PH3520

QUANTUM PHYSICS

January–May 2013

Lecture schedule and meeting hours

- The course will consist of about 43 lectures, including about 8–10 tutorial sessions. However, note that there will be no separate tutorial sessions, and they will be integrated with the lectures.
- The duration of each lecture will be 50 minutes. We will be meeting in HSB 210.
- The first lecture will be on Monday, January 21, and the last lecture will be on Monday, April 29.
- We will meet thrice a week. The lectures are scheduled for 1:00–1:50 PM on Mondays, 10:00–10:50 AM on Thursdays, and 9:00–9:50 AM on Fridays.
- We may also meet during 4:45–5:35 PM on Mondays for either the quizzes or to make up for any lecture that I may have to miss due to, say, travel. Changes in schedule, if any, will be notified sufficiently in advance.
- If you would like to discuss with me about the course outside the lecture hours, you are welcome to meet me at my office (HSB 202A) during 2:00–2:30 PM on Mondays, and during 1:30–2:00 PM on Fridays. In case you are unable to find me in my office on more than occasion, please send me an e-mail at `sriram@physics.iitm.ac.in`.

Information about the course

- I will be distributing hard copies containing information such as the schedule of the lectures, the structure and the syllabus of the course, suitable textbooks and additional references, as well as exercise sheets.
- A PDF file containing these information as well as completed quizzes will also made be available at the link on this course at the following URL:
  `http://www.physics.iitm.ac.in/~sriram/professional/teaching/teaching.html`
  I will keep updating the file as we make progress.

Quizzes, end-of-semester exam and grading

- The grading will be based on three scheduled quizzes and an end-of-semester exam.
- I will consider the best two quizzes for grading, and the two will carry 25% weight each.
- The three quizzes will be on February 18, March 18 and April 15. All these three dates are Mondays, and the quizzes will be held during 4:45–5:35 PM. Note that, we will not be meeting during 1:00–1:50 PM on these three Mondays.
- The end-of-semester exam will be held during 9:00 AM–12:00 NOON on Friday, May 10, and the exam will carry 50% weight.

Syllabus and structure

Quantum physics

1. **Origins of quantum theory and the wave aspects of matter** [$\sim$ 4 lectures]
   (a) Black body radiation – Planck's law
   (b) Photoelectric effect
   (c) Bohr atom model – Frank and Hertz experiment
   (d) de Broglie hypothesis – The Davisson-Germer experiment
   (e) Concept of the wavefunction – The statistical interpretation
   (f) Two-slit experiment – The Heisenberg uncertainty principle

Exercise sheet 1

2. **The postulates of quantum mechanics and the Schrodinger equation** [$\sim$ 4 lectures]
   (a) Observables and operators
   (b) Expectation values and fluctuations
   (c) Measurement and the collapse of the wavefunction
   (d) The time-dependent Schrodinger equation

Exercise sheet 2

3. **The time independent Schrodinger equation in one dimension** [$\sim$ 4 lectures]
   (a) Stationary states – The time-independent Schrodinger equation
   (b) The infinite square well
   (c) Reflection and transmission in potential barriers
   (d) The delta function potential
   (e) The free particle
   (f) Linear harmonic oscillator
   (g) Kronig-Penney model – Energy bands

Exercise sheet 3

Quiz I

4. **Essential mathematical formalism** [$\sim$ 4 lectures]
   (a) Hilbert space
   (b) Observables – Hermitian operators – Eigen functions and eigen values of hermitian operators
   (c) Generalized statistical interpretation
   (d) The generalized uncertainty principle

Exercise sheets 4 and 5

5. **Particle in a central potential** [$\sim$ 4 lectures]
   (a) Motion in a central potential – Orbital angular momentum
   (b) Hydrogen atom – Energy levels
   (c) Degeneracy

Exercise sheet 6

6. **Spin** [$\sim$ 3 lectures]

   (a) Electron spin – Pauli matrices
   (b) Application to magnetic resonance

Exercise sheet 7

7. **Charged particle in uniform constant magnetic field** [$\sim$ 3 lectures]

