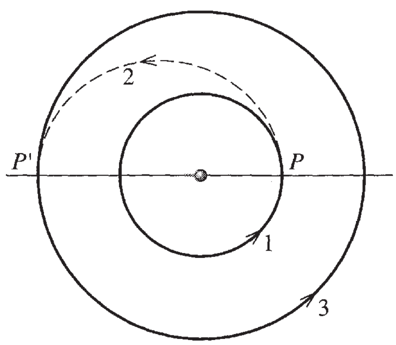
$$r = a\,(1 - \epsilon\cos\xi)\,, \quad t = \sqrt{\frac{m\,a^3}{\alpha}}\,(\xi - \epsilon\sin\xi)\,,$$

   where m is the mass of the particle, a is the semi-major axis of the ellipse, and it has been assumed that the particle is at the perihelion at $\xi = 0$.

5. *Transfer from one circular orbit to another:* A satellite's crew in a circular orbit of radius R_1 wishes to transfer to a circular orbit of radius $2\,R_1$. It is achieved using two successive boosts, as shown in the figure below. Firstly, the satellite is boosted at the point P into the elliptical transfer orbit 2, just large enough to reach out to the larger radius. Secondly, on reaching the required radius (at P', the apogee of the transfer orbit), the satellite is boosted into the desired circular orbit 3. [JRT, Example 8.6]

   (a) What is the factor by which must the speed of the satellite be increased in each of these two boosts?

   (b) What is the factor by which does the satellite's speed increases as a result of the complete maneuver?



Illustrative examples 6 with solutions

Motion in the Keplerian potential

1. *Period of a low orbit Earth satellite:* Use Kepler's third law to estimate the period of a satellite in a circular orbit a few tens of kilometers above the Earth's surface. [JRT, Example 8.5]

   Solution: According to Kepler's third law, the time period T of the satellite will be related to the major axis of the ellipse a as follows
$$T^2 = \frac{4\,\pi^2\,a^3\,\mu_{\rm s}}{G\,M_{\rm E}\,m_{\rm s}},$$
   where $M_{\rm E}$ is the mass of the Earth, while $m_{\rm s}$ is the mass of the satellite and $\mu_{\rm s}$ is its reduced mass. As $\mu_{\rm s} = M_{\rm E}\,m_{\rm s}/(M_{\rm E} + m_{\rm s}) \simeq m_{\rm s}$, we have
$$T^2 \simeq \frac{4\,\pi^2\,R_{\rm E}^3}{G\,M_{\rm E}},$$
   where, since it is a circular trajectory, we have replaced a by $R_{\rm E}$, viz. the Earth's radius, ignoring the height of the satellite above the Earth's surface. Also, since $G\,M_{\rm E}/R_{\rm E}^2 = g$, where g is the acceleration due to gravity, we obtain that
$$T = 2\,\pi\,\sqrt{\frac{R_{\rm E}}{g}}.$$
   Since $R_{\rm E} \simeq 6400\,{\rm km}$ and $g \simeq 10\,{\rm m\,s^{-2}}$, we have
$$T \simeq 2\,\pi\,\sqrt{64\times 10^4}\,{\rm s} \simeq 5025\,{\rm s} \simeq 85\,{\rm minutes}.$$
   In 1961, Vostok 1 had carried Yuri Gagarin, a Soviet cosmonaut, into outer space, making him the first human to do so. Vostok 1 had flown once around the Earth one at a height of about 169 km. Its time period around the Earth was about 89 minutes.

2. *Halley's comet:* Halley's comet is in an elliptic orbit around the Sun. The eccentricity of the orbit is 0.967 and the period is 75.3 years. The mass of the Sun is 1.99×10^{30} kg. [JRT, Example 8,4; KK, Problem 10.9]

   (a) Using these data, determine the distance of Halley's comet from the Sun at perihelion and at aphelion.

   Solution: Recall that, at perihelion and aphelion, we have
$$r_{\rm min} = a\,(1-\epsilon)\,, \quad r_{\rm max} = a\,(1+\epsilon)\,,$$
   where a is the semi-major axis of the ellipse and ϵ is the eccentricity of the orbit. Now, according to Kepler's third law, we have
$$T^2 = \frac{4\,\pi^2\,a^3\,\mu_{\rm c}}{G\,M_{\rm S}\,m_{\rm c}},$$
   where $M_{\rm S}$ is the mass of the Sun, while $m_{\rm c}$ is the mass of the comet and $\mu_{\rm c}$ is its reduced mass. Since, $\mu_{\rm c} = M_{\rm S}\,m_{\rm c}/(M_{\rm S} + m_{\rm c}) \simeq m_{\rm c}$, we have
$$T^2 \simeq \frac{4\,\pi^2\,a^3}{G\,M_{\rm S}}.$$
   or
$$a = \left(\frac{G\,M_{\rm S}\,T^2}{4\,\pi^2}\right)^{1/3}.$$

For the given values, we find that

$$a \simeq 2.66 \times 10^{12}\,\mathrm{m} \simeq 17.9\,\mathrm{AU},$$

where $1\,\mathrm{AU} = 1.49 \times 10^{11}\,\mathrm{m}$ is the so-called astronomical unit which refers to the average distance between the Sun and the Earth.
For $\epsilon = 0.967$, then, for Halley's comet,

$$r_{\mathrm{min}} \simeq 0.591\,\mathrm{AU}, \quad r_{\mathrm{max}} \simeq 35.2\,\mathrm{AU}.$$

It should be noted that r_{min} above is smaller than the average radius of Venus' orbit around the Sun and r_{max} is greater than the average radius of Neptune's orbit.

(b) What is the speed of Halley's comet when it is closest to the Sun?
<u>*Solution:*</u> Since perihelion corresponds to a turning point, the radial velocity of the orbit vanishes so that we have

$$\frac{\mu_{\mathrm{c}}}{2}\, r_{\mathrm{min}}^2\, \dot{\phi}^2 - \frac{G\, M_{\mathrm{S}}\, m_{\mathrm{c}}}{r_{\mathrm{min}}} = \frac{\mu_{\mathrm{c}}}{2}\, v_{\mathrm{min}}^2 - \frac{G\, M_{\mathrm{S}}\, m_{\mathrm{c}}}{r_{\mathrm{min}}} = E$$

where $v_{\mathrm{min}} = r_{\mathrm{min}}\, \dot{\phi}$ denotes the velocity at r_{min}. Therefore, we have

$$v_{\mathrm{min}}^2 = \frac{2\,E}{\mu_{\mathrm{c}}} + \frac{2\,G\,M_{\mathrm{S}}\,m_{\mathrm{c}}}{\mu_{\mathrm{c}}\, r_{\mathrm{min}}}$$

and it is a matter of calculating the corresponding energy E. Note that, we have

$$E = -\frac{(G\,M_{\mathrm{S}}\,m_{\mathrm{c}})^2\,\mu_{\mathrm{c}}}{2\,L^2}\,\left(1-\epsilon^2\right)$$

and

$$a = \frac{L^2}{G\,M_{\mathrm{S}}\,m_{\mathrm{c}}\,\mu_{\mathrm{c}}\,\left(1-\epsilon^2\right)},$$

so that we can write

$$E = -\frac{G\,M_{\mathrm{S}}\,m_{\mathrm{c}}}{2\,a}.$$

Hence, we have

$$v_{\mathrm{min}}^2 = -\frac{G\,M_{\mathrm{S}}\,m_{\mathrm{c}}}{\mu_{\mathrm{c}}\,a} + \frac{2\,G\,M_{\mathrm{S}}\,m_{\mathrm{c}}}{\mu_{\mathrm{c}}\,r_{\mathrm{min}}}$$

and, since, $r_{\mathrm{min}} = a\,(1-\epsilon)$, we obtain that

$$v_{\mathrm{min}}^2 = -\frac{G\,M_{\mathrm{S}}\,m_{\mathrm{c}}}{\mu_{\mathrm{c}}\,a}\,\left(1 - \frac{2}{1-\epsilon}\right) = \frac{G\,M_{\mathrm{S}}\,m_{\mathrm{c}}}{\mu_{\mathrm{c}}\,a}\,\left(\frac{1+\epsilon}{1-\epsilon}\right) \simeq \frac{G\,M_{\mathrm{S}}}{a}\,\left(\frac{1+\epsilon}{1-\epsilon}\right).$$

For the given values of parameters, we find that

$$v_{\mathrm{min}} = 5.45 \times 10^4\,\mathrm{m\,s^{-1}}.$$

Note: The value of the gravitational constant is $G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2/kg^2}$.

3. <u>*Mass of the Moon:*</u> Before landing astronauts on the Moon, the Apollo 11 space vehicle was in an orbit about the Moon. The mass of the vehicle was 9979 kg and the period of the orbit was 120 min. The maximum and minimum distances of the orbits from the center of the Moon were 1861 km and 1838 km. Assuming the Moon to be a uniform spherical body, determine the mass of the Moon according to these data. [KK, Problem 10.11]

<u>*Solution:*</u> Recall that,

$$r_{\mathrm{min}} = a\,(1-\epsilon)\,, \quad r_{\mathrm{max}} = a\,(1+\epsilon)\,,$$

where $r_{\rm min}$ and $r_{\rm max}$ represent the perihelion and aphelion, while a, as before, denote the semi-major axis of the ellipse. Therefore, we have

$$a = \frac{r_{\rm min} + r_{\rm max}}{2},$$

which, in our case, amounts to

$$a = \frac{1861 + 1838}{2}\,{\rm km} \simeq 1.85 \times 10^{6}\,{\rm m}.$$

We also have, according to the Kepler's third law,

$$T^2 = \frac{4\,\pi^2\,a^3\,\mu_{\rm s}}{G\,M_{\rm M}\,m_{\rm s}},$$

where $M_{\rm M}$, $m_{\rm s}$ and $\mu_{\rm s}$ denote the mass of the moon, the mass of the satellite and its reduced mass, respectively. Since, $\mu_{\rm s} \simeq m_{\rm s}$, we have

$$M_{\rm M} \simeq \frac{4\,\pi^2\,a^3}{G\,T^2},$$

which for the values given above, can be determined to be

$$M_{\rm M} \simeq 7.2 \times 10^{22}\,{\rm kg}.$$

4. <u>*Time dependence of bound Keplerian orbits:*</u> Recall that, we had earlier arrived at the orbital trajectory $r(\phi)$ describing a particle moving in the Keplerian central potential $U(r) = -\alpha/r$, with $\alpha > 0$. Solve for the trajectory of the particle as a function of time and show that the elliptical orbits can be expressed parametrically as

$$r = a\,(1 - \epsilon\,\cos\xi)\,, \quad t = \sqrt{\frac{m\,a^3}{\alpha}}\,(\xi - \epsilon\,\sin\xi)\,,$$

where m is the mass of the particle, a is the semi-major axis of the ellipse, and it has been assumed that the particle is at the perihelion at $\xi = 0$. Also, use the result to arrive at Kepler's third law.

<u>*Solution:*</u> Recall that, using the conservation of angular momentum and energy, we had reduced the motion in a central potential to an effective one-dimensional problem with the energy being given by

$$\frac{\mu}{2}\left(\frac{{\rm d}r}{{\rm d}t}\right)^2 + U_{\rm eff}(r) = E,$$

where E is the energy of the particle, mu is its reduces mass, while the effective potential is defined as

$$U_{\rm eff}(r) = U(r) + \frac{L^2}{2\,\mu\,r^2},$$

with $U(r)$ being the original central potential and L is the angular momentum of the particle.

In the case $U(r) = -\alpha/r$, we can write

$$\frac{\mu}{2}\left(\frac{{\rm d}r}{{\rm d}t}\right)^2 = E + \frac{\alpha}{r} - \frac{L^2}{2\,\mu\,r^2}$$

or, equivalently,

$$t - t_0 = \int {\rm d}t = \int \frac{\sqrt{\mu/2}\,{\rm d}r}{\sqrt{E + (\alpha/r) - [L^2/(2\,\mu\,r^2)]}}.$$

Recall that

$$\epsilon = \left(1 + \frac{2\,E\,L^2}{\mu\,\alpha^2}\right)^{1/2}$$

and

$$a = \frac{L^2/(\mu\,\alpha)}{1-\epsilon^2},$$

so that we have

$$E = -\frac{\mu\,\alpha^2}{2\,L^2}\,(1-\epsilon^2) = -\frac{\alpha}{2\,a}$$

Hence, we can write

$$\begin{aligned} t - t_0 &= \int \frac{\sqrt{\mu/2}\,\mathrm{d}r}{\sqrt{-[\alpha/(2\,a)] + (\alpha/r) - [L^2/(2\,\mu\,r^2)]}} = \int \frac{\sqrt{\mu/2}\,\sqrt{2\,\mu\,r^2}\,\mathrm{d}r}{\sqrt{-(\mu\,\alpha\,r^2/a) + 2\,\mu\,\alpha\,r - L^2}} \\ &= \int \frac{\mu\,\mathrm{d}r\,r\,\sqrt{a/(\mu\,\alpha)}}{\sqrt{-r^2 + 2\,a\,r - [a\,L^2/(\mu\,\alpha)]}} = \int \frac{\sqrt{\mu\,a/\alpha}\,\mathrm{d}r\,r}{\sqrt{-r^2 + 2\,a\,r - a^2\,(1-\epsilon^2)}} \\ &= \int \frac{\sqrt{\mu\,a/\alpha}\,\mathrm{d}r\,r}{\sqrt{(a\,\epsilon)^2 - (r-a)^2}} \end{aligned}$$

and, if we set,

$$r - a = -a\,\epsilon\cos\xi,$$

we obtain that

$$t - t_0 = \sqrt{\frac{\mu\,a^3}{\alpha}} \int \mathrm{d}\xi\;(1-\epsilon\cos\xi) = \sqrt{\frac{\mu\,a^3}{\alpha}}\;(\xi - \epsilon\sin\xi)\,.$$

If we now set $t = 0$ at $\xi = 0$, and use the fact that $\mu \simeq m$, we obtain that

$$r = a\;(1-\epsilon\cos\xi)\,, \quad t = \sqrt{\frac{m\,a^3}{\alpha}}\;(\xi - \epsilon\sin\xi)\,,$$

as required.

Evidently, an orbit is complete between, say, $\xi = 0$ and $\xi = 2\,\pi$, i.e. r comes back to its original value. The corresponding time interval is essentially the period of the orbit and is given by

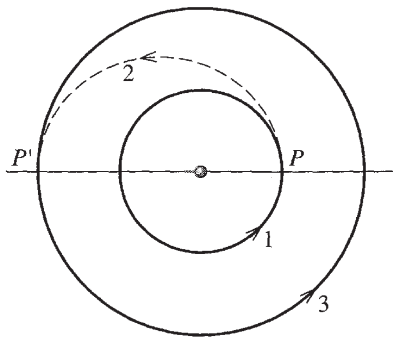
$$T = 2\,\pi\,\sqrt{\frac{m\,a^3}{\alpha}}$$

or, equivalently,

$$T^2 = \frac{4\,\pi^2\,m\,a^3}{\alpha},$$

which is Kepler's third law.

5. *Transfer from one circular orbit to another:* A satellite's crew in a circular orbit of radius R_1 wishes to transfer to a circular orbit of radius $2\,R_1$. It is achieved using two successive boosts, as shown in the figure below. Firstly, the satellite is boosted at the point P into the elliptical transfer orbit 2, just large enough to reach out to the larger radius. Secondly, on reaching the required radius (at P', the apogee of the transfer orbit), the satellite is boosted into the desired circular orbit 3. [JRT, Example 8.6]



(a) What is the factor by which must the speed of the satellite be increased in each of these two boosts?

<u>*Solution:*</u> Note that the satellite transfers from one orbit to another by firing its rockets in the tangential direction, forward or backward, when it is at the perigee of its initial orbit. By our choice of axis, we can choose that this occurs at $\phi = 0$ (and with the initial phase factor δ set to be zero). Also, as the rockets are fired in a tangential direction, the velocity just after firing is still in the same direction, which is perpendicular to the radius from Earth to the satellite. Therefore, the position at which the rockets are fired is also the perigee for the final orbit (with again the initial phase factor δ chosen to be zero).

At perigee, since there is no radial component to the velocity, the angular momentum is $L = \mu_{\mathrm{s}}\, r\, v$. The radius of the satellite's orbit does not change at the instant when the rockets are fired, only its velocity changes, eventually leading to a different orbit than its original one.

Recall that, the orbital equation is given by

$$\frac{1}{r(\phi)} = \frac{G\, M_{\mathrm{E}}\, m_{\mathrm{s}}\, \mu_{\mathrm{s}}}{L^2}\, (1 + \epsilon \cos\phi)\,,$$

where, evidently, M_{E} denotes the mass of the Earth, while m_{s} and μ_{s} represent the mass and the reduced mass of the satellite. Note that the energy of the orbit is related to the eccentricity ϵ as follows:

$$E = -\frac{(G\, M_{\mathrm{E}}\, m_{\mathrm{s}})^2\, \mu_{\mathrm{s}}}{2\, L^2}\, \left(1 - \epsilon^2\right).$$

At $\phi = 0$, from the initial circular and the intermediate elliptical orbits, we have

$$R_1 = \frac{L_2^2}{2\, G\, M_{\mathrm{E}}\, m_{\mathrm{s}}\, \mu_{\mathrm{s}}\, (1 + \epsilon)} = \frac{L_1^2}{2\, G\, M_{\mathrm{E}}\, m_{\mathrm{s}}\, \mu_{\mathrm{s}}},$$

where ϵ is the eccentricity of the intermediate elliptical orbit. The above result implies that

$$L_2^2 = L_1^2\, (1 + \epsilon)\,.$$

and as we have set $L_2 = \lambda\, L_1$, we obtain that

$$\lambda^2 = (1 + \epsilon)\,.$$

Similarly, at $\phi = \pi$, from the intermediate elliptical and the final circular orbits, we have

$$2\, R_1 = \frac{L_2^2}{2\, G\, M_{\mathrm{E}}\, m_{\mathrm{s}}\, \mu_{\mathrm{s}}\, (1 - \epsilon)} = \frac{L_3^2}{2\, G\, M_{\mathrm{E}}\, m_{\mathrm{s}}\, \mu_{\mathrm{s}}}$$

so that

$$L_2^2 = L_3^2\, (1 - \epsilon)\,.$$

Since $L_3 = \lambda'\, L_2$, we obtain that

$$\lambda'^2 = (1 - \epsilon)^{-1}\,.$$

Now, note that, in the case of the elliptical orbit,

$$r_{\mathrm{max}} = 2\, R_1 = a\, (1 + \epsilon)\,, \quad r_{\mathrm{min}} = R_1 = a\, (1 - \epsilon)\,,$$

and, hence,

$$\frac{1 + \epsilon}{1 - \epsilon} = 2,$$

so that $\epsilon = 1/3$. Therefore, $\lambda = \sqrt{4/3}$ and $\lambda' = \sqrt{3/2}$.

(b) What is the factor by which does the satellite's speed increases as a result of the complete maneuver?

<u>*Solution:*</u> In the case of circular orbits of radius R, since there is no radial velocity involved, the velocity remains the same throughout the orbit and is given by (as can be shown from the result obtained in the second exercise above)

$$v^2 = \frac{G\, M_{\rm E}\, m_{\rm s}}{\mu_{\rm s}\, R}.$$

Therefore, in the case of the initial and final circular orbits in our we have

$$v_1^2 = \frac{G\, M_{\rm E}\, m_{\rm s}}{\mu_{\rm s}\, R_1}, \quad v_3^2 = \frac{G\, M_{\rm E}\, m_{\rm s}}{2\, \mu_{\rm s}\, R_1}$$

which implies that

$$v_3 = \frac{v_1}{\sqrt{2}}.$$

In other words, the velocity of the satellite actually decreases, as it goes from the initial circular orbit to the final circular orbit. However, the loss in kinetic energy is suitably compensated by an increase in the potential energy (due to the increased radius) leading to an increase in the total energy of the satellite. Note that $E_3 = -E_1/2$.

Exercise sheet 6

System of particles

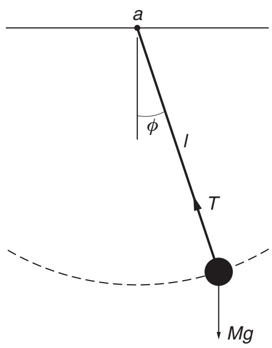
1. (a) *Dependence of angular momentum on the origin:* Show that if the total linear momentum of a system of particles is zero, the angular momentum of the system is the same around all origins.

   (b) *Torque and location of the origin:* Show that if the total force on a system of particles is zero, the torque on the system is the same around all origins. [KK, Problem 7.1]

2. *Moment of inertia of a sphere:* Find the moment of inertia of a uniform sphere of mass M and radius R around an axis through the center. [KK, Problem 7.8]

3. *Angular momenta of a two body system:* Show that in the center of mass frame, the angular momentum $\boldsymbol{L}_1$ of particle 1 is related to the total angular momentum $\boldsymbol{L}$ by $\boldsymbol{L}_1 = (m_2/M)\,\boldsymbol{L}$ and likewise $\boldsymbol{L}_2 = (m_1/M)\,\boldsymbol{L}$, where $M = m_1 + m_2$. Since $\boldsymbol{L}$ is conserved, this shows that the same is true of $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ separately in the centre of mass frame. [JRT, Problem 8.6]

4. *The physical pendulum:* Recall that, in the case of the simple pendulum shown in the figure below, the moment of inertia of the mass is $I_a = M\,l^2$, while the torque on the bob (for small oscillations) is $\tau_a = -M\,g\,l\sin\phi \simeq -M\,g\,l\,\phi$. [KK, Section 7.7]



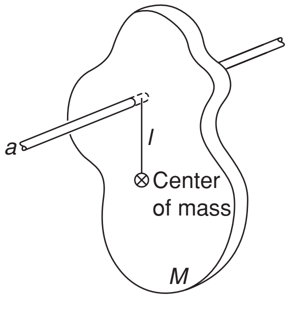
The equation of motion governing the system is

$$I_a\,\ddot{\phi} = \tau_a \simeq -M\,g\,l\,\phi,$$

which implies that the frequency of the pendulum is

$$\omega = \sqrt{\frac{M\,g\,l}{I_a}} = \sqrt{\frac{g}{l}}.$$

Now, consider a physical pendulum as shown in the figure below. Let the moment of inertia of the pendulum about the centre of mass be I_0 and let the center of mass of the system be at a distance l from the pivot point.



(a) Show that the frequency of oscillation of the physical pendulum is

$$\omega = \sqrt{\frac{g\,l}{k^2 + l^2}},$$

where k is called the radius of gyration and is given by

$$k = \sqrt{\frac{I_0}{M}}.$$

(b) Determine the radius of gyration of a rod of length ℓ.

5. *Chasles theorem:* According to the theorem, it is always possible to represent an arbitrary displacement of a rigid body by a translation of its center of mass plus a rotation around its center of mass. Establish the theorem working with a simple example of two bodies connected by a massless rigid rod. [KK, pp. 280–282]

Last updated on September 21, 2017

Exercise sheet 6 with solutions

System of particles

1. (a) *Dependence of angular momentum on the origin:* Show that if the total linear momentum of a system of particles is zero, the angular momentum of the system is the same around all origins.

   Solution: The angular momentum of a collection of N particles is given by

$$\boldsymbol{L} = \sum_{\alpha=1}^{N} m_\alpha\, (\boldsymbol{r}_\alpha \times \boldsymbol{v}_\alpha)\,,$$

   where m_α, $\boldsymbol{r}_\alpha$ and $\boldsymbol{v}_\alpha = \dot{\boldsymbol{r}}_\alpha$ denote the mass, position and velocity of the α-th particle, respectively.

   Let us change the origin by a fixed vector, say, $\boldsymbol{R}$, so that $\boldsymbol{r}_\alpha \to \boldsymbol{r}'_\alpha = \boldsymbol{r}_\alpha + \boldsymbol{R}$. Since $\boldsymbol{R}$ is a constant, the velocities $\boldsymbol{v}_\alpha$ of the particles remain unaffected, i.e. $\boldsymbol{v}_\alpha = \boldsymbol{v}'_\alpha$. Therefore, the angular momentum changes to

$$\begin{aligned}\boldsymbol{L} \to \boldsymbol{L}' &= \sum_{\alpha=1}^{N} m_\alpha\, \left(\boldsymbol{r}'_\alpha \times \boldsymbol{v}'_\alpha\right) = \sum_{\alpha=1}^{N} m_\alpha\, [(\boldsymbol{r}_\alpha + \boldsymbol{R}) \times \boldsymbol{v}_\alpha] \\ &= \sum_{\alpha=1}^{N} m_\alpha\, (\boldsymbol{r}_\alpha \times \boldsymbol{v}_\alpha) + \boldsymbol{R} \times \sum_{\alpha=1}^{N} m_\alpha\, \boldsymbol{v}_\alpha = \boldsymbol{L} + \boldsymbol{R} \times \boldsymbol{P},\end{aligned}$$

   where

$$\boldsymbol{P} = \sum_{\alpha=1}^{N} m_\alpha\, \boldsymbol{v}_\alpha$$

   is the total linear momentum of the system. Clearly, $\boldsymbol{L} = \boldsymbol{L}'$, when $\boldsymbol{P} = 0$, which is the required result.

   (b) *Torque and location of the origin:* Show that if the total force on a system of particles is zero, the torque on the system is the same around all origins. [KK, Problem 7.1]

   Solution: The torque on a system of particles is given by

$$\boldsymbol{T} = \sum_{\alpha=1}^{N} \boldsymbol{r}_\alpha \times \boldsymbol{F}_\alpha,$$

   where $\boldsymbol{F}_\alpha$ is the force on the α-th particle.

   If we change the origin by a fixed vector, say, $\boldsymbol{R}$, so that $\boldsymbol{r}_\alpha \to \boldsymbol{r}'_\alpha = \boldsymbol{r}_\alpha + \boldsymbol{R}$, the torque is modified to

$$\boldsymbol{T} \to \boldsymbol{T}' = \sum_{\alpha=1}^{N} (\boldsymbol{r}_\alpha + \mathrm{R}) \times \boldsymbol{F}_\alpha = \sum_{\alpha=1}^{N} \boldsymbol{r}_\alpha \times \boldsymbol{F}_\alpha + \boldsymbol{R} \times \sum_{\alpha=1}^{N} \boldsymbol{F}_\alpha = \boldsymbol{T} + \boldsymbol{R} \times \boldsymbol{F},$$

   where

$$\boldsymbol{F} = \sum_{\alpha=1}^{N} \boldsymbol{F}_\alpha,$$

   is the total force on the system. Clearly, $\boldsymbol{T} = \boldsymbol{T}'$, when $\boldsymbol{F} = 0$, which is the required result.

2. *Moment of inertia of a sphere:* Find the moment of inertia of a uniform sphere of mass M and radius R around an axis through the center. [KK, Problem 7.8]

<u>*Solution:*</u> Let us choose the axis about which the moment of inertia is to be evaluated to be the z-axis. Consider a differential strip of thickness $\mathrm{d}r$ and $\mathrm{d}\phi$ at radius r and angle θ. In such a case, the moment of inertia of the strip is given by

$$\mathrm{d}I = \rho\,(r^2 - r^2\cos^2\theta)\,r^2\,\mathrm{d}r\sin\theta\,\mathrm{d}\phi,$$

where $r^2\,\mathrm{d}r\sin\theta\,\mathrm{d}\phi$ is the differential volume and ρ is the density of the sphere. Upon integration, the moment of inertia of the complete sphere is given by

$$\begin{aligned} I &= \rho\int_0^R \mathrm{d}r\,r^4 \int_0^\pi \mathrm{d}\theta\sin^3\theta \int_0^{2\pi}\mathrm{d}\phi = \frac{2\,\pi\,\rho}{4}\int_0^R \mathrm{d}r\,r^4\int_0^\pi \mathrm{d}\theta\,\left[3\sin\theta - \sin(3\,\theta)\right] \\ &= \frac{\pi\,\rho}{2}\int_0^R \mathrm{d}r\,r^4\left[-3\cos\theta + \frac{\cos(3\,\theta)}{3}\right]_0^\pi = \frac{\pi\,\rho}{2}\,\frac{16}{3}\,\frac{R^5}{5} = \frac{8\,\pi\,\rho\,R^5}{15} = \frac{2\,M\,R^2}{5}, \end{aligned}$$

where, in arriving at the final expression, we have made use of the fact that the mass of the sphere is $M = (4/3)\,\pi\,R^3\,\rho$.

Actually, we need to derive this in the Cartesian coordinates. In the Cartesian coordinates, we have

$$\begin{aligned} I &= \rho\int_{-R}^R \mathrm{d}x\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\mathrm{d}y\int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}}\mathrm{d}z\,(x^2+y^2) \\ &= 2\,\rho\int_{-R}^R \mathrm{d}x\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\mathrm{d}y\,(x^2+y^2)\,\sqrt{R^2-x^2-y^2} \\ &= 4\,\rho\int_{-R}^R \mathrm{d}x\int_0^{\sqrt{R^2-x^2}}\mathrm{d}y\,(x^2+y^2)\,\sqrt{R^2-x^2-y^2} \end{aligned}$$

and, if we set,

$$y = \sqrt{R^2-x^2}\,\sin\theta,$$

we have

$$\mathrm{d}y = \sqrt{R^2-x^2}\,\cos\theta\,\mathrm{d}\theta,$$

so that

$$\begin{aligned} I &= 4\,\rho\int_{-R}^R \mathrm{d}x\,(R^2-x^2)\int_0^{\pi/2}\mathrm{d}\theta\cos^2\theta\,\left(R^2\sin^2\theta + x^2\cos^2\theta\right) \\ &= 4\,\rho\int_{-R}^R \mathrm{d}x\,(R^2-x^2)\left[R^2\int_0^{\pi/2}\mathrm{d}\theta\cos^2\theta + (x^2-R^2)\int_0^{\pi/2}\mathrm{d}\theta\cos^4\theta\right] \\ &= 4\,\rho\int_{-R}^R \mathrm{d}x\,(R^2-x^2)\left\{\frac{R^2}{2}\int_0^{\pi/2}\mathrm{d}\theta\,\left[1+\cos(2\,\theta)\right] + \frac{(x^2-R^2)}{4}\int_0^{\pi/2}\mathrm{d}\theta\,\left[1+\cos(2\,\theta)\right]^2\right\} \\ &= 4\,\rho\int_{-R}^R \mathrm{d}x\,(R^2-x^2)\left\{\frac{R^2}{2}\,\frac{\pi}{2} + \frac{(x^2-R^2)}{4}\int_0^{\pi/2}\mathrm{d}\theta\,\left[1+2\cos(2\,\theta)+\cos^2(2\,\theta)\right]\right\} \\ &= 4\,\rho\int_{-R}^R \mathrm{d}x\,(R^2-x^2)\left\{\frac{R^2}{2}\,\frac{\pi}{2} + \frac{(x^2-R^2)}{4}\int_0^{\pi/2}\mathrm{d}\theta\,\left[1+2\cos(2\,\theta)+\frac{1}{2}+\frac{1}{2}\cos(4\,\theta)\right]\right\} \\ &= \frac{\pi\,\rho}{4}\int_{-R}^R \mathrm{d}x\,(R^2-x^2)\,\left(3\,x^2+R^2\right) = \frac{\pi\,\rho}{4}\int_{-R}^R \mathrm{d}x\,\left(2\,R^2\,x^2 - 3\,x^4 + R^4\right) \\ &= \frac{\pi\,\rho}{2}\left(\frac{2\,R^5}{3} - \frac{3\,R^5}{5} + R^5\right) = \frac{8\,\pi\,\rho\,R^5}{15} = \frac{2\,M\,R^2}{5}, \end{aligned}$$

where M is the mass of the sphere.

3. *Angular momenta of a two body system:* Show that in the center of mass frame, the angular momentum $\boldsymbol{L}_1$ of particle 1 is related to the total angular momentum L by $\boldsymbol{L}_1 = (m_2/M)\,\boldsymbol{L}$ and likewise $\boldsymbol{L}_2 = (m_1/M)\,\boldsymbol{L}$, where $M = m_1 + m_2$. Since $\boldsymbol{L}$ is conserved, this shows that the same is true of $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ separately in the centre of mass frame. [JRT, Problem 8.6]

   Solution: Let $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ be the position vectors of the two masses m_1 and m_2, with respect to the origin. The location of the center of mass of the system is given by

$$\boldsymbol{R} = \frac{m_1\,\boldsymbol{r}_1 + m_2\,\boldsymbol{r}_2}{m_1 + m_2}.$$

   In the center of mass frame, the position vectors of the two masses are, evidently, $\boldsymbol{r}_1 - \boldsymbol{R}$ and $\boldsymbol{r}_2 - \boldsymbol{R}$. Also, the corresponding velocities are $\dot{\boldsymbol{r}}_1 - \dot{\boldsymbol{R}}$ and $\dot{\boldsymbol{r}}_2 - \dot{\boldsymbol{R}}$. Therefore, the total angular momentum of the system in the centre of mass frame is given by

$$\boldsymbol{L} = \boldsymbol{L}_1 + \boldsymbol{L}_2 = m_1\,(\boldsymbol{r}_1 - \boldsymbol{R}) \times \left(\dot{\boldsymbol{r}}_1 - \dot{\boldsymbol{R}}\right) + m_2\,(\boldsymbol{r}_2 - \boldsymbol{R}) \times \left(\dot{\boldsymbol{r}}_2 - \dot{\boldsymbol{R}}\right).$$

   If we use the above expression for $\boldsymbol{R}$, we obtain that

$$\begin{aligned}\boldsymbol{L}_1 &= \frac{m_1\,m_2^2}{M^2}\,(\boldsymbol{r}_1 - \boldsymbol{r}_2) \times (\dot{\boldsymbol{r}}_1 - \dot{\boldsymbol{r}}_2)\,,\\ \boldsymbol{L}_2 &= \frac{m_1^2\,m_2}{M^2}\,(\boldsymbol{r}_1 - \boldsymbol{r}_2) \times (\dot{\boldsymbol{r}}_1 - \dot{\boldsymbol{r}}_2)\,,\end{aligned}$$

   and, hence,

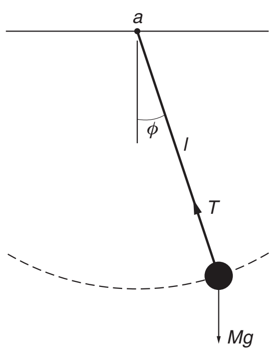
$$\boldsymbol{L} = \left(\frac{m_1\,m_2^2}{M^2} + \frac{m_1^2\,m_2}{M^2}\right)(\boldsymbol{r}_1 - \boldsymbol{r}_2) \times (\dot{\boldsymbol{r}}_1 - \dot{\boldsymbol{r}}_2) = \frac{m_1\,m_2}{M}\,(\boldsymbol{r}_1 - \boldsymbol{r}_2) \times (\dot{\boldsymbol{r}}_1 - \dot{\boldsymbol{r}}_2)\,.$$

   Clearly, we have

$$\boldsymbol{L}_1 = \frac{m_2}{M}\,\boldsymbol{L}, \quad \boldsymbol{L}_2 = \frac{m_1}{M}\,\boldsymbol{L},$$

   as required.

4. *The physical pendulum:* Recall that, in the case of the simple pendulum shown in the figure below, the moment of inertia of the mass is $I_a = M\,l^2$, while the torque on the bob (for small oscillations) is $\tau_a = -M\,g\,l\sin\phi \simeq -M\,g\,l\,\phi$. [KK, Section 7.7]



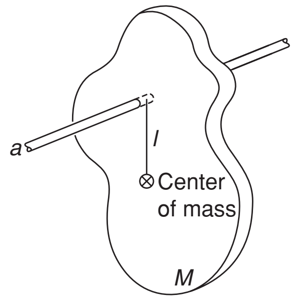
   The equation of motion governing the system is

$$I_a\,\ddot{\phi} = \tau_a \simeq -M\,g\,l\,\phi,$$

   which implies that the frequency of the pendulum is

$$\omega = \sqrt{\frac{M\,g\,l}{I_a}} = \sqrt{\frac{g}{l}}.$$

   Now, consider a physical pendulum as shown in the figure below. Let the moment of inertia of the pendulum about the centre of mass be I_0 and let the center of mass of the system be at a distance l from the pivot point.



(a) Show that the frequency of oscillation of the physical pendulum is

$$\omega = \sqrt{\frac{g\,l}{k^2 + l^2}},$$

where k is called the radius of gyration and is given by

$$k = \sqrt{\frac{I_0}{M}}.$$

<u>Solution:</u> If I_0 is the moment of inertia of the system about the centre of mass, then the moment of inertia about the pivot point is $I = I_0 + M\,l^2$. If the torque on the system is $M\,g\,l\,\phi$ (for small angles), then the equation of motion of the system is

$$(I_0 + M\,l^2)\,\ddot{\phi} + M\,g\,l\,\phi = 0,$$

which implies that the frequency of oscillations is

$$\omega = \sqrt{\frac{M\,g\,l}{I_0 + M\,l^2}}.$$

This can be written as

$$\omega = \sqrt{\frac{g\,l}{(I_0/M) + l^2}} = \sqrt{\frac{g\,l}{k^2 + l^2}},$$

where we have set $k^2 = I_0/M$, which is the required result.

(b) Determine the radius of gyration of a rod of length ℓ.

<u>Solution:</u> Consider a rod of length L, placed along the x-axis with its one end at the origin. The moment of inertia of the rod about a point d from the origin (on the x-axis) is given by

$$I = \lambda \int_0^{\ell} \mathrm{d}x\; (d - x)^2 = \lambda \int_0^{\ell} \mathrm{d}x\; \left(d^2 - 2\,d\,x + x^2\right) = \lambda\, \left(d^2\,\ell - d\,\ell^2 + \frac{\ell^3}{3}\right),$$

where λ is the mass per unit length of the rod. We can write if $M = \lambda\,\ell$ is the mass of the rod,

$$I = M\,\left(d^2 - d\,\ell + \frac{\ell^2}{3}\right)$$

Note that, for a rod of uniform mass per unit length, the centre of mass of the rod is located at $d = \ell/2$. In such a case, we have

$$I_0 = \frac{M\,\ell^2}{12},$$

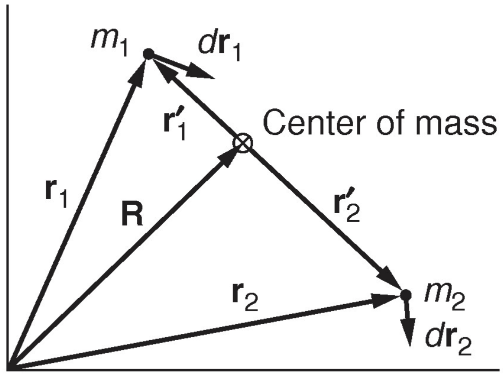
so that the radius of gyration is

$$k = \frac{\ell}{\sqrt{12}}.$$

5. *Chasles theorem:* According to the theorem, it is always possible to represent an arbitrary displacement of a rigid body by a translation of its center of mass plus a rotation around its center of mass. Establish the theorem working with a simple example of two bodies connected by a massless rigid rod. [KK, pp. 280–282]

   Solution: Consider two masses m_1 and m_2 connected by a rigid rod of length l. Let the position vectors of the two masses be $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, and let the position vector of the center of mass of the system be $\boldsymbol{R}$ so that (see accompanying figure)

$$\boldsymbol{R} = \frac{m_1\,\boldsymbol{r}_1 + m_2\,\boldsymbol{r}_2}{m_1 + m_2}.$$



Since the length of the rod is fixed we have

$$(\boldsymbol{r}_1 - \boldsymbol{r}_2)^2 = (\boldsymbol{r}_1 - \boldsymbol{r}_2)\cdot(\boldsymbol{r}_1 - \boldsymbol{r}_2) = l^2,$$

and, hence, upon taking the differential, we obtain

$$(\boldsymbol{r}_1 - \boldsymbol{r}_2)\cdot(\mathrm{d}\boldsymbol{r}_1 - \mathrm{d}\boldsymbol{r}_2) = 0.$$

Note that $\mathrm{d}\boldsymbol{r}_1$ and $\mathrm{d}\boldsymbol{r}_2$ can be considered to be the displacements of the two masses. There are two ways of satisfying the above condition. We can have either

$$\mathrm{d}\boldsymbol{r}_1 = \mathrm{d}\boldsymbol{r}_2,$$

i.e. the two masses are being displaced together or the motion occurs in such a fashion that $(\boldsymbol{r}_1 - \boldsymbol{r}_2)$ is perpendicular to $(\mathrm{d}\boldsymbol{r}_1 - \mathrm{d}\boldsymbol{r}_2)$

Let us consider the first of the above two possibilities for displacement of the two masses. The displacement of the center of mass of the system is given by

$$\mathrm{d}\boldsymbol{R} = \frac{m_1\,\mathrm{d}\boldsymbol{r}_1 + m_2\,\mathrm{d}\boldsymbol{r}_2}{m_1 + m_2}.$$

Let us define

$$\boldsymbol{r}_1' = \boldsymbol{r}_1 - \boldsymbol{R} = \frac{m_2}{M}\,(\boldsymbol{r}_1 - \boldsymbol{r}_2)\,, \quad \boldsymbol{r}_2' = \boldsymbol{r}_2 - \boldsymbol{R} = -\frac{m_1}{M}\,(\boldsymbol{r}_1 - \boldsymbol{r}_2)\,,$$

so that

$$\mathrm{d}\boldsymbol{r}_1' = \mathrm{d}\boldsymbol{r}_1 - \mathrm{d}\boldsymbol{R} = \frac{m_2}{M}\,(\mathrm{d}\boldsymbol{r}_1 - \mathrm{d}\boldsymbol{r}_2)\,, \quad \mathrm{d}\boldsymbol{r}_2' = \mathrm{d}\boldsymbol{r}_2 - \mathrm{d}\boldsymbol{R} = -\frac{m_1}{M}\,(\mathrm{d}\boldsymbol{r}_1 - \mathrm{d}\boldsymbol{r}_2)$$

where $M = m_1 + m_2$. If $\mathrm{d}\boldsymbol{r}_1 = \mathrm{d}\boldsymbol{r}_2$, then $\mathrm{d}\boldsymbol{r}_1' = \mathrm{d}\boldsymbol{r}_2' = 0$, which implies translation of the rigid body without any rotation.

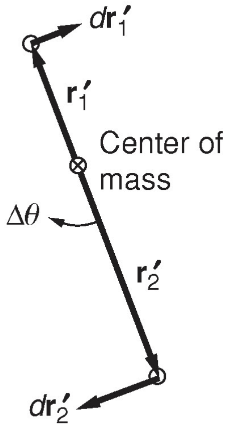
Let us now turn to the second possibility. In such a case, as we mentioned, $(\boldsymbol{r}_1 - \boldsymbol{r}_2)$ is perpendicular to $(\mathrm{d}\boldsymbol{r}_1 - \mathrm{d}\boldsymbol{r}_2)$. Note that $\boldsymbol{r}_1'$ and $\boldsymbol{r}_2'$ denote the position vectors of the two masses about the centre of mass and $\mathrm{d}\boldsymbol{r}_1'$ and $\mathrm{d}\boldsymbol{r}_2'$ represent the corresponding displacements. We have

$$\mathrm{d}\boldsymbol{r}_1' \cdot (\boldsymbol{r}_1' - \boldsymbol{r}_2') = \frac{m_2}{M}\,(\boldsymbol{r}_1 - \boldsymbol{r}_2) \cdot (\mathrm{d}\boldsymbol{r}_1 - \mathrm{d}\boldsymbol{r}_2)\,,$$

which vanishes according to the condition for the rigid rod. Similarly, we have

$$\mathrm{d}\boldsymbol{r}_2' \cdot (\boldsymbol{r}_1' - \boldsymbol{r}_2') = -\frac{m_1}{M}\,(\mathrm{d}\boldsymbol{r}_1 - \mathrm{d}\boldsymbol{r}_2) \cdot (\boldsymbol{r}_1 - \boldsymbol{r}_2),$$

which too vanishes. These two conditions clearly imply that the rod rotates (as illustrated in the accompanying figure).



We also need to establish that the two ends of the rod rotate by the same angle. From the geometry, it should be clear that, we need to show

$$\frac{\mathrm{d}\boldsymbol{r}_1'}{r_1'} = -\frac{\mathrm{d}\boldsymbol{r}_2'}{r_2'}.$$

Since $r_1'/r_2' = m_2/m_1$, we have

$$\frac{\mathrm{d}\boldsymbol{r}_1'}{r_1'} = \frac{m_2}{M}\,\frac{(\mathrm{d}\boldsymbol{r}_1 - \mathrm{d}\boldsymbol{r}_2)}{r_1'} = \frac{m_2}{M}\,\frac{(\mathrm{d}\boldsymbol{r}_1 - \mathrm{d}\boldsymbol{r}_2)}{(m_2/m_1)\,r_2'} = \frac{m_1}{M}\,\frac{(\mathrm{d}\boldsymbol{r}_1 - \mathrm{d}\boldsymbol{r}_2)}{r_2'} = -\frac{\mathrm{d}\boldsymbol{r}_2'}{r_2'},$$

as required.

Exercise sheet 7

Motion in central fields

1. *Trajectory in a repulsive potential:* Consider a particle that is moving under the influence of the repulsive central potential $U(r) = \alpha/r$, where $\alpha > 0$.

   Note: Such a potential can describe, for instance, α-particles being scattered by a nucleus, as in the famous Rutherford scattering experiment.

   (a) What is the allowed energy range for the particle? Can you guess the type of orbital motion that is expected to arise?

   (b) Solve the orbital equation for the problem.

   (c) Obtain the solutions for the trajectory as a function of time.

   (d) Draw the trajectory in, say, x-y plane. How does it compare with the Keplerian case wherein $\alpha < 0$ and energy of the particle is positive?

2. *Possible stable circular orbits:* Consider a particle moving in the potential $U(r) = -A/r^n$, where $A > 0$. What are the values of n which admit stable circular orbits? [KK, Problem 10.4]

3. *Hooke's law in two dimensions:* Consider two particles interacting by the Hooke's law potential energy $U = k\,r^2/2$, where r is their relative position $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$, and subject to no external forces. Show that $\boldsymbol{r}(t)$ describes an ellipse. Hence, show that both particles move on similar ellipses around their common centre of mass. [JRT, Problem 8.11]

4. *Precessing orbits:* A particle of mass m and angular momentum L moves in the field of a central force given by
$$F(r) = -\frac{k}{r^2} + \frac{\lambda}{r^3},$$
   where k and λ are positive. [JRT, Problem 8.23]

   (a) Obtain the corresponding equation of motion governing the orbits in the force field.

   (b) Show that the solution to the orbital equation can be expressed as
$$r(\phi) = \frac{c}{1 + \epsilon\cos(\beta\,\phi)},$$
   where c, β and ϵ are positive constants.

   (c) Find c, β and ϵ in terms of the given parameters, and describe the orbit for the case that $0 < \epsilon < 1$.

   (d) For what values of β is the orbit closed? What happens to the results as $\lambda \to 0$?

5. *Virial theorem for generic bounded orbits:* Earlier, we had established the virial theorem for the case of circular orbits in the central potential $U(r) = k\,r^n$. A more general form of the theorem that applies to any periodic orbit of a particle can be arrived at as follows. [JRT, Problem 8.17]

   (a) Find the time derivative of the quantity $G = \boldsymbol{r}\cdot\boldsymbol{p}$ and, by integrating the quantity from time 0 to t, show that
$$\frac{G(t) - G(0)}{t} = 2\,\langle T\rangle + \langle \boldsymbol{F}\cdot\boldsymbol{r}\rangle,$$
   where $\boldsymbol{F}$ is the net force on the particle, T its kinetic energy and $\langle f\rangle$ denotes the average over time of any quantity f.

   (b) Explain why, if the particle's orbit is periodic and if we make t sufficiently large, we can make the left hand side of the above equation as small as desired. In other words, in such cases, the left side vanishes as t tends to infinity.

(c) Use this result to show that, if $\boldsymbol{F}$ arises due to the potential energy $U = k\, r^n$, then

$$\langle T \rangle = \frac{n\, \langle U \rangle}{2},$$

if $\langle f \rangle$ denotes the time average over a sufficiently long time, larger than the period associated with the bounded trajectory.

(d) What happens when $k > 0$ and $n = 2$?

Note: We had explicitly established this result earlier while studying harmonic oscillators.

Exercise sheet 7 with solutions

Motion in central fields

1. *Trajectory in a repulsive potential:* Consider a particle that is moving under the influence of the repulsive central potential $U(r) = \alpha/r$, where $\alpha > 0$.

   Note: Such a potential can describe, for instance, α-particles being scattered by a nucleus, as in the famous Rutherford scattering experiment.

   (a) What is the allowed energy range for the particle? Can you guess the type of orbital motion that is expected to arise?

   Solution: Since the potential is positive and so is the centrifugal barrier, the effective potential of the system is always positive. Hence, the allowed range of energy is $0 < E < \infty$.

   (b) Solve the orbital equation for the problem.

   Solution: Recall that
$$E = \frac{\mu}{2}\,\dot{r}^2 + \frac{\mu}{2}\,r^2\,\dot{\phi}^2 + U(r),$$
   and $L = \mu\,r^2\,\dot{\phi}$, so that
$$E = \frac{\mu\,\dot{r}^2}{2} + U_{\mathrm{eff}}(r) = \frac{\mu\,\dot{r}^2}{2} + U(r) + \frac{L^2}{2\,\mu\,r^2}$$
   or
$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2\dot{\phi}^2 = \left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2\left(\frac{L}{\mu\,r^2}\right)^2 = \left(\frac{1}{r^2}\,\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2\frac{L^2}{\mu^2} = \frac{2}{\mu}\,\left[E - U_{\mathrm{eff}}(r)\right],$$
   and hence
$$\phi - \phi_1 = \int_{\phi_1}^{\phi}\mathrm{d}\phi = -\int_{x_1}^{x}\frac{\sqrt{L^2/(2\,\mu)}\,\mathrm{d}x}{\sqrt{E - U_{\mathrm{eff}}(x)}},$$
   where $x = 1/r$. We have
$$\begin{aligned}\phi - \phi_1 &= -\int_{x_1}^{x}\frac{\sqrt{L^2/(2\,\mu)}\,\mathrm{d}x}{\sqrt{E - \alpha\,x - L^2\,x^2/(2\,\mu)}} = -\int_{x_1}^{x}\frac{L\,\mathrm{d}x}{\sqrt{2\,\mu\,E - 2\,\mu\,\alpha\,x - L^2\,x^2}}\\ &= -\int_{x_1}^{x}\frac{L\,\mathrm{d}x}{\sqrt{2\,\mu\,E + (\mu\,\alpha/L)^2 - [(\mu\,\alpha/L) + L\,x]^2}}\end{aligned}$$
   and, if we set,
$$\mathcal{A}^2 = 2\,\mu\,E + (\mu\,\alpha/L)^2, \quad y = (\mu\,\alpha/L) + L\,x,$$
   we obtain that
$$\phi - \phi_1 = -\int_{y_1}^{y}\frac{\mathrm{d}y}{\sqrt{\mathcal{A}^2 - y^2}},$$
   which can be integrated to arrive at
$$\phi - \phi_1 = -\sin^{-1}(y/\mathcal{A})$$
   or
$$y = -\mathcal{A}\sin(\phi - \phi_1),$$
   so that for $y_1 = 0$, we have
$$\frac{\mu\,\alpha}{L} + \frac{L}{r} = -\sqrt{2\,\mu\,E + \left(\frac{\mu\,\alpha}{L}\right)^2}\,\sin(\phi - \phi_1),$$
   or
$$\frac{1}{r(\phi)} = \frac{\mu\,\alpha}{L^2}\,\left[-1 - \epsilon\sin(\phi - \phi_1)\right],$$

where

$$\epsilon = \sqrt{1 + \frac{2\,E\,L^2}{\mu\,\alpha^2}}.$$

For $\phi_1 = \pi/2$, the above solution reduces to

$$r(\phi) = \frac{r_0}{\epsilon\cos\phi - 1}$$

where

$$r_0 = \frac{L^2}{\mu\,\alpha}$$

and, it is important to note that $E > 0$ in this case, which implies that $\epsilon > 1$. In other words, only hyperbolic trajectories are possible in this case.

(c) Obtain the solutions for the trajectory as a function of time.

<u>*Solution:*</u> Note that, in terms of time, we have

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \frac{2}{\mu}\,\left[E - U_{\mathrm{eff}}(r)\right] = \frac{2}{\mu}\,\left(E - \frac{\alpha}{r} - \frac{L^2}{2\,\mu\,r^2}\right),$$

which can rewritten as

$$t - t_1 = \int_{t_1}^{t} \mathrm{d}t = \int_{r_1}^{r} \frac{\sqrt{\mu/2}\,\mathrm{d}r}{\sqrt{E - (\alpha/r) - (L^2/2\,\mu\,r^2)}} = \int_{r_1}^{r} \frac{\sqrt{2\,\mu\,r^2}\sqrt{\mu/2}\,\mathrm{d}r}{\sqrt{2\,\mu\,E\,r^2 - 2\,\mu\,\alpha\,r - L^2}}$$

and, since,

$$E = \frac{\mu\,\alpha^2}{2\,L^2}\,\left(\epsilon^2 - 1\right),$$

we can write

$$t - t_0 = \int_{r_0}^{r} \frac{\mu\,r\,\mathrm{d}r}{\sqrt{(\mu\,\alpha\,r/L)^2\,(\epsilon^2 - 1) - 2\,\mu\,\alpha\,r - L^2}} = \int_{r_0}^{r} \frac{\sqrt{b}\,\mu\,r\,\mathrm{d}r/\sqrt{\mu\,\alpha}}{\sqrt{r^2 - 2\,r\,b - b^2\,(\epsilon^2 - 1)}}$$

where we have set

$$a\,(\epsilon^2 - 1) = \frac{L^2}{\mu\,\alpha}.$$

We have

$$t - t_1 = \int_{r_0}^{r} \frac{\sqrt{a\,\mu/\alpha}\,r\,\mathrm{d}r}{\sqrt{(r - a)^2 - (a\,\epsilon)^2}}$$

and, if we set,

$$r - a = a\,\epsilon\cosh\xi,$$

we obtain that

$$t - t_1 = \sqrt{\frac{\mu\,a^3}{\alpha}}\,\int_{\xi_1}^{\xi} \mathrm{d}\xi\,(1 + \epsilon\cosh\xi) = \sqrt{\frac{\mu\,a^3}{\alpha}}\,\left(\xi + \epsilon\sinh\xi\right),$$

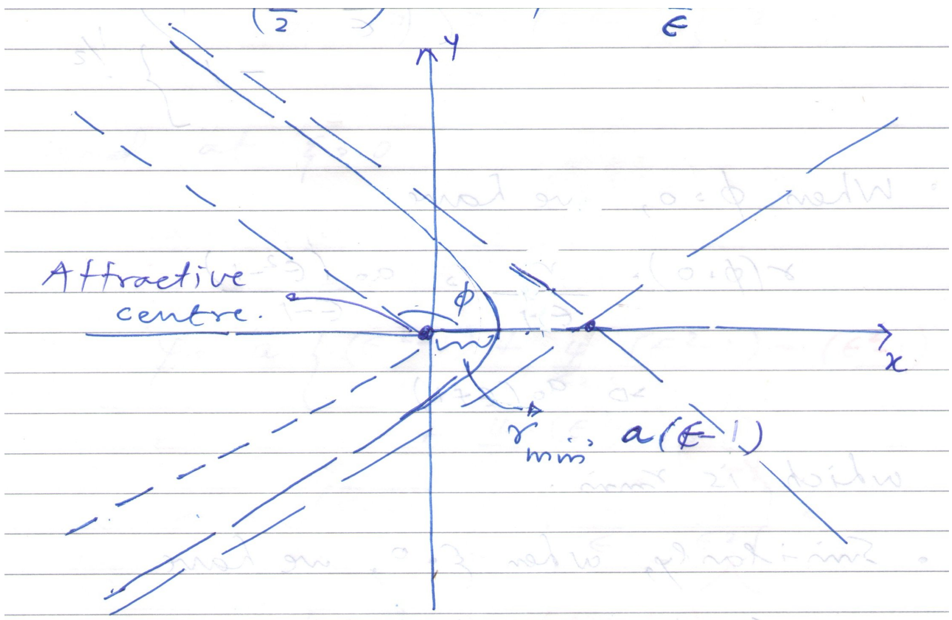
where we have set $\xi_1 = 0$. If we choose $t_0 = 1$, then we have

$$r = a\,\left(1 + \epsilon\cosh\xi\right), \quad t = \sqrt{\frac{\mu\,a^3}{\alpha}}\,\left(\xi + \epsilon\sinh\xi\right),$$

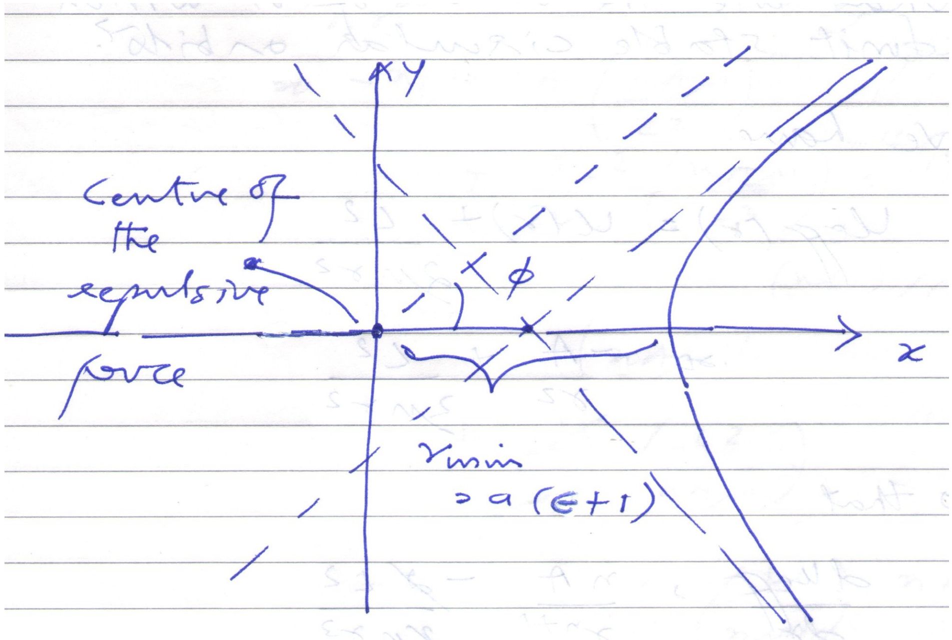
which is the required result.

(d) Draw the trajectory in, say, x-y plane. How does it compare with the Keplerian case wherein $\alpha < 0$ and energy of the particle is positive?

<u>*Solution:*</u> The trajectory in the attractive case will be as shown in the figure below.



Whereas, in the repulsive case, the trajectory will be as shown below.



2. *Possible stable circular orbits:* Consider a particle moving in the potential $U(r) = -A/r^n$, where $A > 0$. What are the values of n which admit stable circular orbits? [KK, Problem 10.4]

Solution: The effective potential of the system is given by

$$U_{\text{eff}}(r) = -\frac{A}{r^n} + \frac{L^2}{2\,\mu\,r^2},$$

so that

$$\frac{\mathrm{d}U_{\text{eff}}}{\mathrm{d}r} = \frac{n\,A}{r^{n+1}} - \frac{L^2}{\mu\,r^3}$$

and

$$\frac{\mathrm{d}^2U_{\text{eff}}}{\mathrm{d}r^2} = -\frac{n\,(n+1)\,A}{r^{n+2}} + \frac{3\,L^2}{\mu\,r^4}.$$

Clearly, $\mathrm{d}U_{\text{eff}}/\mathrm{d}r = 0$ when

$$r_* = \left(\frac{n\,\mu\,A}{L^2}\right)^{1/(n-2)}$$

and, since, $A > 0$, r_* is real and positive only when $n > 0$. In such cases, we have

$$\begin{aligned}
\left(\frac{\mathrm{d}^2 U_{\mathrm{eff}}}{\mathrm{d}r^2}\right)_{r=r_*} &= -n\,(n+1)\,A\,\left(\frac{L^2}{n\,\mu\,A}\right)^{(n+2)/(n-2)} + \frac{3\,L^2}{\mu}\,\left(\frac{L^2}{n\,\mu\,A}\right)^{4/(n-2)}\\
&= -n\,(n+1)\,A\,\left(\frac{L^2}{n\,\mu\,A}\right)^{(n+2)/(n-2)} + 3\,n\,A\,\left(\frac{L^2}{n\,\mu\,A}\right)\,\left(\frac{L^2}{n\,\mu\,A}\right)^{4/(n-2)}\\
&= A\,\left(\frac{L^2}{n\,\mu\,A}\right)^{(n+2)/(n-2)}\,\left[-n\,(n+1) + 3\,n\right]\\
&= A\,\left(\frac{L^2}{n\,\mu\,A}\right)^{(n+2)/(n-2)}\,(2-n)\,n,
\end{aligned}$$

and, since we have already demanded n to be positive (so that r_* is positive and real), the quantity $\mathrm{d}^2 U_{\mathrm{eff}}/\mathrm{d}r^2$ will be positive (i.e. a minimum and hence a stable point) only if $n < 2$.

3. *Hooke's law in two dimensions:* Consider two particles interacting by the Hooke's law potential energy $U = k\,r^2/2$, where r is their relative position $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$, and subject to no external forces. Show that $\boldsymbol{r}(t)$ describes an ellipse. Hence, show that both particles move on similar ellipses around their common centre of mass. [JRT, Problem 8.11]

   Solution: As earlier, we have

$$\begin{aligned}
t - t_1 &= \int_{t_1}^{t} \mathrm{d}t = \int_{r_1}^{r} \frac{\sqrt{\mu/2}\,\mathrm{d}r}{\sqrt{E - U_{\mathrm{eff}}(r)}} = \int_{r_1}^{r} \frac{\sqrt{\mu/2}\,\mathrm{d}r}{\sqrt{E - (k\,r^2/2) - [L^2/(2\,\mu\,r^2)]}}\\
&= \int_{r_1}^{r} \frac{\sqrt{2\,\mu\,r^2}\,\sqrt{\mu/2}\,\mathrm{d}r}{\sqrt{2\,\mu\,E\,r^2 - \mu\,k\,r^4 - L^2]}} = \int_{x_1}^{x} \frac{(\mu/2)\,\mathrm{d}x}{\sqrt{2\,\mu\,E\,x - \mu\,k\;x^2 - L^2]}},
\end{aligned}$$

where we have now set $x = r^2$. We obtain that

$$\begin{aligned}
t - t_1 &= \int_{x_1}^{x} \frac{(\mu/2)\,\mathrm{d}x/\sqrt{\mu\,k}}{\sqrt{(2\,E\,x/k) - x^2 - [L^2/(\mu\,k)]}} = \int_{x_1}^{x} \frac{\sqrt{\mu/k}\,\mathrm{d}x/2}{\sqrt{(E^2/k^2) - [L^2/(\mu\,k)] - [x-(E/k)]^2}}\\
&= \int_{y_1}^{y} \frac{\sqrt{\mu/k}\,\mathrm{d}y/2}{\sqrt{\mathcal{A}^2 - y^2}} = \frac{\sqrt{\mu/k}}{2}\,\left[\sin^{-1}(y/\mathcal{A})\right]_{y_1}^{y},
\end{aligned}$$

where we have set

$$y = x - (E/k), \quad \mathcal{A}^2 = \frac{E^2}{k^2} - \frac{L^2}{\mu\,k}.$$

Therefore, we obtain that

$$y = x - \frac{E}{k} = r^2 - \frac{E}{k} = \mathcal{A}\sin\left[\sqrt{(4\,k)/\mu}\,(t-t_1)\right]$$

or

$$\begin{aligned}
r^2(t) &= \frac{E}{k} + \sqrt{\frac{E^2}{k^2} - \frac{L^2}{\mu\,k}}\;\sin\left[\sqrt{(4\,k)/\mu}\,(t-t_1)\right]\\
&= \frac{E}{k}\left\{1 + \sqrt{1 - \frac{L^2\,k}{\mu\,E^2}}\;\sin\left[2\,\omega\,(t-t_1)\right]\right\},
\end{aligned}$$

where $\omega = \sqrt{k/\mu}$, which is the required solution. Note that we can write

$$r^2(t) = \frac{E}{k} + \mathcal{A}'\cos(2\,\omega\,t) + \mathcal{B}'\sin(2\,\omega\,t),$$

where

$$\mathcal{A}' = -\sqrt{\frac{E^2}{k^2} - \frac{L^2}{\mu\, k}}\,\sin{(2\,\omega\, t_1)}, \quad \mathcal{B}' = \sqrt{\frac{E^2}{k^2} - \frac{L^2}{\mu\, k}}\,\cos{(2\,\omega\, t_1)},$$

so that

$$\mathcal{A}'^2 + \mathcal{B}'^2 = \frac{E^2}{k^2} - \frac{L^2}{\mu\, k}.$$

We now need to establish that this, in general, describes, an ellipse. In general, the solution along the x and y-directions on the plane can be written as

$$x(t) = x_0 \cos{(\omega\, t)} + \frac{v_{x0}}{\omega} \sin{(\omega\, t)}, \quad y(t) = y_0 \cos{(\omega\, t)} + \frac{v_{y0}}{\omega} \sin{(\omega\, t)},$$

where (x_0, y_0) and (v_{x0}, v_{y0}) denote the initial conditions on the positions and velocities along the two directions. Note that the energy E and angular momentum L can be written as

$$\begin{aligned} E &= \frac{\mu}{2}\,\left(v_{x0}^2 + v_{y0}^2\right) + \frac{k}{2}\,\left(x_0^2 + y_0^2\right), \\ L &= \mu\,\left(x_0\, v_{y0} - y_0\, v_{x0}\right). \end{aligned}$$

In terms of these quantities, one finds that

$$r^2(t) = x^2(t) + y^2(t) = \frac{E}{k} + \mathcal{A} \cos{(2\,\omega\, t)} + \mathcal{B} \sin{(2\,\omega\, t)},$$

where

$$\mathcal{A} = \frac{1}{2}\,\left[(x_0^2 + y_0^2) - \frac{1}{\omega^2}\,\left(v_{x0}^2 + v_{y0}^2\right)\right], \quad \mathcal{B} = \frac{1}{\omega}\,\left(x_0\, v_{x0} + y_0\, v_{y0}\right),$$

and one can show that

$$\mathcal{A}'^2 + \mathcal{B}'^2 = \mathcal{A}^2 + \mathcal{B}^2,$$

as required.

Now, the solutions

$$x(t) = x_0 \cos{(\omega\, t)} + \frac{v_{x0}}{\omega} \sin{(\omega\, t)}, \quad y(t) = y_0 \cos{(\omega\, t)} + \frac{v_{y0}}{\omega} \sin{(\omega\, t)},$$

describe an ellipse whose axes are, in general, titled with respect to the x-axis at an angle. For instance, if $v_{x0} = v_{y0} = 0$, then $L = 0$ and $y = (y_0/x_0)\, x$, which described a straight line, as is consistent with zero angular momentum. If $v_{x0} = 0$ and $y_0 = 0$, then we have

$$\left(\frac{x}{x_0}\right)^2 + \left(\frac{y}{v_{y0}/\omega}\right)^2 = \cos^2(\omega\, t) + \sin^2(\omega\, t) = 1,$$

which, evidently, describes an ellipse. For any given initial condition, we can reorient our x and y axis such that these conditions are satisfied, which suggests that, the orbit is, in general an ellipse.

4. *Precessing orbits:* A particle of mass m and angular momentum L moves in the field of a central force given by

$$F(r) = -\frac{k}{r^2} + \frac{\lambda}{r^3},$$

where k and λ are positive. [JRT, Problem 8.23]

(a) Obtain the corresponding equation of motion governing the orbits in the force field.

Solution: Since we have

$$E = \frac{\mu}{2}\,\dot{r}^2 + U_{\rm eff}(r) = \frac{\mu}{2}\,\left(\frac{{\rm d}r}{{\rm d}\phi}\right)^2\,\left(\frac{{\rm d}\phi}{{\rm d}t}\right)^2 + U_{\rm eff}(r),$$

and since $L = \mu\, r^2\, \dot{\phi}$, we obtain that

$$E = \frac{L^2}{2\,\mu}\,\left(\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 + U_{\rm eff}(r) = \frac{L^2}{2\,\mu}\,\left(\frac{\mathrm{d}x}{\mathrm{d}\phi}\right)^2 + U_{\rm eff}(x) = \frac{L^2}{2\,\mu}\,\left(\frac{\mathrm{d}x}{\mathrm{d}\phi}\right)^2 + U(x) + \frac{L^2}{2\,\mu}\,x^2,$$

where we have set $x = 1/r$. If we differentiate this equation with respect to ϕ, since E is a constant, we obtain that

$$\frac{\mathrm{d}x}{\mathrm{d}\phi}\,\left[\frac{L^2}{\mu}\,\frac{\mathrm{d}^2x}{\mathrm{d}\phi^2} + \frac{\mathrm{d}U}{\mathrm{d}x} + \frac{L^2}{\mu}\,x\right] = 0.$$

As $\mathrm{d}x/\mathrm{d}\phi$ is not in general zero, we have

$$\frac{\mathrm{d}^2x}{\mathrm{d}\phi^2} + x = -\frac{\mu}{L^2}\,\frac{\mathrm{d}U}{\mathrm{d}x}.$$

(b) Show that the solution to the orbital equation can be expressed as

$$r(\phi) = \frac{c}{1 + \epsilon\cos(\beta\,\phi)},$$

where c, β and ϵ are positive constants.

Solution: The potential corresponding to the above force is given by

$$U(r) = -\frac{k}{r} + \frac{\lambda}{2\,r^2}.$$

Therefore, we have

$$U(x) = -k\,x + \frac{\lambda}{2}\,x^2,$$

so that

$$\frac{\mathrm{d}U}{\mathrm{d}x} = -k + \lambda\,x$$

and hence the equation of motion governing the system can be expressed as

$$\frac{\mathrm{d}^2x}{\mathrm{d}\phi^2} + x = \frac{\mu}{L^2}\,(k - \lambda\,x)$$

or, equivalently,

$$\frac{\mathrm{d}^2x}{\mathrm{d}\phi^2} + \left(1 + \frac{\mu\,\lambda}{L^2}\right)\,x = \frac{\mu\,k}{L^2}.$$

Note that, in the absence of the right hand side, the equation resembles a simple harmonic oscillator equation. Also, the right hand side corresponds to a constant 'force'. It is straightforward to check that the solution can be expressed as

$$x = \frac{1}{r} = \frac{\mu\,k}{\beta^2\,L^2} - \alpha\,\sin\left[\beta\,(\phi - \phi_1)\right],$$

where α is a constant and

$$\beta = \sqrt{1 + \frac{\mu\,\lambda}{L^2}}.$$

If we now choose $\phi_1 = \pi/2$, we obtain that

$$\frac{1}{r} = \frac{\mu\,k}{\beta^2\,L^2} + \alpha\,\frac{2\,\mu\,\mathrm{E}\,\mathrm{c}^2}{\beta^2\,\mathrm{L}^2}\cos(\beta\,\phi),$$

or, equivalently,

$$r = \frac{c}{1 + \epsilon\cos(\beta\,\phi)},$$

where c and ϵ are given by

$$c = \frac{\beta^2\,L^2}{\mu\,k}, \quad \epsilon = \frac{\alpha\,\beta^2\,L^2}{\mu\,k}.$$

(c) Find c, β and ϵ in terms of the given parameters, and describe the orbit for the case that $0 < \epsilon < 1$.

<u>*Solution:*</u> We have already been able to express c in terms of the parameters describing the problem. Note that ϵ depends on the constant α. The quantity α and, hence, ϵ, can be determined by substituting the above solution for x in the following expression for energy we had arrived at earlier

$$E = \frac{L^2}{2\,\mu}\left(\frac{\mathrm{d}x}{\mathrm{d}\phi}\right)^2 + U(x) + \frac{L^2}{2\,\mu}\,x^2.$$

In our case, we have

$$E = \frac{L^2}{2\,\mu}\left(\frac{\mathrm{d}x}{\mathrm{d}\phi}\right)^2 - k\,x + \left(\frac{L^2}{2\,\mu} + \frac{\lambda}{2}\right)x^2 = \frac{L^2}{2\,\mu}\left(\frac{\mathrm{d}x}{\mathrm{d}\phi}\right)^2 - k\,x + \frac{\beta^2\,L^2}{2\,\mu}\,x^2$$

and

$$x = \frac{1}{c}\,[1 + \epsilon\cos(\beta\,\phi)]\,,$$

so that we obtain that

$$\begin{aligned}
E &= \frac{\beta^2\,L^2}{2\,\mu\,c^2}\,\epsilon^2\sin^2(\beta\,\phi) - \frac{k}{c}\,[1 + \epsilon\cos(\beta\,\phi)] + \frac{\beta^2\,L^2}{2\,\mu\,c^2}\,[1 + \epsilon\cos(\beta\,\phi)]^2\,,\\
&= \frac{\beta^2\,L^2}{2\,\mu\,c^2}\,\epsilon^2\sin^2(\beta\,\phi) - \frac{k}{c}\,[1 + \epsilon\cos(\beta\,\phi)] + \frac{\beta^2\,L^2}{2\,\mu\,c^2}\left[1 + 2\,\epsilon\cos(\beta\,\phi) + \epsilon^2\cos^2(\beta\,\phi)\right]\\
&= \frac{\beta^2\,L^2}{2\,\mu\,c^2}\,\epsilon^2 - \frac{\beta^2\,L^2}{\mu\,c^2} - \cancel{\frac{\beta^2\,L^2}{\mu\,c^2}\,\epsilon\cos(\beta\,\phi)} + \frac{\beta^2\,L^2}{2\,\mu\,c^2} + \cancel{\frac{\beta^2\,L^2}{\mu\,c^2}\,\epsilon\cos(\beta\,\phi)}\\
&= \frac{\beta^2\,L^2}{2\,\mu\,c^2}\,\epsilon^2 - \frac{\beta^2\,L^2}{2\,\mu\,c^2}
\end{aligned}$$

or

$$\frac{\beta^2\,L^2}{2\,\mu\,c^2}\,\epsilon^2 = E + \frac{\beta^2\,L^2}{2\,\mu\,c^2}$$

or

$$\epsilon = \sqrt{1 + \frac{2\,\mu\,E\,c^2}{\beta^2\,L^2}} = \sqrt{1 + \frac{2\,E\,\beta^2\,L^2}{\mu\,k^2}}.$$

Note that, when $-\mu\,k^2/(2\,\beta^2\,L^2) < E < 0$, $0 < \epsilon < 1$. For these energies, the orbits correspond to precessing ellipses for a general β.

(d) For what values of β is the orbit closed? What happens to the results as $\lambda \to 0$?

<u>*Solution:*</u> Note that the orbits will be closed whenever β is a rational number. The $\lambda \to 0$ case clearly corresponds to the standard Kepler problem we have discussed earlier.

5. <u>*Virial theorem for generic bounded orbits:*</u> Earlier, we had established the virial theorem for the case of circular orbits in the central potential $U(r) = k\,r^n$. A more general form of the theorem that applies to any periodic orbit of a particle can be arrived at as follows. [JRT, Problem 8.17]

(a) Find the time derivative of the quantity $G = \boldsymbol{r}\cdot\boldsymbol{p}$ and, by integrating the quantity from time 0 to t, show that

$$\frac{G(t) - G(0)}{t} = 2\,\langle T\rangle + \langle \boldsymbol{F}\cdot\boldsymbol{r}\rangle,$$

where $\boldsymbol{F}$ is the net force on the particle, T its kinetic energy and $\langle f\rangle$ denotes the average over time of any quantity f.

<u>*Solution:*</u> Note that

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\cdot\boldsymbol{p} + \boldsymbol{r}\cdot\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = 2\,T + \boldsymbol{r}\cdot\boldsymbol{F},$$

since $T = (\boldsymbol{p} \cdot \boldsymbol{v})/2$ and $\boldsymbol{F} = \mathrm{d}\boldsymbol{p}/\mathrm{d}t$. Upon averaging over a time interval t, we, evidently, obtain

$$\frac{1}{t}\int_0^t \mathrm{d}t\, \frac{\mathrm{d}G}{\mathrm{d}t} = \frac{G(t) - G(0)}{t} = 2\,\langle T\rangle + \langle \boldsymbol{r}\cdot\boldsymbol{F}\rangle.$$

(b) Explain why, if the particle's orbit is periodic and if we make t sufficiently large, we can make the left hand side of the above equation as small as desired. In other words, in such cases, the left side vanishes as t tends to infinity.

Solution: If the orbit is bounded, then $\boldsymbol{r}$ and $\boldsymbol{v}$ are bounded and so will G be. In such a case, if we choose t to be sufficiently large, then the left hand side of the above expression will be arbitrarily small so that we will have

$$2\,\langle T\rangle + \langle \boldsymbol{r}\cdot\boldsymbol{F}\rangle = 0$$

or

$$\langle T\rangle = -\frac{\langle \boldsymbol{r}\cdot\boldsymbol{F}\rangle}{2}.$$

(c) Use this result to show that, if $\boldsymbol{F}$ arises due to the potential energy $U = k\,r^n$, then

$$\langle T\rangle = \frac{n\,\langle U\rangle}{2},$$

if $\langle f\rangle$ denotes the time average over a sufficiently long time, larger than the period associated with the bounded trajectory.

Solution: In such a case,

$$\boldsymbol{F} = -\boldsymbol{\nabla} U = -\frac{\mathrm{d}U}{\mathrm{d}r}\,\hat{\boldsymbol{r}} = -k\,n\,r^{n-1}\,\hat{\boldsymbol{r}} = -k\,n\,r^{n-2}\,\boldsymbol{r}$$

and, hence,

$$\boldsymbol{r}\cdot\boldsymbol{F} = -k\,n\,r^{n-2}\,\boldsymbol{r}\cdot\boldsymbol{r} = -k\,n\,r^n = -n\,U,$$

so that

$$-\langle \boldsymbol{r}\cdot\boldsymbol{F}\rangle = n\,\langle U\rangle.$$

Therefore, the virial theorem reduces to

$$\langle T\rangle = \frac{n}{2}\,\langle U\rangle.$$

(d) What happens when $k > 0$ and $n = 2$?

Solution: When $k > 0$ and $n = 2$, we have

$$\langle T\rangle = \langle U\rangle,$$

which is the result we had discussed earlier in the case of the one-dimensional oscillator.

Note: We had explicitly established this result earlier while studying harmonic oscillators.

Illustrative examples 7

Introductory vector calculus

1. *Properties of the gradient:* Use $\mathrm{d}f = \boldsymbol{\nabla} f \cdot \mathrm{d}\boldsymbol{r}$ to show that [JRT, Problem 4.18]

   (a) The vector $\boldsymbol{\nabla} f$ at any point $\boldsymbol{r}$ is perpendicular to the surface of constant f through $\boldsymbol{r}$.

   (b) The direction of $\boldsymbol{\nabla} f$ at any point $\boldsymbol{r}$ is the direction in which f increases fastest as we move away from $\boldsymbol{r}$.

2. *The gradient in curvilinear coordinates:* Show that the gradient operator in cylindrical polar coordinates is
$$\boldsymbol{\nabla} = \hat{\boldsymbol{\rho}}\frac{\partial}{\partial \rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial \phi} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z} \; .$$
[HMS, p141-143]

3. *Work done by a non-conservative force:* Evaluate the work done by the two dimensional force $\boldsymbol{F} = y\hat{\boldsymbol{x}} + 2x\hat{\boldsymbol{y}}$ going from the origin O to point $P = (1, 1)$ along the paths

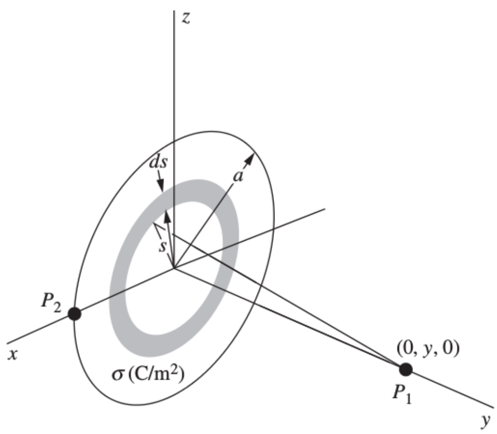
   (a) from O to $Q = (1, 0)$ then from Q straight up to P,

   (b) straight along the line $y = x$ and

   (c) around a quarter circle centered on Q.

   [JRT, Example 4.1]

4. *The potential of a charged disk:* Find the potential at a point P_1 on the axis of symmetry of a uniformly charged disk with charge density σ and radius a. The disk is in the xz plane with its centre at the origin such that the axis of symmetry is aligned with the y axis as shown in the figure. Therefore, $P_1 = (0, y, 0)$. Sketch the potential as a function of y and compare it to that of a point charge at the origin with the same charge as the disk. [PM p68-70]



5. *The centre-of-mass of a hemisphere:* Use spherical polar coordinates r, θ, ϕ to find the centre-of-mass of a uniform density solid hemisphere of radius R, whose flat face lies in the xy plane with its centre at the origin. [JRT Problem 3.22]

Illustrative examples 7 with solutions

Introductory vector calculus

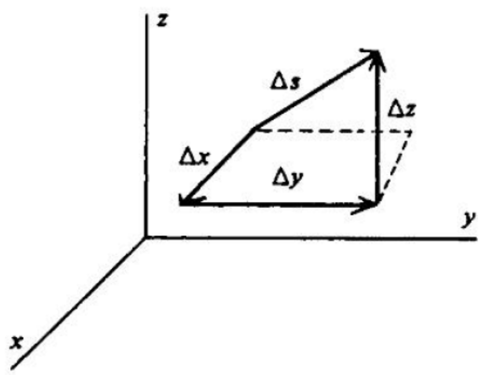
1. <u>*Properties of the gradient:*</u> Use $\mathrm{d}f = \boldsymbol{\nabla} f \cdot \mathrm{d}\boldsymbol{r}$ to show that [JRT, Problem 4.18]

   (a) The vector $\boldsymbol{\nabla} f$ at any point $\boldsymbol{r}$ is perpendicular to the surface of constant f through $\boldsymbol{r}$.
   <u>*Solution:*</u> If f is constant on a given surface, then $\mathrm{d}f = 0$ on the surface. If $\mathrm{d}\boldsymbol{r}$ is an infinitesimal vector on the surface, then $\mathrm{d}f = 0$ implies that $\boldsymbol{\nabla} f \cdot \mathrm{d}\boldsymbol{r} = 0$ or, equivalently, $\boldsymbol{\nabla} f$ is perpendicular to $\mathrm{d}\boldsymbol{r}$.

   (b) The direction of $\boldsymbol{\nabla} f$ at any point $\boldsymbol{r}$ is the direction in which f increases fastest as we move away from $\boldsymbol{r}$.
   <u>*Solution:*</u> In general
   $$df = \boldsymbol{\nabla} f \cdot d\boldsymbol{r}$$
   where $d\boldsymbol{r} \equiv \epsilon \hat{\boldsymbol{u}}$ where ϵ is fixed and small. Therefore, we can write
   $$df = |\boldsymbol{\nabla} f| \epsilon \cos\theta \ ,$$
   where θ is the angle between $\hat{\boldsymbol{u}}$ and $\boldsymbol{\nabla} f$. Evidently df is a maximum when $\theta = 0$, which is equivalent to the displacement in the direction $\boldsymbol{\nabla} f$ as required.

2. <u>*The gradient in curvilinear coordinates:*</u> Show that the gradient operator in cylindrical polar coordinates is
   $$\boldsymbol{\nabla} = \hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z} \ .$$
   [HMS, p141-143]

   <u>*Solution:*</u> First recall in Cartesian coordinates that for a function $f(x, y, z)$
   $$\Delta f = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z = \left(\hat{\boldsymbol{x}}\frac{\partial f}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial f}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial f}{\partial z}\right) \cdot \Delta\boldsymbol{s} \ ,$$
   where $\Delta\boldsymbol{s} = \Delta x\hat{\boldsymbol{x}} + \Delta y\hat{\boldsymbol{y}} + \Delta z\hat{\boldsymbol{z}} = \Delta s\hat{\boldsymbol{u}}$ (see figure below).



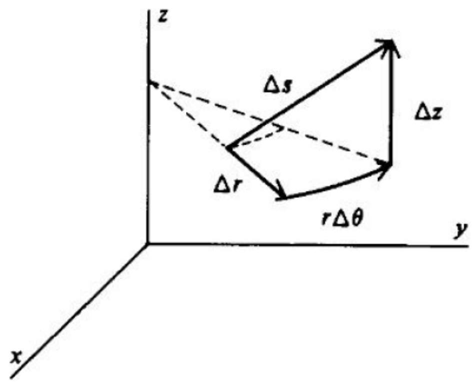
   Then taking the limit of $\Delta s \to 0$
   $$\lim_{\Delta s \to 0}\frac{\Delta f}{\Delta s} = \frac{df}{ds} = \left(\hat{\boldsymbol{x}}\frac{\partial f}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial f}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial f}{\partial z}\right) \cdot \hat{\boldsymbol{u}} \equiv \boldsymbol{\nabla} f \cdot \hat{\boldsymbol{u}} \ .$$

   So to find the gradient in cylindrical polar coordinates we use the same analysis for a function $f(\rho, \phi, z)$. So we write
   $$\Delta f = \frac{\partial f}{\partial\rho}\Delta\rho + \frac{\partial f}{\partial\phi}\Delta\phi + \frac{\partial f}{\partial z}\Delta z$$

and

$$\Delta \boldsymbol{s} = \Delta\rho\hat{\boldsymbol{\rho}} + (\rho + \Delta\rho)\Delta\phi\hat{\boldsymbol{\phi}} + \Delta z\hat{\boldsymbol{z}} = \Delta\rho\hat{\boldsymbol{\rho}} + \rho\Delta\phi\hat{\boldsymbol{\phi}} + \Delta z\hat{\boldsymbol{z}} \ ,$$

where we drop the term $\Delta\rho\,\Delta\phi$ and approximate the arc to a straight line (chord) in the limit of $\Delta\rho \to 0$ and $\Delta\phi \to 0$. (See figure below where $\rho \to r$ and $\phi \to \theta$.)



Therefore, we can write

$$\begin{aligned}
\Delta f &= \left(\hat{\boldsymbol{\rho}}\frac{\partial f}{\partial \rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial f}{\partial \phi} + \hat{\boldsymbol{z}}\frac{\partial f}{\partial z}\right)\cdot \Delta\boldsymbol{s} \\
\Rightarrow \lim_{\Delta s\to 0}\frac{\Delta f}{\Delta s} &= \frac{df}{ds} = \left(\hat{\boldsymbol{\rho}}\frac{\partial f}{\partial \rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial f}{\partial \phi} + \hat{\boldsymbol{z}}\frac{\partial f}{\partial z}\right)\cdot \hat{\boldsymbol{u}} \equiv \boldsymbol{\nabla} f\cdot\hat{\boldsymbol{u}} \ ,
\end{aligned}$$

where, $\boldsymbol{\nabla}$ has the required form.

Alternatively, we can use the chain rule and the definitions of the unit vectors to explicitly move from the cartesian to the cylindrical coordinates. To begin with we use the chain rule to write the derivatives in terms of ρ and ϕ

$$\begin{aligned}
\boldsymbol{\nabla} &= \hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z} \\
&= \hat{\boldsymbol{x}}\left(\frac{\partial\rho}{\partial x}\frac{\partial}{\partial\rho} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi}\right) + \hat{\boldsymbol{y}}\left(\frac{\partial\rho}{\partial y}\frac{\partial}{\partial\rho} + \frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi}\right) + \hat{\boldsymbol{z}}\frac{\partial}{\partial z} \ .
\end{aligned}$$

Using the definitions $\rho = \sqrt{x^2+y^2}$ and $\phi = \tan^{-1}\frac{y}{x}$ we have

$$\begin{aligned}
\frac{\partial\rho}{\partial x} &= \frac{x}{\sqrt{x^2+y^2}} = \cos\phi \\
\frac{\partial\rho}{\partial y} &= \frac{y}{\sqrt{x^2+y^2}} = \sin\phi \\
\frac{\partial\phi}{\partial x} &= -\frac{y}{x^2}\frac{1}{1+\frac{y^2}{x^2}} = -\frac{y}{x^2+y^2} = -\frac{\sin\phi}{\rho} \\
\frac{\partial\phi}{\partial y} &= \frac{1}{x}\frac{1}{1+\frac{y^2}{x^2}} = \frac{x}{x^2+y^2} = \frac{\cos\phi}{\rho} \ .
\end{aligned}$$

We insert the above into our expression for the gradient

$$\begin{aligned}
\boldsymbol{\nabla} &= (\cos\phi\hat{\boldsymbol{x}} + \sin\phi\hat{\boldsymbol{y}})\frac{\partial}{\partial\rho} + (-\sin\phi\hat{\boldsymbol{x}} + \cos\phi\hat{\boldsymbol{y}})\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z} \\
&= \hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z} \ ,
\end{aligned}$$

where the definitions of $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ are used in the last step.

3. *Work done by a non-conservative force:* Evaluate the work done by the two dimensional force $\boldsymbol{F} = y\hat{\boldsymbol{x}} + 2x\hat{\boldsymbol{y}}$ going from the origin O to point $P = (1, 1)$ along the paths

   (a) from O to $Q = (1, 0)$ then from Q straight up to P,

   (b) straight along the line $y = x$ and

   (c) around a quarter circle centered on Q.

   [JRT, Example 4.1]

   Solution:

   (a) Work done is $W = \int \boldsymbol{F}\cdot d\boldsymbol{r}$, where $d\boldsymbol{r}$ is the infinitesimal line element *i.e.* $d\boldsymbol{r} = dx\hat{\boldsymbol{x}} + dy\hat{\boldsymbol{y}} + dz\hat{\boldsymbol{z}}$ in Cartesian coordinates. Therefore,

   $$dW = y\,dx + 2x\,dy\ ,$$

   for the force given in the problem. For the first path we split it into two parts OQ and QP such that

   $$W = \int_{OQ} y\,dx + \int_{QP} 2x\,dy\ ,$$

   where we have used the fact that $dy = 0$ and $dx = 0$ along OQ and OP, respectively. Now $y = 0$ along OQ so the first term is zero and $x = 1$ along OP so we have:

   $$W = \int_0^1 2dy = 2\ .$$

   (b) For the straight path OP $y = x$. Therefore, we can use the relation to eliminate x or y from the integrand to perform the integral with respect to just one coordinate. Taking $dy = dx$ and $y = x$ we have

   $$W = \int_0^1 (x\,dx + 2x\,dx) = 3\int_0^1 x\,dx = \frac{3}{2}\ .$$

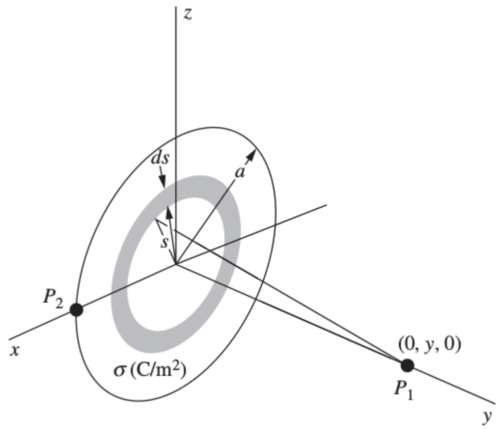
   (c) For the circular path OP we can write $(x, y) = (1 - \cos\theta, \sin\theta)$, where $0 \le \theta \le \frac{\pi}{2}$. So we first have to rewrite $d\boldsymbol{r}$ in terms of θ

   $$d\boldsymbol{r} = \frac{dx}{d\theta}d\theta\hat{\boldsymbol{x}} + \frac{dy}{d\theta}d\theta\hat{\boldsymbol{y}} = \sin\theta d\theta\hat{\boldsymbol{x}} + \cos\theta d\theta\hat{\boldsymbol{y}}\ .$$

   This leads to the work done being

   $$\begin{aligned} W &= \int_0^{\pi/2} \left[\sin^2\theta + 2\,(1 - \cos\theta)\cos\theta\right]\,d\theta \\ &= \int_0^{\pi/2} \left(2\cos\theta - \frac{3}{2}\cos 2\theta - \frac{1}{2}\right)\,d\theta \\ &= \left[2\sin\theta - \frac{3}{4}\sin 2\theta - \frac{\theta}{2}\right]_0^{\pi/2} \\ &= 2 - \frac{\pi}{4} \approx 1.21\ . \end{aligned}$$

4. *The potential of a charged disk:* Find the potential at a point P_1 on the axis of symmetry of a uniformly charged disk with charge density σ and radius a. The disk is in the xz plane with its centre at the origin such that the axis of symmetry is aligned with the y axis as shown in the figure. Therefore, $P_1 = (0, y, 0)$. Sketch the potential as a function of y and compare it to that of a point charge at the origin with the same charge as the disk. [PM p68-70]



Solution: An infinitesimal region of area dA on the surface of the disk a distance ρ (s in figure) from the origin will contribute an amount

$$dV = \frac{\sigma \, dA}{4\pi\epsilon_0 \sqrt{\rho^2 + y^2}}$$

to the potential at P_1. In plane polar coordinates in the xz plane $dA = \rho \, d\rho \, d\phi$. Therefore, the potential is given by

$$\begin{aligned} V &= \frac{\sigma}{4\pi\epsilon_0} \int_0^a \int_0^{2\pi} \frac{\rho}{\sqrt{\rho^2 + y^2}} d\rho \, d\phi \\ &= \frac{\sigma}{2\epsilon_0} \left[\sqrt{\rho^2 + y^2} \right]_0^a \\ &= \frac{\sigma}{2\epsilon_0} \left[\sqrt{y^2 + a^2} - y \right] \end{aligned}$$

To extend to negative y it is clear the whole system is symmetric w.r.t. a reflection in the xz plane. Therefore, to ensure $V(-y) = V(y)$ for $y < 0$, we take $y \to -y$ in the solution for positive y, which leads to

$$V = \frac{\sigma}{2\epsilon_0} \left[\sqrt{y^2 + a^2} + y \right] \ (y < 0) \ .$$

(This is equivalent to the opposite choice of sign of $\sqrt{y^2}$ in the solution for positive y. Only one option makes physical sense for positive and negative y.)

To sketch the potential it is best to consider what happens as $y \to 0$ and $y \to \infty$. In the former case, we can rewrite the solution as

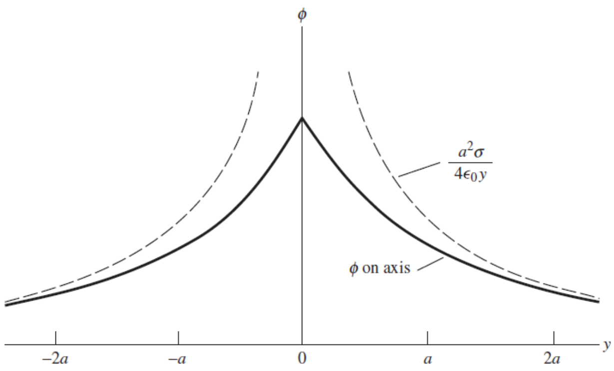
$$V = \frac{\sigma \, a}{2\epsilon_0} \left[\sqrt{1 + \left(\frac{y}{a}\right)^2} - \frac{y}{a} \right] \approx \frac{\sigma \, a}{2\epsilon_0} \left[1 - \frac{y}{a} \right] \ ,$$

where we have kept only the leading order in y in the last expression. There are two things to note: (1) that when $y \ll a$ the expression leads to a uniform $\boldsymbol{E}$ with a magnitude $\sigma/2\epsilon_0$ as found for an infinite charged sheet with uniform density; and (2) the potential is not differentiable at $y = 0$ because the solution for $y < 0$, $y \approx \frac{\sigma \, a}{2\epsilon_0} \left[1 + \frac{y}{a}\right]$, has a first derivate of opposite sign to that for positive y.

As for $y \gg a$ we have

$$V = \frac{\sigma \, a}{2\epsilon_0} \left[\frac{y}{a} \sqrt{\left(\frac{a}{y}\right)^2 + 1} - \frac{y}{a} \right] \approx \frac{\sigma \, \pi a^2}{4\pi\epsilon_0 y} \ .$$

This is the expression for the potential of a point charge at the origin with a total charge $\sigma \, \pi a^2$, which is the total charge of the disk. Finally with these inputs it is straightforward to sketch the potential, where $\phi = V$ below.



5. <u>*The centre-of-mass of a hemisphere:*</u> Use spherical polar coordinates r, θ, ϕ to find the centre-of-mass of a uniform density solid hemisphere of radius R, whose flat face lies in the xy plane with its centre at the origin. [JRT Problem 3.22]

   <u>*Solution:*</u> The general expression for the centre-of-mass is

$$\boldsymbol{R}_{\text{CoM}} = \frac{\int \boldsymbol{r}\rho(\boldsymbol{r})\, dV}{\int \rho(\boldsymbol{r})\, dV} \ ,$$

   where ρ is the density. So for a uniform density hemisphere radius R with mass $M = \frac{2}{3}\rho\pi R^3$ this gives

$$\boldsymbol{R}_{\text{CoM}} = \frac{\rho}{M}\int \boldsymbol{r}\, dV = \frac{3}{2\pi R^3}\int_0^R \int_0^{\frac{\pi}{2}} \int_0^{2\pi} [\sin\theta\cos\phi\hat{\boldsymbol{x}} + \sin\theta\sin\phi\hat{\boldsymbol{y}} + \cos\theta\hat{\boldsymbol{z}}]\, r^3 \sin\theta\, dr\, d\theta d\phi \ ,$$

   where we have used the expressions for $\boldsymbol{r}$ and dV in spherical polar coordinates. Performing the ϕ integral removes the terms dependent on $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ because $\int_0^{2\pi}\cos\phi\, d\phi = \int_0^{2\pi}\sin\phi\, d\phi = 0$. Also performing the r integral we have

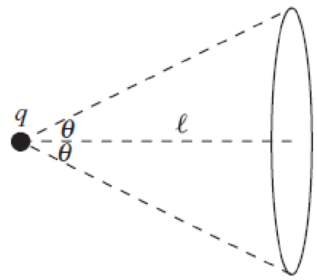
$$\boldsymbol{R}_{\text{CoM}} = \frac{3R}{4}\left[\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta d\theta\right]\hat{\boldsymbol{z}} = \frac{3R}{8}\hat{\boldsymbol{z}} \ .$$

Illustrative examples 8

Divergence, Gauss Law and continuity

1. *The divergence in spherical polar coordinates:* Show that the divergence in spherical polar coordinates is
$$\boldsymbol{\nabla} \cdot \boldsymbol{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta F_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi} ,$$
where $\boldsymbol{F} = F_r\left(r,\theta,\phi\right)\hat{\boldsymbol{r}} + F_\theta\left(r,\theta,\phi\right)\hat{\boldsymbol{\theta}} + F_\phi\left(r,\theta,\phi\right)\hat{\boldsymbol{\phi}}$. [HMS p41-43]

2. *Flux through a circle:* A point charge q is located at the origin. Consider the electric field flux through a circle a distance l from q, subtending an angle 2θ, as shown in the figure.



Since there are no charges except at the origin, any surface that is bounded by the circle and that stays to the right of the origin must contain the same flux. (Why?) Calculate this flux by taking the surface to be:

   (a) the flat disk bounded by the circle;
   (b) the spherical cap (with the sphere centered at the origin) bounded by the circle.

[PM Problem 1.15]

3. *Journey to the centre of the Earth:*

   (a) Use Gauss' law for gravitation
$$\boldsymbol{\nabla} \cdot \boldsymbol{g} = -4\pi G\rho ,$$
   where ρ is the density, to find the gravitational field $\boldsymbol{g}$ both inside and outside the Earth. Assume the Earth's density is uniform.
   (b) Sketch $|\boldsymbol{g}|$ as a function of r.
   (c) Ignoring the various logistical problems, an elevator shaft is drilled through the centre of the earth. An elevator is released from the surface. How long will it take to return?

4. *Electric field of a charged sheet and slab*

   (a) Use Gauss' law and symmetry to find the electrostatic field as a function of position for an infinite uniform plane of charge. Let the charge lie in the yz-plane and denote the charge per unit area by σ.
   (b) Repeat part (a) for an infinite slab of charge parallel to the yz-plane whose charge density is given by
$$\rho(x) = \begin{cases} \rho_0 & \text{if } -b < x < b \ , \\ 0 & \text{if } |x| \geq b \ , \end{cases}$$
   where ρ_0 and b are constants.
   (c) Repeat part (b) with $\rho\left(x\right) = \rho_0 e^{-|x/b|}$. [HMS Problem II-11]

5. *The equation of continuity in an fluid:* In the spatial description, the equation of continuity for a fluid is

$$\frac{d\rho}{dt} + \rho \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0 \ ,$$

where $\rho(\boldsymbol{r}, t)$ and $\boldsymbol{v}(\boldsymbol{r}, t)$ are the density and velocity, respectively. Show that the continuity equation can be equivalently written as

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0 \ .$$

[JRT Problem 16.34]

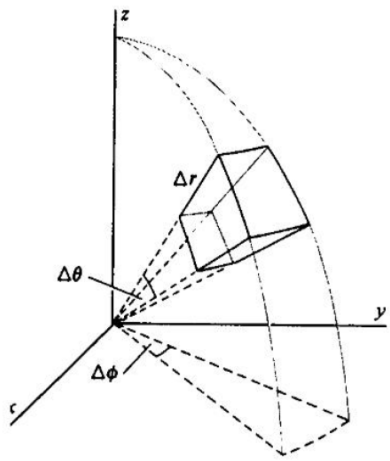
Last updated on October 1, 2017

Illustrative examples 8 solutions

Divergence, Gauss Law and continuity

1. *The divergence in spherical polar coordinates:* Show that the divergence in spherical polar coordinates is
$$\boldsymbol{\nabla}\cdot\boldsymbol{F}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2F_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta F_\theta\right)+\frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}\ ,$$
where $\boldsymbol{F}=F_r\left(r,\theta,\phi\right)\hat{\boldsymbol{r}}+F_\theta\left(r,\theta,\phi\right)\hat{\boldsymbol{\theta}}+F_\phi\left(r,\theta,\phi\right)\hat{\boldsymbol{\phi}}$. [HMS p41-43]

Solution: The divergence is defined as
$$\boldsymbol{\nabla}\cdot\boldsymbol{F}=\lim_{\Delta V\to 0}\frac{1}{\Delta V}\iint_S \boldsymbol{F}\cdot\hat{\boldsymbol{n}}\,dS$$
where ΔV is an infinitesimal volume about $\boldsymbol{r}$ with surface S. In spherical polar coordinates we consider the volume shown in the figure below, which has faces corresponding to $\hat{\boldsymbol{n}}=\pm\hat{\boldsymbol{r}}$, $\pm\hat{\boldsymbol{\theta}}$ and $\pm\hat{\boldsymbol{\phi}}$. We consider the contributions of the pairs of faces with opposite normals to calculate the divergence.

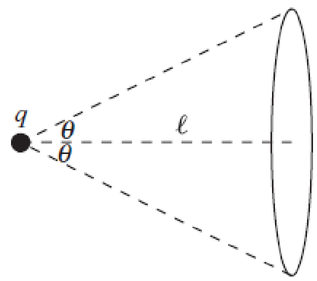


We will start by evaluating the flux through the two faces S_1 and S_2 with $\hat{\boldsymbol{n}}=\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{n}}=-\hat{\boldsymbol{r}}$, respectively. For S_1 we have
$$\begin{aligned}\iint_{S_1}\boldsymbol{F}\cdot\hat{\boldsymbol{n}}\,dS &= \iint_{S_1}F_r\,dS\\ &= F_r\left(r+\tfrac{\Delta r}{2},\theta,\phi\right)(r+\tfrac{\Delta r}{2})^2\sin\theta\Delta\theta\Delta\phi\ ,\end{aligned}$$
because the area of S_1 is $\left[\left(r+\frac{\Delta r}{2}\right)\Delta\theta\right]\times\left[\left(r+\frac{\Delta r}{2}\right)\sin\theta\Delta\phi\right]$ when we approximate the arcs to chords for infinitesimal side lengths. Similarly for S_2 we have
$$\begin{aligned}\iint_{S_2}\boldsymbol{F}\cdot\hat{\boldsymbol{n}}\,dS &= -\iint_{S_2}F_r\,dS\\ &= -F_r\left(r-\tfrac{\Delta r}{2},\theta,\phi\right)(r-\tfrac{\Delta r}{2})^2\sin\theta\Delta\theta\Delta\phi\ .\end{aligned}$$

Next we can calculate the contribution to the divergence from the sum of these two sides
$$\begin{aligned}&\lim_{\Delta V\to 0}\frac{1}{\Delta V}\iint_{S_1,S_2}\boldsymbol{F}\cdot\hat{\boldsymbol{n}}\,dS\\ =\ &\lim_{\Delta V\to 0}\frac{1}{r^2\sin\theta\Delta r\Delta\theta\Delta\phi}\left[F_r\left(r+\tfrac{\Delta r}{2},\theta,\phi\right)(r+\tfrac{\Delta r}{2})^2-F_r\left(r-\tfrac{\Delta r}{2},\theta,\phi\right)(r-\tfrac{\Delta r}{2})^2\right]\sin\theta\Delta\theta\Delta\phi\\ =\ &\frac{1}{r^2}\lim_{\Delta r\to 0}\frac{F_r\left(r+\frac{\Delta r}{2},\theta,\phi\right)(r+\frac{\Delta r}{2})^2-F_r\left(r-\frac{\Delta r}{2},\theta,\phi\right)(r-\frac{\Delta r}{2})^2}{\Delta r}\\ =\ &\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2F_r\right)\ .\end{aligned}$$

One can perform similar analyses for the pairs of sides with $\hat{\boldsymbol{n}} = \pm\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{n}} = \pm\hat{\boldsymbol{\phi}}$ to find the other two contributions that lead to the required result.

2. *Flux through a circle:* A point charge q is located at the origin. Consider the electric field flux through a circle a distance l from q, subtending an angle 2θ, as shown in the figure.



Since there are no charges except at the origin, any surface that is bounded by the circle and that stays to the right of the origin must contain the same flux. (Why?) Calculate this flux by taking the surface to be:

(a) the flat disk bounded by the circle;
Solution We will work in spherical polar coordinates (r, θ', ϕ) with the charge at the origin and the z axis along the line to the centre of the circle. (The prime is introduced as θ is a fixed angle in the problem.) Therefore, the electric field is

$$\boldsymbol{E} = \frac{q}{4\pi\epsilon_0\, r^2}\hat{\boldsymbol{r}}$$

The flux through the disk will be given by

$$\Phi = \iint_{\text{disk}} \boldsymbol{E}\cdot\hat{\boldsymbol{z}}\, dA = \frac{q}{4\pi\epsilon_0}\iint_{\text{disk}} \frac{\cos\theta' dA}{r^2} \; .$$

Now $dA = \rho\, d\rho\, d\phi = l^2 \tan\theta' \sec^2\theta' d\theta' d\phi$, where $\rho = l\tan\theta'$ is the distance from the centre of the disk. Also, $r^2 = l^2 + \rho^2 = l^2\left(1+\tan^2\theta'\right) = l^2\sec^2\theta'$. So we can write

$$\Phi = \frac{q}{4\pi\epsilon_0}\int_0^{\theta}\int_0^{2\pi} \sin\theta'\, d\theta' d\phi = \frac{q}{2\epsilon_0}\left(1-\cos\theta\right) \; .$$

(b) the spherical cap (with the sphere centered at the origin) bounded by the circle.
Solution: For the spherical cap we have

$$\Phi = \iint_{\text{cap}} \boldsymbol{E}\cdot\hat{\boldsymbol{r}}\, dA = \frac{q}{4\pi\epsilon_0}\iint_{\text{cap}} \frac{dA}{r^2} = \frac{q}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\theta} \frac{r^2\sin\theta'}{r^2}\, d\theta' d\phi = \frac{q}{2\epsilon_0}\left(1-\cos\theta\right) \; ,$$

where we have used $dA = r^2\sin\theta' d\theta' d\phi$ on a spherical surface.

[PM Problem 1.15]

3. *Journey to the centre of the Earth:*

(a) Use Gauss' law for gravitation

$$\boldsymbol{\nabla}\cdot\boldsymbol{g} = -4\pi\, G\rho \; ,$$

where ρ is the density, to find the gravitational field $\boldsymbol{g}$ both inside and outside the Earth. Assume the Earth's density is uniform.

Solution: Consider a spherical Gaussian surface radius r with its centre corresponding to the centre of the earth We write Gauss' Law in integral form for volume enclosed by this spherical surface

$$\begin{aligned}\iiint \boldsymbol{\nabla}\cdot\boldsymbol{g}\,dV &= -4\pi\,G\iiint \rho\,dV\\ \Rightarrow \iint \boldsymbol{g}\cdot\hat{\boldsymbol{r}}\,dA &= -4\pi\,G\iiint \rho\,dV\\ \Rightarrow g(r)4\pi r^2 &= -4\pi\,G\iiint \rho\,dV\end{aligned}$$

where in the last step we have used the spherical symmetry such that $\boldsymbol{g} = g(r)\hat{\boldsymbol{r}}$. When $r < R_\oplus$, and assuming a uniform density $\rho = M_\oplus/(\frac{4}{3}\pi R_\oplus^3)$ we have

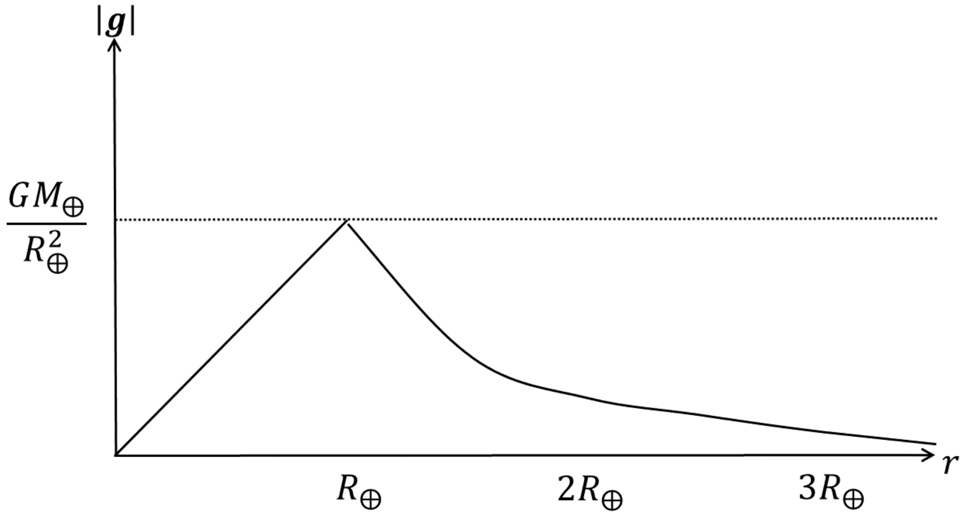
$$g(r)r^2 = -\frac{3GM_\oplus}{4\pi R_\oplus^3}\frac{4}{3}\pi r^3 \Rightarrow \boldsymbol{g} = -\frac{GM_\oplus}{R_\oplus^3}r\,\hat{\boldsymbol{r}}\ .$$

Then for $r \geq R_\oplus$

$$g(r)r^2 = -\frac{3GM_\oplus}{4\pi R_\oplus^3}\frac{4}{3}\pi R_\oplus^3 \Rightarrow \boldsymbol{g} = -\frac{GM_\oplus}{r^2}\,\hat{\boldsymbol{r}}\ .$$

(b) Sketch $|\boldsymbol{g}|$ as a function of r.

Solution:



(c) Ignoring the various logistical problems, an elevator shaft is drilled through the centre of the earth. An elevator is released from the surface. How long will it take to return?

Solution: The force $m\boldsymbol{g}$ is Hookean, so ignoring air resistance, the elevator will perform simple harmonic motion through the shaft. The force constant is $k = GmM_\oplus/R_\oplus^3$ so the period is

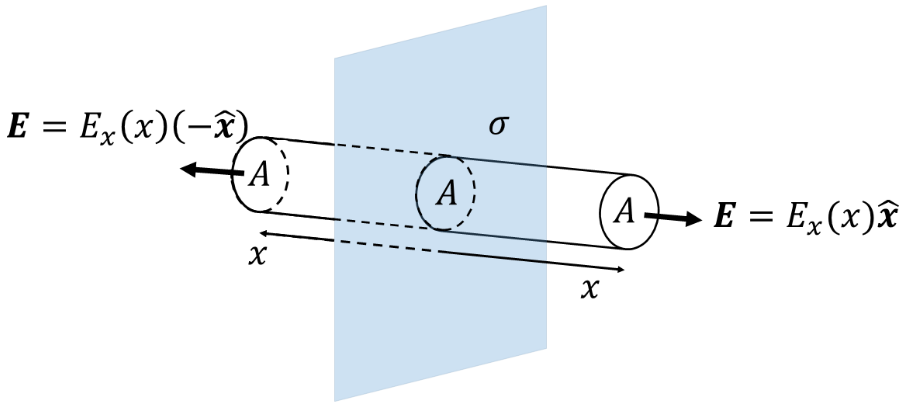
$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{R_\oplus^3}{GM_\oplus}} = 85 \text{ mins}\ ,$$

where $G = 6.67\times10^{-11}$ m^{3}kg^{-1}s^{-2}, $R_\oplus = 6380$ km and $M_\oplus = 5.97\times10^{24}$ kg.

4. *Electric field of a charged sheet and slab*

   (a) Use Gauss' law and symmetry to find the electrostatic field as a function of position for an infinite uniform plane of charge. Let the charge lie in the yz-plane and denote the charge per unit area by σ.

   Solution: Firstly, because the system is invariant in translations in the yz-plane it means that the electric field can only be in the x direction. So one draws a Gaussian pillbox about the surface extending a distance x on either size. (See figure below.)



   Then Gauss' Law in integral form gives

$$\iint_{\text{pillbox}} \boldsymbol{E}\cdot\hat{\boldsymbol{n}} = 2E_x(x)\,A = \frac{Q_{\text{enc}}}{\epsilon_0}\,.$$

   Here, $Q_{\text{enc}} = \sigma A$ so

$$\boldsymbol{E} = \begin{cases} \frac{\sigma}{2\epsilon_0}\hat{\boldsymbol{x}} & \text{if } x > 0\,, \\ -\frac{\sigma}{2\epsilon_0}\hat{\boldsymbol{x}} & \text{if } x < 0\,. \end{cases}$$

   (b) Repeat part (a) for an infinite slab of charge parallel to the yz-plane whose charge density is given by

$$\rho(x) = \begin{cases} \rho_0 & \text{if } -b < x < b\,, \\ 0 & \text{if } |x| \geq b\,, \end{cases}$$

   where ρ_0 and b are constants.

   Solution: Again the translational symmetry allows us to select Gaussian pill box such that

$$\iint_{\text{pillbox}} \boldsymbol{E}\cdot\hat{\boldsymbol{n}} = 2E_x(x)\,A = \frac{Q_{\text{enc}}}{\epsilon_0}\,,$$

   However, in this case the charge enclosed will depend on whether $x < b$ or $x \geq b$. In the first case $Q_{\text{enc}} = 2\rho_0 Ax$ whereas in the second case $Q_{\text{enc}} = 2Ab\rho_0$, which leads to

$$\boldsymbol{E} = \begin{cases} \frac{\rho_0}{\epsilon_0}x\hat{\boldsymbol{x}} & \text{if } -b < x < b\,, \\ \frac{\rho_0 b}{\epsilon_0}\hat{\boldsymbol{x}} & \text{if } x \geq b\,, \\ -\frac{\rho_0 b}{\epsilon_0}\hat{\boldsymbol{x}} & \text{if } x \leq -b\,, \end{cases}$$

   (c) Repeat part (b) with $\rho(x) = \rho_0 e^{-|x/b|}$. [HMS Problem II-11]

Solution: Again the only difference to (a) and (b) is in determining the enclosed charge

$$Q_{\rm enc} = A\rho_0 \left(\int_0^x e^{-x'/b}\, dx' + \int_{-x}^0 e^{x'/b}\, dx' \right) = 2bA\rho_0 \left(1 - e^{-x/b}\right)$$

So we find

$$\boldsymbol{E} = \begin{cases} \frac{b\rho_0}{\epsilon_0}\left(1 - e^{-x/b}\right)\hat{\boldsymbol{x}} & \text{if } x > 0\ , \\ -\frac{b\rho_0}{\epsilon_0}\left(1 - e^{x/b}\right)\hat{\boldsymbol{x}} & \text{if } x < 0\ . \end{cases}$$

5. *The equation of continuity in an fluid:* In the spatial description, the equation of continuity for a fluid is

$$\frac{d\rho}{dt} + \rho \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0\ ,$$

where $\rho(\boldsymbol{r}, t)$ and $\boldsymbol{v}(\boldsymbol{r}, t)$ are the density and velocity, respectively. Show that the continuity equation can be equivalently written as

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0\ .$$

[JRT Problem 16.34]

Solution: We first rewrite the first term on the LHS as the stream derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dz}{dt}\frac{\partial}{\partial z} = \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} = \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}\ .$$

Therefore, the continuity equation becomes

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}\rho + \rho \boldsymbol{\nabla} \cdot \boldsymbol{v} &= 0 \\ \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) &= 0\ . \end{aligned}$$

The moment of inertia tensor

Consider a collection of particles that form a rigid body. The angular momentum of the α-th particle is given by

$$\boldsymbol{L}_\alpha = m_\alpha\,(\boldsymbol{r}_\alpha \times \boldsymbol{v}_\alpha)\,,$$

where $\boldsymbol{r}_\alpha$ and $\boldsymbol{v}_\alpha$ are the position and velocity vectors of the particle with respect to a given origin. The total angular momentum of the rigid body can then be written as

$$\boldsymbol{L} = \sum_{\alpha=1}^{N} \boldsymbol{L}_\alpha = \sum_{\alpha=1}^{N} m_\alpha\,(\boldsymbol{r}_\alpha \times \boldsymbol{v}_\alpha)\,,$$

where N is the total number of particles which constitute the rigid body.

If the rigid body is rotating with an angular velocity $\boldsymbol{\omega}$, then the velocity of the α-th particle will be given by

$$\boldsymbol{v}_\alpha = \boldsymbol{\omega} \times \boldsymbol{r}_\alpha.$$

Therefore, the total angular momentum of the system can be written as

$$\boldsymbol{L} = \sum_{\alpha=1}^{N} m_\alpha\,[\boldsymbol{r}_\alpha \times (\boldsymbol{\omega} \times \boldsymbol{r}_\alpha)]$$

and since

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}\,(\boldsymbol{A}\cdot\boldsymbol{C}) - \boldsymbol{C}\,(\boldsymbol{A}\cdot\boldsymbol{B})\,,$$

we can write

$$\begin{aligned}
\boldsymbol{L} &= \sum_{\alpha=1}^{N} m_\alpha\,\left[r_\alpha^2\,\boldsymbol{\omega} - \boldsymbol{r}_\alpha\,(\boldsymbol{r}_\alpha\cdot\boldsymbol{\omega})\right]\\
&= \sum_{\alpha=1}^{N} m_\alpha\,\left[\left(x_\alpha^2+y_\alpha^2+z_\alpha^2\right)\,(\omega_x\,\hat{\boldsymbol{x}}+\omega_y\,\hat{\boldsymbol{y}}+\omega_z\,\hat{\boldsymbol{z}}) - (x_\alpha\,\hat{\boldsymbol{x}}+y_\alpha\,\hat{\boldsymbol{y}}+z_\alpha\,\hat{\boldsymbol{z}})\,(x_\alpha\,\omega_x+y_\alpha\,\omega_y+z_\alpha\,\omega_z)\right]\\
&= \sum_{\alpha=1}^{N} m_\alpha\,\Big\{\left[\left(y_\alpha^2+z_\alpha^2\right)\,\omega_x - x_\alpha\,y_\alpha\,\omega_y - x_\alpha\,z_\alpha\,\omega_z\right]\,\hat{\boldsymbol{x}} + \left[-x_\alpha\,y_\alpha\,\omega_x + \left(x_\alpha^2+z_\alpha^2\right)\,\omega_x - y_\alpha\,z_\alpha\,\omega_z\right]\,\hat{\boldsymbol{y}}\\
&\quad + \left[-x_\alpha\,z_\alpha\,\omega_x - y_\alpha\,z_\alpha\,\omega_y + \left(x_\alpha^2+y_\alpha^2\right)\,\omega_z\right]\,\hat{\boldsymbol{z}}\Big\}\\
&= (I_{xx}\,\omega_x + I_{xy}\,\omega_y + I_{xz}\,\omega_z)\,\hat{\boldsymbol{x}} + (I_{yx}\,\omega_x + I_{yy}\,\omega_y + I_{yz}\,\omega_z)\,\hat{\boldsymbol{y}} + (I_{zx}\,\omega_x + I_{zy}\,\omega_y + I_{zz}\,\omega_z)\,\hat{\boldsymbol{z}},
\end{aligned}$$

where

$$\begin{aligned}
I_{xx} &= \sum_{\alpha=1}^{N} m_\alpha\,\left(y_\alpha^2+z_\alpha^2\right), & I_{xy} &= I_{yx} = -\sum_{\alpha=1}^{N} m_\alpha\,x_\alpha\,y_\alpha, & I_{xz} &= I_{zx} = -\sum_{\alpha=1}^{N} m_\alpha\,x_\alpha\,z_\alpha,\\
I_{yx} = I_{xy} &= -\sum_{\alpha=1}^{N} m_\alpha\,y_\alpha\,x_\alpha, & I_{yy} &= \sum_{\alpha=1}^{N} m_\alpha\,\left(x_\alpha^2+z_\alpha^2\right), & I_{yz} &= I_{zy} = -\sum_{\alpha=1}^{N} m_\alpha\,y_\alpha\,z_\alpha,\\
I_{zx} = I_{xz} &= -\sum_{\alpha=1}^{N} m_\alpha\,x_\alpha\,z_\alpha, & I_{zy} &= I_{yz} = -\sum_{\alpha=1}^{N} m_\alpha\,y_\alpha\,z_\alpha, & I_{zz} &= \sum_{\alpha=1}^{N} m_\alpha\,\left(x_\alpha^2+y_\alpha^2\right).
\end{aligned}$$

The quantity I_{ij} with $(i, j) = (1, 2, 3)$ forms a 3×3 symmetric matrix which is known as the moment of inertia tensor. Note that, in terms of the moment of inertia tensor I_{ij}, we can express the components of the angular momentum L_i of the rigid body as

$$L_i = \sum_{j=1}^{3} I_{ij}\,\omega_j,$$

where ω_i represent the components of the angular velocity of the body.

It should be noted that only in some special symmetric cases that the angular momentum and angular velocity can be in the same direction. This can occur, for instance, when all the the diagonal components are equal and all the non-diagonal components are zero, as in the case of a sphere, *and* all the components of the angular velocities are equal. Otherwise, even a symmetric object such as a sphere will exhibit precession (i.e. a rotation of the axis of rotation), apart from the simpler rotation.

Last updated on October 30, 2017

Exercise sheet 8

Introductory vector calculus

1. *Topography of a hill:* The height of a certain hill (in m) is given by

$$h(x, y) = 10\left(2xy - 3x^2 - 4y^2 - 18x + 28y + 12\right) \ ,$$

where x and y are the distances (in km) east and north of Shimla, respectively.

(a) Where is the top of the hill?

(b) How high is the hill?

(c) How steep is the slope (in m km^{-1}) at a point 1 km north and 1 km east of Shimla? In which direction is the slope the steepest, at that point? [DG Problem 1.12]

2. *A special motion in the field of an electric dipole:* The potential of an ideal electric dipole $\boldsymbol{p} = p\hat{\boldsymbol{z}}$ located at the origin is

$$\Phi(r, \theta, \phi) = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2} \ ,$$

in spherical polar coordinates (r, θ, ϕ).

(a) Calculate the electric field $\boldsymbol{E}$.

(b) Consider a bead mass m charge q released from rest in the xy plane a distance R from the origin along the x axis. It is constrained to move on a semicircular wire with radius R that passes through $(0, 0, -R)$ with its ends at $(R, 0, 0)$ and $(-R, 0, 0)$. Ignoring gravity, calculate the speed of the bead at a point along the wire.

(c) Show that the wire is not necessary to constrain the bead such that it follows this path.

[Based on DG Problem 3.49]

3. *Gravitational potential of a ring:* Consider a thin uniform circular ring of radius a and mass M. A mass m is placed in the plane of the ring. Find a position of equilibrium and determine whether it is stable? (Assume this experiment is performed in a weightless environment.) [TM Example 5.3]

4. *Moment of inertia of a triangle:* Find the moment of inertia of a thin sheet of mass M in the shape of an equilateral triangle about an axis through a vertex, perpendicular to the sheet. The length of each side is L. [KK Problem 7.7]

5. *Generalized moment of inertia:* In general the angular momentum of a rigid body is given by

$$\boldsymbol{L} = \boldsymbol{I}\boldsymbol{\omega} \ ,$$

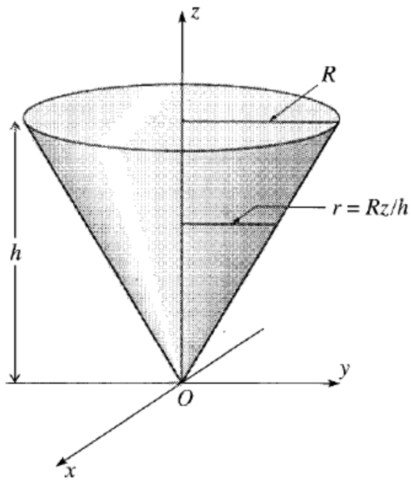
where $\boldsymbol{I}$ is a 3×3 matrix called the *moment of inertia tensor* and $\boldsymbol{\omega}$ is the angular velocity. The diagonal components of $\boldsymbol{I}$ are given by

$$I_{xx} = \int \varrho(y^2 + z^2)\, dV \ ,$$

and similar expressions for I_{yy} and I_{zz}. Here ϱ is the density. The off-diagonal elements of $\boldsymbol{I}$ are given by

$$I_{xy} = I_{yx} = -\int \varrho xy \, dV \ ,$$

and similar expressions for I_{xz} and I_{yz}. Find the moment of inertia tensor $\boldsymbol{I}$ for a spinning top that is a uniform solid cone (mass M, height h and base radius R). Choose the z axis along the axis of symmetry of the cone as shown in the figure.



For an arbitrary angular velocity $\boldsymbol{\omega}$, what is the top's angular momentum $\boldsymbol{L}$? [JRT Example 10.3]

Last updated on October 6, 2017

Exercise sheet 8 Solution

Introductory vector calculus

1. *Topography of a hill:* The height of a certain hill (in m) is given by

$$h(x,y) = 10\left(2xy - 3x^2 - 4y^2 - 18x + 28y + 12\right) ,$$

where x and y are the distances (in km) east and north of Shimla, respectively.

(a) Where is the top of the hill?

Solution:

The top of the hill will correspond to an extrema of the function, which will be at a point where $\boldsymbol{\nabla} h = 0$.

$$\boldsymbol{\nabla} h = 10\left([2y - 6x - 18]\,\hat{\boldsymbol{x}} + [2x - 8y + 28]\,\hat{\boldsymbol{y}}\right) .$$

Therefore, when $\boldsymbol{\nabla} h|_{(x_0,y_0)} = 0$ we have

$$\begin{aligned} 2y_0 - 6x_0 &= 18 \\ 2x_0 - 8y_0 &= -28 , \end{aligned}$$

which results in $(x_0, y_0) = (-2, 3)$ km. This is a maximum because $\frac{\partial^2 h}{\partial x^2} = -60$ and $\frac{\partial^2 h}{\partial y^2} = -80$ for all values of (x, y).

(b) How high is the hill?

Solution:

Evaluate $h(-2, 3) = 720$ m.

(c) How steep is the slope (in m km^{-1}) at a point 1 km north and 1 km east of Shimla? In which direction is the slope the steepest, at that point? [DG Problem 1.12]

Solution:

Evaluate $\boldsymbol{\nabla} h\,(1, 1)$ then

$$|\boldsymbol{\nabla} h\,(1,1)| \;\text{ and }\; \frac{\boldsymbol{\nabla} h\,(1,1)}{|\boldsymbol{\nabla} h\,(1,1)|} ,$$

gives the magnitude of the steepest slope and its direction, respectively. Therefore, as

$$\boldsymbol{\nabla} h\,(1,1) = -220\hat{\boldsymbol{x}} + 220\hat{\boldsymbol{y}} ,$$

the slope is $\sqrt{2} \times 220$ m km^{-1} ≈ 310 m km^{-1} in a direction $\frac{1}{\sqrt{2}}\left(-\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}\right)$, which is equivalent to NW.

2. *A special motion in the field of an electric dipole:* The potential of an ideal electric dipole $\boldsymbol{p} = p\hat{\boldsymbol{z}}$ located at the origin is

$$\Phi(r,\theta,\phi) = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2} ,$$

in spherical polar coordinates (r, θ, ϕ).

(a) Calculate the electric field $\boldsymbol{E}$.

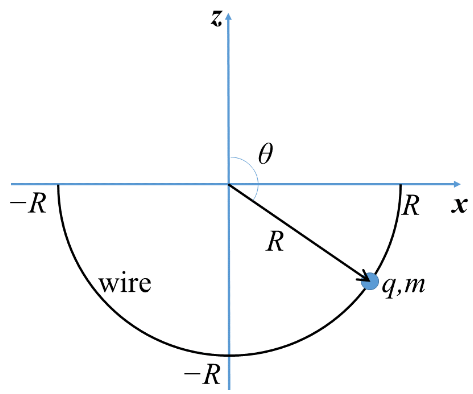
Solution:

The electric field $\boldsymbol{E}$ is given by $-\boldsymbol{\nabla}\Phi$. Therefore,

$$\begin{aligned} \boldsymbol{E} &= -\left(\hat{\boldsymbol{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\Phi \\ &= -\frac{p}{4\pi\epsilon_0}\left(-\frac{2\cos\theta}{r^3}\hat{\boldsymbol{r}} - \frac{\sin\theta}{r^3}\hat{\boldsymbol{\theta}}\right) = \frac{p}{4\pi\epsilon_0\, r^3}\left(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right) . \end{aligned}$$

(b) Consider a bead mass m charge q released from rest in the xy plane a distance R from the origin along the x axis. It is constrained to move on a semicircular wire with radius R that passes through $(0, 0, -R)$ with its ends at $(R, 0, 0)$ and $(-R, 0, 0)$. Ignoring gravity, calculate the speed of the bead at a point along the wire.

<u>Solution:</u> The figure shows the wire, the bead and coordinate definitions.



In spherical polar coordinates $(x, y, z) = (R, 0, 0)$ equivalent to $(r, \theta, \phi) = (R, \frac{\pi}{2}, 0)$. Here, the force on the charge is in the $\hat{\boldsymbol{\theta}}(\frac{\pi}{2}, 0) = -\hat{\boldsymbol{z}}$ direction, so it starts to move along the wire. The speed v is easily computed from the total energy as the bead moves along the wire

$$E = T + q\Phi = \frac{m}{2}v^2 + \frac{qp}{4\pi\epsilon_0\, R^2}\cos\theta\ .$$

Using the initial conditions $v = 0$ and $\theta = \frac{\pi}{2}$, we find that $E = 0$. Therefore,

$$v = \sqrt{-\frac{qp}{2\pi\epsilon_0\, mR^2}\cos\theta}\ .$$

This means that the bead reaches its maximum speed at $(0, 0, -R)$ when $\theta = \pi$, then it will decelerate as it moves toward $(-R, 0, 0)$ where it will come to a standstill. Then it will start to move back along the wire as the field is in the $-\hat{\boldsymbol{z}}$ direction at this point as well. So the bead will oscillate back and forth along the wire.

Alternatively, one can solve the θ component of the equation of motion in spherical polar coordinates.

$$\begin{aligned}
m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta\right) &= qE_\theta = \frac{qp}{4\pi\epsilon_0\, r^3}\sin\theta \\
\Rightarrow mR\ddot{\theta} &= \frac{qp}{4\pi\epsilon_0\, R^3}\sin\theta \quad (\because \dot{r} = \dot{\phi} = 0) \\
\Rightarrow 2\ddot{\theta}\dot{\theta} &= \frac{qp}{2\pi\epsilon_0\, mR^4}\sin\theta\,\dot{\theta} \\
\Rightarrow \frac{d}{dt}\dot{\theta}^2 &= \frac{qp}{2\pi\epsilon_0\, mR^4}\sin\theta\frac{d\theta}{dt} \\
\Rightarrow \dot{\theta}^2 &= \frac{qp}{2\pi\epsilon_0\, mR^4}\int_{\pi/2}^{\theta}\sin\theta' d\theta' \\
\Rightarrow R^2\dot{\theta}^2 &= v^2 = -\frac{qp}{2\pi\epsilon_0\, mR^2}\cos\theta\ ,
\end{aligned}$$

as before.

(c) Show that the wire is not necessary to constrain the bead such that it follows this path.

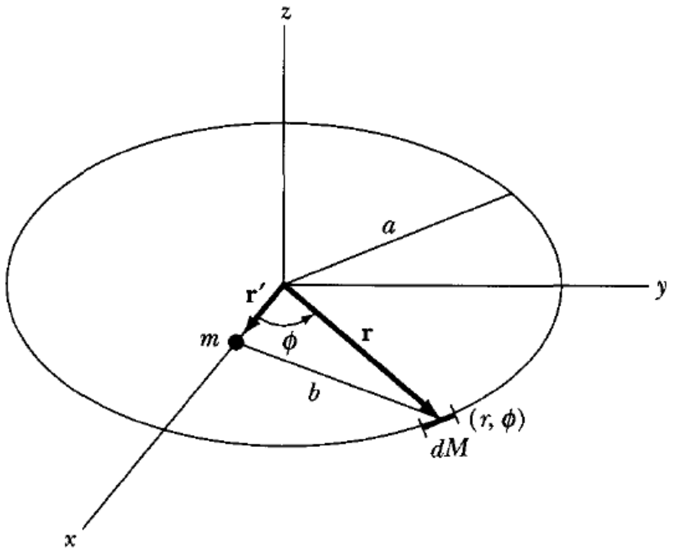
<u>Solution:</u> If we remove the wire we must consider the radial part of the equation of motion. Note $\dot{\phi} = 0$ still because there is no force in the azimuthal direction we have

$$m\left(\ddot{r} - R\dot{\theta}^2\right) = qE_r = \frac{qp}{2\pi\epsilon_0\, R^3}\cos\theta$$

substituting for $\dot{\theta}^2$ we see that for $\ddot{r} = 0$ for motion along the wire direction. This means that the wire is not required to keep the bead at fixed radius, as $\dot{r}(t = 0) = 0$, so the semicircular motion would occur without it. [Based on DG Problem 3.49]

3. *Gravitational potential of a ring:* Consider a thin uniform circular ring of radius a and mass M. A mass m is placed in the plane of the ring. Find a position of equilibrium and determine whether it is stable? (Assume this experiment is performed in a weightless environment.) [TM Example 5.3]

Solution: Intuition tells us the centre of the ring will be a point of equilibrium but we need to prove this and find whether it is stable or not. Placing m a distance r' from the centre and make this the x axis as shown in the figure below.



We can then write the contribution to the potential from a small segment of the ring dM that subtends an angle $d\phi$ as

$$d\Phi = -G\frac{dM}{b} = -\frac{Ga\varrho}{b}\,d\phi$$

where $\varrho = M/(2\pi a)$ and b is the distance from m to dM. Now we define the position vectors to dM and m as $\boldsymbol{r}$ and $\boldsymbol{r}'$, respectively. Such that b can be written as

$$\begin{aligned} b &= \left|\boldsymbol{r} - \boldsymbol{r}'\right| = \left|a\cos\phi\hat{\boldsymbol{x}} + a\sin\phi\hat{\boldsymbol{y}} - r'\hat{\boldsymbol{x}}\right| \\ &= \sqrt{(a\cos\phi - r')^2 + a^2\sin^2\phi} \\ &= \sqrt{a^2 + r'^2 - 2ar'\cos\phi} = a\sqrt{1 + \left(\frac{r'}{a}\right)^2 - \frac{2r'}{a}\cos\phi}\,. \end{aligned}$$

This expression for b can then be used to find $\Phi(r')$

$$\begin{aligned} \Phi(r') &= -\varrho a G \int_0^{2\pi} \frac{d\phi}{b} \\ &= -\varrho G \int_0^{2\pi} \frac{d\phi}{\sqrt{1 + \left(\frac{r'}{a}\right)^2 - \frac{2r'}{a}\cos\phi}}\,. \end{aligned}$$

Let us consider points close to the origin such that $r' \ll a$ such that we can expand the integrand using the Maclaurin series for $(1+x)^n \approx 1 + nx + \frac{1}{2}n(n-1)x^2 + \ldots$ *i.e.*

$$\begin{aligned}
\frac{1}{\sqrt{1+\left(\frac{r'}{a}\right)^2 - \frac{2r'}{a}\cos\phi}} &= 1 - \frac{1}{2}\left[\left(\frac{r'}{a}\right)^2 - \frac{2r'}{a}\cos\phi\right] + \frac{3}{8}\left[\left(\frac{r'}{a}\right)^2 - \frac{2r'}{a}\cos\phi\right]^2 + \ldots \\
&= 1 + \frac{r'}{a}\cos\phi + \frac{1}{2}\left(\frac{r'}{a}\right)^2\left(3\cos^2\phi - 1)\right) + \ldots\ ,
\end{aligned}$$

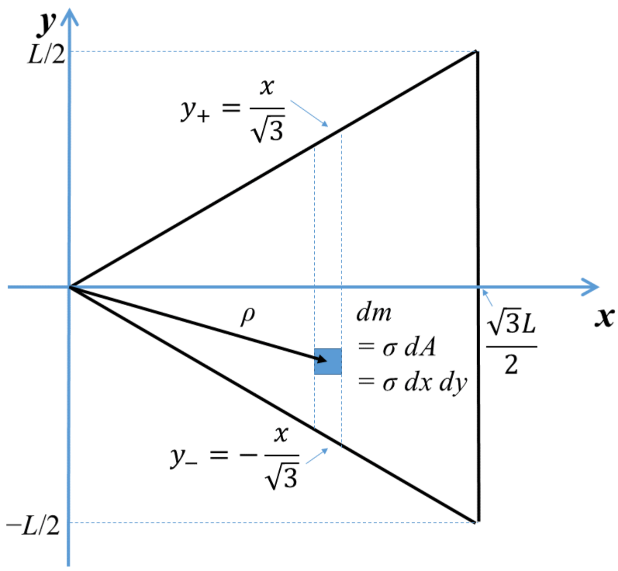
where we have truncated the series at $(r'/a)^2$. We can now evaluate the potential

$$\begin{aligned}
\Phi &= -\varrho G \int_0^{2\pi}\left[1 + \frac{r'}{a}\cos\phi + \frac{1}{2}\left(\frac{r'}{a}\right)^2\left(3\cos^2\phi - 1)\right) + \ldots\right] d\phi \\
&\approx -\varrho G\left[2\pi + \frac{1}{2}\left(\frac{r'}{a}\right)^2 \int_0^{2\pi}\left\{\frac{3}{2}(1+\cos 2\phi) - 1\right\} d\phi\right] \\
&\approx -\varrho G\left[2\pi + \frac{2\pi}{4}\left(\frac{r'}{a}\right)^2\right] \propto -r'^{\,2}\ .
\end{aligned}$$

So about the origin the potential is that of the inverted oscillator ($\propto -r'^{\,2}$), which we know is a point of **unstable** (!) equilibrium. (Displacements in the z direction are stable *c.f.* Exercise Sheet 5 Question 2.)

4. *Moment of inertia of a triangle:* Find the moment of inertia of a thin sheet of mass M in the shape of an equilateral triangle about an axis through a vertex, perpendicular to the sheet. The length of each side is L. [KK Problem 7.7]

Solution: Let us set up our coordinates as shown in the figure, where the z axis, pointing out of the page, is that of rotation.



Taking a small element with $dm = \sigma dx dy$, where σ is the mass per unit area, its contribution to the moment of inertia is

$$dI = \rho^2\, dm = \sigma\, \left(x^2 + y^2\right)\, dm\ .$$

So to get the total moment of inertia we will integrate over y with x dependent limits $y_\pm = \pm x/\sqrt{3}$

then x between 0 and $\frac{\sqrt{3}}{2}L$ as follows:

$$\begin{aligned} I &= \sigma \int_0^{\frac{\sqrt{3}}{2}L} dx \int_{-\frac{x}{\sqrt{3}}}^{+\frac{x}{\sqrt{3}}} dy \left(x^2+y^2\right) \\ &= \sigma \int_0^{\frac{\sqrt{3}}{2}L} dx \left[yx^2+\frac{y^3}{3}\right]_{-\frac{x}{\sqrt{3}}}^{+\frac{x}{\sqrt{3}}} = 2\sigma \int_0^{\frac{\sqrt{3}}{2}L} dx \left[\frac{x^3}{\sqrt{3}}+\frac{x^3}{9\sqrt{3}}\right] \\ &= \frac{20}{9\sqrt{3}}\sigma \left[\frac{x^4}{4}\right]_0^{\frac{\sqrt{3}}{2}L} \\ &= \frac{5}{9\sqrt{3}}\frac{9}{16}\sigma L^4 = \frac{5}{16\sqrt{3}}\sigma L^4 \ . \end{aligned}$$

Now

$$\sigma = \frac{M}{\text{Area}} = \frac{M}{\frac{1}{2}\times L \times \frac{\sqrt{3}}{2}L} = \frac{4M}{\sqrt{3}L^2} \ ,$$

which leads to

$$I = \frac{5}{12}ML^2 \ ,$$

upon substitution.

5. *Generalized moment of inertia:* In general the angular momentum of a rigid body is given by

$$\boldsymbol{L} = \boldsymbol{I}\boldsymbol{\omega} \ ,$$

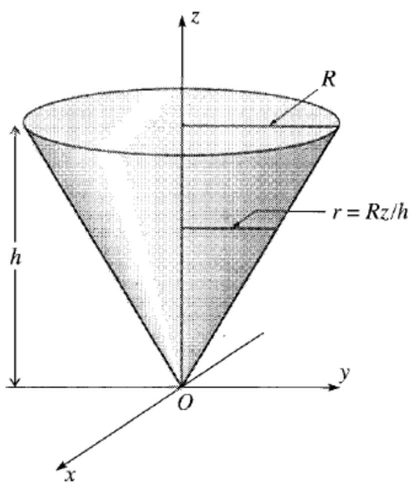
where $\boldsymbol{I}$ is a 3×3 matrix called the *moment of inertia tensor* and $\boldsymbol{\omega}$ is the angular velocity. The diagonal components of $\boldsymbol{I}$ are given by

$$I_{xx} = \int \varrho(y^2+z^2)\, dV \ ,$$

and similar expressions for I_{yy} and I_{zz}. Here ϱ is the density. The off-diagonal elements of $\boldsymbol{I}$ are given by

$$I_{xy} = I_{yx} = -\int \varrho xy\, dV \ ,$$

and similar expressions for I_{xz} and I_{yz}. Find the moment of inertia tensor $\boldsymbol{I}$ for a spinning top that is a uniform solid cone (mass M, height h and base radius R). Choose the z axis along the axis of symmetry of the cone as shown in the figure.



Solution:

First we note that the density of the cone is

$$\varrho = \frac{M}{\text{Volume}} = \frac{3M}{\pi R^2 h} \ .$$

Next we will calculate the I_{zz} component, which is simply done in cylindrical polar coordinates because $\rho^2 = x^2 + y^2$,

$$I_{zz} = \varrho \int_V dV \rho^2 = \varrho \int_0^h dz \int_0^{2\pi} d\phi \int_0^{r(z)} \rho \, d\rho \rho^2 \ ,$$

where $r(z) = \frac{Rz}{h}$ the radius of the cone at height z above the tip as shown in the figure. Therefore,

$$\begin{aligned} I_{zz} &= \frac{2\pi}{4} \varrho \int_0^h dz \, r(z)^4 = \frac{\pi}{2} \left(\frac{R}{h}\right)^4 \varrho \int_0^h dz \, z^4 \\ &= \frac{\pi}{10} R^4 h \varrho = \frac{3}{10} M R^2 \ . \end{aligned}$$

Next we evaluate I_{xx}, but first we note that $I_{xx} = I_{yy}$ because of the rotational symmetry about the z axis. (A rotation through 90° about the z-axis leaves the cone unchanged but $I_{xx} \longleftrightarrow I_{yy}$ because $x \to y$ and $y \to -x$.) To evaluate I_{xx} we write

$$I_{xx} = \varrho \int_V dV \left(y^2 + z^2\right) = \varrho \int_V dV y^2 + \varrho \int_V dV z^2 \ ,$$

where we note that the first term is the same as the second in the I_{zz} integral. Because of the invariance under rotations about the z axis, which was noted when equating $I_{xx} = I_{yy}$, we can interchange the two terms in the integral hence they have to be equal to one another. (Can be explicitly shown by inserting $y = \rho \sin \phi$ and recomputing the integral.) Therefore, the first term is $\frac{1}{2} I_{zz} = \frac{3}{20} M R^2$. Now we evaluate the second term

$$\begin{aligned} \varrho \int_V dV z^2 &= \varrho \int_0^h dz \int_0^{2\pi} d\phi \int_0^{r(z)} \rho \, d\rho \, z^2 = \pi \varrho \int_0^h dz \, r(z)^2 \, z^2 \\ &= \pi \varrho \left(\frac{R}{h}\right)^2 \int_0^h dz \, z^4 = \frac{\pi}{5} \varrho R^2 h^3 = \frac{3}{5} M h^2 \ , \end{aligned}$$

therefore

$$I_{xx} = I_{yy} = \frac{3}{20} M \left(R^2 + 4h^2\right) \ .$$

The terms $I_{xy} = I_{xz} = I_{yz} = 0$ because of the reflection symmetry of the cone about the xz and yz planes. To appreciate this consider the contribution at (x_0, y_0, z_0) to the total,

$$dI_{xy}|_{(x_0,y_0,z_0)} = -x_0 y_0 \, \varrho \, dM \ ,$$

which is canceled by that at $(x_0, -y_0, z_0)$

$$dI_{xy}|_{(x_0,-y_0,z_0)} = x_0 y_0 \, \varrho \, dM = -dI_{xy}|_{(x_0,y_0,z_0)} \ .$$

So the whole calculation can be split into such pairs leading to $I_{xy} = 0$. Identical arguments apply for I_{xz} and I_{yz}. Therefore, we can write the full moment of inertia tensor

$$\boldsymbol{I} = \frac{3}{20} M \begin{bmatrix} R^2 + 4h^2 & 0 & 0 \\ 0 & R^2 + 4h^2 & 0 \\ 0 & 0 & 2R^2 \end{bmatrix} \ .$$

For an arbitrary angular velocity $\boldsymbol{\omega}$, what is the top's angular momentum $\boldsymbol{L}$? [JRT Example 10.3]

<u>*Solution:*</u> The angular velocity is given by $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$ Therefore

$$\boldsymbol{L} = \boldsymbol{I}\boldsymbol{\omega} = \frac{3}{20}M\left[\left(R^2 + 4h^2\right)(\omega_x + \omega_y) + 2R^2\omega_z\right] \ .$$

If $\boldsymbol{\omega} = \omega\hat{\boldsymbol{z}}$ you recover the usual result of $L = I\omega$ for a cone spinning about its axis of symmetry.

Last updated on October 6, 2017

Department of Physics
Indian Institute of Technology Madras

Quiz II

From phase portraits to introductory vector calculus

Date: October 24, 2017 Time: 08:00 – 08:50 AM

Name: **Roll No.:**

Instructor: **Batch:**

Instructions

1. Begin by completing the information requested above. Please write your complete name, your roll number, the name of your instructor, and your batch number (out of I–XII). **The answer sheet will not be evaluated unless both your name and roll number are written.**
2. This question paper cum answer sheet booklet contains **six** single-sided pages. Please check right away that all the pages are present.
3. As we had announced earlier, this quiz consists of 3 true/false questions (for 1 mark each), 3 multiple choice questions with one correct option (for 1 mark each), 4 fill in the blanks (for 1 mark each), two questions involving detailed calculations (for 3 marks each) and one question involving some plotting (for 4 marks), adding to a total of 20 marks.
4. You are expected to answer **all** the questions. There are no negative marks.
5. **The answers have to be written in the boxes provided. Answers written elsewhere in the booklet will not be evaluated.**
6. **Kindly write the answers, including sketches, with a blue or black pen. Note that answers written with pencils or pens of other colors will not be evaluated.**
7. You can use the empty reverse sides for rough work. No extra sheets will be provided.
8. You are not allowed to use a calculator or any other electronic device during the quiz. Please note that you will not be permitted to continue with the quiz if you are found with any such device.
9. Make sure that you return question paper cum answer sheet booklet before you leave the examination hall.

For use by examiners
(Do not write in this space)

Q1–Q10	Q11	Q12	Q13	Total

✦ **True or false** (1 mark each, write True (T)/False (F) in the box provided)

1. A planet moves with the maximum speed at its aphelion (i.e. at the maximum distance from the sun).

2. An extended object has a mass density that is independent of the cylindrical polar coordinate ϕ. The moment of inertia of the object will be the same about the x, y and z-axes.

3. The electrostatic potential in a region is given by $V(\rho, \phi) = \rho^2 + 4\,\rho\cos\phi + 4$, where ρ and ϕ are the plane polar coordinates. The lines of constant positive V are circles.

✦ **Multiple choice questions** (1 mark each, write the one correct option in the box provided)

4. A particle moves only under the gravitational attraction of an infinitely long massive wire that has uniform density and lies along the z-axis. Which of the following sets contains **ALL** the conserved kinematic quantities of the particle? (Note that, E, $\vec{\boldsymbol{p}}$ and $\vec{\boldsymbol{L}}$ denote the energy, momentum and angular momentum of the particle, respectively.)

   [**A**] E [**B**] $\vec{\boldsymbol{L}},\ p_z$ [**C**] $L_z,\ \vec{\boldsymbol{p}}$ [**D**] $E,\ L_z,\ p_z$

5. The orbit of a particle in a central force is a circle which passes through the origin described by $r = r_0\cos\phi$, where r_0 is a constant. The central force $\vec{\boldsymbol{F}}(r)$ is proportional to

   [**A**] $-\vec{\boldsymbol{r}}/r^6$ [**B**] $-\vec{\boldsymbol{r}}/r^5$ [**C**] $-\vec{\boldsymbol{r}}/r^4$ [**D**] $-\vec{\boldsymbol{r}}/r^3$

6. A satellite moving in a circular orbit of radius R is given a forward thrust leading to an elliptical orbit of eccentricity $\epsilon = \frac{1}{2}$. The maximum distance $r_{\rm max}$ reached by the satellite will be

   [**A**] $2\,R$ [**B**] $3\,R$ [**C**] $4\,R$ [**D**] $R/2$

✦ **Fill in the blanks** (1 mark each, write the answer in the box provided)

7. A particle moves in a plane along the logarithmic spiral $\rho = {\rm e}^{\phi}$, where ρ and ϕ denote the plane polar coordinates. What is the angle between the position and the velocity vectors of the particle?

8. A particle moving under the influence of the central force $U(r) = \alpha\, r^2$, where $\alpha > 0$, starts with the initial conditions $(x, y) = (x_0, y_0)$ with $(v_x, v_y) = (0, 0)$. Express the trajectory of the particle in the x-y-plane as the function $y(x)$.

9. A mountaineer is attempting to climb Mount Normal. The height of Mount Normal is given by $h(x, y) = h_0 \exp - \left[(x-1)^2 + \frac{1}{4}\,(y-2)^2 + x\,y + 1\right]$, where h_0 is a constant and (x, y) are the positions with respect to the base camp. The mountaineer is at the position $(x, y) = (1, 1)$. In what direction in the x-y-plane should she move to climb the steepest slope?

10. Given the force $\vec{\boldsymbol{F}} = x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}}$, evaluate the work done to move a particle along the line $y = -x + 1$ from the point $(x, y) = (1, 0)$ to the point $(0, 1)$.

✦ **Questions with detailed answers** (write the calculations and answers within the boxes provided)

11. *Angular velocity of a laminar sheet:* A square laminar sheet of mass M has its corners at $\left(0, -\frac{a}{2}, 0\right)$, $\left(a, -\frac{a}{2}, 0\right)$, $\left(a, \frac{a}{2}, 0\right)$ and $\left(0, \frac{a}{2}, 0\right)$. Its mass per unit area is given by $\sigma(x) = \sigma_0 \left(1 - \frac{x}{a}\right)$. (a) Determine the moment of inertia of the sheet about the y-axis in terms of M and a. (b) The sheet is initially stationary but can rotate about the y-axis. A small piece of putty with mass $\frac{M}{2}$ is fired at the sheet with velocity $\vec{\boldsymbol{v}} = \frac{v_0}{\sqrt{2}} (\hat{\boldsymbol{x}} - \hat{\boldsymbol{z}})$ from $(0, 0, \frac{a}{2})$, where v_0 denotes the speed of the putty. On impact with the sheet, the putty sticks and the sheet begins to rotate with angular velocity $\omega\, \hat{\boldsymbol{y}}$. Evaluate ω in terms of v_0, M and a. (**Ignore gravity and air resistance.**) $\boxed{\text{1+2 marks}}$

12. *Kepler problem in velocity space:* Recall that the orbit of a particle moving under the influence of the central force $U(r) = -\alpha/r$, where $\alpha > 0$, is given by $r(\phi) = r_0/(\epsilon\cos\phi + 1)$, where $r_0 = L^2/(\mu\,\alpha)$ and $\epsilon = \sqrt{1 + 2\,E\,L^2/(\mu\,\alpha^2)}$, where μ, E and L denote the reduced mass, energy and angular momentum of the particle. (a) Express the velocities v_x and v_y of the particle in terms of ϕ. (b) Show that the particle describes a circle in the (v_x, v_y) space. 2+1 marks

13. *Trajectories in phase space:* Consider a particle moving in the following one-dimensional potential: $U(x) = -\alpha\,(x^2 - x_0^2)^2$, where $\alpha > 0$. (a) Draw the potential $U(x)$, specifically marking the values of the extrema. (b) Determine the range of energy for which the system can exhibit bounded motion. (c) Draw the following phase space trajectories indicating the direction of motion with arrows: (i) bounded motion and (ii) unbounded motion for a positive as well as a negative value of energy. $\boxed{\text{1.5+0.5+2 marks}}$

Solutions to Quiz II

From phase portraits to introductory vector calculus

✦ True or false

1. A planet moves with the maximum speed at its aphelion (i.e. at the maximum distance from the sun).

   Solution: False. As angular momentum L is a constant and the velocity is perpendicular to the radial vector at aphelion $r_{\rm max}$ and perihelion $r_{\rm min}$, we have

   $$L = \mu\, r_{\rm max}\, v_{\rm min} = \mu\, r_{\rm min}\, v_{\rm max},$$

   where μ is the reduced mass, while $v_{\rm min}$ and $v_{\rm max}$ are the minimum and maximum speeds. Therefore, the particle will be moving at its minimum speed ($v_{\rm min}$) at the aphelion ($r_{\rm max}$).

2. An extended object has a mass density that is independent of the cylindrical polar coordinate ϕ. The moment of inertia of the object will be the same about the x, y and z-axes.

   Solution: False. Clearly, the moment of inertia about the z-axis will be distinct when compared with the moment of inertia about the other two axes.

3. The electrostatic potential in a region is given by $V(\rho,\phi) = \rho^2 + 4\,\rho\cos\phi + 4$, where ρ and ϕ denote the plane polar coordinates. The lines of constant positive V are circles.

   Solution: True. Note that

   $$V(\rho,\phi) = \rho^2 + 4\,\rho\cos\phi + 4 = x^2 + y^2 + 4\,x + 4 = (x+2)^2 + y^2,$$

   which, evidently, describes a circle if V is a positive constant.

✦ Multiple choice questions

4. A particle moves only under the gravitational attraction of an infinitely long massive wire that has uniform density and lies along the z-axis. Which of the following sets contains **ALL** the conserved kinematic quantities of the particle? (Note that, E, $\vec{\boldsymbol{p}}$ and $\vec{\boldsymbol{L}}$ denote the energy, momentum and angular momentum of the particle, respectively.)

   [**A**] E [**B**] $\vec{\boldsymbol{L}},\ p_z$ [**C**] $L_z,\ \vec{\boldsymbol{p}}$ [**D**] $E,\ L_z,\ p_z$

   Solution: D. The system is independent of time, so E is conserved. The system is invariant under translations in the z-direction, so p_z is conserved. The system is invariant under rotations about the z-axis and hence L_z is conserved.

5. The orbit of a particle in a central force is a circle which passes through the origin described by $r = r_0\cos\phi$, where r_0 is a constant. The central force $\vec{\boldsymbol{F}}(r)$ is proportional to

   [**A**] $-\vec{\boldsymbol{r}}/r^6$ [**B**] $-\vec{\boldsymbol{r}}/r^5$ [**C**] $-\vec{\boldsymbol{r}}/r^4$ [**D**] $-\vec{\boldsymbol{r}}/r^3$

   Solution: A. Recall that

   $$E = \frac{\mu}{2}\,\dot{r}^2 + \frac{L^2}{2\,\mu\,r^2} + U(r),$$

   so that on differentiation, we obtain that

   $$\frac{\mu}{2}\,2\,\dot{r}\,\ddot{r} + \frac{{\rm d}U}{{\rm d}r}\,\dot{r} - \frac{2\,L^2}{2\,\mu\,r^3}\,\dot{r} = 0$$

   and, since, $F_r = -{\rm d}U/{\rm d}r$, we have

   $$F_r = \mu\,\ddot{r} - \frac{L^2}{\mu\,r^3}.$$

As $r = r_0 \cos\phi$ and $\dot{\phi} = L/(\mu\, r^2)$, we have

$$\dot{r} = -r_0 \sin\phi\, \dot{\phi}, \quad \ddot{r} = -r_0 \cos\phi\, \dot{\phi}^2 - r_0 \sin\phi\, \ddot{\phi}$$

and

$$\ddot{\phi} = -\frac{2\,L}{\mu\, r^2}\, \dot{r} = \frac{2\,L\, r_0}{\mu\, r^3} \sin\phi\, \dot{\phi},$$

so that

$$\begin{aligned} F_r &= -\mu\, r_0 \cos\phi\, \dot{\phi}^2 - \mu\, r_0 \sin\phi\, \ddot{\phi} - \frac{L^2}{\mu\, r^3} = -\frac{L^2}{\mu\, r^3} - \frac{2\,L^2}{\mu\, r^5}\, r_0^2 \sin^2\!\phi - \frac{L^2}{\mu\, r^3} \\ &= -\frac{2\,L^2}{\mu\, r^3} - \frac{2\,L^2}{\mu\, r^5}\,(r_0^2 - r^2) = -\frac{2\,L^2\, r_0^2}{\mu\, r^5}, \end{aligned}$$

which is the required result.

This can also be presented in a simpler fashion. We can write

$$E = \frac{\mu}{2}\left(\dot{r}^2 + r^2\, \dot{\phi}^2\right) + U(r) = \frac{\mu}{2}\left[\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 + r^2\right] \dot{\phi}^2 + U(r),$$

and as $r = r_0 \cos\phi$ and $\dot{\phi} = L/(\mu\, r^2)$, we have

$$E = \frac{\mu\, r_0^2}{2}\, \dot{\phi}^2 + U(r) = \frac{L^2\, r_0^2}{2\,\mu\, r^4} + U(r),$$

which, upon differentiation, leads to

$$F_r = -\frac{\mathrm{d}U}{\mathrm{d}r} = -\frac{2\,L^2\, r_0^2}{\mu\, r^5}.$$

6. A satellite moving in a circular orbit of radius R is given a forward thrust leading to an elliptical orbit of eccentricity $\epsilon = \frac{1}{2}$. The maximum distance r_{max} reached by the satellite will be

   [**A**] $2\,R$ [**B**] $3\,R$ [**C**] $4\,R$ [**D**] $R/2$

   Solution: B. We have, for the elliptical orbit,

   $$r_{\mathrm{min}} = a\,(1-\epsilon), \quad r_{\mathrm{max}} = a\,(1+\epsilon),$$

   and, as $\epsilon = \frac{1}{2}$, we obtain that

   $$r_{\mathrm{min}} = \frac{a}{2} = R, \quad r_{\mathrm{max}} = \frac{3\,a}{2} = 3\,R.$$

✦ Fill in the blanks

7. A particle moves in a plane along the logarithmic spiral $\rho = \mathrm{e}^{\phi}$, where ρ and ϕ denote the plane polar coordinates. What is the angle between the position and the velocity vectors of the particle?

   Solution We have

   $$\vec{\boldsymbol{r}} = \mathrm{e}^{\phi}\, \hat{\boldsymbol{\rho}}, \quad \vec{\boldsymbol{v}} = \frac{\mathrm{d}\vec{\boldsymbol{r}}}{\mathrm{d}t} = \mathrm{e}^{\phi}\, \dot{\phi}\, \hat{\boldsymbol{\rho}} + \mathrm{e}^{\phi}\, \frac{\mathrm{d}\hat{\boldsymbol{\rho}}}{\mathrm{d}\phi}\, \dot{\phi} = \mathrm{e}^{\phi}\, \dot{\phi}\, (\hat{\boldsymbol{\rho}} + \hat{\boldsymbol{\phi}}),$$

   since $\mathrm{d}\hat{\boldsymbol{\rho}}/\mathrm{d}\phi = \hat{\boldsymbol{\phi}}$. Clearly, the angle between $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{v}}$ of the particle is 45°.

8. A particle moving under the influence of the central force $U(r) = \alpha\, r^2$, where $\alpha > 0$, starts with the initial conditions $(x, y) = (x_0, y_0)$ with $(v_x, v_y) = (0, 0)$. Express the trajectory of the particle in the x-y-plane as the function $y(x)$.

   Solution: The solutions along the x and y-directions can be expressed as

$$x(t) = x_0 \cos(\omega\, t) + \frac{v_{x0}}{\omega} \sin(\omega\, t), \quad y(t) = y_0 \cos(\omega\, t) + \frac{v_{y0}}{\omega} \sin(\omega\, t),$$

   where $\omega = (2\,\alpha/\mu)^{1/2}$, with μ being the reduced mass of the particle. Since, $(v_{x0}, v_{y0}) = (0, 0)$, we have $y(x) = (y_0/x_0)\, x$.

9. A mountaineer is attempting to climb Mount Normal. The height of Mount Normal is given by $h(x, y) = h_0 \exp - \left[(x-1)^2 + \frac{1}{4}(y-2)^2 + x\, y + 1\right]$, where h_0 is a constant and (x, y) are the positions with respect to, say, the base camp. The mountaineer is at the position $(x, y) = (1, 1)$. In what direction in the x-y-plane should she move to climb the steepest slope?

   Solution: The direction of the gradient gives the steepest slope. The gradient of $h(x, y)$ is

$$\vec{\boldsymbol{\nabla}} h = \hat{\boldsymbol{x}}\,\frac{\partial h}{\partial x} + \hat{\boldsymbol{y}}\,\frac{\partial h}{\partial y} = -h_0 \left[(2\,x + y - 2)\; \hat{\boldsymbol{x}} + \left(\frac{y}{2} + x - 1\right)\; \hat{\boldsymbol{y}}\right]\, \mathrm{e}^{-\left[(x-1)^2 + \frac{1}{4}(y-2)^2 + x\, y + 1\right]}.$$

   The value of the $\vec{\boldsymbol{\nabla}} h$ at $(x, y) = (1, 1)$ is

$$\vec{\boldsymbol{\nabla}} h\Big|_{(1,1)} = -h_0 \left(\hat{\boldsymbol{x}} + \frac{1}{2}\,\hat{\boldsymbol{y}}\right)\, \mathrm{e}^{-\frac{9}{4}}.$$

   So the unit vector in the direction of the steepest slope at $(x, y) = (1, 1)$ is

$$\frac{\vec{\boldsymbol{\nabla}} h\Big|_{(1,1)}}{\left|\vec{\boldsymbol{\nabla}} h\Big|_{(1,1)}\right|} = -\frac{2}{\sqrt{5}} \left(\hat{\boldsymbol{x}} + \frac{1}{2}\,\hat{\boldsymbol{y}}\right).$$

   Any vector proportional to this is fine.

10. Given the force $\vec{\boldsymbol{F}} = x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}}$, evaluate the work done to move a particle along the line $y = -x + 1$ from the point $(x, y) = (1, 0)$ to the point $(0, 1)$.

    Solution: Note that

$$\vec{\boldsymbol{F}} \cdot \mathrm{d}\vec{\boldsymbol{r}} = F_x\, \mathrm{d}x + F_y\, \mathrm{d}y$$

    and, since, in our case, $(F_x, F_y) = (x, y$, we obtain that

$$\vec{\boldsymbol{F}} \cdot \mathrm{d}\vec{\boldsymbol{r}} = x\, \mathrm{d}x + y\, \mathrm{d}y.$$

    Also, on the line $y = -x + 1$, we have $\mathrm{d}y = -\mathrm{d}x$, so that

$$\int_{(1,0)}^{(0,1)} \vec{\boldsymbol{F}} \cdot \mathrm{d}\vec{\boldsymbol{r}} = \int_{(1,0)}^{(0,1)} (x\, \mathrm{d}x + y\, \mathrm{d}y) = \int_1^0 [x\, \mathrm{d}x + (-x+1)\,(-\mathrm{d}x)] = \int_1^0 \mathrm{d}x\,(2\,x - 1) = 0.$$

✦ Questions with detailed answers

11. _Angular velocity of a laminar sheet:_ A square laminar sheet of mass M has its corners at $\left(0, -\frac{a}{2}, 0\right)$, $\left(a, -\frac{a}{2}, 0\right)$, $\left(a, \frac{a}{2}, 0\right)$ and $\left(0, \frac{a}{2}, 0\right)$. Its mass per unit area is given by $\sigma(x) = \sigma_0 \left(1 - \frac{x}{a}\right)$. (a) Determine the moment of inertia of the sheet about the y-axis in terms of M and a. (b) The sheet is initially stationary but can rotate about the y-axis. A very small piece of putty with mass $\frac{M}{2}$ is fired at the sheet with velocity $\vec{\boldsymbol{v}} = \frac{v_0}{\sqrt{2}}\,(\hat{\boldsymbol{x}} - \hat{\boldsymbol{z}})$ from $(0, 0, \frac{a}{2})$, where v_0 denotes the speed of the

putty. On impact with the sheet, the putty sticks and the sheet begins to rotate with angular velocity $\omega\,\hat{\boldsymbol{y}}$? Evaluate ω in terms of v_0, M and a. (**Ignore gravity and air resistance.**)

Solution: The contribution to the moment of inertia I of an infinitesimal region a distance x from the y axis is

$$\mathrm{d}I = \sigma(x)\,x^2\,\mathrm{d}x\,\mathrm{d}y,$$

which we can then integrate over the sheet to obtain

$$I = \int_{-\frac{a}{2}}^{\frac{a}{2}} \mathrm{d}y \int_0^a \mathrm{d}x\,\sigma_0\left(1-\frac{x}{a}\right)x^2 = a\,\sigma_0\left(\frac{x^3}{3}-\frac{x^4}{4\,a}\right)_0^a = \frac{1}{12}\,\sigma_0\,a^4.$$

The total mass of the sheet is

$$M = \int_{-\frac{a}{2}}^{\frac{a}{2}} \mathrm{d}y \int_0^a \mathrm{d}x\,\sigma_0\left(1-\frac{x}{a}\right) = \frac{1}{2}\,\sigma_0\,a^2.$$

(This can also be stated by saying the average σ over the interval 0 to a is $\frac{\sigma_0}{2}$, due to the linear variation from σ_0 to 0.) Therefore, we can write **[1 mark]**

$$I = \frac{M\,a^2}{6}.$$

The angular momentum before and after the putty sticks is conserved. The initial angular momentum is solely due to the putty, which can be calculated from the initial position $\vec{\boldsymbol{r}}_0 = \left(0, 0, \frac{a}{2}\right)$ and momentum $\vec{\boldsymbol{p}} = \frac{M}{2}\vec{\boldsymbol{v}} = \frac{M\,v_0}{2\sqrt{2}}\,(\hat{\boldsymbol{x}} - \hat{\boldsymbol{z}})$: **[0.5 mark]**

$$\vec{\boldsymbol{L}}_{\text{initial}} = \vec{\boldsymbol{r}}_0 \times \vec{\boldsymbol{p}} = \frac{M\,v_0}{2\sqrt{2}}\begin{vmatrix}\hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}}\\ 0 & 0 & \frac{a}{2}\\ 1 & 0 & -1\end{vmatrix} = \frac{M\,a\,v_0}{4\sqrt{2}}\,\hat{\boldsymbol{y}}.$$

The putty intersects with the centre of the sheet so once stuck it has a moment of inertia $I_{\text{putty}} = \frac{M\,a^2}{8}$ about the y-axis. Therefore, the angular momentum after collision is **[0.5 mark]**

$$\vec{\boldsymbol{L}}_{\text{final}} = (I + I_{\text{putty}})\,\omega\,\hat{\boldsymbol{y}} = \frac{7\,M\,a^2\,\omega}{24}\,\hat{\boldsymbol{y}}.$$

Equating the initial and final angular momenta, we get

$$\frac{7\,M\,a^2\,\omega}{24} = \frac{M\,a\,v_0}{4\sqrt{2}},$$

which implies that **[1 mark]**

$$\omega = \frac{6\,v_0}{7\sqrt{2}\,a}.$$

12. *Kepler problem in velocity space:* Recall that the orbit of a particle moving under the influence of the central force $U(r) = -\alpha/r$, where $\alpha > 0$, is given by $r(\phi) = r_0/(\epsilon\cos\phi + 1)$, where $r_0 = L^2/(\mu\,\alpha)$ and $\epsilon = \sqrt{1 + 2\,E\,L^2/(\mu\,\alpha^2)}$, where μ, E and L denote the reduced mass, energy and angular momentum of the particle. (a) Express the velocities v_x and v_y of the particle in terms of ϕ. (b) Show that the particle describes a circle in the (v_x, v_y) space.

Solution: Since

$$r = \frac{r_0}{1 + \epsilon\cos\phi},$$

we have

$$x = r\cos\phi = \frac{r_0\cos\phi}{1+\epsilon\cos\phi}, \quad y = r\sin\phi = \frac{r_0\sin\phi}{1+\epsilon\cos\phi},$$

so that

$$v_x = \dot{x} = -\frac{r_0\sin\phi\,\dot{\phi}}{(1+\epsilon\cos\phi)^2}, \quad v_y = \dot{y} = \frac{r_0\,\dot{\phi}}{(1+\epsilon\cos\phi)^2}\,(\epsilon+\cos\phi)$$

and, since, $\dot{\phi} = L/(\mu\,r^2)$, we can write

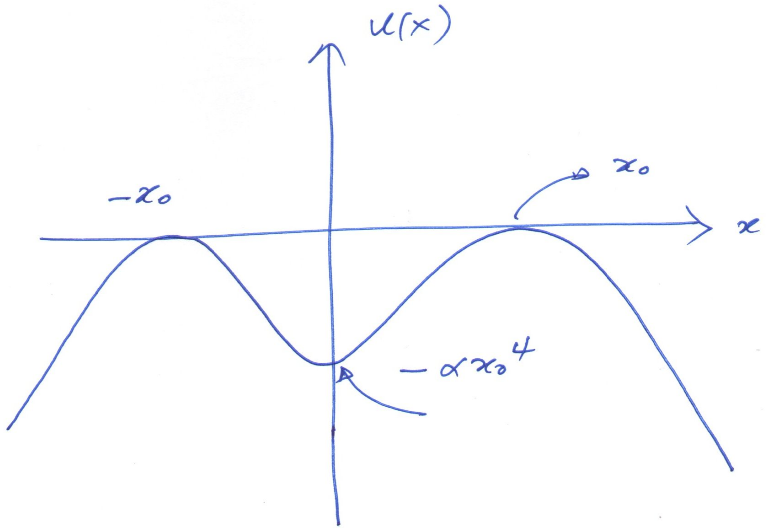
$$v_x = -\frac{L}{\mu\,r_0}\sin\phi, \quad v_y = \frac{L}{\mu\,r_0}\,(\epsilon+\cos\phi)\,,$$

leading to

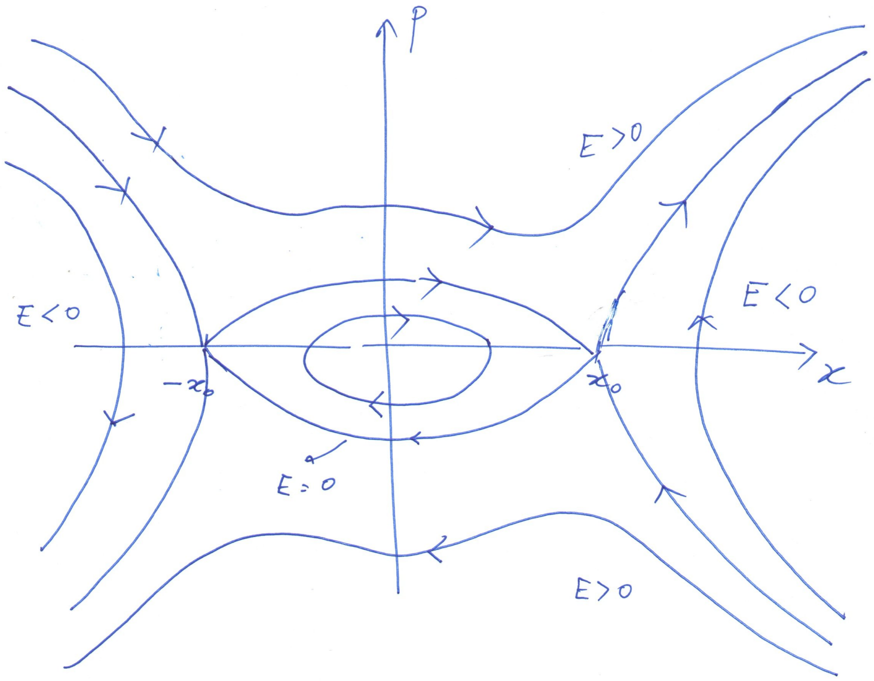
$$v_x^2 + \left(v_y - \frac{\epsilon\,L}{\mu\,r_0}\right)^2 = \frac{L^2}{\mu^2\,r_0^2}.$$

13. *Trajectories in phase space:* Consider a particle moving in the following one-dimensional potential: $U(x) = -\alpha\,(x^2 - x_0^2)^2$, where $\alpha > 0$. (a) Draw the potential $U(x)$, specifically marking the values of the extrema. (b) Determine the range of energy for which the system can exhibit bounded motion. (c) Draw the following phase space trajectories indicating the direction of motion with arrows: (i) bounded motion and (ii) unbounded motion for a positive as well as a negative value of energy. $\boxed{\text{1.5+0.5+2 marks}}$

Solution: The potential $U(x)$ is illustrated in the figure below.



The particle exhibits bounded motion if starts between $-x_0$ and x_0 with energy $-\alpha\,x_0^4 < E < 0$. The resulting phase trajectories are illustrated in the figure below.



p
x
E>0
E<0
E<0
$-x_0$
x_0
E=0
E>0

Department of Physics
Indian Institute of Technology Madras

Quiz II – Make up

From phase portraits to introductory vector calculus

Date: January 11, 2018 Time: 02:00 – 02:50 PM

Name:		**Roll No.:**	
Instructor:		**Batch:**	

Instructions

1. Begin by completing the information requested above. Please write your complete name, your roll number, the name of your instructor, and your batch number (out of I–XII). **The answer sheet will not be evaluated unless both your name and roll number are written.**
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4. You are expected to answer **all** the questions. There are no negative marks.
5. **The answers have to be written in the boxes provided. Answers written elsewhere in the booklet will not be evaluated.**
6. **Kindly write the answers, including sketches, with a blue or black pen. Note that answers written with pencils or pens of other colors will not be evaluated.**
7. You can use the empty reverse sides for rough work. No extra sheets will be provided.
8. You are not allowed to use a calculator or any other electronic device during the quiz. Please note that you will not be permitted to continue with the quiz if you are found with any such device.
9. Make sure that you return question paper cum answer sheet booklet before you leave the examination hall.

For use by examiners
(Do not write in this space)

Q1–Q10	Q11	Q12	Q13	Total

✦ **True or false** (1 mark each, write True (T)/False (F) in the box provided)

1. A comet moving around the Sun in a hyperbolic orbit sweeps equal areas in equal intervals of time.
2. A particle moves in a plane along the logarithmic spiral $\rho = \mathrm{e}^{\phi}$, where ρ and ϕ denote the plane polar coordinates. The angle between the position and the velocity vectors of the particle at any instant is $60°$.
3. Consider the electric field $\vec{\boldsymbol{E}} = -(x+2)\,\hat{\boldsymbol{x}} - y\,\hat{\boldsymbol{y}}$. The electrostatic lines corresponding to the field on the x-y-plane are circles.

✦ **Multiple choice questions** (1 mark each, write the one correct option in the box provided)

4. A particle is moving under the influence of the electric field generated by an infinite plane of positive charge. The normal to the plane is along the positive z-direction. Which of the following sets contains **ALL** the conserved kinematic quantities of the particle? (Note that, E, $\vec{\boldsymbol{p}}$ and $\vec{\boldsymbol{L}}$ denote the energy, momentum and angular momentum of the particle, respectively.)

   [**A**] E [**B**] $p_z,\ \vec{\boldsymbol{L}}$ [**C**] $\vec{\boldsymbol{p}},\ L_z$ [**D**] $E,\ p_x,\ p_y,\ L_z$

5. The orbit of a particle in a central force is a circle which passes through the origin described by $r = r_0 \cos\phi$, where r_0 is a constant. The central force $\vec{\boldsymbol{F}}(r)$ is proportional to

   [**A**] $-\vec{\boldsymbol{r}}/r^6$ [**B**] $-\vec{\boldsymbol{r}}/r^5$ [**C**] $-\vec{\boldsymbol{r}}/r^4$ [**D**] $-\vec{\boldsymbol{r}}/r^3$

6. A satellite moving in a circular orbit of radius R is given a forward thrust leading to a parabolic orbit. If M and m denote the masses of the Earth and satellite, the energy imparted to the satellite will be

   [**A**] $G\,M\,m/R$ [**B**] $G\,M\,m/(2\,R)$ [**C**] $2\,G\,M\,m/R$ [**D**] $4\,G\,M\,m/R$

✦ **Fill in the blanks** (1 mark each, write the answer in the box provided)

7. A particle is moving along the trajectory $\vec{\boldsymbol{r}}(t) = A\,t\,\hat{\boldsymbol{x}} + B\,t^2\,\hat{\boldsymbol{y}}$ in the x-y-plane, where A and B are constants. Express the velocity and acceleration of particle in terms of the plane polar coordinates ρ and ϕ and the corresponding unit vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$.

8. A particle is moving on a circular orbit under the influence of the central force $U(r) = \alpha\,r^2$, where $\alpha > 0$. The radius and energy of the orbit will be

9. The height of a sand hill is described by the function $h(x,y) = h_0 \exp - [(x-1)^2 + (y-2)^2]$, where h_0 is a constant and (x,y) are the positions with respect to a convenient origin. An ant at the position $(x,y) = (1,2)$ intends to climb down. In what direction in the x-y-plane should it move to come down the steepest slope?

10. Given the force $\vec{\boldsymbol{F}} = x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}}$, evaluate the work done to move a particle from the point $(x,y) = (1,0)$ to the point $(0,1)$.

✦ **Questions with detailed answers** (write the calculations and answers within the boxes provided)

11. *Angular velocity of a laminar sheet:* A square laminar sheet of mass M has its corners at $\left(0,-\frac{a}{2},0\right)$, $\left(a,-\frac{a}{2},0\right)$, $\left(a,\frac{a}{2},0\right)$ and $\left(0,\frac{a}{2},0\right)$. Its mass per unit area is given by $\sigma\left(x\right)=\sigma_0\left(1-\frac{x}{a}\right)$. (a) Determine the moment of inertia of the sheet about the y-axis in terms of M and a. (b) The sheet is initially stationary but can rotate about the y-axis. A small piece of putty with mass $\frac{M}{2}$ is fired at the sheet with velocity $\vec{\boldsymbol{v}}=\frac{v_0}{\sqrt{2}}\left(\hat{\boldsymbol{x}}-\hat{\boldsymbol{z}}\right)$ from $\left(0,0,\frac{a}{2}\right)$, where v_0 denotes the speed of the putty. On impact with the sheet, the putty sticks and the sheet begins to rotate with angular velocity $\omega\,\hat{\boldsymbol{y}}$. Evaluate ω in terms of v_0, M and a. (**Ignore gravity and air resistance.**) $\boxed{\text{1+2 marks}}$

12. *Motion in a repulsive potential:* Consider a particle moving under the influence of the repulsive central potential $U(r) = \alpha/r$, where $\alpha > 0$. (a) Express the energy of the particle in terms of the radial velocity of the particle, its angular momentum and the potential $U(r)$. (b) Using conservation of angular momentum, integrate the equation suitably to obtain the orbital trajectory $r(\phi)$ of the particle. $\boxed{\text{1+2 marks}}$

13. *Trajectories in phase space:* Consider a particle moving in the following one-dimensional potential: $U(x) = (\alpha\, x^2 - \beta\, x^3)\, \mathrm{e}^{-\gamma\, x}$, where $\alpha = \beta = \gamma = 1$. (a) Draw the potential $U(x)$, specifically marking the values of the extrema. (b) Determine the range of energy and position for which the system can exhibit bounded motion. (c) Draw the following phase space trajectories indicating the direction of motion with arrows: (i) bounded motion for a positive and negative value of energy, and (ii) unbounded motion for a positive as well as negative value of energy. $\boxed{\text{1.5+0.5+2 marks}}$

Surface and volume elements, and Jacobians

Transformation of an area element

Consider an elemental area in two dimensions, say, in the x-y-plane. The area of the element can be expressed as

$$\mathrm{d}\boldsymbol{A} = \mathrm{d}\boldsymbol{x} \times \mathrm{d}\boldsymbol{y} = (\mathrm{d}x\,\hat{\boldsymbol{x}}) \times (\mathrm{d}y\,\hat{\boldsymbol{y}}) = \mathrm{d}x\,\mathrm{d}y\,(\hat{\boldsymbol{x}} \times \hat{\boldsymbol{y}}) = \mathrm{d}x\,\mathrm{d}y\,\hat{\boldsymbol{z}} = \mathrm{d}A\,\hat{\boldsymbol{z}},$$

where we have set $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$. Let us now evaluate $\mathrm{d}\boldsymbol{A}$ in the plane polar coordinates (ρ, ϕ) defined as

$$x = \rho\cos\phi, \quad y = \rho\sin\phi.$$

We can write

$$\begin{aligned}
\mathrm{d}\boldsymbol{x} &= \frac{\partial x}{\partial \rho}\,\mathrm{d}\boldsymbol{\rho} + \frac{\partial x}{\partial \phi}\,\mathrm{d}\boldsymbol{\phi} = \frac{\partial x}{\partial \rho}\,\mathrm{d}\rho\,\hat{\boldsymbol{\rho}} + \frac{\partial x}{\partial \phi}\,\mathrm{d}\phi\,\hat{\boldsymbol{\phi}},\\
\mathrm{d}\boldsymbol{y} &= \frac{\partial y}{\partial \rho}\,\mathrm{d}\boldsymbol{\rho} + \frac{\partial y}{\partial \phi}\,\mathrm{d}\boldsymbol{\phi} = \frac{\partial y}{\partial \rho}\,\mathrm{d}\rho\,\hat{\boldsymbol{\rho}} + \frac{\partial y}{\partial \phi}\,\mathrm{d}\phi\,\hat{\boldsymbol{\phi}}
\end{aligned}$$

where, recall that, $(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}})$ are the unit vectors in the polar coordinates. Hence, we have

$$\mathrm{d}\boldsymbol{A} = \mathrm{d}\boldsymbol{x} \times \mathrm{d}\boldsymbol{y} = \left(\frac{\partial x}{\partial \rho}\,\frac{\partial y}{\partial \phi} - \frac{\partial x}{\partial \phi}\,\frac{\partial y}{\partial \rho}\right)(\mathrm{d}\boldsymbol{\rho} \times \mathrm{d}\boldsymbol{\phi}) = J(\rho,\phi)\,(\mathrm{d}\boldsymbol{\rho} \times \mathrm{d}\boldsymbol{\phi}) = \rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\left(\hat{\boldsymbol{\rho}} \times \hat{\boldsymbol{\phi}}\right) = \rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\hat{\boldsymbol{z}},$$

where we have made use of the fact that $(\hat{\boldsymbol{\rho}} \times \hat{\boldsymbol{\phi}}) = \hat{\boldsymbol{z}}$. The quantity

$$J(\rho,\phi) = \left(\frac{\partial x}{\partial \rho}\,\frac{\partial y}{\partial \phi} - \frac{\partial x}{\partial \phi}\,\frac{\partial y}{\partial \rho}\right) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \cos\phi & -\rho\sin\phi \\ \sin\phi & \rho\cos\phi \end{vmatrix} = \rho$$

is known as the Jacobian of the transformation from the Cartesian to the plane polar coordinates. Therefore, we have $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y = J(\rho,\phi)\,\mathrm{d}\rho\,\mathrm{d}\phi = \rho\,\mathrm{d}\rho\,\mathrm{d}\phi$.

Transformation of a volume element

Now consider an infinitesimal volume element in three dimensions. In terms of the Cartesian coordinates, the volume element can be written as

$$\mathrm{d}V = \mathrm{d}\boldsymbol{z} \cdot (\mathrm{d}\boldsymbol{x} \times \mathrm{d}\boldsymbol{y}) = \mathrm{d}z\,\hat{\boldsymbol{z}} \cdot (\mathrm{d}x\,\hat{\boldsymbol{x}} \times \mathrm{d}y\,\hat{\boldsymbol{y}}) = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,(\hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{z}}) = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Let us first evaluate the corresponding volume element in cylindrical polar coordinates. Since we have already evaluated $(\mathrm{d}\boldsymbol{x} \times \mathrm{d}\boldsymbol{y})$, we immediately obtain that

$$\mathrm{d}V = \mathrm{d}\boldsymbol{z} \cdot (\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\hat{\boldsymbol{z}}) = \rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}z\,(\hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{z}}) = \rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}z.$$

It is useful to note that the corresponding Jacobian of the transformation

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z,$$

is given by

$$J(\rho,\phi,z) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho.$$

Let us now consider the case of the spherical polar coordinates. In such a case, we have

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\phi,$$

so that

$$\begin{aligned}
\mathrm{d}\boldsymbol{x} &= \frac{\partial x}{\partial r}\,\mathrm{d}\boldsymbol{r} + \frac{\partial x}{\partial \theta}\,\mathrm{d}\boldsymbol{\theta} + \frac{\partial x}{\partial \phi}\,\mathrm{d}\boldsymbol{\phi} = \frac{\partial x}{\partial r}\,\mathrm{d}r\,\hat{\boldsymbol{r}} + \frac{\partial x}{\partial \theta}\,\mathrm{d}\theta\,\hat{\boldsymbol{\theta}} + \frac{\partial x}{\partial \phi}\,\mathrm{d}\phi\,\hat{\boldsymbol{\phi}},\\
\mathrm{d}\boldsymbol{y} &= \frac{\partial x}{\partial r}\,\mathrm{d}\boldsymbol{r} + \frac{\partial y}{\partial \theta}\,\mathrm{d}\boldsymbol{\theta} + \frac{\partial y}{\partial \phi}\,\mathrm{d}\boldsymbol{\phi} = \frac{\partial y}{\partial r}\,\mathrm{d}r\,\hat{\boldsymbol{r}} + \frac{\partial y}{\partial \theta}\,\mathrm{d}\theta\,\hat{\boldsymbol{\theta}} + \frac{\partial y}{\partial \phi}\,\mathrm{d}\phi\,\hat{\boldsymbol{\phi}},\\
\mathrm{d}\boldsymbol{z} &= \frac{\partial x}{\partial r}\,\mathrm{d}\boldsymbol{r} + \frac{\partial z}{\partial \theta}\,\mathrm{d}\boldsymbol{\theta} + \frac{\partial z}{\partial \phi}\,\mathrm{d}\boldsymbol{\phi} = \frac{\partial z}{\partial r}\,\mathrm{d}r\,\hat{\boldsymbol{r}} + \frac{\partial z}{\partial \theta}\,\mathrm{d}\theta\,\hat{\boldsymbol{r}} + \frac{\partial z}{\partial \phi}\,\mathrm{d}\phi\,\hat{\boldsymbol{\phi}},
\end{aligned}$$

where $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}})$ are the orthonormal unit vectors in the spherical polar coordinates. We find that

$$\begin{aligned}
\mathrm{d}\boldsymbol{x}\times\mathrm{d}\boldsymbol{y} &= \left(\frac{\partial x}{\partial \theta}\,\frac{\partial y}{\partial \phi} - \frac{\partial x}{\partial \phi}\,\frac{\partial y}{\partial \theta}\right)\mathrm{d}\theta\,\mathrm{d}\phi\,\hat{\boldsymbol{r}} + \left(\frac{\partial x}{\partial \phi}\,\frac{\partial y}{\partial r} - \frac{\partial x}{\partial r}\,\frac{\partial y}{\partial \phi}\right)\mathrm{d}r\,\mathrm{d}\phi\,\hat{\boldsymbol{\theta}}\\
&\quad + \left(\frac{\partial x}{\partial r}\,\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\,\frac{\partial y}{\partial r}\right)\mathrm{d}r\,\mathrm{d}\theta\,\hat{\boldsymbol{\phi}}
\end{aligned}$$

and hence

$$\begin{aligned}
\mathrm{d}V &= \mathrm{d}\boldsymbol{z}\cdot(\mathrm{d}\boldsymbol{x}\times\mathrm{d}\boldsymbol{y})\\
&= \left[\frac{\partial z}{\partial r}\left(\frac{\partial x}{\partial \theta}\,\frac{\partial y}{\partial \phi} - \frac{\partial x}{\partial \phi}\,\frac{\partial y}{\partial \theta}\right) + \frac{\partial z}{\partial \theta}\left(\frac{\partial x}{\partial \phi}\,\frac{\partial y}{\partial r} - \frac{\partial x}{\partial r}\,\frac{\partial y}{\partial \phi}\right) + \frac{\partial z}{\partial \phi}\left(\frac{\partial x}{\partial r}\,\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\,\frac{\partial y}{\partial r}\right)\right]\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\\
&= J(r,\theta,\phi)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi,
\end{aligned}$$

where the quantity $J(r,\theta,\phi)$ is given by

$$J(r,\theta,\phi) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi\\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi\\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta,$$

so that

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

Evidently, the quantity $J(\rho,\theta,\phi)$ is the Jacobian of the transformation from the Cartesian to spherical polar coordinates.

Exercise sheet 9

Divergence, flux and continuity

1. *Visualizing and testing the divergence theorem:*

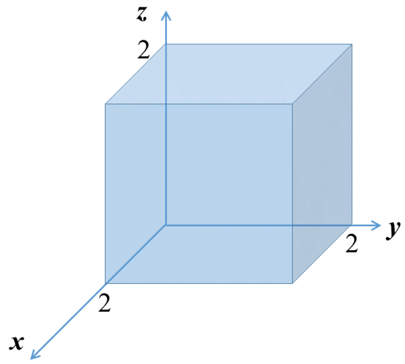
   (a) Sketch example vector fields $\boldsymbol{v}(\boldsymbol{r}, t)$ where the field lines are parallel and have (i) positive, (ii) negative and (iii) zero divergence.

   (b) Sketch example vector fields $\boldsymbol{v}(\boldsymbol{r}, t)$ where the field lines are **not** parallel and have (i) positive, (ii) negative and (iii) zero divergence.

   (c) Test the divergence theorem for the function

   $$\boldsymbol{v} = (xy)\,\hat{\boldsymbol{x}} + (2yz)\,\hat{\boldsymbol{y}} + (3xz)\,\hat{\boldsymbol{z}}\ .$$

   Take as your volume the cube shown in the figure, with side length 2. [DG Problem 1.32]



2. *Field from a spherical shell, right and wrong*

   (a) Use Gauss' Law to show that the electric field outside and an infinitesimal distance away from a uniformly charged spherical shell, with radius R and surface charge density σ, is given by

   $$\boldsymbol{E} = \frac{\sigma}{\epsilon_0}\hat{\boldsymbol{r}}\ .$$

   Also, show that inside the shell the field is zero.

   (b) Now derive this result in the following way, slice the the shell into rings (symmetrically located with respect to the point in question), and then integrate the field contributions from all the rings. You should obtain the incorrect result

   $$\boldsymbol{E} = \frac{\sigma}{2\epsilon_0}\hat{\boldsymbol{r}}\ !$$

   (c) Why isn't this result correct? Explain how to modify it to obtain the correct result. *Hint:* You could very well have performed the above integral in an effort to obtain the electric field an infinitesimal distance inside the shell, where we know the field is zero. Does the integration provide a good description of what's going on for points on the shell that are very close to the point in question? [PM Problem 1.22]

3. *Finding the mass distribution:* The gravitational potential of some configuration of mass is given by

   $$\Phi(r) = GM_0\frac{e^{-\lambda r} - 2}{r}\ ,$$

   where M_0 and λ are constants of appropriate dimensions. Find $\boldsymbol{g}(\boldsymbol{r})$, $\rho(\boldsymbol{r})$ and the total mass M of the configuration. *Hint: You will need to read DG Sec. 1.5.* [Based on DG Problem 2.46]

4. *Boundary conditions for $\boldsymbol{E}$ and $\boldsymbol{B}$ fields:* Use Gauss' Law to show how an electric field $\boldsymbol{E}$ normally incident to an interface with a surface charge density σ changes as it crosses the interface. Similarly find out how a magnetic field $\boldsymbol{B}$ normally incident on the surface behaves. You will need to use the equation $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ that indicates that magnetic monopoles do not exist.

5. *Continuity equation in electromagnetism:*

   (a) A current I is uniformly distributed over a wire of circular cross section, with a radius a. Find the volume current density, $\boldsymbol{J}$?

   (b) If the current density is proportional to the distance from the axis of the wire,

   $$|\boldsymbol{J}| = k\rho \ ,$$

   find the total current in the wire?

   (c) By considering a slice of the wire show that

   $$\oint_{\mathcal{S}} \boldsymbol{J} \cdot d\boldsymbol{A} = 0 \ ,$$

   when you integrate over the whole surface of the slice $\mathcal{S}$.

   (d) From this result motivate that the continuity equation for electromagnetism is

   $$\boldsymbol{\nabla} \cdot \boldsymbol{J} + \frac{\partial \varrho}{\partial t} = 0 \ ,$$

   where ϱ is the charge density. This is the local description of charge conservation. [Based on DG Example 5.4]

Last updated on October 18, 2017

Exercise sheet 9 with solutions

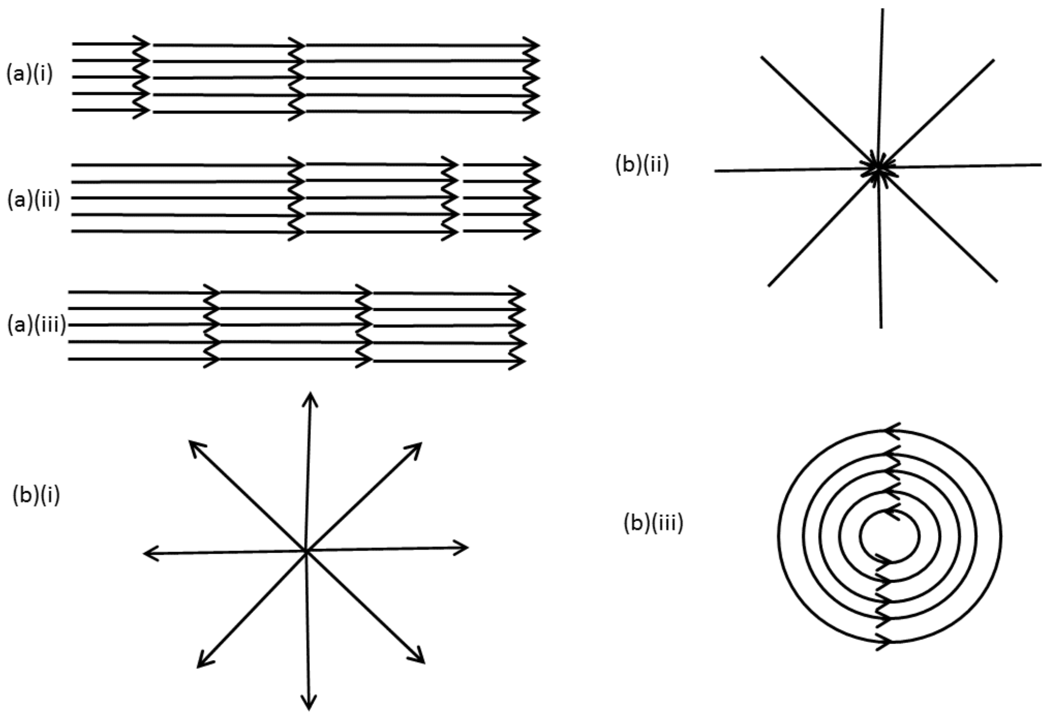
Divergence, flux and continuity

1. *Visualizing and testing the divergence theorem:*

   (a) Sketch example vector fields $\boldsymbol{v}(\boldsymbol{r},t)$ where the field lines are parallel and have (i) positive, (ii) negative and (iii) zero divergence.

   (b) Sketch example vector fields $\boldsymbol{v}(\boldsymbol{r},t)$ where the field lines are **not** parallel and have (i) positive, (ii) negative and (iii) zero divergence.

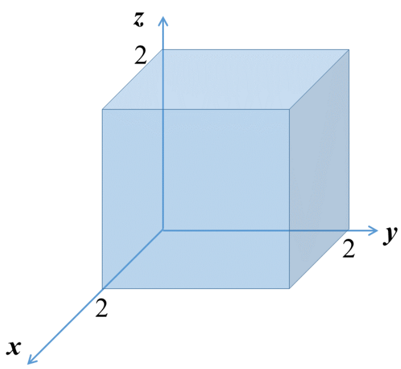
   Solution:



   (c) Test the divergence theorem for the function

$$\boldsymbol{v} = (xy)\,\hat{\boldsymbol{x}} + (2yz)\,\hat{\boldsymbol{y}} + (3xz)\,\hat{\boldsymbol{z}}\ .$$

   Take as your volume the cube shown in the figure, with side length 2. [DG Problem 1.32]



Solution:

The divergence theorem is:

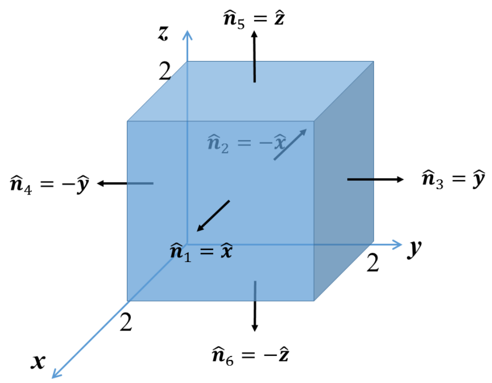
$$\iiint_{\mathcal{V}} \boldsymbol{\nabla}\cdot\boldsymbol{v}\, dV = \oint_{\mathcal{S}} \boldsymbol{v}\cdot d\boldsymbol{a}\ ,$$

where $\mathcal{S}$ is the surface that bounds a volume $\mathcal{V}$. We will calculate the volume integral first. For that we need

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(3xz) = y + 2z + 3x\ .$$

Therefore

$$\begin{aligned}\int_0^2 dx \int_0^2 dy \int_0^2 dx \boldsymbol{\nabla}\cdot\boldsymbol{v} &= \int_0^2 dx \int_0^2 dy \int_0^2 dx\,(y+2z+3x) \\ &= 4\left(\left[\frac{y^2}{2}\right]_0^2 + \left[z^2\right]_0^2 + \left[\frac{3x^2}{2}\right]_0^2\right) \\ &= 4\,(2+4+6) = 48\ .\end{aligned}$$



Now for the surface integral we have to consider each side $\mathcal{S}_i$ separately with the normal vectors $\hat{\boldsymbol{n}}_i$ as shown in the figure above. Therefore,

$$\begin{aligned}\oint_{\mathcal{S}} \boldsymbol{v}\cdot d\boldsymbol{a} &= \iint_{\mathcal{S}_1} \boldsymbol{v}\cdot\hat{\boldsymbol{x}}\, dy\, dz + \iint_{\mathcal{S}_2} \boldsymbol{v}\cdot -\hat{\boldsymbol{x}}\, dy\, dz \\ &+ \iint_{\mathcal{S}_3} \boldsymbol{v}\cdot\hat{\boldsymbol{y}}\, dx\, dz + \iint_{\mathcal{S}_4} \boldsymbol{v}\cdot -\hat{\boldsymbol{y}}\, dx\, dz \\ &+ \iint_{\mathcal{S}_5} \boldsymbol{v}\cdot\hat{\boldsymbol{z}}\, dx\, dy + \iint_{\mathcal{S}_6} \boldsymbol{v}\cdot -\hat{\boldsymbol{z}}\, dx\, dy \\ &= \iint_{\mathcal{S}_1} xy\, dy\, dz - \iint_{\mathcal{S}_2} xy\, dy\, dz \\ &+ \iint_{\mathcal{S}_3} 2yz\, dx\, dz - \iint_{\mathcal{S}_4} 2yz\, dx\, dz \\ &+ \iint_{\mathcal{S}_5} 3xz\, dx\, dy - \iint_{\mathcal{S}_6} 3xz\, dx\, dy\ ,\end{aligned}$$

the second, fourth and sixth terms are all zero because $x = 0$, $y = 0$ and $z = 0$ on the surfaces $\mathcal{S}_2$, $\mathcal{S}_4$ and $\mathcal{S}_6$, respectively. Therefore, we are left with the terms for the integrals over $\mathcal{S}_1$, $\mathcal{S}_3$ and $\mathcal{S}_5$, which have $x = 2$, $y = 2$ and $z = 2$ on the surfaces, respectively. Therefore, we have

$$\begin{aligned}\oint_{\mathcal{S}} \boldsymbol{v}\cdot d\boldsymbol{a} &= \int_0^2\int_0^2 2y\, dy\, dz + \int_0^2\int_0^2 4z\, dx\, dz + \int_0^2\int_0^2 6x\, dx\, dy \\ &= \int_0^2 4y\, dy + \int_0^2 8z\, dz + \int_0^2 12x\, dx \\ &= \left[2y^2\right]_0^2 + \left[4z^2\right]_0^2 + \left[6z^2\right]_0^2 = 12\times 4 = 48\ .\end{aligned}$$

2. <u>*Field from a spherical shell, right and wrong*</u>

(a) Use Gauss' Law to show that the electric field outside and an infinitesimal distance away from a uniformly charged spherical shell, with radius R and surface charge density σ, is given by

$$\boldsymbol{E} = \frac{\sigma}{\epsilon_0}\hat{\boldsymbol{r}} \, .$$

Also, show that inside the shell the field is zero.

<u>Solution:</u> If you consider a spherical Gaussian surface a radial distance r outside the shell, then using spherical symmetry, to assume $\boldsymbol{E} = E_r(r)\hat{\boldsymbol{r}}$, use Gauss' Law

$$\begin{aligned} \oint_S \boldsymbol{E}\cdot d\boldsymbol{A} &= \frac{Q_{\rm enc}}{\epsilon_0} \\ \Rightarrow E_r(r)(4\pi r^2) &= \frac{4\pi R^2\sigma}{\epsilon_0} \\ \Rightarrow \boldsymbol{E} &= \frac{\sigma R^2}{\epsilon_0 r^2}\hat{\boldsymbol{r}} \, . \end{aligned}$$

Therefore, when considering the limit $r \to R$, just outside the surface we get

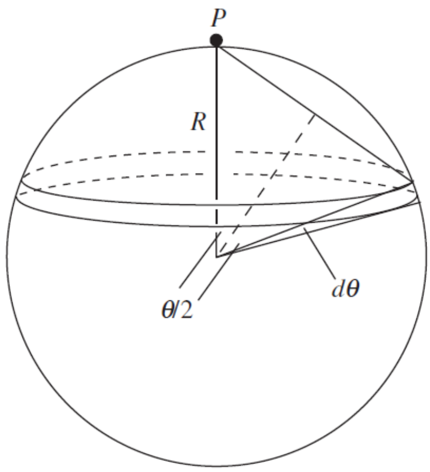
$$\boldsymbol{E}_{\text{just outside}} = \lim_{r\to R}\frac{\sigma R^2}{\epsilon_0 r^2}\hat{\boldsymbol{r}} = \frac{\sigma}{\epsilon_0}\hat{\boldsymbol{r}} \, .$$

For a Gaussian surface inside the shell the enclosed charge $Q_{\rm enc} = 0$ therefore $E_r(r) = 0$.

(b) Now derive this result in the following way, slice the the shell into rings (symmetrically located with respect to the point in question), and then integrate the field contributions from all the rings. You should obtain the incorrect result

$$\boldsymbol{E} = \frac{\sigma}{2\epsilon_0}\hat{\boldsymbol{r}} \; !$$

<u>Solution</u> Let the rings be parameterized by the angle θ down from the top of the sphere, as shown in the Figure.



The width of a ring is $Rd\theta$, and its circumference is $2\pi R\sin\theta$. So its area is $2\pi R^2\sin\theta d\theta$. All points on the ring are a distance $2R\sin\frac{\theta}{2}$ from the given point P, which is infinitesimally close to the top of the shell. Only the vertical component of the field survives, and this brings in a factor of $\sin\frac{\theta}{2}$. The total field at the top of the shell is therefore apparently equal to

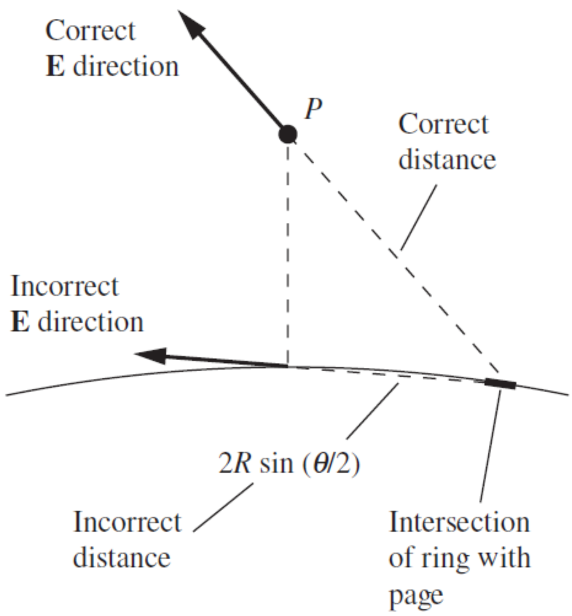
$$\begin{aligned} \frac{1}{4\pi\epsilon_0}\int_0^\pi \frac{\sigma\, 2\pi\, R^2\, \sin\theta\sin\frac{\theta}{2}\, d\theta}{\left(2R\,\sin\frac{\theta}{2}\right)^2} &= \frac{\sigma}{4\epsilon_0}\int_0^\pi \cos\frac{\theta}{2}\, d\theta \\ &= \frac{\sigma}{2\epsilon_0}\left[\sin\tfrac{\theta}{2}\right]_0^\pi = \frac{\sigma}{2\epsilon_0} \, , \end{aligned}$$

where we have used $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$. So this gives the required incorrect result.

(c) Why isn't this result correct? Explain how to modify it to obtain the correct result. *Hint:* You could very well have performed the above integral in an effort to obtain the electric field an infinitesimal distance inside the shell, where we know the field is zero. Does the integration provide a good description of what's going on for points on the shell that are very close to the point in question? [PM Problem 1.22]

<u>*Solution:*</u> As noted in the statement of the problem, it is no surprise that the above result is incorrect, because the same calculation would supposedly yield the field just inside the shell too, where we know it equals zero instead of σ/ϵ_0. The calculation does give the average of these two values as explained below.

The reason why the calculation is invalid is that it doesn't correctly describe the field arising from points on the shell very close to the point P, that is, for rings characterized by $\theta \approx 0$. It is incorrect for two reasons. The closeup view in the figure below shows that the distance from a ring to the given point P is not equal to $2R\sin\frac{\theta}{2}$. Additionally, it shows that the field does not point along the line from the particular point on the ring to the top of the shell. It points more vertically, toward P, so the extra factor of $\sin\frac{\theta}{2}$ we inserted is not correct. No matter how close P is to the shell, we can always zoom in close enough so that the picture looks like the one in the figure below. The only difference is that the more we need to zoom in, the straighter the arc of the circle is. In the limit where P is very close to the shell, the arc is essentially a straight line.



What is true is that if we remove a tiny circular patch from the top of the shell (whose radius is much larger than the distance from P to the shell, but much smaller than the radius of the shell), then the integral in part (b) is valid for the remaining part of the shell. From the form of the integrand, we see that the tiny patch contributes negligibly to the integral. So we can say that the field due to the remaining part of the shell is essentially equal to the above result of $\sigma/2\epsilon_0$. By superposition, the total field due to the entire shell equals this field, plus the field due to the tiny circular patch. But if the point in question is infinitesimally close to the shell, then this tiny patch looks like an infinite plane, the field of which we know is $\sigma/2\epsilon_0$. The desired total field is therefore

$$E_{\text{outside}} = E_{\text{shell minus patch}} + E_{\text{patch}} = \frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0} = \frac{\sigma}{\epsilon_0}\,.$$

By superposition we also obtain the correct field just inside the shell:

$$E_{\text{inside}} = E_{\text{shell minus patch}} - E_{\text{patch}} = \frac{\sigma}{2\epsilon_0} - \frac{\sigma}{2\epsilon_0} = 0\,.$$

The relative minus sign arises because the field from the shell-minus-patch is continuous across the hole, but the field from the patch is not; it points in different directions on either side of the patch.

3. *Finding the mass distribution:* The gravitational potential of some configuration of mass is given by

$$\Phi(r) = GM_0 \frac{e^{-\lambda r} - 2}{r} ,$$

where M_0 and λ are constants of appropriate dimensions. Find $\boldsymbol{g}(\boldsymbol{r})$, $\rho(\boldsymbol{r})$ and the total mass M of the configuration. *Hint: You will need to read DG Sec. 1.5.* [Based on DG Problem 2.46]

Solution: First we find $\boldsymbol{g}$

$$\boldsymbol{g} = -\boldsymbol{\nabla}\Phi = -\frac{\partial \Phi}{\partial r}\hat{\boldsymbol{r}} = GM_0 \left[\left(\frac{1}{r^2} + \frac{\lambda}{r} \right) e^{-\lambda r} - \frac{2}{r^2} \right] \hat{\boldsymbol{r}} .$$

Next we use Gauss' Law for gravitation

$$\begin{aligned}
\rho &= -\frac{1}{4\pi G} \boldsymbol{\nabla} \cdot \boldsymbol{g} \\
&= \frac{M_0}{4\pi} \left[2\boldsymbol{\nabla} \cdot \left(\frac{\hat{\boldsymbol{r}}}{r^2} \right) - \boldsymbol{\nabla} \cdot \left(e^{-\lambda r} \frac{\hat{\boldsymbol{r}}}{r^2} \right) - \boldsymbol{\nabla} \cdot \left(\frac{\lambda e^{-\lambda r}}{r} \hat{\boldsymbol{r}} \right) \right] \\
&= \frac{M_0}{4\pi} \left[2\boldsymbol{\nabla} \cdot \left(\frac{\hat{\boldsymbol{r}}}{r^2} \right) - e^{-\lambda r} \boldsymbol{\nabla} \cdot \left(\frac{\hat{\boldsymbol{r}}}{r^2} \right) - \frac{\hat{\boldsymbol{r}}}{r^2} \cdot \boldsymbol{\nabla} e^{-\lambda r} - \boldsymbol{\nabla} \cdot \left(\frac{\lambda e^{-\lambda r}}{r} \hat{\boldsymbol{r}} \right) \right] ,
\end{aligned}$$

where we have used $\boldsymbol{\nabla} \cdot (f\boldsymbol{A}) = f\boldsymbol{\nabla} \cdot \boldsymbol{A} + \boldsymbol{A} \cdot \boldsymbol{\nabla} f$ to expand the middle term. Now just considering the expression

$$\boldsymbol{\nabla} \cdot \left(\frac{\hat{\boldsymbol{r}}}{r^2} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{1}{r^2} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} (1) = 0 ,$$

and $\iiint_{\mathcal{V}} \boldsymbol{\nabla} \cdot \left(\frac{\hat{\boldsymbol{r}}}{r^2} \right) \, dV = 0$. But if we use the divergence theorem over a spherical surface radius R

$$\oint \frac{\hat{\boldsymbol{r}}}{r^2} \cdot d\boldsymbol{A} = \frac{1}{R^2} 4\pi R^2 = 4\pi \; !$$

This paradox is resolved by introducing the Dirac delta function $\delta^{(3)}(\boldsymbol{r})$ that describes an infinitely high and infinitesimally narrow distribution with a unit volume at the origin. Therefore,

$$\int f(\boldsymbol{r}) \delta^{(3)}(\boldsymbol{r}) \, dV = f(\boldsymbol{r} = 0) .$$

The Dirac delta function has many uses. Remember we defined a point mass as an object with radius tending to zero and density tending to infinity with finite mass m, so mathematically we can describe this as $m\delta^{(3)}(\boldsymbol{r})$ if it is at the origin. Similarly we can define a point charge in this way. Hence we can fix the divergence theorem by defining

$$\boldsymbol{\nabla} \cdot \frac{\hat{\boldsymbol{r}}}{r^2} = 4\pi \delta^{(3)}(\boldsymbol{r}) ,$$

as the divergence in this case is zero everywhere but the origin where it is infinite. (Much more detail can be found in DG Sec. 1.5.)

Now we can proceed from here to find ρ and the total mass.

$$\begin{aligned}
\rho &= \frac{M_0}{4\pi} \left[8\pi \delta^{(3)}(\boldsymbol{r}) - 4\pi e^{-\lambda r} \delta^{(3)}(\boldsymbol{r}) - \frac{\hat{\boldsymbol{r}}}{r^2} \cdot \left(-\lambda e^{-\lambda r} \hat{\boldsymbol{r}} \right) - \frac{1}{r^2} \frac{\partial}{\partial r} \left(r \lambda e^{-\lambda r} \right) \right] \\
&= \frac{M_0}{4\pi} \left[8\pi \delta^{(3)}(\boldsymbol{r}) - 4\pi e^{0} \delta^{(3)}(\boldsymbol{r}) + \frac{\lambda e^{-\lambda r}}{r^2} - \frac{\lambda e^{-\lambda r}}{r^2} + \frac{\lambda^2 e^{-\lambda r}}{r} \right] \\
&= M_0 \delta^{(3)}(\boldsymbol{r}) + M_0 \frac{\lambda^2 e^{-\lambda r}}{4\pi r} ,
\end{aligned}$$

where in the penultimate term we have used the fact that $f(\boldsymbol{r})\delta^{(3)}(\boldsymbol{r}) = f(0)\delta^{(3)}(\boldsymbol{r})$ because $\delta^{(3)}(\boldsymbol{r}) = 0$ everywhere other than $\boldsymbol{r} = 0$.

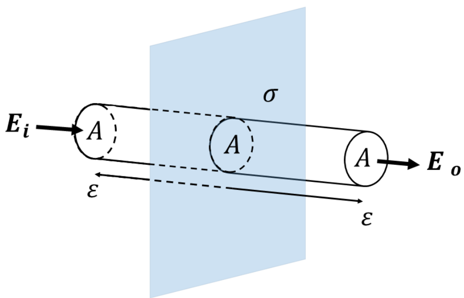
Now the total mass can be found by integrating ρ

$$\begin{aligned} M &= \iiint_{\text{All space}} \rho\, dV = M_0 + \frac{M_0\lambda^2}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^{\infty} dr\, r\, e^{-\lambda r} \\ &= M_0 + M_0\lambda^2 \left(\left[-\frac{1}{\lambda} r e^{-\lambda r} \right]_0^{\infty} + \frac{1}{\lambda}\int_0^{\infty} dr\, e^{-\lambda r} \right) \\ &= M_0 + M_0\lambda^2 \left[-\frac{1}{\lambda^2} e^{-\lambda r} \right]_0^{\infty} = 2M_0 \ , \end{aligned}$$

where the second integral is done by parts.

4. *Boundary conditions for* $\boldsymbol{E}$ *and* $\boldsymbol{B}$ *fields:* Use Gauss' Law to show how an electric field $\boldsymbol{E}$ normally incident to an interface with a surface charge density σ changes as it crosses the interface. Similarly find out how a magnetic field $\boldsymbol{B}$ normally incident on the surface behaves. You will need to use the equation $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ that indicates that magnetic monopoles do not exist.

Solution: Consider a small cylindrical Gaussian volume that spans the surface with distance height ϵ either side and ends of area A as shown in the figure.



Now with a normal field the integral form of Gauss' Law gives

$$\oint_S \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$\Rightarrow \quad \boldsymbol{E}_i \cdot (-A\hat{\boldsymbol{n}}) + \boldsymbol{E}_o \cdot (A\hat{\boldsymbol{n}}) = \frac{\sigma A}{\epsilon_0} \ ,$$

where we have used the fact that the flux over the curved surface of the Gaussian volume is zero,

$$|\boldsymbol{E}_o| - |\boldsymbol{E}_i| = \frac{\sigma}{\epsilon_0}$$

(*c.f* an infinite flat sheet). So it is discontinuous by an amount $\frac{\sigma}{\epsilon_0}$.

Similarly applying Gauss theorem to $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ will lead to

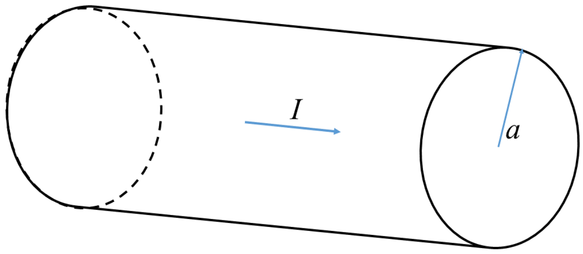
$$|\boldsymbol{B}_o| - |\boldsymbol{B}_i| = 0 \ ,$$

in other words the magnetic field is continuous. These are important results in electro and magnetostatics.

Further, the result can be modified for non-normal incidence, in that the normal component obeys these equations as $\epsilon \to 0$ means there is no contribution from the tangential component to the flux through the curved side of the Gaussian volume.

5. *Continuity equation in electromagnetism:*

   (a) A current I is uniformly distributed over a wire of circular cross section, with a radius a. Find the volume current density, $\boldsymbol{J}$? *Solution:* A sketch of the wire is shown in the figure below.



   As the area perpendicular to the current flow is πa^2 the current density is

$$J = \frac{I}{\pi\, a^2}$$

   (b) If the current density is proportional to the distance from the axis of the wire,

$$|\boldsymbol{J}| = k\rho\ ,$$

   find the total current in the wire?
   Solution: If $\mathcal{S}$ is a surface perpendicular to the flow in the $\hat{\boldsymbol{z}}$ direction, we have

$$\begin{aligned} I &= \int_{\mathcal{S}} \boldsymbol{J} \cdot d\boldsymbol{A} = \int_0^{2\pi} d\phi \int_0^a d\rho\, k\rho^2 \quad (\because d\boldsymbol{A} = \rho d\rho d\phi \hat{\boldsymbol{z}}) \\ &= 2\pi k \left[\frac{\rho^3}{3}\right]_0^a = \frac{2\pi k a^3}{3}\ . \end{aligned}$$

   (c) By considering a slice of the wire show that

$$\oint_{\mathcal{S}} \boldsymbol{J} \cdot d\boldsymbol{A} = 0\ ,$$

   when you integrate over the whole surface of the slice $\mathcal{S}$.
   Solution: First we note that there is no current flowing out of the curved surface so we just consider the planar faces. The only difference is the sign of the normal when we integrate over the surfaces, so the two contributions cancel resulting in zero net current flowing into or out of the slice.

   (d) From this result motivate that the continuity equation for electromagnetism is

$$\boldsymbol{\nabla} \cdot \boldsymbol{J} + \frac{\partial \varrho}{\partial t} = 0\ ,$$

   where ϱ is the charge density. This is the local description of charge conservation. [Based on DG Example 5.4]
   Solution: Considering an arbitrary volume $\mathcal{V}$ with a surface $\mathcal{S}$ the net current in or out I_{total} is

$$I_{\text{total}} = \oint_{\mathcal{S}} \boldsymbol{J} \cdot d\boldsymbol{A} = \iiint_{\mathcal{V}} \boldsymbol{\nabla} \cdot \boldsymbol{J}\, dV\ ,$$

where we have used the divergence theorem. Now in terms of the enclosed charge Q_{enc} its rate of change would be equal in magnitude to I_{total} but of opposite sign,

$$I_{\text{total}} = -\frac{dQ_{\text{enc}}}{dt} = -\frac{d}{dt}\iiint_{\mathcal{V}} \varrho \, dV = \iiint_{\mathcal{V}} -\frac{\partial \varrho}{\partial t} \, dV \ .$$

So we have two separate volume integrals to describe the total current through the surface of **any** volume $\mathcal{V}$. Therefore the integrands have to be equal as well leading to

$$\boldsymbol{\nabla} \cdot \boldsymbol{J} = -\frac{\partial \varrho}{\partial t} \Rightarrow \boldsymbol{\nabla} \cdot \boldsymbol{J} + \frac{\partial \varrho}{\partial t} = 0 \ ,$$

as required.

Last updated on October 17, 2017

Illustrative examples 9

Circulation of vector fields

1. *Visualizing curl I:*

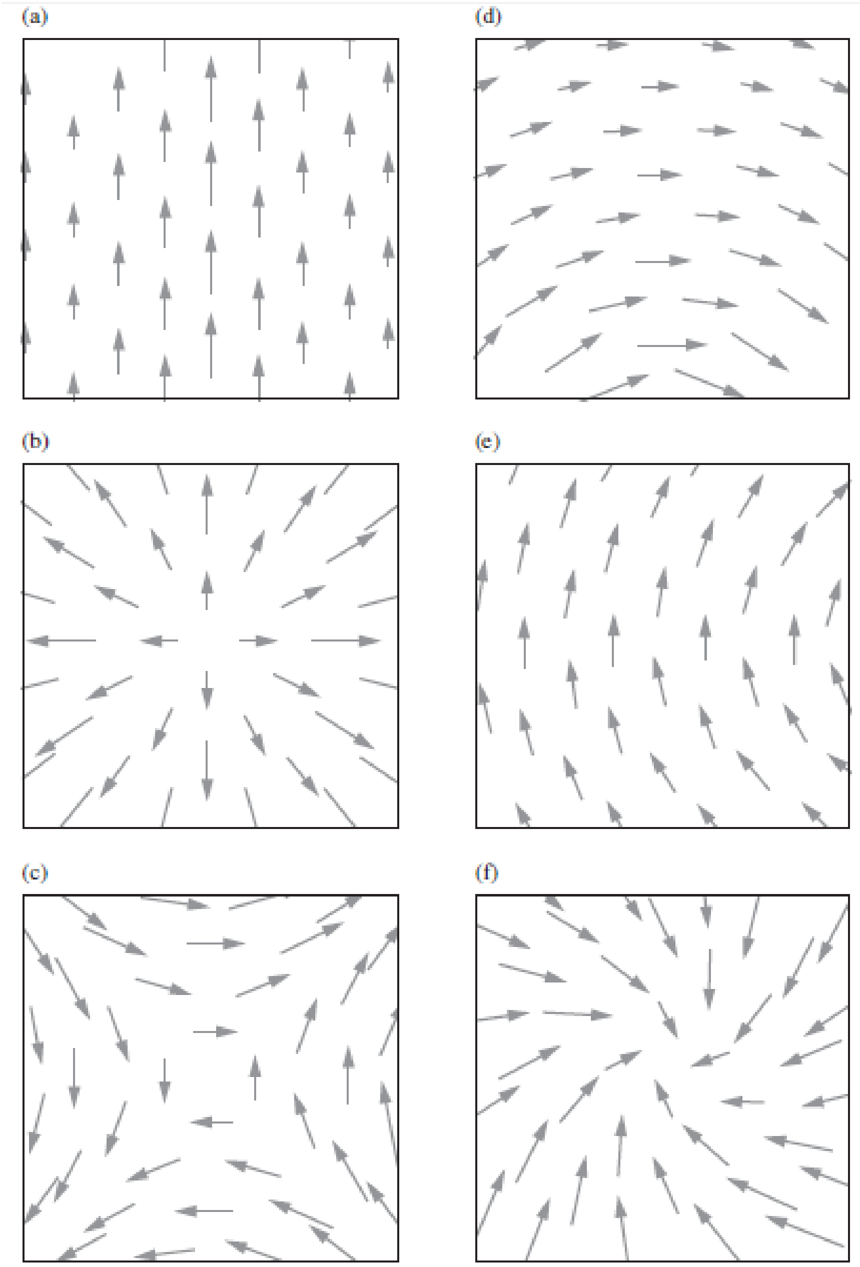
   (a) Consider water moving in a circular path around the origin with some constant angular velocity ω. Sketch the flow and find the curl of the velocity in terms of ω.

   (b) Consider two different velocity fields

   $$\boldsymbol{v}_1 = v_0 e^{-(y/\lambda)^2}\hat{\boldsymbol{y}} \quad \text{and} \quad \boldsymbol{v}_2 = v_0 e^{-(x/\lambda)^2}\hat{\boldsymbol{y}} \ ,$$

   where v_0 and λ are constants of appropriate dimension. Sketch the flow in the $x-y$ plane and calculate $\boldsymbol{\nabla}\times\boldsymbol{v}_{1,2}$. Explain the values of the curl by considering a small paddle wheel with its blades in the $x-y$ plane. [HMS pp85-90]

2. *Visualizing curl II:* Look at the six different $\boldsymbol{E}$ fields below. Three are conservative and three are not. Identify which are which with reasons. [PM pp 98-99]



3. *Curl in curvilinear coordinates:* Show that the curl in cylindrical polar coordinates is given by

$$\nabla \times \boldsymbol{v} = \left[\frac{1}{\rho}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\boldsymbol{\rho}} + \left[\frac{\partial v_\rho}{\partial z} - \frac{\partial v_z}{\partial \rho}\right]\hat{\boldsymbol{\phi}} + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho v_\phi) - \frac{\partial v_\rho}{\partial \phi}\right]\hat{\boldsymbol{z}} \, .$$

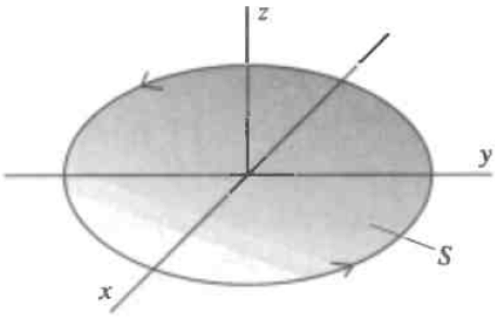
[HMS pp82-84 and Problem III-8]

4. *Is the Coulomb Force Conservative?* Consider the force $\boldsymbol{F}$ on a charge q due to a fixed charge Q at the origin. Use the curl to show that it is conservative and find the corresponding potential energy U. Check that $-\nabla U = \boldsymbol{F}$. [JRT, Example 4.5]

5. *Checking Stokes' theorem:* Confirm Stokes' theorem

$$\oint_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_{\mathcal{S}} \nabla \times \boldsymbol{F} \cdot d\boldsymbol{a} \, ,$$

where $\boldsymbol{F}(x, y, z) = z\hat{\boldsymbol{x}} + x\hat{\boldsymbol{y}} - x\hat{\boldsymbol{z}}$, $\mathcal{C}$ is a circle of radius 1 centered at the origin and lying in the xy-plane, and $\mathcal{S}$ the part of the xy-plane enclosed by the circle (see figure below).



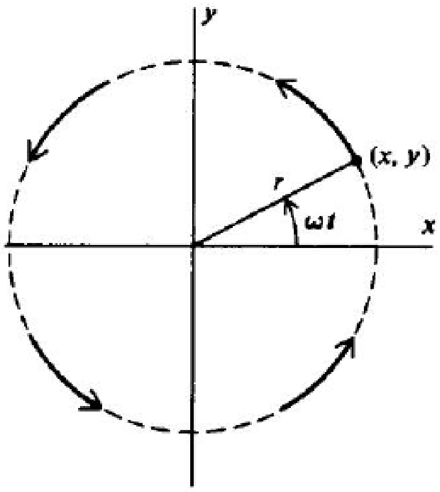
Illustrative examples 9 with solutions

Circulation of vector fields

1. *Visualizing curl I:*

   (a) Consider water moving in a circular path around the origin with some constant angular velocity ω. Sketch the flow and find the curl of the velocity in terms of ω.

   Solution: A sketch of the flow is below.



The velocity is

$$\boldsymbol{v} = \frac{dx}{dt}\hat{\boldsymbol{x}} + \frac{dy}{dt}\hat{\boldsymbol{y}}$$

where $x = r\cos\omega t$ and $y = r\sin\omega t$ and $\dot{r} = 0$. Therefore,

$$\boldsymbol{v} = r\omega\left(-\sin\omega t\hat{\boldsymbol{x}} + \cos\omega t\hat{\boldsymbol{y}}\right) = \omega\left(-y\hat{\boldsymbol{x}} + x\hat{\boldsymbol{y}}\right) \ .$$

Then we can find the curl of the velocity

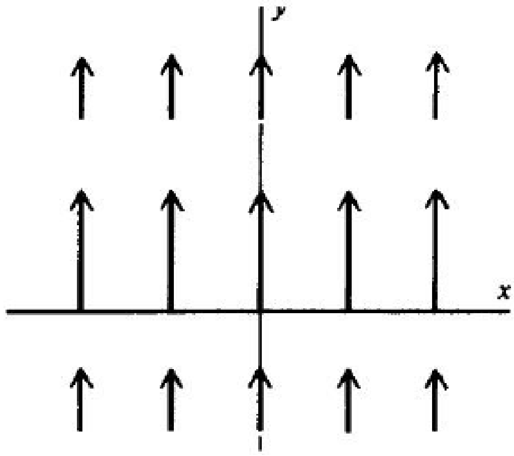
$$\boldsymbol{\nabla}\times\boldsymbol{v} = \omega\begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = 2\omega\hat{\boldsymbol{z}} \ .$$

   (b) Consider two different velocity fields

$$\boldsymbol{v}_1 = v_0 e^{-(y/\lambda)^2}\hat{\boldsymbol{y}} \quad \text{and} \quad \boldsymbol{v}_2 = v_0 e^{-(x/\lambda)^2}\hat{\boldsymbol{y}} \ ,$$

where v_0 and λ are constants of appropriate dimension. Sketch the flow in the $x-y$ plane and calculate $\boldsymbol{\nabla}\times\boldsymbol{v}_{1,2}$. Explain the values of the curl by considering a small paddle wheel with its blades in the $x-y$ plane. [HMS pp85-90]

Solution: A sketch of $\boldsymbol{v}_1$ is below.

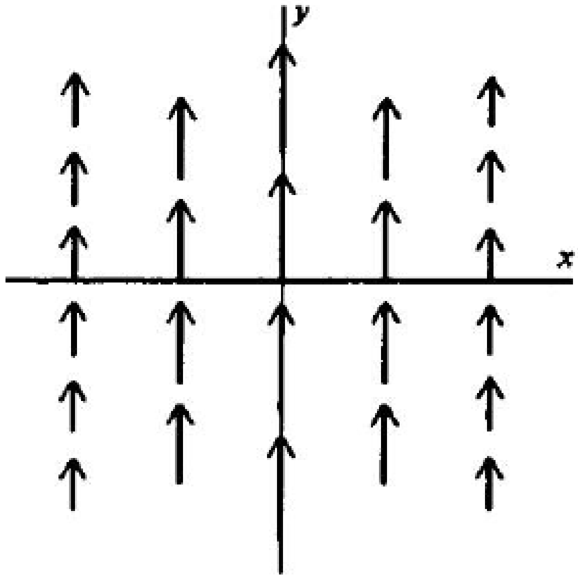


The curl of $\boldsymbol{v}_1$ is

$$\boldsymbol{\nabla} \times \boldsymbol{v}_1 = \omega \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & v_0 e^{-(y/\lambda)^2} & 0 \end{vmatrix} = 0 \, .$$

If you were to place a small paddle wheel anywhere in the flow the torque would always be zero due to the contributions of opposite paddles canceling out.

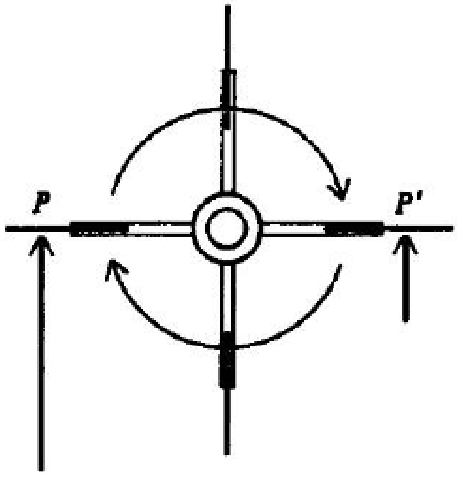
A sketch of $\boldsymbol{v}_2$ is below.



The curl of $\boldsymbol{v}_2$ is

$$\boldsymbol{\nabla} \times \boldsymbol{v}_2 = \omega \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & v_0 e^{-(x/\lambda)^2} & 0 \end{vmatrix} = -\frac{2v_0 x}{\lambda^2} e^{-(x/\lambda)^2} \, \hat{\boldsymbol{z}} \, .$$

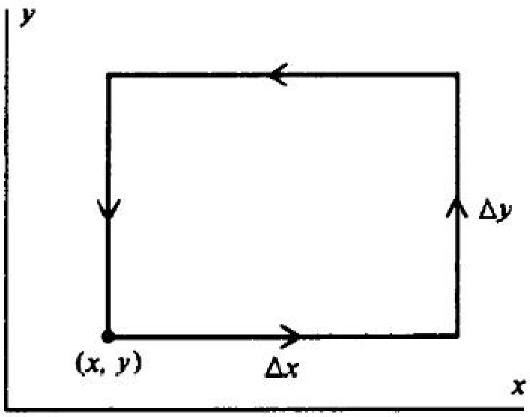
A small paddle wheel placed in this flow pattern would spin, even though the water is moving in the same direction everywhere. The reason this happens can be understood is that the velocity of the water varies with x, so that it strikes one of the paddles (P) in the figure below



with greater velocity than the other (P'). Thus there will be a net torque. The direction of rotation will be clockwise for $x > 0$ and anticlockwise when $x < 0$, which reflected in the sign change of the curl when move x is positive or negative.

In more mathematical terms, the line integral of $\boldsymbol{v}_2$ around a small rectangle (as shown in the figure below) will be different from zero, for while

$$\int_{\text{bottom}} \boldsymbol{v}_2 \cdot d\boldsymbol{r} = \int_{\text{top}} \boldsymbol{v}_2 \cdot d\boldsymbol{r} = 0 \ ,$$



the contributions from the other two sides are

$$\int_{\text{right}} \boldsymbol{v}_2 \cdot d\boldsymbol{r} = v_{2,x}\left(x + \Delta x\right) \Delta y \ ,$$

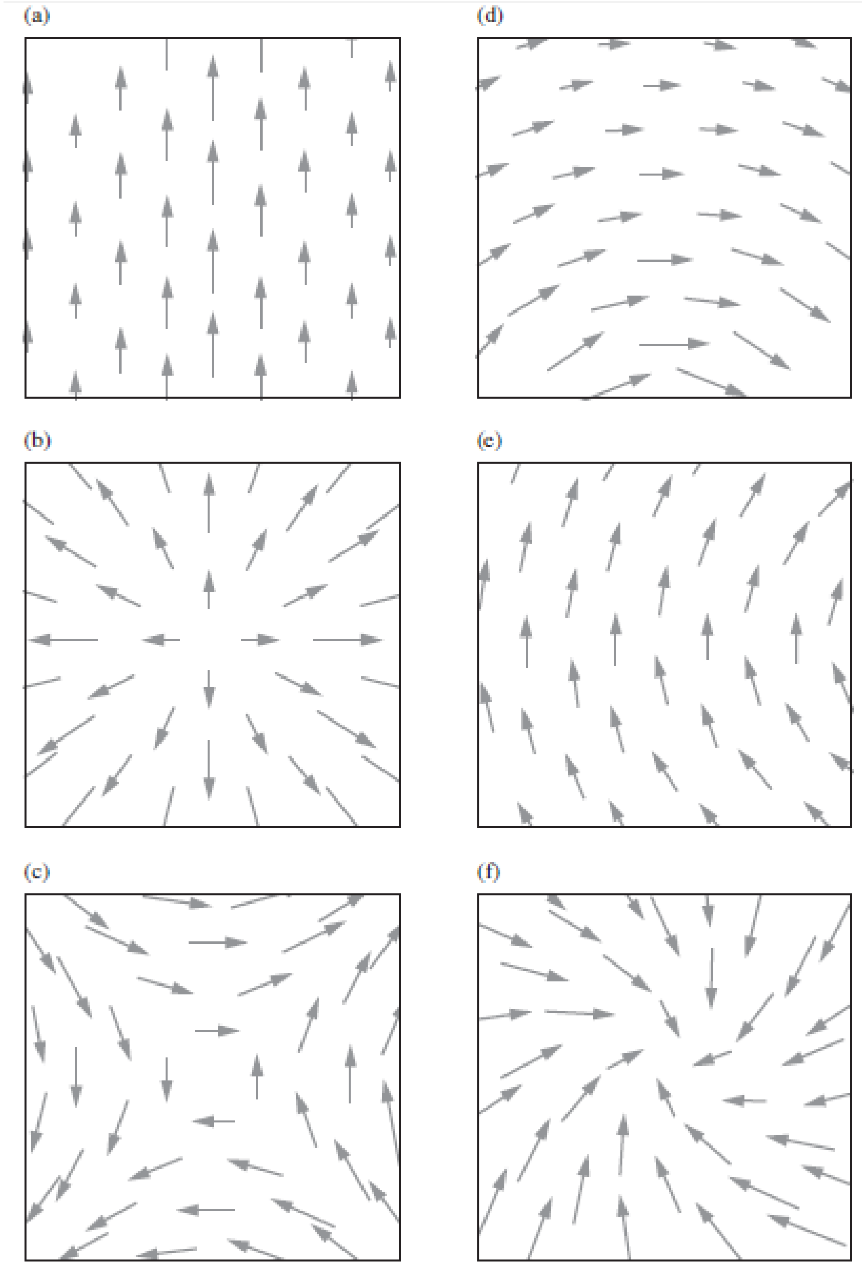
and

$$\int_{\text{left}} \boldsymbol{v}_2 \cdot d\boldsymbol{r} = -v_{2,x}\left(x\right) \Delta y \ .$$

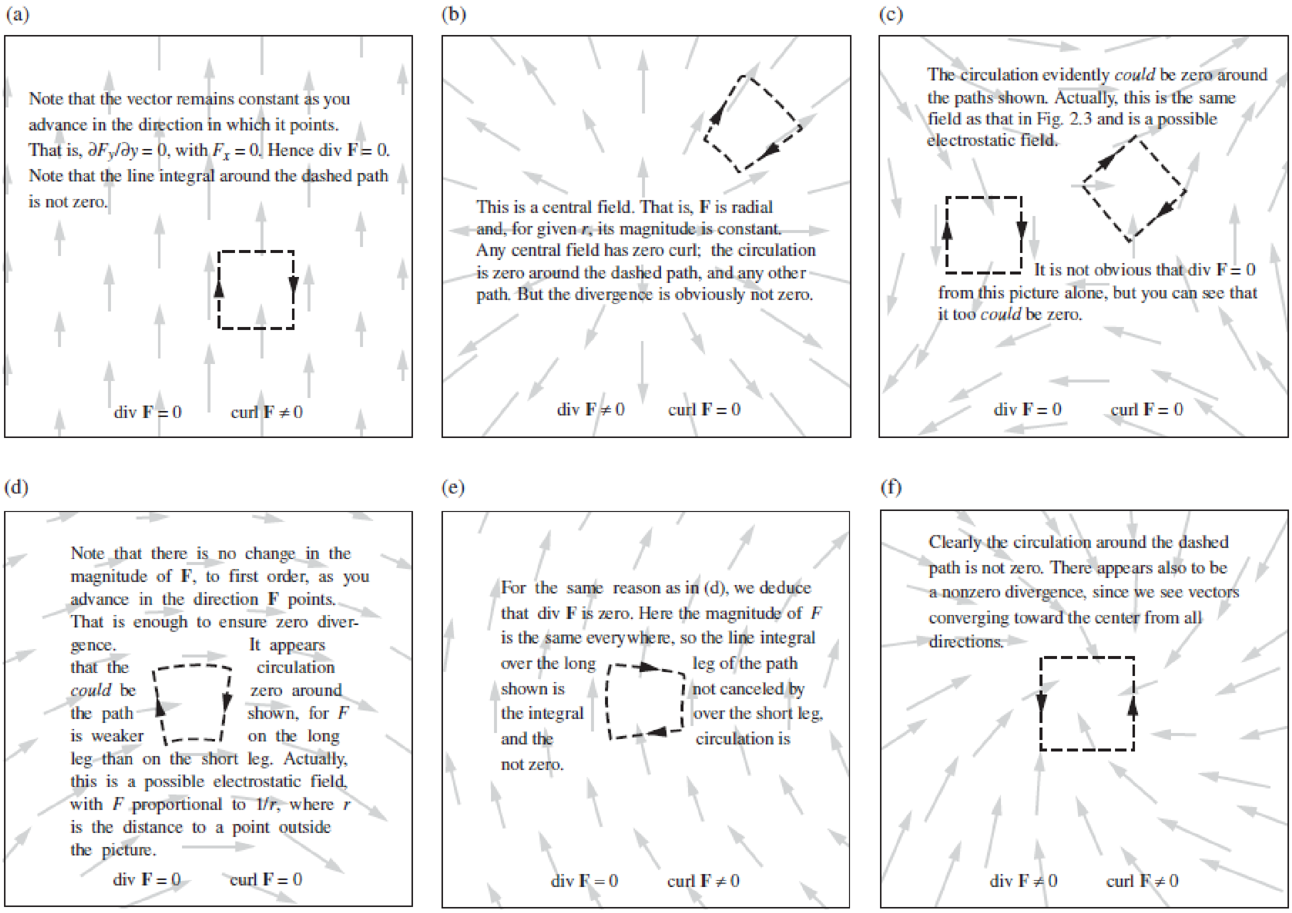
These do not cancel because $v_{2,x}\left(x + \Delta x\right) \neq v_{2,x}\left(x\right)$. Therefore,

$$\oint \boldsymbol{v}_2 \cdot d\boldsymbol{r} \neq 0 \ .$$

2. *Visualizing curl II:* Look at the six different $\boldsymbol{E}$ fields below. Three are conservative and three are not. Identify which are which with reasons. [PM pp 98-99]



Solution:

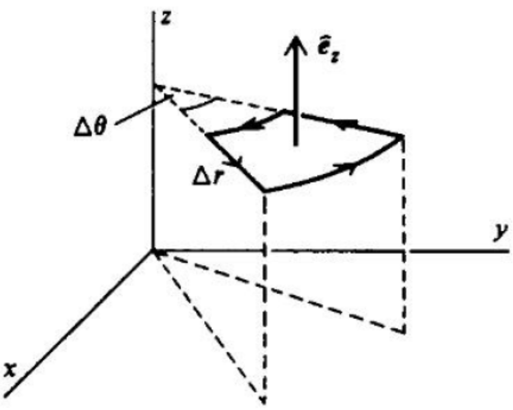


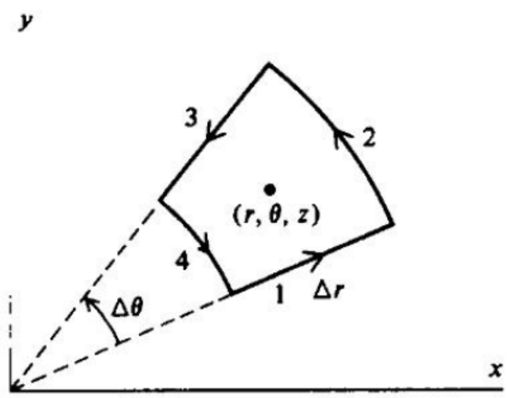
3. *Curl in curvilinear coordinates:* Show that the curl in cylindrical polar coordinates is given by

$$\boldsymbol{\nabla} \times \boldsymbol{v} = \left[\frac{1}{\rho}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\boldsymbol{\rho}} + \left[\frac{\partial v_\rho}{\partial z} - \frac{\partial v_z}{\partial \rho}\right]\hat{\boldsymbol{\phi}} + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho v_\phi) - \frac{\partial v_\rho}{\partial \phi}\right]\hat{\boldsymbol{z}}\ .$$

[HMS pp82-84 and Problem III-8]

Solution: As for the Cartesian derivation we will look at paths that lie in a plane that has one of the unit vectors as its normal. We will start with the z component and consider the path shown in the figure below (here $\hat{\boldsymbol{e}}_z = \hat{\boldsymbol{z}}$, $r = \rho$ and $\theta = \phi$).





The path is such that only one coordinate is varying along each segment *i.e.* ρ along 1 and 3 and ϕ along 2 and 4. The vector field is

$$\boldsymbol{v} = v_\rho \hat{\boldsymbol{\rho}} + v_\phi \hat{\boldsymbol{\phi}} + v_z \hat{\boldsymbol{z}} \ .$$

We will first consider the contribution from segment 1

$$\int_1 \boldsymbol{v} \cdot \hat{\boldsymbol{\rho}} \, ds \approx v_\rho \left(\rho, \phi - \tfrac{\Delta\phi}{2}, z\right) \ \Delta\rho \ ,$$

while along segment 3 it is

$$\int_1 \boldsymbol{v} \cdot (-\hat{\boldsymbol{\rho}}) \ ds \approx -v_\rho \left(\rho, \phi + \tfrac{\Delta\phi}{2}, z\right) \ \Delta\rho \ .$$

The area enclosed by the path is $\Delta S = \rho d\rho d\phi$ so

$$\lim_{\Delta S \to 0} \int_{1+2} \boldsymbol{v} \cdot \boldsymbol{ds} \approx = \lim_{\Delta S \to 0} -\frac{\Delta\rho}{\rho\Delta\rho\Delta\phi} \left[v_\rho \left(\rho, \phi + \tfrac{\Delta\phi}{2}, z\right) - v_\rho \left(\rho, \phi - \tfrac{\Delta\phi}{2}, z\right) \right] = -\frac{1}{\rho} \frac{\partial v_\rho}{\partial \phi} \ ,$$

where the partial derivative is evaluated at (ρ, ϕ, z).

Now we consider segment 2

$$\int_3 \boldsymbol{v} \cdot \hat{\boldsymbol{\phi}} \, ds \approx v_\phi \left(\rho + \tfrac{\Delta\rho}{2}, \phi, z\right) \ \left(\rho + \tfrac{\Delta\rho}{2}\right) \Delta\phi \ ,$$

and segment 4

$$\int_4 \boldsymbol{v} \cdot \left(-\hat{\boldsymbol{\phi}}\right) \ ds \approx -v_\phi \left(\rho - \tfrac{\Delta\rho}{2}, \phi, z\right) \ \left(\rho - \tfrac{\Delta\rho}{2}\right) \Delta\phi \ .$$

Thus

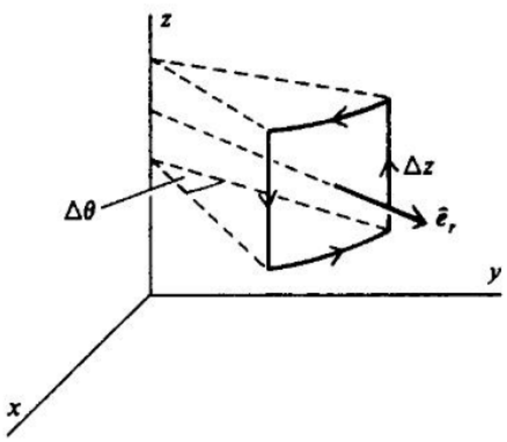
$$\begin{aligned} \lim_{\Delta S \to 0} \int_{3+4} \boldsymbol{v} \cdot \boldsymbol{ds} \approx \ &= \ \lim_{\Delta S \to 0} \frac{\Delta\phi}{\rho\Delta\rho\Delta\phi} \left[v_\phi \left(\rho + \tfrac{\Delta\rho}{2}, \phi, z\right) \ \left(\rho + \tfrac{\Delta\rho}{2}\right) - v_\phi \left(\rho - \tfrac{\Delta\rho}{2}, \phi, z\right) \ \left(\rho - \tfrac{\Delta\rho}{2}\right) \right] \\ &= \ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho v_\phi\right) \ , \end{aligned}$$

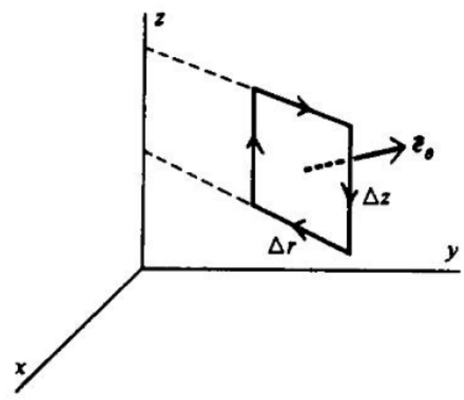
where the partial derivative is evaluated at (ρ, ϕ, z). Therefore,

$$\left(\boldsymbol{\nabla} \times \boldsymbol{v}\right)_z = \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} \left(\rho v_\phi\right) - \frac{\partial v_\rho}{\partial \phi} \right] \ ,$$

as required.

By considering the left and the right figures below one can calculate the ρ and ϕ components of the curl, respectively.





4. Is the Coulomb Force Conservative? Consider the force $\boldsymbol{F}$ on a charge q due to a fixed charge Q at the origin. Use the curl to show that it is conservative and find the corresponding potential energy U. Check that $-\boldsymbol{\nabla}U = \boldsymbol{F}$. [JRT, Example 4.5]

   Solution: The force in question is

$$\boldsymbol{F} = \frac{qQ}{4\pi\epsilon_0 r^2}\,\hat{\boldsymbol{r}} = \frac{\gamma}{r^3}\,\boldsymbol{r}\ ,$$

   where we have introduced the constant $\gamma = qQ/4\pi\epsilon_0$ for convenience. From this we can find the $\boldsymbol{\nabla}\times\boldsymbol{F}$

$$\boldsymbol{\nabla}\times\boldsymbol{F} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\gamma x}{r^3} & \frac{\gamma y}{r^3} & \frac{\gamma z}{r^3} \end{vmatrix}\ .$$

   We will first just consider the x component

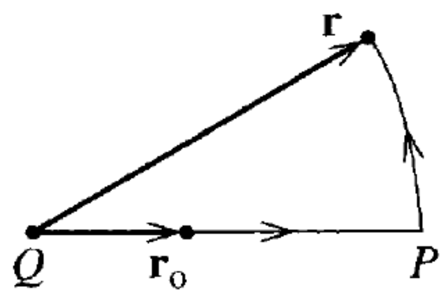
$$(\boldsymbol{\nabla}\times\boldsymbol{F})_x = \gamma z\left(\frac{\partial}{\partial y}r^{-3}\right) - \gamma y\left(\frac{\partial}{\partial z}r^{-3}\right) = \gamma z\left(-3yr^{-5}\right) - \gamma y\left(-3zr^{-5}\right) = 0\ ,$$

   where we have used $r = \sqrt{x^2+y^2+z^2}$ when using the chain rule to calculate the derivatives. Similarly $(\boldsymbol{\nabla}\times\boldsymbol{F})_y = (\boldsymbol{\nabla}\times\boldsymbol{F})_z = 0$; therefore, Coulomb force is conservative.

   Now that we have shown the force is path independent we can workout $U(\boldsymbol{r})$ by choosing a convenient path between $\boldsymbol{r}_0$ and $\boldsymbol{r}$ where $U(\boldsymbol{r}_0) = 0$:

$$U(\boldsymbol{r}) = -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{F}(\boldsymbol{r}')\cdot d\boldsymbol{r}'$$

   We can just choose a radial path between some point at the same radius as $\boldsymbol{r}$ then move radially around the circle. For the first part of the path it is parallel to the force with $dr\hat{\boldsymbol{r}}$ and the second is orthogonal, hence it doesn't make a contribution the integral (See figure below).



   Therefore, we have along this path

$$U(\boldsymbol{r}) = -\int_{r_0}^{r} \frac{\gamma}{\boldsymbol{r}'^2}dr' = \frac{\gamma}{r} - \frac{\gamma}{r_0} = \frac{\gamma}{r}\ ,$$

   where we have taken $r_0 \to \infty$ as the reference point.

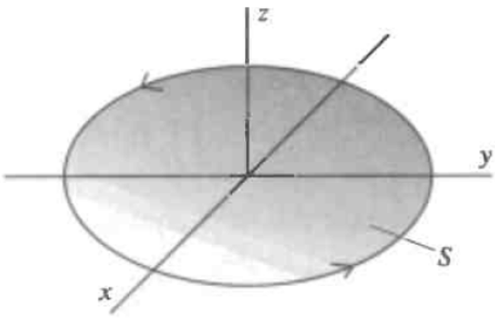
   Finally, we can close the circle and show that $\boldsymbol{F} = -\boldsymbol{\nabla}U$:

$$\boldsymbol{F} = -\boldsymbol{\nabla}U = -\frac{\partial U}{\partial x}\hat{\boldsymbol{x}} - \frac{\partial U}{\partial y}\hat{\boldsymbol{y}} - \frac{\partial U}{\partial z}\hat{\boldsymbol{z}} = \frac{\gamma x}{r^3}\hat{\boldsymbol{x}} + \frac{\gamma y}{r^3}\hat{\boldsymbol{y}} + \frac{\gamma z}{r^3}\hat{\boldsymbol{z}} = \frac{\gamma}{r^2}\,\hat{\boldsymbol{r}}\ .$$

5. Checking Stokes' theorem: Confirm Stokes' theorem

$$\oint_{\mathcal{C}} \boldsymbol{F}\cdot d\boldsymbol{r} = \iint_{\mathcal{S}} \boldsymbol{\nabla}\times\boldsymbol{F}\cdot d\boldsymbol{a}\ ,$$

   where $\boldsymbol{F}(x,y,z) = z\hat{\boldsymbol{x}} + x\hat{\boldsymbol{y}} - x\hat{\boldsymbol{z}}$, $\mathcal{C}$ is a circle of radius 1 centered at the origin and lying in the xy-plane, and $\mathcal{S}$ the part of the xy-plane enclosed by the circle (see figure below).



Solution: We will first calculate the line integral. Around a circular path we should use cylindrical polar coordinates with

$$d\boldsymbol{r} = d\rho\hat{\boldsymbol{\rho}} + \rho d\phi\boldsymbol{\phi} + dz\hat{\boldsymbol{z}} = (\cos\phi d\rho - \rho\sin\phi d\phi)\,\hat{\boldsymbol{x}} + (\sin\phi d\rho + \rho\cos\phi d\phi)\,\hat{\boldsymbol{y}} + dz\hat{\boldsymbol{z}}\ ,$$

and

$$\boldsymbol{F} = z\hat{\boldsymbol{x}} + \rho\cos\phi\hat{\boldsymbol{y}} - \rho\cos\phi\hat{\boldsymbol{z}}\ .$$

Therefore

$$\begin{aligned}
\oint_{\mathcal{C}} \boldsymbol{F}\cdot d\boldsymbol{r} &= \oint_{\mathcal{C}} \left(z\left[\cos\phi d\rho - \rho\sin\phi d\phi\right] + \rho\cos\phi\left[\sin\phi d\rho + \rho\cos\phi d\phi\right] + \rho\cos\phi dz\right) \\
&= \int_0^{2\pi} \cos^2\phi\, d\phi \quad \because dz = d\rho = z = 0 \text{ and } \rho = 1 \text{ on path } \mathcal{C} \\
&= \frac{1}{2}\int_0^{2\pi} (1+\cos 2\phi)\, d\phi \\
&= \pi\ .
\end{aligned}$$

To compute the righthand side of Stokes' theorem we first find the curl of $\boldsymbol{F}$

$$\boldsymbol{\nabla}\times\boldsymbol{F} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & -x \end{vmatrix} = 2\hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}\ .$$

Therefore,

$$\iint_{\mathcal{S}} \boldsymbol{\nabla}\times\boldsymbol{F}\cdot d\boldsymbol{a} = \iint_{\mathcal{S}} (-2\hat{\boldsymbol{y}} + \hat{\boldsymbol{z}})\cdot(\hat{\boldsymbol{z}}da) = \int_0^{2\pi} d\phi \int_0^1 d\rho\rho = \pi\ ,$$

where we have used $da = \rho d\rho d\phi$. Hence Stokes' theorem is verified.

Exercise Sheet 10

Circulation of vector fields and fluids

1. *Identifying conservative forces* Which of the following forces is/are conservative?

   (a) $\boldsymbol{F} = k\,(x, 2y, 3z)$, where k is a constant.

   (b) $\boldsymbol{F} = k\,(y, x, 0)$.

   (c) $\boldsymbol{F} = k\,(-y, x, 0)$.

   For those that are conservative, find the corresponding potential energy U, and verify by direct differentiation that $\boldsymbol{F} = -\boldsymbol{\nabla} U$. [JRT Problem 4.23]

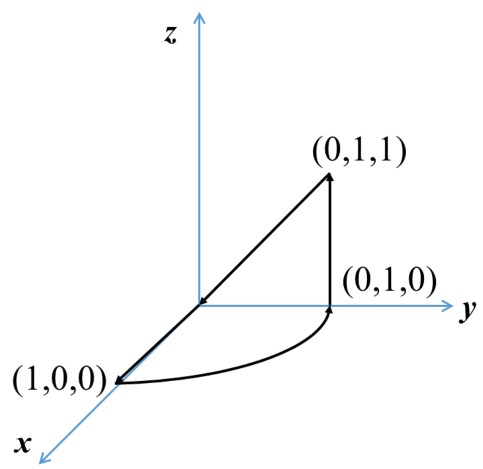
2. *Some useful results involving the curl*

   (a) Show that
   $$\boldsymbol{\nabla} \times \left(\frac{\boldsymbol{A} \times \boldsymbol{r}}{2}\right) = \boldsymbol{A}\,.$$
   where $\boldsymbol{A}$ is a constant vector. [HMS Problem III.6]

   (b) Show that $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{F}) = 0$. [HMS Problem III-7]

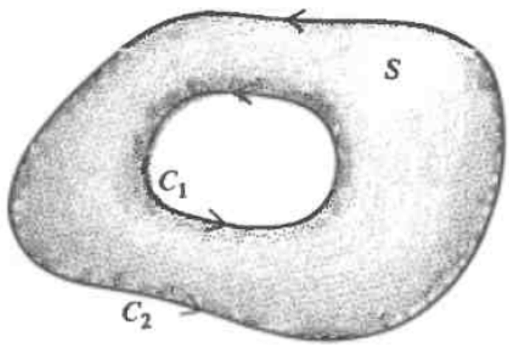
   (c) Show that any central force of the form $\boldsymbol{F}\,(r) = f\,(r)\,\hat{\boldsymbol{r}}$ is irrotational (that is, $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$). [HMS Problem III-12]

3. *Verifying Stokes' theorem* Compute the line integral of
   $$\boldsymbol{v} = \left(r\cos^2\theta\right)\hat{\boldsymbol{r}} - \left(r\cos\theta\sin\theta\right)\hat{\boldsymbol{\theta}} + 3r\hat{\boldsymbol{\phi}}\,,$$
   around the path shown in the figure below. (Points are labeled with Cartesian coordinates.) The curved section in the xy plane is part of a circle with unit radius. Use either spherical or cylindrical polar coordinates. Check your answer with Stokes' theorem. [DG Problem 1.56]



4. *Derivation of Ampère's Law*

   (a) Consider a vector function with the property $\nabla \times \boldsymbol{F} = 0$ everywhere on two closed curves C_1 and C_2, as well as on any capping surface S of the region enclosed by them (see the figure).



   Show the circulation of $\boldsymbol{F}$ around C_1 equals the circulation of $\boldsymbol{F}$ around C_2. In calculating the circulations direct the curves as in the figure.

   (b) The magnetic field due to an infinitely long straight wire along the z axis carrying a uniform current I is
$$\boldsymbol{B} = \frac{\mu_0 I}{2\pi\rho}\hat{\boldsymbol{\phi}} \ .$$
   Show that $\nabla \times \boldsymbol{B} = 0$ everywhere apart from $\rho = 0$.

   (c) Prove Ampère's circuital law for the field of the wire given in part (b). [*Hint* Use the result of (b) to find the circulation of $\boldsymbol{B}$ about a circle with the wire passing through its centre and normal to its plane. Then use the result of part (a) to relate this circulation around an arbitrary curve enclosing the current.] [HMS Problem III-16]

5. *Euler's equation and Bernoulli's theorem*

   (a) Consider a volume of inviscid (zero viscosity) fluid. The total force acting on the surface of the volume is
$$\boldsymbol{F}_{\text{sur}} = -\oint p \, d\boldsymbol{a} \ ,$$
   where p is the pressure and the integral is over the surface of the volume. Show that
$$-\oint p \, d\boldsymbol{a} = -\iiint \nabla p \, dV \ .$$
   [*Hint:* Multiply p by a constant vector then use the divergence theorem.]

   (b) Use Newton's II Law to show that for a small volume dV of the inviscid fluid with density ρ, subject to the surface force and gravity, follows
$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{v} = -\frac{\nabla p}{\rho} + \boldsymbol{g} \ ,$$
   where $\boldsymbol{g} = g\hat{\boldsymbol{z}}$. This is Euler's equation.

   (c) Use Euler's equation for an incompressible fluid (i.e. ρ constant), to show that for steady flow (*i.e.* p and $\boldsymbol{v}$ are constant at any point $\boldsymbol{r}$)
$$\Psi = \frac{1}{2}\rho v^2 + \rho g z + p \ ,$$
   is constant. This is Bernoulli's theorem.

Exercise sheet 10 with solutions

Circulation of vector fields and fluids

1. *Identifying conservative forces* Which of the following forces is/are conservative?

   (a) $\boldsymbol{F} = k\,(x, 2y, 3z)$, where k is a constant.

   (b) $\boldsymbol{F} = k\,(y, x, 0)$.

   (c) $\boldsymbol{F} = k\,(-y, x, 0)$.

   For those that are conservative, find the corresponding potential energy U, and verify by direct differentiation that $\boldsymbol{F} = -\boldsymbol{\nabla} U$. [JRT Problem 4.23]

   Solution

   (a) To find whether the force is conservative or not we use the curl

   $$\boldsymbol{\nabla} \times \boldsymbol{F} = k \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 2y & 3z \end{vmatrix} = 0\ .$$

   As the curl is zero we conclude it is conservative. Therefore, we can find $U(x, y, z)$ by considering the components starting with that in x

   $$F_x = kx = -\frac{\partial U}{\partial x} \Rightarrow U = -\tfrac{1}{2}kx^2 + f(y, z)\ ,$$

   where $f(y, z)$ is an arbitrary function of y and z. Next we can consider the y component

   $$F_y = 2ky = -\frac{\partial U}{\partial y} = -\frac{\partial f}{\partial y} \Rightarrow f = -ky^2 + g(z) \Rightarrow U = -k\left(\tfrac{1}{2}x^2 + y^2\right) + g(z)\ ,$$

   where $g(z)$ is an arbitrary function of z. Finally, we consider the z component

   $$F_z = 3kz = -\frac{\partial U}{\partial z} = -\frac{\partial g}{\partial z} \Rightarrow g = -\tfrac{3}{2}kz^2 + C \Rightarrow U = -k\left(\tfrac{1}{2}x^2 + y^2 + \tfrac{3}{2}z^2\right) + C\ ,$$

   where C is an arbitrary constant.
   We can finally check that this $U(x, y, z)$ gives the correct force by computing

   $$\boldsymbol{F} = -\boldsymbol{\nabla} U = -\frac{\partial U}{\partial x}\hat{\boldsymbol{x}} - \frac{\partial U}{\partial y}\hat{\boldsymbol{y}} - \frac{\partial U}{\partial z}\hat{\boldsymbol{z}} = k\left(x\hat{\boldsymbol{x}} + 2y\hat{\boldsymbol{y}} + 3z\hat{\boldsymbol{z}}\right)\ ,$$

   as required.

   (b) As before we first compute the curl

   $$\boldsymbol{\nabla} \times \boldsymbol{F} = k \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & 0 \end{vmatrix} = k\left[\frac{\partial y}{\partial y} - \frac{\partial x}{\partial x}\right]\hat{\boldsymbol{z}} = (1-1)\hat{\boldsymbol{z}} = 0\ .$$

   Therefore, the force is conservative. As before we can find $U(x, y, z)$ by considering the components, starting with that in x

   $$F_x = ky = -\frac{\partial U}{\partial x} \Rightarrow U = -kxy + f(y, z)\ ,$$

   where $f(y, z)$ is an arbitrary function of y and z. Next we can consider the y component

   $$F_y = kx = -\frac{\partial U}{\partial y} = kx - \frac{\partial f}{\partial y} \Rightarrow \frac{\partial f}{\partial y} = 0 \Rightarrow U = -kxy + f(z)\ ,$$

where f is only a function of z or constant. Now considering the z component

$$F_z = 0 = -\frac{\partial U}{\partial z} = -\frac{\partial f}{\partial z} \Rightarrow U = -kxy + C \ ,$$

as f is independent of z. We now make the check that $U(x,y,z)$ gives the correct force by computing

$$\boldsymbol{F} = -\boldsymbol{\nabla} U = -\frac{\partial U}{\partial x}\hat{\boldsymbol{x}} - \frac{\partial U}{\partial y}\hat{\boldsymbol{y}} - \frac{\partial U}{\partial z}\hat{\boldsymbol{z}} = y\hat{\boldsymbol{x}} + x\hat{\boldsymbol{y}} \ ,$$

as required.

(c) As in (a) and (b) we first compute the curl

$$\boldsymbol{\nabla} \times \boldsymbol{F} = k \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & 0 \end{vmatrix} = k \left[\frac{\partial(-y)}{\partial y} - \frac{\partial x}{\partial x} \right] \hat{\boldsymbol{z}} = (-1-1)\hat{\boldsymbol{z}} = -2\hat{\boldsymbol{z}} \ .$$

Therefore, the force is not conservative.

2. <u>*Some useful results involving the curl*</u>

(a) Show that

$$\boldsymbol{\nabla} \times \left(\frac{\boldsymbol{A} \times \boldsymbol{r}}{2} \right) = \boldsymbol{A} \ ,$$

where $\boldsymbol{A}$ is a constant vector. [HMS Problem III.6]

<u>*Solution:*</u> First let us compute $\boldsymbol{A} \times \boldsymbol{r}$

$$\boldsymbol{A} \times \boldsymbol{r} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ A_x & A_y & A_z \\ x & y & z \end{vmatrix} = [A_y z - A_z y]\,\hat{\boldsymbol{x}} + [A_z x - A_x z]\,\hat{\boldsymbol{y}} + [A_x y - A_y x]\,\hat{\boldsymbol{z}} \ .$$

Next we compute

$$\begin{aligned}
\boldsymbol{\nabla} \times \left(\frac{\boldsymbol{A} \times \boldsymbol{r}}{2} \right) &= \frac{1}{2} \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_y z - A_z y & A_z x - A_x z & A_x y - A_y x \end{vmatrix} \\
&= \tfrac{1}{2} \left[\frac{\partial}{\partial y}(A_x y - A_y x) - \frac{\partial}{\partial z}(A_z x - A_x y) \right] \hat{\boldsymbol{x}} \\
&+ \tfrac{1}{2} \left[\frac{\partial}{\partial z}(A_y z - A_z y) - \frac{\partial}{\partial x}(A_x y - A_y x) \right] \hat{\boldsymbol{y}} \\
&+ \tfrac{1}{2} \left[\frac{\partial}{\partial x}(A_z x - A_x z) - \frac{\partial}{\partial y}(A_y z - A_z y) \right] \hat{\boldsymbol{z}} \\
&= \tfrac{1}{2}[A_x - (-A_x)]\,\hat{\boldsymbol{x}} + \tfrac{1}{2}[A_y - (-A_y)]\,\hat{\boldsymbol{y}} + \tfrac{1}{2}[A_z - (-A_z)]\,\hat{\boldsymbol{z}} \\
&= A_x\,\hat{\boldsymbol{x}} + A_y\,\hat{\boldsymbol{y}} + A_z\,\hat{\boldsymbol{z}} = \boldsymbol{A} \ ,
\end{aligned}$$

as required.

(b) Show that $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{F}) = 0$. [HMS Problem III-7]

<u>*Solution*</u> We write out the curl in components

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \left[\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right] \hat{\boldsymbol{x}} + \left[\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right] \hat{\boldsymbol{y}} + \left[\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right] \hat{\boldsymbol{z}} \ .$$

Therefore,

$$\begin{aligned}
\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{F}) &= \frac{\partial}{\partial x}\left[\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right] + \frac{\partial}{\partial y}\left[\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right] + \frac{\partial}{\partial z}\left[\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right] \\
&= \frac{\partial^2 F_z}{\partial x \partial y} - \frac{\partial^2 F_y}{\partial x \partial z} + \frac{\partial^2 F_x}{\partial y \partial z} - \frac{\partial^2 F_z}{\partial y \partial x} + \frac{\partial^2 F_y}{\partial z \partial x} - \frac{\partial^2 F_x}{\partial z \partial y} = 0 \ ,
\end{aligned}$$

because $\frac{\partial^2 F_z}{\partial x \partial y} = \frac{\partial^2 F_z}{\partial y \partial x}$ (*i.e.* the order of partial differentiation does not matter) and similar for the other second-order derivatives.

(c) Show that any central force of the form $\boldsymbol{F}(r) = f(r)\,\hat{\boldsymbol{r}}$ is irrotational (that is, $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$). [HMS Problem III-12]

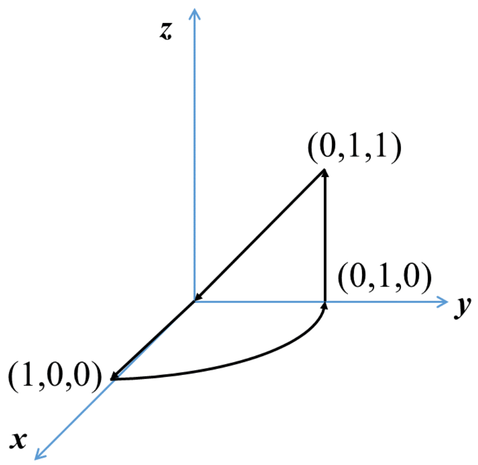
Solution This is most simply shown using the curl in spherical polar coordinates:

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \begin{vmatrix} \frac{\hat{\boldsymbol{r}}}{r^2 \sin\theta} & \frac{\hat{\boldsymbol{\theta}}}{r\sin\theta} & \frac{\hat{\boldsymbol{\phi}}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ f(r) & 0 & 0 \end{vmatrix} = 0\,.$$

3. *Verifying Stokes' theorem* Compute the line integral of

$$\boldsymbol{v} = \left(r\cos^2\theta\right)\hat{\boldsymbol{r}} - \left(r\cos\theta\sin\theta\right)\hat{\boldsymbol{\theta}} + 3r\hat{\boldsymbol{\phi}}\,,$$

around the path shown in the figure below. (Points are labeled with Cartesian coordinates.) The curved section in the xy plane is part of a circle with unit radius. Use either spherical or cylindrical polar coordinates. Check your answer with Stokes' theorem. [DG Problem 1.56]



Solution: For the line integral we perform it along the four different segments, starting from the origin, then add the contributions.

(1) The path direction is radial so $\boldsymbol{dl} = dr\,\hat{\boldsymbol{r}}$, with $\theta = \frac{\pi}{2}$ and $\phi = 0$ along the x axis. Therefore,

$$\boldsymbol{v}\cdot\boldsymbol{dl} = v_r\,dr = r\,\cos^2\theta\,dr = 0 \quad \because \theta = \tfrac{\pi}{2}\,.$$

(2) For the arc in the xy plane the path is azimuthal with $\boldsymbol{dl} = r\,\sin\theta d\phi\,\hat{\boldsymbol{\phi}}$ and $r = 1$, $\theta = \frac{\pi}{2}$ and $\phi\,: 0 \to \frac{\pi}{2}$. Therefore,

$$\boldsymbol{v}\cdot\boldsymbol{dl} = v_\phi\, r\,\sin\theta\,dr = 3r^2\sin\theta\,d\phi = 3\,d\phi \quad \because \theta = \tfrac{\pi}{2} \text{ and } r = 1\,,$$

which when integrated along the path gives

$$\int_2 \boldsymbol{v}\cdot\boldsymbol{dl} = 3\int_0^{\frac{\pi}{2}} \mathrm{d}\phi = \frac{3\pi}{2}\,.$$

(3) For the straight line from $(0,1,0)$ to $(0,1,1)$ we are in the yz plane so $\phi = \frac{\pi}{2}$, therefore $y = r\sin\theta\sin\phi = r\sin\theta = 1$ in this plane. Furthermore, the path has $\theta\,:\frac{\pi}{2} \to \frac{\pi}{4}$. We can use $r = 1/\sin\theta$ to derive an equation for dr in terms of θ

$$dr = -\frac{\cos\theta}{\sin^2\theta}\,d\theta\,.$$

We can now calculate the line integrand

$$\begin{aligned}
\boldsymbol{v}\cdot d\boldsymbol{l} &= v_r\, dr + v_\theta\, r d\theta \quad \because d\boldsymbol{l} = dr\,\hat{\boldsymbol{r}} + r\, d\theta\hat{\boldsymbol{\theta}} \\
&= r\,\cos^2\theta\, dr - r^2\cos\theta\sin\theta\, d\theta = \frac{\cos^2\theta}{\sin\theta}\left(-\frac{\cos\theta}{\sin^2\theta}\,d\theta\right) - \frac{\cos\theta\sin\theta}{\sin^2\theta}d\theta \\
&= -\left(\frac{\cos^3\theta}{\sin^3\theta} + \frac{\cos\theta}{\sin\theta}\right)\,d\theta = -\frac{\cos\theta}{\sin\theta}\left(\frac{\cos^2\theta+\sin^2\theta}{\sin^2\theta}\right)\,d\theta = -\frac{\cos\theta}{\sin^3\theta}\,d\theta\,.
\end{aligned}$$

Therefore,

$$\int_3 \boldsymbol{v}\cdot d\boldsymbol{l} = -\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \frac{\cos\theta\, d\theta}{\sin^3\theta} = -\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \frac{d(\sin\theta)}{\sin^3\theta} = \left[\frac{1}{2\sin^2\theta}\right]_{\frac{\pi}{2}}^{\frac{\pi}{4}} = 1 - \frac{1}{2} = \frac{1}{2}\,.$$

(4) The last segment is radial but with $r : \sqrt{2} \to 0$ and $\theta = \frac{\pi}{4}$ and $\phi = \frac{\pi}{2}$, so $\boldsymbol{v}\cdot d\boldsymbol{l} = r\cos^2\theta dr = \frac{1}{2}r dr$. Therefore,

$$\int_4 \boldsymbol{v}\cdot d\boldsymbol{l} = \frac{1}{2}\int_{\sqrt{2}}^{0} r\, dr = \left[\frac{1}{2}\frac{r^2}{2}\right]_{\sqrt{2}}^{0} = -\frac{1}{4}\cdot 2 = -\frac{1}{2}\,.$$

We now sum the segments to get the total about the path

$$\oint \boldsymbol{v}\cdot d\boldsymbol{l} = 0 + \frac{3\pi}{2} + \frac{1}{2} - \frac{1}{2} = \frac{3\pi}{2}\,.$$

To check the solution with Stokes' theorem we first compute the curl

$$\begin{aligned}
\boldsymbol{\nabla}\times\boldsymbol{v} &= \begin{vmatrix} \frac{\hat{\boldsymbol{r}}}{r^2\sin\theta} & \frac{\hat{\boldsymbol{\theta}}}{r\sin\theta} & \frac{\hat{\boldsymbol{\phi}}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ v_r & r\,v_\theta & r\sin\theta\, v_\phi \end{vmatrix} = \begin{vmatrix} \frac{\hat{\boldsymbol{r}}}{r^2\sin\theta} & \frac{\hat{\boldsymbol{\theta}}}{r\sin\theta} & \frac{\hat{\boldsymbol{\phi}}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ r\cos^2\theta & -r^2\cos\theta\sin\theta & 3r^2\sin\theta \end{vmatrix} \\
&= \frac{\hat{\boldsymbol{r}}}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}3r^2\sin\theta - \frac{\partial}{\partial\phi}\left(-r^2\cos\theta\sin\theta\right)\right] \\
&\quad + \frac{\hat{\boldsymbol{\theta}}}{r\sin\theta}\left[\frac{\partial}{\partial\phi}r\cos^2\theta - \frac{\partial}{\partial r}3r^2\sin\theta\right] \\
&\quad + \frac{\hat{\boldsymbol{\phi}}}{r}\left[\frac{\partial}{\partial r}\left(-r^2\cos\theta\sin\theta\right) - \frac{\partial}{\partial\theta}r\cos^2\theta\right] \\
&= \frac{3\cos\theta}{\sin\theta}\hat{\boldsymbol{r}} - 6\hat{\boldsymbol{\theta}}\,.
\end{aligned}$$

We can separate the surface integral into two parts: one in the xy plane and one in the yz plane. In the yz plane the area element is $d\boldsymbol{a} = -r\, dr\, d\theta\, \hat{\boldsymbol{\phi}}$ which is orthogonal to the curl so

$$\oint_{yz\ \text{plane}} (\boldsymbol{\nabla}\times\boldsymbol{v})\cdot d\boldsymbol{a} = 0\,.$$

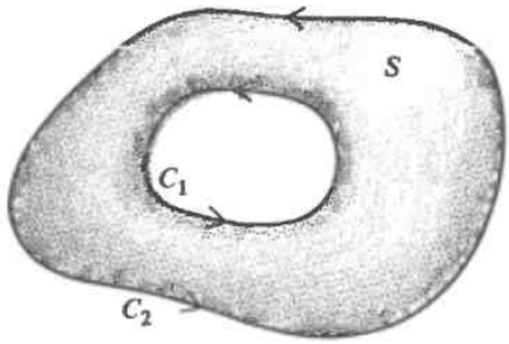
In the xy plane $\theta = \frac{\pi}{2}$ so $\boldsymbol{\nabla}\times\boldsymbol{v} = -6\hat{\boldsymbol{\theta}}$ and $d\boldsymbol{a} = -r\sin\theta\, dr\, d\phi\, \hat{\boldsymbol{\theta}} = -r\, dr\, d\phi\, \hat{\boldsymbol{\theta}}$. The integral of the curl is

$$\oint_{xy\ \text{plane}} (\boldsymbol{\nabla}\times\boldsymbol{v})\cdot d\boldsymbol{a} = \int_0^1\int_0^{\frac{\pi}{2}} 6r\, dr\, d\phi = \frac{\pi}{2}\left[3r^2\right]_0^1 = \frac{3\pi}{2}\,,$$

as required.

4. Derivation of Ampère's Law

(a) Consider a vector function with the property $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$ everywhere on two closed curves C_1 and C_2, as well as on any capping surface S of the region enclosed by them (see the figure).



Show the circulation of $\boldsymbol{F}$ around C_1 equals the circulation of $\boldsymbol{F}$ around C_2. In calculating the circulations direct the curves as in the figure.

Solution: First we state Stokes' theorem for C_1

$$\oint_{C_1} \boldsymbol{F} \cdot d\boldsymbol{l} = \iint_{S_1} \boldsymbol{\nabla} \times \boldsymbol{F} \cdot d\boldsymbol{a} \,,$$

where S_1 is a capping surface enclosed by C_1. Then we consider Stokes' theorem for C_2

$$\oint_{C_2} \boldsymbol{F} \cdot d\boldsymbol{l} = \iint_{S_2} \boldsymbol{\nabla} \times \boldsymbol{F} \cdot d\boldsymbol{a} \,,$$

where S_2 s a capping surface enclosed by C_2. We can split S_2 into S_1 and S because the direction of circulation is the same for C_1 and C_2. (Otherwise the normals would not have the same sense in the two regions.) Therefore,

$$\oint_{C_2} \boldsymbol{F} \cdot d\boldsymbol{l} = \iint_{S_1} \boldsymbol{\nabla} \times \boldsymbol{F} \cdot d\boldsymbol{a} + \iint_{S} \boldsymbol{\nabla} \times \boldsymbol{F} \cdot d\boldsymbol{a} = \iint_{S_1} \boldsymbol{\nabla} \times \boldsymbol{F} \cdot d\boldsymbol{a} = \oint_{C_1} \boldsymbol{F} \cdot d\boldsymbol{l} \,,$$

where he have used the fact the $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$ everywhere on S. So the circulation about C_1 and C_2 is the same.

(b) The magnetic field due to an infinitely long straight wire along the z axis carrying a uniform current I is

$$\boldsymbol{B} = \frac{\mu_0 I}{2\pi\rho} \hat{\boldsymbol{\phi}} \,.$$

Show that $\boldsymbol{\nabla} \times \boldsymbol{B} = 0$ everywhere apart from $\rho = 0$.

Solution: We use cylindrical polar coordinates to calculate the curl of $\boldsymbol{B}$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{\mu_0 I}{2\pi} \begin{vmatrix} \frac{\hat{\boldsymbol{\rho}}}{\rho} & \hat{\boldsymbol{\phi}} & \frac{\hat{\boldsymbol{z}}}{\rho} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ 0 & \rho(1/\rho) & 0 \end{vmatrix} = \frac{\mu_0 I}{2\pi} \frac{1}{\rho} \frac{\partial}{\partial\rho}(1) = 0 \,,$$

except at $\rho = 0$ where we have $0/0$ which is undefined. (This is similar to the problems related to Gauss' law when discussing a point charge c.f. Ex. Sheet 9 Problem 3.)

(c) Prove Ampère's circuital law for the field of the wire given in part (b). [*Hint* Use the result of (b) to find the circulation of $\boldsymbol{B}$ about a circle with the wire passing through its centre and normal to its plane. Then use the result of part (a) to relate this circulation around an arbitrary curve enclosing the current.] [HMS Problem III-16]

Solution: For a circular path of fixed radius R with the wire at the centre and normal to it, we have $d\boldsymbol{l} = Rd\phi\hat{\boldsymbol{\phi}}$, so

$$\oint \boldsymbol{B}\cdot d\boldsymbol{l} = B_\phi(R)[2\pi R] = \frac{\mu_0 I}{2\pi R}[2\pi R] = \mu_0 I \ .$$

So as $\boldsymbol{\nabla}\times\boldsymbol{B} = 0$ apart from the origin any path outside this, like C_2 in part (a) will have the same circulation. Using the principle of superposition (see PM Sec. 6.2 for details) we can generalize this result to

$$\oint_{\mathcal{C}} \boldsymbol{B}\cdot d\boldsymbol{l} = \mu_0 I_{\text{enc}} = \mu_0 \iint_{\mathcal{S}} \boldsymbol{J}\cdot d\boldsymbol{A} \ ,$$

for any arbitrary surface $\mathcal{S}$ bound by $\mathcal{C}$ and I_{enc} is the total current flowing through a surface enclosed by the path. Then using Stokes' theorem on the lefthand side we get

$$\iint_{\mathcal{S}} (\boldsymbol{\nabla}\times\boldsymbol{B})\cdot d\boldsymbol{A} = \mu_0 \iint_{\mathcal{S}} \boldsymbol{J}\cdot d\boldsymbol{A} \ ,$$

therefore as the surfaces are the same the integrands must be equal, which leads to Ampère's circulation law

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0 \boldsymbol{J} \ .$$

5. *Euler's equation and Bernoulli's theorem*

(a) Consider a volume of inviscid (zero viscosity) fluid. The total force acting on the surface of the volume is

$$\boldsymbol{F}_{\text{sur}} = -\oint p\, d\boldsymbol{a} \ ,$$

where p is the pressure and the integral is over the surface of the volume. Show that

$$-\oint p\, d\boldsymbol{a} = -\iiint \boldsymbol{\nabla} p\, dV \ .$$

[*Hint:* Multiply p by a constant vector then use the divergence theorem.]

Solution: We apply the divergence theorem to $p\boldsymbol{c}$, where c is an arbitrary constant vector,

$$\begin{aligned} \oint p\boldsymbol{c}.d\boldsymbol{a} &= \iiint \boldsymbol{\nabla}\cdot(p\boldsymbol{c})\, dV \\ \Rightarrow \boldsymbol{c}\cdot\left(\oint p d\boldsymbol{a}\right) &= \iiint [\boldsymbol{c}\cdot(\boldsymbol{\nabla}p) + p\boldsymbol{\nabla}\cdot\boldsymbol{c}]\, dV \\ &= \boldsymbol{c}\cdot\left(\iiint \boldsymbol{\nabla}p\, dV\right) \ \because \boldsymbol{\nabla}\cdot\boldsymbol{c} = 0 \ . \end{aligned}$$

Therefore, as this is true for any $\boldsymbol{c}$ we have, when multiplying both sides by -1,

$$-\oint p\, d\boldsymbol{a} = -\iiint \boldsymbol{\nabla} p\, dV \ ,$$

as required.

(b) Use Newton's II Law to show that for a small volume dV of the inviscid fluid with density ρ, subject to the surface force and gravity, follows

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{v} = -\frac{\boldsymbol{\nabla}p}{\rho} + \boldsymbol{g} \ ,$$

where $\boldsymbol{g} = g\hat{\boldsymbol{z}}$. This is Euler's equation.

<u>*Solution:*</u> For a small volume of fluid dV with density ρ, which is not experiencing any viscous forces, Newton's II Law gives

$$\begin{aligned}
\rho dV \frac{d\boldsymbol{v}}{dt} &= \boldsymbol{F}_{\text{sur}} + \boldsymbol{F}_{\text{grav}} \\
&= -\boldsymbol{\nabla} p \, dV + \rho dV \boldsymbol{g} \\
\Rightarrow \frac{d\boldsymbol{v}}{dt} &= -\frac{\boldsymbol{\nabla} p}{\rho} + \boldsymbol{g} \\
\Rightarrow \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\, \boldsymbol{v} &= -\frac{\boldsymbol{\nabla} p}{\rho} + \boldsymbol{g} \, ,
\end{aligned}$$

where we have used the stream derivative (c.f. Ill. Ex. 8.5) to rewrite the lefthand side.

(c) Use Euler's equation for an incompressible fluid (i.e. ρ constant), to show that for steady flow (*i.e.* p and $\boldsymbol{v}$ are constant at any point $\boldsymbol{r}$)

$$\Psi = \frac{1}{2}\rho v^2 + \rho g z + p \, ,$$

is constant. This is Bernoulli's theorem.

<u>*Solution:*</u> We will use Euler's formula in the form, multiplied by ρ, stated in the penultimate line in part (*b*). We take its dot product with $\boldsymbol{v}$

$$\begin{aligned}
\rho \boldsymbol{v} \cdot \frac{d\boldsymbol{v}}{dt} + \boldsymbol{v} \cdot \boldsymbol{\nabla} p - \rho \boldsymbol{v} \cdot \boldsymbol{g} &= 0 \\
\frac{1}{2}\rho \frac{d}{dt}(\boldsymbol{v} \cdot \boldsymbol{v}) + \boldsymbol{v} \cdot \boldsymbol{\nabla} p + \rho \boldsymbol{v} \cdot \boldsymbol{\nabla} g z &= 0 \ \because \boldsymbol{g} = -\boldsymbol{\nabla}(gz) \, .
\end{aligned}$$

The stream derivative for a function $f(\boldsymbol{r})$ is

$$\boldsymbol{v} \cdot \boldsymbol{\nabla} f = \frac{df}{dt} - \frac{\partial f}{\partial t} = \frac{df}{dt} \, .$$

This is the case for p in steady flow (only depends on position) and gz. Therefore, we can write

$$\begin{aligned}
\frac{1}{2}\rho \frac{d}{dt}\left(v^2\right) + \frac{dp}{dt} + \rho \frac{d}{dt}(gz) &= 0 \\
\frac{d}{dt}\left(\frac{1}{2}\rho v^2 + \rho g z + p\right) &= 0 \ \because \frac{d\rho}{dt} = 0 \\
\Rightarrow \Psi = \frac{1}{2}\rho v^2 + \rho g z + p &= \text{constant} \, ,
\end{aligned}$$

as required.

Department of Physics
Indian Institute of Technology Madras

End-of-semester exam

From Newton's laws to circulation of vector fields

Date: November 17, 2017 Time: 09:00 AM – 12:00 NOON

Name:

Roll No.:

Instructor:

Batch:

Instructions

1. Begin by completing the information requested above. Please write your complete name, your roll number, the name of your instructor, and your batch number (out of I–XII). **The answer sheet will not be evaluated unless both your name and roll number are written.**
2. This question paper cum answer sheet booklet contains **fourteen** single-sided pages. Please check right away that all the pages are present.
3. As we had announced earlier, this exam consists of 4 true/false questions (for 1 mark each), 4 multiple choice questions with more than one correct option (for 2 marks each), 8 fill in the blanks (for 1 mark each), 5 questions involving detailed calculations (for 3 marks each) and 3 questions involving some plotting (for 5 marks each), adding to a total of 50 marks.
4. You are expected to answer **all** the questions. There are no negative marks.
5. **The answers have to be written in the boxes provided. Answers written elsewhere in the booklet will not be evaluated.**
6. **Kindly write the answers, including sketches, with a blue or black pen. Note that answers written with pencils or pens of other colors will not be evaluated.**
7. You can use the empty reverse sides for rough work. No extra sheets will be provided.
8. You are not allowed to use a calculator or any other electronic device during the exam. Please note that you will not be permitted to continue with the exam if you are found with any such device.
9. Make sure that you return question paper cum answer sheet booklet before you leave the examination hall.

For use by examiners
(Do not write in this space)

Q1–Q8	Q9-16	Q17-21	Q22-24	Total

✦ **True or false** (1 mark each, write True (T)/False (F) in the box provided)

1. The position vector of a particle in motion is given as $\vec{\boldsymbol{r}}(t) = r_0\,\hat{\boldsymbol{r}}$, where r_0 is a constant. This implies that the components of the velocity along the $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ directions are always zero.

2. Kepler's second law, viz. that a planet sweeps equal areas in equal intervals of time, is actually applicable to any trajectory in any central potential.

3. The orbital solution $r = r_0/\cos\phi$ describes the motion of a free particle, where r_0 is the radius of closest approach to the origin and ϕ is the angle measured with respect to the radius vector at the point of closest approach.

4. The force $\vec{\boldsymbol{F}} = k\,(y\,\hat{\boldsymbol{x}} + x\,\hat{\boldsymbol{y}} - 3\,z\,\hat{\boldsymbol{z}})$, where k is a constant, is conservative.

✦ **Multiple choice questions** (2 marks each, write the correct option(s) in the box provided. Note that there can be more than one correct option and zero marks will be given if you choose an incorrect option.)

5. A charged particle is moving in the field of a constant and uniform magnetic field. In general, the particle's

   [**A**] Speed is constant [**B**] Energy is conserved
   [**C**] Trajectory can be a helix [**D**] Trajectory can be a parabola

6. A particle moves under the influence of the gravitational force due to an infinite sheet of mass which lies confined to the x-y-plane. Which of the following kinematic quantities of the particle are conserved? (Note that E and $\vec{\boldsymbol{p}} = (p_x, p_y, p_z)$ denote the energy and momentum of the particle, respectively.)

   [**A**] E [**B**] p_x [**C**] p_y [**D**] p_z

7. A particle moves in the central potential $U(r) = k\,r^4$, where $k > 0$. If the particle has a reduced mass μ, angular momentum L and follows a circular orbit, the radius r_0 and energy E_0 of the particle are

   [**A**] $r_0 = \;[L^2/(4\,k\,\mu)]^{1/6}$ [**B**] $E_0 = (3/2)\,(L^4\,k/2\,\mu^2)^{1/3}$
   [**C**] $r_0 = 2\,[L^2/(4\,k\,\mu)]^{1/6}$ [**D**] $E_0 = (5/2)\,(L^4\,k/2\,\mu^2)^{1/3}$

8. A region has a uniform magnetic field $\vec{\boldsymbol{B}} = B_0\,\hat{\boldsymbol{z}}$ throughout. Consider a cone of height h, base radius R and axis of symmetry along the z-axis. Let the vertex of the cone be at the origin and the outward normal to the flat face be along the positive z-direction. If $\Phi_{\rm F}$ and $\Phi_{\rm C}$ denote the magnetic flux through the flat and the curved surfaces of the cone, then

   [**A**] $\Phi_{\rm F} = B_0\,\pi\,R^2$ [**B**] $\Phi_{\rm F} = -B_0\,\pi\,R^2$
   [**C**] $\Phi_{\rm C} = B_0\,\pi\,R^2$ [**D**] $\Phi_{\rm C} = -B_0\,\pi\,R^2$

✦ **Fill in the blanks** (1 mark each, write the answer in the box provided)

9. Consider the elastic collision of two different masses in one dimension. The relation between the relative velocity of the two masses before (say, $v_1 - v_2$) and after (say, $v_1' - v_2'$) the collision is

10. A particle exhibits bounded motion in the one-dimensional potential $U(x) = \alpha\, x^6$, where $\alpha > 0$. What is the relation between kinetic and potential energies of the particle, when averaged over one period?

11. A particle is moving on a plane with the velocity $\vec{\boldsymbol{v}}(t) = c\, t\, \hat{\boldsymbol{\phi}}$, where c is a constant. The acceleration $\vec{\boldsymbol{a}}$ of the particle is

12. A central potential allows a particle to move in a spiral orbit as follows: $r(\phi) = k\, \phi^2$, where k is a constant. Express the force $\vec{\boldsymbol{F}}(r)$ that gives rise to such a trajectory in terms of the angular momentum L, k and the reduced mass μ.

13. Recently, two neutron stars, one having mass $1.2\, M_{\rm Sun}$ and another $1.6\, M_{\rm Sun}$ were observed to have moved in an elliptical orbit about their common center of mass. If their maximum separation is 400 km, estimate the time period of their orbit. (In the central potential $U(r) = -\alpha/r$, according to Kepler's third law, $T^2 = 4\,\pi^2\,\mu\, a^3/\alpha$. Note that, $G = 6.7 \times 10^{-11}\,{\rm m^2\,kg^{-1}\,s^{-2}}$ and $M_{\rm Sun} = 2 \times 10^{30}\,{\rm kg}$.)

14. The infinite plane $z = a\,x + b\,y + c$ has a constant surface charge density σ. The electric field above and below the plane are given by

15. A sphere of radius 0.5 m has its centre at the origin and a current density $\vec{\boldsymbol{J}} = 2\,\hat{\boldsymbol{r}}\,{\rm A/m^2}$ over its surface. By how much does the amount of charge within the sphere change in one minute? (Note: The unit A denotes Ampère, i.e. Coulomb per unit time.)

16. A conservative force field is given by $\vec{\boldsymbol{F}} = f_0\,[-\rho\cos(2\,\phi)\,\hat{\boldsymbol{\rho}} + \rho\sin(2\,\phi)\,\hat{\boldsymbol{\phi}} + z\,\hat{\boldsymbol{z}}]$, where f_0 is a constant. Calculate the work done in moving a particle from $(\rho, \phi, z) = (4, \pi/2, 2)$ to $(4, \pi, 1)$.

✦ **Questions with detailed answers** (write the calculations and answers within the boxes provided)

17. *Integrating vectors:* Evaluate the following integrals: 1.5+1.5 marks

$$\text{(a)} \int \mathrm{d}t \left(\vec{\boldsymbol{A}} \times \frac{\mathrm{d}^2\vec{\boldsymbol{A}}}{\mathrm{d}t^2}\right), \quad \text{(b)} \int \mathrm{d}t \left(\frac{1}{r}\frac{\mathrm{d}\vec{\boldsymbol{r}}}{\mathrm{d}t} - \frac{\vec{\boldsymbol{r}}}{r^2}\frac{\mathrm{d}r}{\mathrm{d}t}\right)$$

18. *Frequency of small oscillations:* Consider the one-dimensional potential

$$U(x) = U_0 \left(\frac{x}{d}\right)^{2\,n} \mathrm{e}^{-x/d},$$

where U_0 and d are positive constants and $n \geq 1$ is an integer. (a) Plot the potential as a function of (x/d) for $n = 1$. (b) Determine the frequency of small oscillations about the minima for $n \geq 1$ (and integer n). $\boxed{\text{1.5+1.5 marks}}$

19. *Parabolic trajectory in the Kepler problem:* Consider a particle moving in the central Kepler potential $U(r) = -\alpha/r$, where $\alpha > 0$. (a) Write down the equation of motion governing the radial trajectory of the particle. (b) Obtain the solution to the above equation of motion when the energy of particle is zero. (Note that it would be easier to express t in terms of r.) $\boxed{\text{1+2 marks}}$

20. An electric dipole of moment $\vec{\boldsymbol{p}} = p\,\hat{\boldsymbol{z}}$ is located at the origin. The dipole creates the electrostatic potential

$$\phi(\boldsymbol{r}) = \frac{\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{r}}}{4\,\pi\,\epsilon_0\,r^3}.$$

(a) Determine the corresponding electric field $\vec{\boldsymbol{E}}$ in terms of $\vec{\boldsymbol{p}}$, $\hat{\boldsymbol{r}}$ and r. (b) Evaluate the work done by the electric field to move a charge q from $z = 1$ to $z = \infty$. $\boxed{\text{2+1 marks}}$

21. *Vortex in a fluid and Stokes' theorem:* A fluid flowing in the x-y-plane has the following velocity field: $\vec{\boldsymbol{v}} = a\,(-y\,\hat{\boldsymbol{x}} + x\,\hat{\boldsymbol{y}})/(x^2 + y^2)^n$, where a is a constant. (a) Evaluate the vorticity field $\vec{\boldsymbol{\Omega}}$ corresponding to the above velocity field (recall that $\vec{\boldsymbol{\Omega}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}}$). (b) Assuming $n < 1$, verify the Stokes' theorem for the velocity field $\vec{\boldsymbol{v}}$ by considering a circular path in the x-y-plane, with its centre at the origin. $\boxed{\text{1+2 marks}}$

22. *Behavior of the damped, driven oscillator:* Recall that, a damped, driven oscillator satisfies the following equation of motion:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 2\,\beta\,\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2\,x = f_0\cos(\omega\,t).$$

Assume that the oscillator is underdamped, say, $\beta/\omega_0 \simeq 10^{-2}$. (a) Obtain the solution to the above equation for times such that $\beta\,t \gg 1$, expressing the amplitude (say, A) and phase (say, δ) of the solution in terms of the parameters ω_0, β, f_0 and ω. (b) Plot the square of the amplitude A^2 and phase δ as a function of ω/ω_0. (c) Assuming $\omega = \omega_0$, plot the position $x(t)$, the velocity $v(t) = (\mathrm{d}x/\mathrm{d}t)$ and the forcing term $f(t) = f_0\cos(\omega\,t)$ as a function of $(\omega\,t)$, in particular highlighting the difference in phases between these quantities. 1+2+2 marks

23. *Trajectories in phase space:* A particle moves in the one-dimensional potential $U(x) = \alpha\, x^2\, \mathrm{e}^{-x^2}$, where $\alpha > 0$. (a) Plot the potential, specifically indicating the locations of the maxima and minima. (b) What is the allowed range of energy, and what is the range of energy (and position) for which the particle exhibits bounded motion? (c) Illustrate each type of bounded and unbounded trajectories that are possible in the phase space, indicating the direction of motion with arrows. $\boxed{\text{2+1+2 marks}}$

24. *Electric field in and around a charged rod:* A very long rod of radius R carries a charge per unit length λ that is uniformly distributed throughout the rod. It is surrounded by a coaxial cylindrical shell of radius $3\,R$ which is oppositely charged and carries the charge per unit length $-\lambda$. (a) Calculate the electric field at the following locations: (i) inside the charged rod, (ii) between the charged rod and the coaxial shell and (iii) outside the coaxial shell. (b) Plot the magnitude of the electric field as a function of the radial distance from the centre of the rod (i.e. from 0) to $5\,R$. $\boxed{\text{3+2 marks}}$

Solutions to the end-of-semester exam

From Newton's laws to circulation of vector fields

✦ True or false

1. The position vector of a particle in motion is given as $\vec{\boldsymbol{r}}(r,\theta,\phi) = r_0\,\hat{\boldsymbol{r}}$, where r_0 is a constant. This implies that the components of the velocity along the $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ directions are always zero.

   <u>*Solution:*</u> **False**. The velocity is

$$\frac{\mathrm{d}\vec{\boldsymbol{r}}}{\mathrm{d}t} = r_0\,\frac{\mathrm{d}\hat{\boldsymbol{r}}}{\mathrm{d}t} = r_0\,\left(\dot{\theta}\,\hat{\boldsymbol{\theta}} + \dot{\phi}\,\sin\theta\,\hat{\boldsymbol{\phi}}\right).$$

   In general $\dot{\theta}$ and $\dot{\phi}$ are not equal to zero and hence the statement is false. Note that the motion is that of a particle constrained to move on the surface of a sphere of radius r_0.

2. Kepler's second law, viz. that a planet sweeps equal areas in equal intervals of time, is actually applicable to any trajectory in any central potential.

   <u>*Solution:*</u> **True.** The derivation of the second law only assumes the conservation of angular momentum, which is true for any central potential.

3. The orbital solution $r = r_0/\cos\phi$ describes the motion of a free particle, where r_0 is the radius of closest approach to the origin and ϕ is the angle measured with respect to the radius vector at the point of closest approach.

   <u>*Solution:*</u> **True**. Define the point of closest approach to be along the x axis so that

$$x = r\cos\phi = r_0\ ,\quad y = r\sin\phi = r_0\tan\phi,$$

   which parametrically describes the straight line $x = r_0$ with $\phi = (-\frac{\pi}{2}, \frac{\pi}{2})$.

4. The force $\vec{\boldsymbol{F}} = k\,(y\,\hat{\boldsymbol{x}} + x\,\hat{\boldsymbol{y}} - 3\,z\,\hat{\boldsymbol{z}})$, where k is a constant, is conservative.

   <u>*Solution:*</u> **True**. We take the curl to check if the force is conservative:

$$\boldsymbol{\nabla}\times\boldsymbol{F} = k\begin{vmatrix}\hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y & x & -3z\end{vmatrix} = \left(\frac{\partial}{\partial x}x - \frac{\partial}{\partial y}y\right)\hat{\boldsymbol{z}} = 0.$$

   Therefore, the force is conservative.

✦ Multiple choice questions (2 marks each, write the correct option(s) in the box provided. Note that there can be more than one correct option and zero marks will be given if you choose an incorrect option.)

5. A charged particle is moving in the field of a constant and uniform magnetic field. In general, the particle's

   [**A**] Speed is constant [**B**] Energy is conserved

   [**C**] Trajectory can be a helix [**D**] Trajectory can be a parabola

   <u>*Solution:*</u> **A**, **B** and **C**. The magnetic force is conservative but does no work. Therefore, the energy and kinetic energy are constant and hence **A** and **B** are true. It is well known that a charged particle moves in a circle in the plane perpendicular to the magnetic field. If there is any velocity component in the direction of the field, the trajectory is a helix of constant pitch, so **C** is correct.

6. A particle moves under the influence of the gravitational force due to an infinite sheet of mass which lies confined to the x-y-plane. Which of the following kinematic quantities of the particle

are conserved? (Note that E and $\vec{\boldsymbol{p}} = (p_x, p_y, p_z)$ denote the energy and momentum of the particle, respectively.)

[**A**] E [**B**] p_x [**C**] p_y [**D**] p_z

<u>*Solution:*</u> **A**, **B** and **C**. As the potential is time-independent, the energy is conserved. The gravitational field is uniform and in the $\pm z$ direction. Therefore, the potential only depends on z, so it is invariant under translation in the x and y directions. Hence, the momentum components in those directions are conserved.

7. A particle moves in the central potential $U(r) = k\,r^4$, where $k > 0$. If the particle has a reduced mass μ, angular momentum L and follows a circular orbit, the radius r_0 and energy E_0 of the particle are

[**A**] $r_0 = \;[L^2/(4\,k\,\mu)]^{1/6}$ [**B**] $E_0 = (3/2)\,(L^4\,k/2\,\mu^2)^{1/3}$

[**C**] $r_0 = 2\,[L^2/(4\,k\,\mu)]^{1/6}$ [**D**] $E_0 = (5/2)\,(L^4\,k/2\,\mu^2)^{1/3}$

<u>*Solution:*</u> **A** and **B**. The effective potential for this motion is

$$U_{\rm eff} = k\,r^4 + \frac{L^2}{2\,\mu\,r^2}$$

and hence

$$\frac{\mathrm{d}U_{\rm eff}}{\mathrm{d}r} = 4\,k\,r^3 - \frac{L^2}{\mu\,r^3}.$$

The circular orbit will occur at the radius r_0 when $(\mathrm{d}U_{\rm eff}/\mathrm{d}r)_{r_0} = 0$. Therefore, we have

$$r_0 = \left(\frac{L^2}{4\,\mu\,k}\right)^{1/6},$$

so **A** is correct. In a circular orbit $\dot{r} = 0$, so that

$$\begin{aligned} E_0 &= U_{\rm eff}(r_0) = k\,r_0^4 + \frac{L^2}{2\,\mu\,r_0^2} \\ &= k\left(\frac{L^2}{4\,\mu\,k}\right)^{2/3} + \frac{L^2}{2\,\mu}\left(\frac{4\,\mu\,k}{L^2}\right)^{1/3} \\ &= \left(\frac{L^4 k}{\mu^2}\right)^{1/3}\left(\frac{1}{4^{2/3}} + \frac{4^{1/3}}{2}\right) = \frac{3}{2}\left(\frac{L^4\,k}{2\,\mu^2}\right)^{1/3}, \end{aligned}$$

and hence the correct option is **B**.

8. A region has a uniform magnetic field $\vec{\boldsymbol{B}} = B_0\,\hat{\boldsymbol{z}}$ throughout. Consider a cone of height h, base radius R and axis of symmetry along the z-axis. Let the vertex of the cone be at the origin and the outward normal to the flat face be along the positive z-direction. If $\Phi_{\rm F}$ and $\Phi_{\rm C}$ denote the magnetic flux through the flat and the curved surfaces of the cone, then

[**A**] $\Phi_{\rm F} = B_0\,\pi\,R^2$ [**B**] $\Phi_{\rm F} = -B_0\,\pi\,R^2$

[**C**] $\Phi_{\rm C} = B_0\,\pi\,R^2$ [**D**] $\Phi_{\rm C} = -B_0\,\pi\,R^2$

<u>*Solution:*</u> **A** and **D**. According to Gauss' divergence theorem, for magnetic fields, we have

$$\oint \vec{\boldsymbol{B}}\cdot\mathrm{d}\vec{\boldsymbol{A}} = 0$$

over any closed surface, since $\vec{\nabla}\cdot\vec{\boldsymbol{B}} = 0$. Therefore,

$$\Phi_{\rm C} + \Phi_{\rm F} = \int_{\rm C}\vec{\boldsymbol{B}}\cdot\mathrm{d}\vec{\boldsymbol{A}} + \int_{\rm F}\vec{\boldsymbol{B}}\cdot\mathrm{d}\vec{\boldsymbol{A}} = 0$$

and hence

$$\Phi_{\rm C} = -\Phi_{\rm F} = -\int_0^R {\rm d}\rho\, \rho \int_0^{2\pi} {\rm d}\phi\, (B_0\, \hat{\boldsymbol{z}}) \cdot \hat{\boldsymbol{z}} = -B_0\, \pi\, R^2,$$

so that

$$\Phi_{\rm F} = B_0\, \pi\, R^2.$$

✦ **Fill in the blanks** (1 mark each, write the answer in the box provided)

9. Consider the elastic collision of two different masses in one dimension. The relation between the relative velocity of the two masses before (say, $v_1 - v_2$) and after (say, $v_1' - v_2'$) the collision is

   Solution: From conservation of momentum and energy, we have

$$m_1\, v_1 + m_2\, v_2 = m_1\, v_1' + m_2\, v_2',$$

   and

$$\frac{m_1\, v_1^2}{2} + \frac{m_2\, v_2^2}{2} = \frac{m_1\, v_1'^2}{2} + \frac{m_2\, v_2'^2}{2}.$$

   These can be written as

$$m_1\, \left(v_1 - v_1'\right) = m_2\, \left(v_2' - v_2\right)$$

   and

$$\frac{m_1}{2}\, \left(v_1^2 - v_1'^2\right) = \frac{m_1}{2}\, \left(v_1 + v_1'\right)\, \left(v_1 - v_1'\right) = \frac{m_2}{2}\, \left(v_2'^2 - v_2^2\right) = \frac{m_2}{2}\, \left(v_2 + v_2'\right)\, \left(v_2' - v_2\right),$$

   so that, upon using these two expressions, we arrive at

$$v_1 + v_1' = v_2 + v_2'$$

   or

$$v_1 - v_2 = v_2' - v_1',$$

   which is the required result.

10. A particle exhibits bounded motion in the one-dimensional potential $U(x) = \alpha\, x^6$, where $\alpha > 0$. What is the relation between kinetic and potential energies of the particle, when averaged over one period?

    Solution: The virial theorem for a $U(x) = \alpha\, x^n$ is

$$\langle T \rangle = (n/2)\, \langle U \rangle.$$

    Therefore, for $n = 6$, we have

$$\langle T \rangle = 3\, \langle U \rangle.$$

11. A particle is moving on a plane with the velocity $\vec{\boldsymbol{v}}(t) = c\, t\, \hat{\boldsymbol{\phi}}$, where c is a constant. The acceleration $\vec{\boldsymbol{a}}$ of the particle is

    Solution: First we find the velocity, i.e.

$$\vec{\boldsymbol{a}} = \dot{\vec{\boldsymbol{v}}} = c\, \left(\hat{\boldsymbol{\phi}} - \dot{\phi}\, t\, \hat{\boldsymbol{\rho}}\right).$$

12. A central potential allows a particle to move in a spiral orbit as follows: $r = k\, \phi^2$, where k is a constant. Express the force $\vec{\boldsymbol{F}}(r)$ that gives rise to such a trajectory in terms of the angular momentum L, k and the reduced mass μ.

    Solution: We write the energy

$$E = \frac{\mu}{2}\, \left(\dot{r}^2 + r^2\, \dot{\phi}^2\right) + U(r)$$

which implies that

$$U(r) = -\frac{\mu}{2}\,(\dot{r}^2 + r^2\,\dot{\phi}^2) + E.$$

Now,

$$\dot{r} = 2\,k\,\phi\,\dot{\phi} = 2\,\sqrt{k\,r}\,\dot{\phi}$$

and $\dot{\phi} = L/(\mu\,r^2)$ so that

$$U(r) = -\frac{L^2}{2\,\mu\,r^4}\,\left(4\,k\,r + r^2\right) + E = -\frac{L^2}{2\,\mu}\,\left(\frac{4\,k}{r^3} + \frac{1}{r^2}\right) + E,$$

and, hence, the force is

$$F_r = -\frac{\mathrm{d}U}{\mathrm{d}r} = -\frac{L^2}{\mu}\,\left(\frac{1}{r^3} + \frac{6\,k}{r^4}\right).$$

13. Recently, two neutron stars, one having mass $1.2\,M_{\mathrm{Sun}}$ and another $1.6\,M_{\mathrm{Sun}}$ were observed to have moved in an elliptical orbit about their common center of mass. If their maximum separation is 400 km, estimate the time period of their orbit. (In the central potential $U(r) = -\alpha/r$, according to Kepler's third law, $T^2 = 4\,\pi^2\,\mu\,a^3/\alpha$. Note that, $G = 6.7\times 10^{-11}\,\mathrm{m}^2\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$ and $M_{\mathrm{Sun}} = 2\times 10^{30}\,\mathrm{kg}$.)

    Solution: First we note $\alpha = G\,M_1\,M_2$ and $\mu = M_1\,M_2/(M_1 + M_2)$, so that

    $$T^2 = \frac{4\,\pi^2\,M_1\,M_2\,a^3}{(M_1+M_2)\,G\,M_1\,M_2} = \frac{4\,\pi^2\,a^3}{2.8\,M_{\mathrm{Sun}}\,G} = \frac{4^4\,\pi^2\times 10^{15}}{5.6\times 10^{30}\times 6.7\times 10^{-11}} = \frac{4^4\pi^2}{37.5\times 10^4}\ \mathrm{s},$$

    which leads to

    $$T \approx \sqrt{\frac{4^3}{10^4}} \approx \frac{8}{10^2} = 0.08\,\mathrm{s}.$$

14. The infinite plane $z = a\,x + b\,y + c$ has a constant surface charge density σ. The electric field above and below the plane are given by

    Solution: The electric field will be given by $\vec{\boldsymbol{E}} = \pm(\sigma/2\,\epsilon_0),\,\hat{\boldsymbol{n}}$, where $\hat{\boldsymbol{n}}$ is the normal to the plane. Given that the plane can be described as $f(x,y,z) = c = z - a\,x - b\,y$, we have the normal to be

    $$\hat{\boldsymbol{n}} = \frac{\boldsymbol{\nabla} f}{|\boldsymbol{\nabla} f|} = \frac{-a\,\hat{\boldsymbol{x}} - b\,\hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}}{\sqrt{a^2+b^2+1}},$$

    so that

    $$\vec{\boldsymbol{E}} = \pm\frac{\sigma}{2\,\epsilon_0}\,\left(\frac{-a\,\hat{\boldsymbol{x}} - b\,\hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}}{\sqrt{a^2+b^2+1}}\right).$$

15. A sphere of radius 0.5 m has its centre at the origin and a current density $\vec{\boldsymbol{J}} = 2\,\hat{\boldsymbol{r}}\,\mathrm{A/m^2}$ over its surface. By how much does the amount of charge within the sphere change in one minute? (Note: The unit A denotes Ampère, i.e. Coulomb per unit time.)

    Solution: We will use the integrated continuity equation, viz.

    $$\oint \vec{\boldsymbol{J}}\cdot \mathrm{d}\vec{\boldsymbol{A}} = -\frac{\mathrm{d}Q_{\mathrm{enc}}}{\mathrm{d}t}$$

    which implies that

    $$\frac{\mathrm{d}Q_{\mathrm{enc}}}{\mathrm{d}t} = -\int_0^{\pi}\mathrm{d}\theta\int_0^{2\,\pi}\mathrm{d}\phi\ (2\,\hat{\boldsymbol{r}})\cdot\left(R^2\,\sin\,\theta\,\hat{\boldsymbol{r}}\right) = -8\,\pi\,R^2 = -2\,\pi\,\mathrm{C/s}$$

    because $R = 0.5\,\mathrm{m}$. Therefore, the change in charge in one minute is $60\times(\mathrm{d}Q_{\mathrm{enc}}/\mathrm{d}t) = -120\,\pi\,\mathrm{C} = -372\,\mathrm{C}$.

16. A conservative force field is given by $\vec{\boldsymbol{F}} = f_0\,[-\rho\cos(2\,\phi)\,\hat{\boldsymbol{\rho}} + \rho\sin(2\,\phi)\,\hat{\boldsymbol{\phi}} + z\,\hat{\boldsymbol{z}}]$, where f_0 is a constant. Calculate the work done in moving a particle from $(\rho, \phi, z) = (4, \pi/2, 2)$ to $(4, \pi, 1)$.

 Solution: Note that
$$\mathrm{d}\vec{\boldsymbol{l}} = \mathrm{d}\rho\,\hat{\boldsymbol{\rho}} + \rho\,\mathrm{d}\phi\,\hat{\boldsymbol{\phi}} + \mathrm{d}z\,\hat{\boldsymbol{z}},$$
 so that
$$\int \vec{\boldsymbol{F}}\cdot\mathrm{d}\vec{\boldsymbol{l}} = f_0 \int_{(4,\frac{\pi}{2},2)}^{(4,\pi,1)} \left[-\rho\cos(2\,\phi)\,\mathrm{d}\rho + \rho^2\sin(2\,\phi)\,\mathrm{d}\phi + z\,\mathrm{d}z\right]$$
 and, since the path is along a constant ρ, viz. $\rho = 4$, we have
$$\begin{aligned}\int \vec{\boldsymbol{F}}\cdot\mathrm{d}\vec{\boldsymbol{l}} &= f_0\left(16\int_{\pi/2}^{\pi}\mathrm{d}\phi\,\sin(2\,\phi) + \int_2^1 \mathrm{d}z\,z\right)\\ &= f_0\left\{8\,\left[-\cos(2\,\phi)\right]_{\pi/2}^{\pi} + \left(\frac{z^2}{2}\right)_2^1\right\} = f_0\,\left(-16 - \frac{3}{2}\right) = -\frac{35}{2}\,f_0.\end{aligned}$$

✦ **Questions with detailed answers** (write the calculations and answers within the boxes provided)

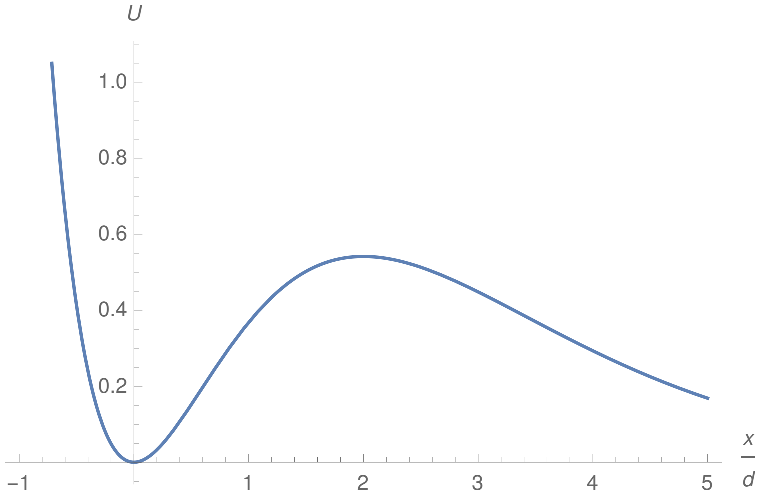
17. *Integrating vectors:* Evaluate the following integrals:
$$\text{(a)}\ \int \mathrm{d}t\,\left(\vec{\boldsymbol{A}}\times\frac{\mathrm{d}^2\vec{\boldsymbol{A}}}{\mathrm{d}t^2}\right),\quad \text{(b)}\ \int \mathrm{d}t\,\left(\frac{1}{r}\frac{\mathrm{d}\vec{\boldsymbol{r}}}{\mathrm{d}t} - \frac{\vec{\boldsymbol{r}}}{r^2}\frac{\mathrm{d}r}{\mathrm{d}t}\right)$$

 Solution: (a) Note that
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{A}\times\dot{\boldsymbol{A}}\right) = \dot{\boldsymbol{A}}\times\dot{\boldsymbol{A}} + \boldsymbol{A}\times\ddot{\boldsymbol{A}} = \boldsymbol{A}\times\ddot{\boldsymbol{A}},$$
 because the vector product of a vector with itself is zero. Therefore,
$$\int \mathrm{d}t\,\left(\boldsymbol{A}\times\ddot{\boldsymbol{A}}\right) = \int \mathrm{d}t\,\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{A}\times\dot{\boldsymbol{A}}\right) = \boldsymbol{A}\times\dot{\boldsymbol{A}} + \boldsymbol{C},$$
 where $\boldsymbol{C}$ is a constant vector.

 (b) Note that, we have
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\vec{\boldsymbol{r}}}{r}\right) = \frac{1}{r}\frac{\mathrm{d}\vec{\boldsymbol{r}}}{\mathrm{d}t} - \frac{\vec{\boldsymbol{r}}}{r^2}\frac{\mathrm{d}r}{\mathrm{d}t},$$
 using the chain rule. Therefore, we obtain that
$$\int \mathrm{d}t\,\left(\frac{1}{r}\frac{\mathrm{d}\vec{\boldsymbol{r}}}{\mathrm{d}t} - \frac{\vec{\boldsymbol{r}}}{r^2}\frac{\mathrm{d}r}{\mathrm{d}t}\right) = \frac{\vec{\boldsymbol{r}}}{r} + \boldsymbol{C}.$$

18. *Frequency of small oscillations:* Consider the one-dimensional potential
$$U(x) = U_0\,\left(\frac{x}{d}\right)^{2\,n}\,\mathrm{e}^{-x/d},$$
 where U_0 and d are positive constants and $n \geq 1$. (a) Plot the potential as a function of (x/d) for $n = 1$. (b) Determine the frequency of small oscillations about the minima for $n \geq 1$.

 Solution: The potential for $n = 1$ behaves as follows:



We find that

$$\frac{\mathrm{d}U}{\mathrm{d}x} = \frac{U_0}{d}\left(\frac{x}{d}\right)^{2n} \mathrm{e}^{-x/d}\left(2n - \frac{x}{d}\right),$$

and

$$\frac{\mathrm{d}^2U}{\mathrm{d}x^2} = \frac{U_0}{d^2}\left(\frac{x}{d}\right)^{2n-2} \mathrm{e}^{-x/d}\left[-\frac{x}{d}\left(2n - \frac{x}{d}\right) - \frac{x}{d} + (2n-1)\left(2n - \frac{x}{d}\right)\right].$$

But, all these are not needed! It is clear that the minimum occurs at $x = 0$ for all $n \geq 1$. If we Taylor expand the potential about $x = 0$, we obtain that

$$U(x) \simeq U_0\left(\frac{x}{d}\right)^{2n}.$$

The potential has a point of inflection for $n > 1$. For $n = 1$, the frequency ω of small oscillations is determined by the relation

$$\frac{m\,\omega^2}{2} = \frac{U_0}{d^2},$$

or

$$\omega = \sqrt{\frac{2\,U_0}{m\,d^2}}.$$

19. *Parabolic trajectory in the Kepler problem:* Consider a particle moving in the central Kepler potential $U(r) = -\alpha/r$, where $\alpha > 0$. (a) Write down the equation of motion governing the radial trajectory of the particle. (b) Obtain the solution $r(t)$ to the equation of motion when the energy of particle is zero.

Solution: The energy equation is given by

$$E = \frac{\mu}{2}\left(\dot{r}^2 + r^2\,\dot{\phi}^2\right) + U(r) = \frac{\mu}{2}\,\dot{r}^2 + \frac{L^2}{2\,\mu\,r^2} + U(r) = \frac{\mu}{2}\,\dot{r}^2 + U_{\mathrm{eff}}(r),$$

which upon differentiation leads to

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = -\frac{\alpha}{r^2} + \frac{L^2}{\mu\,r^3}.$$

For $E = 0$, we have

$$\frac{\mu}{2}\,\dot{r}^2 = \frac{\alpha}{r} - \frac{L^2}{2\,\mu\,r^2}$$

so that

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \sqrt{\frac{2}{\mu}}\left(\frac{\alpha}{r} - \frac{L^2}{2\,\mu\,r^2}\right)^{1/2}$$

or

$$t - t_0 = \int \mathrm{d}t = \int \frac{\mathrm{d}r\,\sqrt{\mu/2}}{\sqrt{\frac{\alpha}{r} - \frac{L^2}{2\,\mu\,r^2}}} = \mu \int \frac{\mathrm{d}r\;r}{\sqrt{2\,\mu\,\alpha\,r - L^2}}$$

and, if we set

$$x = \sqrt{2\,\mu\,\alpha\,r - L^2}, \quad \mathrm{d}x = \frac{\mathrm{d}r\;(\mu\,\alpha)}{\sqrt{2\,\mu\,\alpha\,r - L^2}},$$

we obtain

$$t - t_0 = \frac{1}{2\,\mu\,\alpha^2} \int \mathrm{d}x\;\left(x^2 - L^2\right) = \frac{1}{2\,\mu\,\alpha^2}\left(\frac{x^3}{3} + L^2\,x\right) = \frac{x}{2\,\mu\,\alpha^2}\left(\frac{x^2}{3} + L^2\right)$$

which can be expressed as

$$t - t_0 = \frac{1}{2\,\mu\,\alpha^2}\sqrt{2\,\mu\,\alpha\,r - L^2}\left(\frac{2}{3}\,\mu\,\alpha\,r - \frac{L^2}{3} + L^2\right) = \frac{1}{3\,\mu\,\alpha^2}\sqrt{2\,\mu\,\alpha\,r - L^2}\;\left(\mu\,\alpha\,r + L^2\right).$$

20. An electric dipole of moment $\vec{\boldsymbol{p}} = p\,\hat{\boldsymbol{z}}$ is located at the origin. The dipole creates the electrostatic potential

$$\phi(\boldsymbol{r}) = \frac{\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{r}}}{4\,\pi\,\epsilon_0\,r^3}.$$

(a) Determine the corresponding electric field $\vec{\boldsymbol{E}}$ in terms of $\vec{\boldsymbol{p}}$, $\hat{\boldsymbol{r}}$ and r. (b) Evaluate the work done by the electric field to move a charge q along the z-axis from $z = 1$ to $z = \infty$.

<u>*Solution:*</u> (a) As $\boldsymbol{p} = p\,\hat{\boldsymbol{z}}$, we have

$$\begin{aligned}
\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}\phi \quad &= \quad -\frac{p\,z}{4\,\pi\,\epsilon_0}\frac{\partial}{\partial x}\left[\left(x^2+y^2+z^2\right)^{-3/2}\right]\hat{\boldsymbol{x}} - \frac{p\,z}{4\,\pi\,\epsilon_0}\frac{\partial}{\partial y}\left[\left(x^2+y^2+z^2\right)^{-3/2}\right]\hat{\boldsymbol{y}} \\
&\quad -\frac{p}{4\,\pi\,\epsilon_0}\frac{\partial}{\partial z}\left[z\,\left(x^2+y^2+z^2\right)^{-3/2}\right]\hat{\boldsymbol{z}} \\
&= \quad \frac{3\,p\,x\,z}{4\,\pi\,\epsilon_0}\frac{\hat{\boldsymbol{x}}}{(x^2+y^2+z^2)^{5/2}} + \frac{3\,p\,y\,z}{4\,\pi\,\epsilon_0}\frac{\hat{\boldsymbol{y}}}{(x^2+y^2+z^2)^{5/2}} + \frac{3\,p\,z^2}{4\,\pi\,\epsilon_0}\frac{\hat{\boldsymbol{z}}}{(x^2+y^2+z^2)^{5/2}} \\
&\quad -\frac{p}{4\,\pi\,\epsilon_0}\,\frac{\hat{\boldsymbol{z}}}{(x^2+y^2+z^2)^{3/2}} \\
&= \quad \frac{1}{4\,\pi\,\epsilon_0\,r^3}\left[3\,(\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}})\,\hat{\boldsymbol{r}} - \boldsymbol{p}\right].
\end{aligned}$$

(b) The work done is

$$W = -\Delta\phi = -q\,\phi(0,0,\infty) + q\,\phi(0,0,1) = \left.\frac{q\,p\,z}{4\,\pi\,\epsilon_0\,z^3}\right|_{z=1} = \frac{q\,p}{4\,\pi\,\epsilon_0}.$$

21. <u>*Vortex in a fluid and Stokes' theorem:*</u> A fluid flowing in the x-y-plane has the following velocity field $\vec{\boldsymbol{v}} = a\,(-y\,\hat{\boldsymbol{x}} + x\,\hat{\boldsymbol{y}})/(x^2+y^2)^n$, where a is a constant. (a) Evaluate the vorticity field $\vec{\boldsymbol{\Omega}}$ corresponding to the above velocity field (recall that $\vec{\boldsymbol{\Omega}} = \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{v}}$). (b) Assuming $n < 1$, verify the Stokes' theorem for the velocity field $\vec{\boldsymbol{v}}$ by considering a circular path in the x-y-plane, with its centre at the origin. $\boxed{\text{1+2 marks}}$

<u>*Solution:*</u> The vorticity field is given by

$$\vec{\boldsymbol{\Omega}} = a\begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \partial_x & \partial_y & \partial_z \\ \frac{x}{(x^2+y^2)^n} & \frac{-y}{(x^2+y^2)^n} & 0 \end{vmatrix} = \frac{2\,a\,(1-n)}{(x^2+y^2)^n}\,\hat{\boldsymbol{z}},$$

and, if we write in terms of cylindrical polar coordinates, we have

$$\vec{\boldsymbol{\Omega}} = \frac{2\,a\,(1-n)}{\rho^{2\,n}}\,\hat{\boldsymbol{z}}.$$

Note that, we can also write

$$\vec{\boldsymbol{v}} = \frac{a}{\rho^{2\,n-1}}\,\hat{\boldsymbol{\phi}}.$$

If we calculate the line integral along a circle of radius R, we have

$$\oint \vec{\boldsymbol{v}}\cdot \mathrm{d}\vec{\boldsymbol{l}} = \frac{2\,\pi\,a\,R}{R^{2\,n-1}} = \frac{2\,\pi\,a}{R^{2\,n-2}}.$$

Similarly, we have

$$\begin{aligned}
\int_{\mathrm{S}} \vec{\boldsymbol{\Omega}}\cdot \mathrm{d}\vec{\boldsymbol{A}} &= \int_0^R \mathrm{d}\rho\,\rho \int_0^{2\,\pi} \mathrm{d}\phi\,\frac{2\,a\,(1-n)}{\rho^{2\,n}}\,\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{z}} = 4\,\pi\;a\,(1-n)\int_0^R \frac{\mathrm{d}\rho}{\rho^{2\,n-1}}\\
&= 4\,\pi\;a\,(1-n)\left(\frac{\rho^{2-2\,n}}{2-2\,n}\right)_0^R = \frac{4\,\pi\;a\,(1-n)}{2\,(1-n)}\,R^{2-2\,n} = \frac{2\,\pi\,a}{R^{2\,n-2}},
\end{aligned}$$

for $n<1$, which is the result we have obtained above from the line integral.

22. *Behavior of the damped, driven oscillator:* Recall that, a damped, driven oscillator satisfies the following equation of motion:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 2\,\beta\,\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2\,x = f_0\cos{(\omega\,t)}.$$

Assume that the oscillator is underdamped, i.e. $\beta \ll \omega_0$. (a) Obtain the solution to the above equation for times such that $\beta\,t \gg 1$, expressing the amplitude (say, A) and phase (say, δ) of the solution in terms of the parameters ω_0, β, f_0 and ω. (b) Plot the square of the amplitude A^2 and phase δ as a function of ω/ω_0. (c) Assuming $\omega=\omega_0$, plot the position $x(t)$, the velocity $v(t) = (\mathrm{d}x/\mathrm{d}t)$ and the forcing term $f(t) = f_0\cos{(\omega\,t)}$ as a function of $(\omega\,t)$, in particular highlighting the difference in phases between these quantities.

Solution: Recall that the solution to the damped oscillator at late times (i.e. when $\beta\,t\gg 1$) can be expressed as

$$x(t) = A\cos{(\omega\,t-\delta)},$$

where $f_0 =$, the amplitude A is given by

$$A^2 = \frac{f_0^2}{(\omega_0^2-\omega^2)^2 + 4\,\beta^2\,\omega_0^2}$$

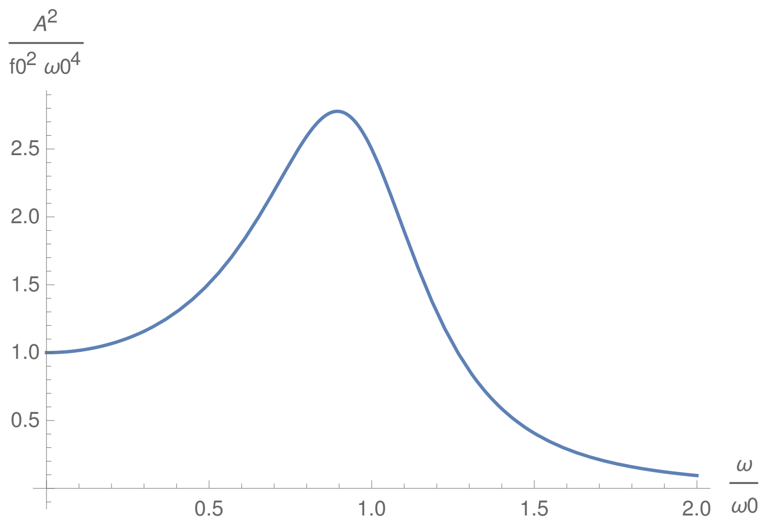
and the phase δ is defined through the relation

$$\tan\delta = \frac{2\,\beta\,\omega}{\omega_0^2-\omega^2}.$$

Note that, we can write

$$\frac{A^2}{f_0^2/\omega_0^4} = \frac{1}{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2 + 4\,\frac{\beta^2}{\omega_0^2}\,\frac{\omega^2}{\omega_0^2}}$$

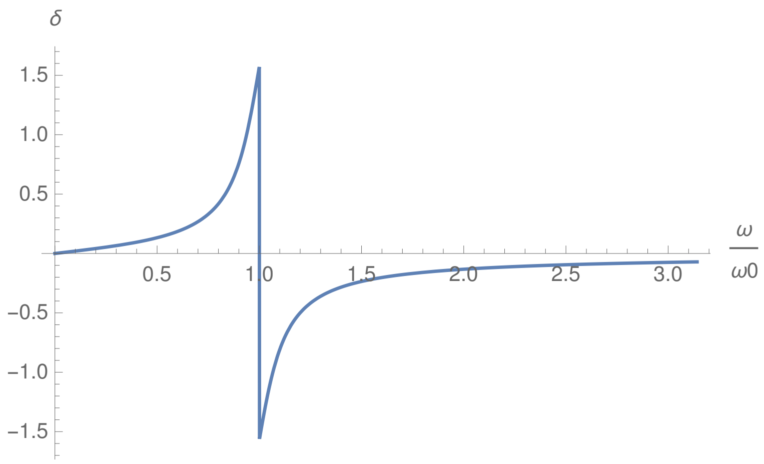
which is plotted for $\beta/\omega_0 = 0.1$ in the figure below.



Similarly, we have

$$\tan\delta = \frac{2\,(\beta/\omega_0)\,(\omega/\omega_0)}{1-\frac{\omega^2}{\omega_0^2}},$$

which is plotted in the figure below for $\beta/\omega_0 = 0.1$



Note that when $\delta = \pi/2$, we have

$$x(t) = A\sin(\omega\,t).$$

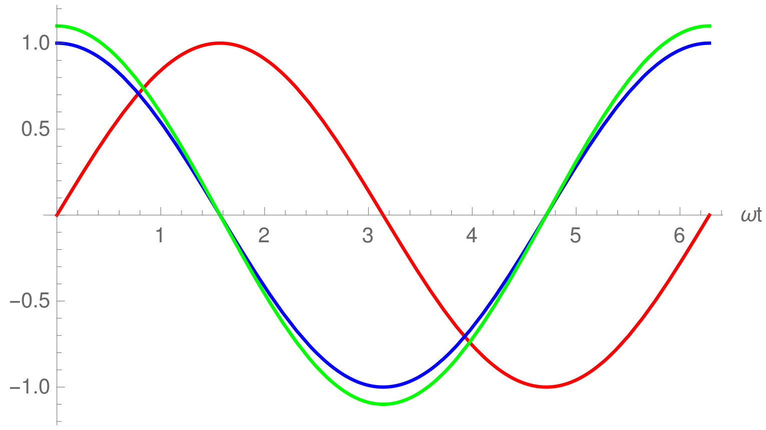
The velocity of the oscillator is given by

$$v(t) = \dot{x}(t) = -\omega\,A\sin(\omega\,t - \delta).$$

Now, when $\omega = \omega_0$, $\tan\delta$ is infinite or $\delta = \pi/2$ and hence we obtain that

$$v(t) = \omega\,A\cos(\omega\,t),$$

which is exactly in phase with the forcing term. The quantities $x(t)$ (in red), $v(t)$ (in blue) and $f(t)$ (in green) are plotted below.



23. *Trajectories in phase space:* A particle moves in the one-dimensional potential $U(x) = \alpha\, x^2\, \mathrm{e}^{-x^2}$, where $\alpha > 0$. (a) Plot the potential, specifically indicating the locations of the maxima and minima. (b) What is the allowed range of energy, and what is the range of energy (and position) for which the particle exhibits bounded motion? (c) Illustrate each type of bounded and unbounded trajectories that are possible in the phase space, indicating the direction of motion with arrows.

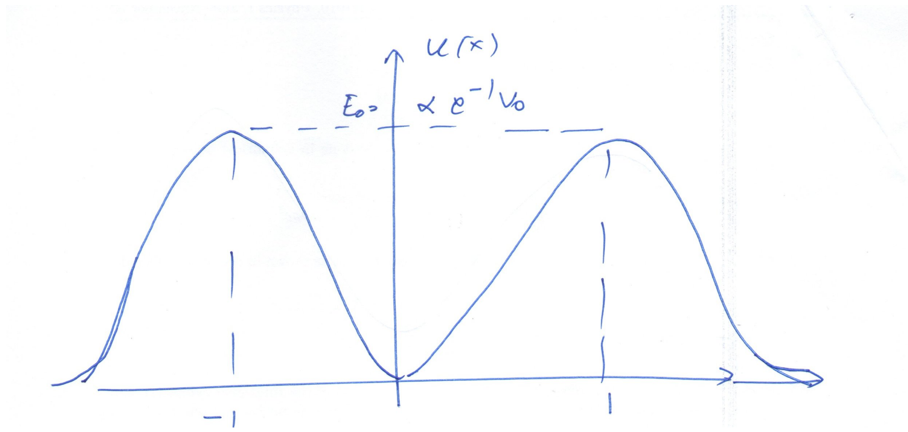
Solution: One finds that

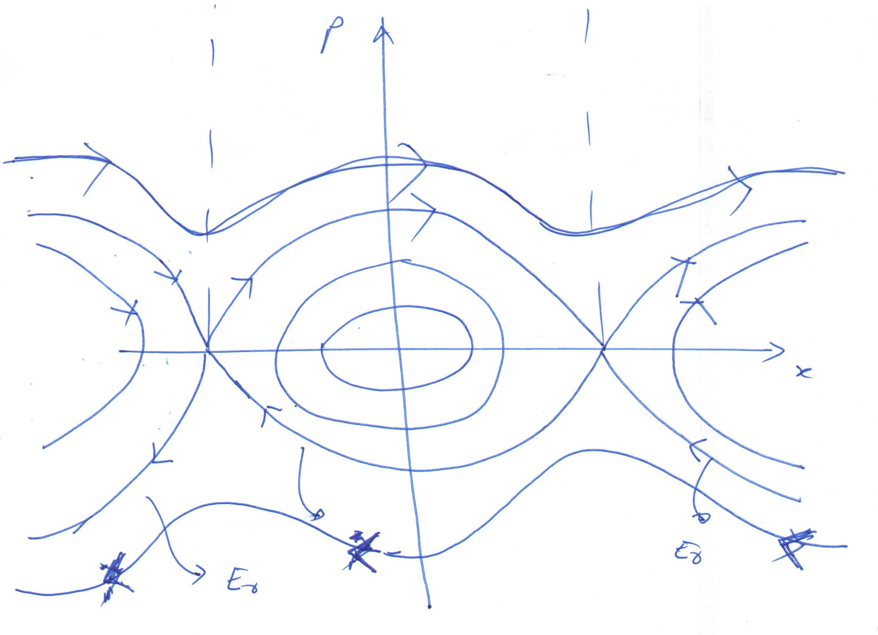
$$\frac{\mathrm{d}U}{\mathrm{d}x} = 2\, U_0\, \mathrm{e}^{-x^2}\, x\,(1 - x^2)$$

and

$$\frac{\mathrm{d}^2U}{\mathrm{d}x^2} = 2\, U_0\, \mathrm{e}^{-x^2}(1 - 5\,x^2 + 2\,x^4)$$

so that, clearly, the extrema are located at $x = 0$ and $x = 1$. It is also evident that $x = 0$ is a minimum, while $x = 1$ is a maximum. The potential and the phase portrait are illustrated in the two figures below.





24. *Electric field in and around a charged rod:* A very long rod of radius R carries a charge per unit length λ that is uniformly distributed throughout the rod. It is surrounded by a coaxial tube of radius $3\,R$ which is oppositely charged and carries the charge per unit length $-\lambda$. (a) Calculate the electric field at the following locations: (i) inside the charged rod, (ii) between the charged rod and the coaxial tube and (iii) outside the coaxial tube. (b) Plot the magnitude of the electric field as a function of the radial distance from the centre of the tube (i.e. from 0) to $5\,R$. $\boxed{\text{3+2 marks}}$

Solution: (a) We use Gauss' law and exploit the cylindrical symmetry to compute the $\vec{\boldsymbol{E}} = E(\rho)\,\hat{\boldsymbol{\rho}}$. We will use a cylinder of length L about the axis of symmetry as the Gaussian volume. Also note that the charge density in the rod is $\varrho = \lambda/(\pi R^2)$. Therefore, since

$$\iint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{a}} = \frac{Q_{\rm enc}}{\epsilon_0},$$

(i) when $\rho < R$, we have

$$E(\rho)\,(2\,\pi\,\rho\,L) = \frac{\varrho\,\pi\,\rho^2\,L}{\epsilon_0},$$

which implies that

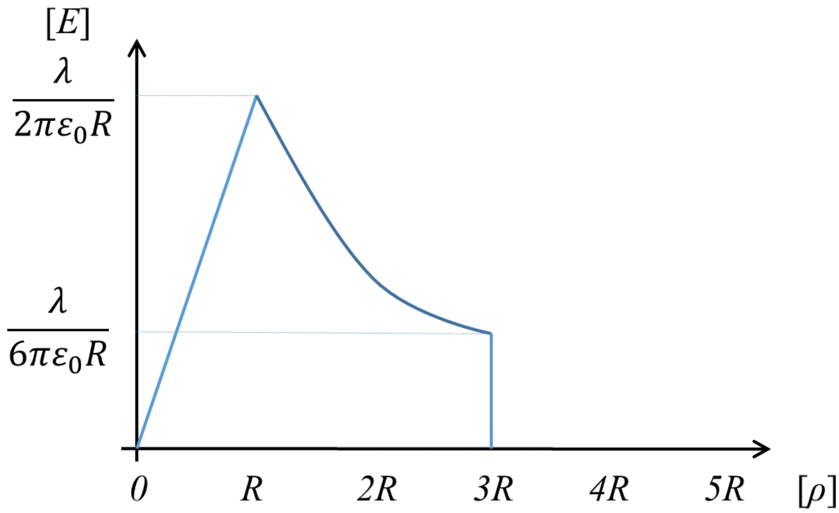
$$\vec{\boldsymbol{E}} = \frac{\lambda\,\rho}{2\,\pi\,\epsilon_0\,R^2}\,\hat{\boldsymbol{\rho}}.$$

(ii) And, when $R \le \rho < 3\,R$, we have

$$E(\rho)(2\,\pi\,\rho L) = \frac{\lambda L}{\epsilon_0} \Rightarrow \boldsymbol{E} = \frac{\lambda}{2\pi\rho\epsilon_0}\hat{\boldsymbol{\rho}}.$$

(iii) $\rho \ge 3R$

$$E(\rho)(2\pi\rho L) = \frac{\lambda L - \lambda L}{\epsilon_0} \Rightarrow \boldsymbol{E} = 0\,.$$


$[E]$
$\frac{\lambda}{2\pi\varepsilon_0 R}$
$\frac{\lambda}{6\pi\varepsilon_0 R}$
0
R
2R
3R
4R
5R
$[\rho]$

Department of Physics
Indian Institute of Technology Madras

End-of-semester exam – Make up

From Newton's laws to circulation of vector fields

Date: January 11, 2018 Time: 09:00 AM – 12:00 Noon

Name: **Roll No.:**

Instructor: **Batch:**

Instructions

1. Begin by completing the information requested above. Please write your complete name, your roll number, the name of your instructor, and your batch number (out of I–XII). **The answer sheet will not be evaluated unless both your name and roll number are written.**
2. This question paper cum answer sheet booklet contains **fourteen** single-sided pages. Please check right away that all the pages are present.
3. As we had announced earlier, this exam consists of 4 true/false questions (for 1 mark each), 4 multiple choice questions with more than one correct option (for 2 marks each), 8 fill in the blanks (for 1 mark each), 5 questions involving detailed calculations (for 3 marks each) and 3 questions involving some plotting (for 5 marks each), adding to a total of 50 marks.
4. You are expected to answer **all** the questions. There are no negative marks.
5. **The answers have to be written in the boxes provided. Answers written elsewhere in the booklet will not be evaluated.**
6. **Kindly write the answers, including sketches, with a blue or black pen. Note that answers written with pencils or pens of other colors will not be evaluated.**
7. You can use the empty reverse sides for rough work. No extra sheets will be provided.
8. You are not allowed to use a calculator or any other electronic device during the exam. Please note that you will not be permitted to continue with the exam if you are found with any such device.
9. Make sure that you return question paper cum answer sheet booklet before you leave the examination hall.

For use by examiners
(Do not write in this space)

Q1–Q8	Q9-16	Q17-21	Q22-24	Total

✦ **True or false** (1 mark each, write True (T)/False (F) in the box provided)

1. A particle moves in a plane along the logarithmic spiral $\rho = e^{\phi}$, where ρ and ϕ denote the plane polar coordinates. The angle between the position and the velocity vectors of the particle at any instant is $60°$.

2. The value of the product of the position and velocity of an undamped, one-dimensional oscillator, when averaged over one period, is zero.

3. A comet moving around the Sun in a hyperbolic orbit sweeps equal areas in equal intervals of time.

4. The force $\vec{\boldsymbol{F}} = k\,(y\,\hat{\boldsymbol{x}} + x\,\hat{\boldsymbol{y}} - 3\,z\,\hat{\boldsymbol{z}})$, where k is a constant, is conservative.

✦ **Multiple choice questions** (2 marks each, write the correct option(s) in the box provided. Note that there can be more than one correct option and zero marks will be given if you choose an incorrect option.)

5. A particle is moving under the influence of the electric field generated by an infinite plane of positive charge. The normal to the plane is along the positive z-direction. Which of the following kinematic quantities associated with the particle are conserved? (Note that, E, $\vec{\boldsymbol{p}}$ and $\vec{\boldsymbol{L}}$ denote the energy, momentum and angular momentum of the particle, respectively.)

   [A] E **[B]** $p_x,\ p_y$ **[C]** $p_z,\ L_z$ **[D]** L_z

6. A particle is moving on a circular orbit under the influence of the central force $U(r) = \alpha\,r^2$, where $\alpha > 0$. If the particle has a reduced mass μ and angular momentum L, the radius r_0 and energy E_0 of the particle are

   [A] $r_0 = [L^2/(2\,\alpha\,\mu)]^{1/4}$ **[B]** $E_0 = (2\,L^2\,\alpha/\mu)^{1/2}$
   [C] $r_0 = 2\,[L^2/(2\,\alpha\,\mu)]^{1/4}$ **[D]** $E_0 = 2\,(2\,L^2\,\alpha/\mu)^{1/2}$

7. A satellite moving in a circular orbit of radius R is given a forward thrust leading to a parabolic orbit. If M and m denote the masses of the Earth and satellite, the energy imparted to the satellite will be

   [A] $G\,M\,m/R$ **[B]** $G\,M\,m/(2\,R)$ **[C]** $2\,G\,M\,m/R$ **[D]** $4\,G\,M\,m/R$

8. A region has a uniform magnetic field $\vec{\boldsymbol{B}} = B_0\,\hat{\boldsymbol{z}}$ throughout. Consider a cone of height h, base radius R and axis of symmetry along the z-axis. Let the vertex of the cone be at the origin and the outward normal to the flat face be along the positive z-direction. If $\Phi_{\rm F}$ and $\Phi_{\rm C}$ denote the magnetic flux through the flat and the curved surfaces of the cone, then

   [A] $\Phi_{\rm F} = B_0\,\pi\,R^2$ **[B]** $\Phi_{\rm F} = -B_0\,\pi\,R^2$
   [C] $\Phi_{\rm C} = B_0\,\pi\,R^2$ **[D]** $\Phi_{\rm C} = -B_0\,\pi\,R^2$

✦ **Fill in the blanks** (1 mark each, write the answer in the box provided)

9. A particle is moving along the trajectory $\vec{\boldsymbol{r}}(t) = A\,t\,\hat{\boldsymbol{x}} + B\,t^2\,\hat{\boldsymbol{y}}$ in the x-y-plane, where A and B are constants. Express the velocity and acceleration of particle in terms of the plane polar coordinates ρ and ϕ and the corresponding unit vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$.

10. Consider the elastic collision of two different masses in one dimension. The relation between the relative velocity of the two masses before (say, $v_1 - v_2$) and after (say, $v_1' - v_2'$) the collision is

11. Express the solution to the critically damped, one-dimensional oscillator in terms of the initial position x_0 and initial velocity v_0.

12. The orbit of a particle in a central force is a circle which passes through the origin described by $r = r_0 \cos\phi$, where r_0 is a constant. What is the force $\vec{\boldsymbol{F}}(r)$ that gives rise to such a trajectory?

13. Recently, two neutron stars, one having mass $1.2\, M_{\rm Sun}$ and another $1.6\, M_{\rm Sun}$ were observed to have moved in an elliptical orbit about their common center of mass. If their maximum separation is $400\,{\rm km}$, estimate the time period of their orbit. (In the central potential $U(r) = -\alpha/r$, according to Kepler's third law, $T^2 = 4\,\pi^2\,\mu\,a^3/\alpha$. Note that, $G = 6.7 \times 10^{-11}\,{\rm m^2\,kg^{-1}\,s^{-2}}$ and $M_{\rm Sun} = 2 \times 10^{30}\,{\rm kg}$.)

14. The infinite plane $z = a\,x + b\,y + c$ has a constant surface charge density σ. The electric field above and below the plane are given by

15. A sphere of radius $0.5\,{\rm m}$ has its centre at the origin and a current density $\vec{\boldsymbol{J}} = 2\,\hat{\boldsymbol{r}}\,{\rm A/m^2}$ over its surface. By how much does the amount of charge within the sphere change in one minute? (Note: The unit A denotes Ampère, i.e. Coulomb per unit time.)

16. A conservative force field is given by $\vec{\boldsymbol{F}} = f_0\,[-\rho\cos(2\,\phi)\,\hat{\boldsymbol{\rho}} + \rho\sin(2\,\phi)\,\hat{\boldsymbol{\phi}} + z\,\hat{\boldsymbol{z}}]$, where f_0 is a constant. Calculate the work done in moving a particle from $(\rho, \phi, z) = (4, \pi/2, 2)$ to $(4, \pi, 1)$.

✦ **Questions with detailed answers** (write the calculations and answers within the boxes provided)

17. *Integrating vectors:* Given $\vec{\boldsymbol{v}} = \mathrm{d}\vec{\boldsymbol{r}}/\mathrm{d}t$, evaluate the following integrals: $\boxed{\text{1.5+1.5 marks}}$

$$\text{(a)} \int \mathrm{d}t\,\left(\vec{\boldsymbol{r}} \times \frac{\mathrm{d}\vec{\boldsymbol{v}}}{\mathrm{d}t}\right), \quad \text{(b)} \int \mathrm{d}t\,\left(\frac{\vec{\boldsymbol{v}}}{r^n} - \frac{n\,\vec{\boldsymbol{r}}}{r^{n+1}}\,\frac{\mathrm{d}r}{\mathrm{d}t}\right).$$

18. *Frequency of small oscillations:* Consider the one-dimensional potential

$$U(x) = U_0 \left(\frac{x^2}{d^2} + \frac{d^2}{x^2} \right),$$

where U_0 and d are positive constants. (a) Plot the quantity $U(x)/U_0$ against x/d. (b) Determine the frequency of small oscillations about the minima. $\boxed{\text{1.5+1.5 marks}}$

19. *Kepler problem in velocity space:* Recall that the orbit of a particle moving under the influence of the central force $U(r) = -\alpha/r$, where $\alpha > 0$, is given by $r(\phi) = r_0/(\epsilon \cos\phi + 1)$, where $r_0 = L^2/(\mu\,\alpha)$ and $\epsilon = \sqrt{1 + 2\,E\,L^2/(\mu\,\alpha^2)}$, where μ, E and L denote the reduced mass, energy and angular momentum of the particle. (a) Express the velocities v_x and v_y of the particle in terms of ϕ. (b) Show that the particle describes a circle in the (v_x, v_y) space. 2+1 marks

20. An electric dipole of moment $\vec{\boldsymbol{p}} = p\,\hat{\boldsymbol{z}}$ is located at the origin. The dipole creates the electrostatic potential

$$\phi(\boldsymbol{r}) = \frac{\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{r}}}{4\,\pi\,\epsilon_0\,r^3}.$$

(a) Determine the corresponding electric field $\vec{\boldsymbol{E}}$ in terms of $\vec{\boldsymbol{p}}$, $\hat{\boldsymbol{r}}$ and r. (b) Evaluate the work done by the electric field to move a charge q from $z = 1$ to $z = \infty$. $\boxed{\text{2+1 marks}}$

21. *Vortex in a fluid and Stokes' theorem:* A fluid flowing in the x-y-plane has the following velocity field: $\vec{\boldsymbol{v}} = a\,(-y\,\hat{\boldsymbol{x}} + x\,\hat{\boldsymbol{y}})$, where a is a constant. (a) Evaluate the vorticity field $\vec{\boldsymbol{\Omega}}$ corresponding to the above velocity field (recall that $\vec{\boldsymbol{\Omega}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}}$). (b) Verify the Stokes' theorem for the velocity field $\vec{\boldsymbol{v}}$ by considering a circular path in the x-y-plane, with its centre at the origin. $\boxed{\text{1+2 marks}}$

22. *Behavior of the damped, driven oscillator:* Recall that, a damped, driven oscillator satisfies the following equation of motion:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 2\,\beta\,\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2\,x = f_0\cos(\omega\,t).$$

Assume that the oscillator is underdamped, say, $\beta/\omega_0 \simeq 10^{-2}$. (a) Obtain the solution to the above equation for times such that $\beta\,t \gg 1$, expressing the amplitude (say, A) and phase (say, δ) of the solution in terms of the parameters ω_0, β, f_0 and ω. (b) Plot the square of the amplitude A^2 and phase δ as a function of ω/ω_0. (c) Assuming $\omega = \omega_0$, plot the position $x(t)$, the velocity $v(t) = (\mathrm{d}x/\mathrm{d}t)$ and the forcing term $f(t) = f_0\cos(\omega\,t)$ as a function of $(\omega\,t)$, in particular highlighting the difference in phases between these quantities. $\boxed{\text{1+2+2 marks}}$

23. *Trajectories in phase space:* Consider a particle moving in the following one-dimensional potential: $U(x) = (\alpha\, x^2 - \beta\, x^3)\, \mathrm{e}^{-\gamma\, x}$, where $\alpha = \beta = \gamma = 1$. (a) Draw the potential $U(x)$, specifically marking the values of the extrema. (b) Determine the range of energy and position for which the system can exhibit bounded motion. (c) Draw the following phase space trajectories indicating the direction of motion with arrows: (i) bounded motion for a positive and negative value of energy, and (ii) unbounded motion for a positive as well as negative value of energy. $\boxed{\text{2+1+2 marks}}$

24. *Electric field in and around a charged shell:* A very long cylindrical shell of radius R carries a charge per unit length λ that is uniformly distributed throughout the shell. It is surrounded by a coaxial cylindrical shell of radius $5\,R$ which is oppositely charged and carries the charge per unit length $-\lambda$. (a) Calculate the electric field at the following locations: (i) inside the original charged shell, (ii) between the inner shell and the coaxial shell and (iii) outside the coaxial shell. (b) Plot the magnitude of the electric field as a function of the radial distance from the centre of the original shell (i.e. from 0) to $7\,R$. $\boxed{\text{3+2 marks}}$