   (a) Landau levels – Wavefunctions
   (b) Elements of the quantum Hall effect

Exercise sheet 8

Quiz II

8. **Time-independent perturbation theory** [$\sim$ 4 lectures]

   (a) Non-degenerate and degenerate cases
   (b) Application to Zeeman and Stark effects

Exercise sheet 9

9. **Time-dependent perturbation theory** [$\sim$ 4 lectures]

   (a) Transition probabilities
   (b) Fermi's Golden Rule
   (c) Decay of excited states of atoms in the dipole approximation

Exercise sheet 10

10. **Identical particles and spin** [$\sim$ 3 lectures]

    (a) Fermions and bosons
    (b) Free electron gas
    (c) Blackbody radiation

Exercise sheet 11

Quiz III

11. **Semi-classical theory of radiation** [$\sim$ 3 lectures]

    (a) Spontaneous and stimulated emission
    (b) Einstein's A and B coefficients
    (c) Population inversion
    (d) Maxwell-Bloch equations
    (e) Laser action

End-of-semester exam

Basic textbooks

1. S. Gasiorowicz, *Quantum Physics*, Third edition (John Wiley and Sons, New York, 2003).
2. R. L. Liboff, *Introductory Quantum Mechanics*, Fourth Edition (Pearson Education, Delhi, 2003).
3. W. Greiner, *Quantum Mechanics*, Fourth edition (Springer, Delhi, 2004).
4. D. J. Griffiths, *Introduction to Quantum Mechanics*, Second edition (Pearson Education, Delhi, 2005).
5. R. Shankar, *Principles of Quantum Mechanics*, Second edition (Springer, Delhi, 2008).

Advanced textbooks

1. L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Course of Theoretical Physics, Volume 3), Third Edition (Pergamon Press, New York, 1977).
2. J. J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley, Singapore, 1994).

Other references

1. D. Danin, *Probabilities of the Quantum World* (Mir Publishers, Moscow, 1983).
2. G. Gamow, *Thirty Years that Shook Physics: The Story of Quantum Theory* (Dover Publications, New York, 1985).
3. R. P. Crease and C. C. Mann, *The Second Creation: Makers of the Revolution in Twentieth-Century Physics* (Rutgers University Press, New Jersey, U.S.A., 1996), Chapters 1–4.
4. M. S. Longair, *Theoretical Concepts in Physics*, Second Edition (Cambridge University Press, Cambridge, England, 2003), Chapters 11–15.

Exercise sheet 1

Origins of quantum theory and the wave aspects of matter

1. *Black body radiation and Planck's law:* Consider a black body maintained at the temperature T. According to Planck's radiation law, the energy per unit volume within the frequency range ν and $\nu + \mathrm{d}\nu$ associated with the electromagnetic radiation emitted by the black body is given by

$$u_\nu\, \mathrm{d}\nu = \frac{8\,\pi\,h}{c^3}\, \frac{\nu^3\, \mathrm{d}\nu}{\exp\left(h\,\nu/k_{\mathrm{B}}\,T\right) - 1},$$

where h and k_{B} denote the Planck and the Boltzmann constants, respectively, while c represents the speed of light.

   (a) Arrive at the Wien's law, viz. that $\lambda_{\mathrm{max}}\, T = b =$ constant, from the above Planck's radiation law. Note that λ_{max} denotes the wavelength at which the energy density of radiation from the black body is the maximum.

   (b) The total energy emitted by the black body is described by the integral

$$u = \int_0^\infty d\nu\; u_\nu.$$

   Using the above expression for u_ν, evaluate the integral and show that

$$u = \frac{4\,\sigma}{c}\, T^4,$$

   where σ denotes the Stefan constant given by

$$\sigma = \frac{\pi^2\, k_{\mathrm{B}}^4}{60\,\hbar^3\, c^2},$$

   with $\hbar = h/(2\,\pi)$.

   (c) The experimentally determined values of the Stefan's constant σ and the Wien's constant b are found to be

$$\sigma = 5.67 \times 10^{-8}\, \mathrm{J\, m^{-2}\, s^{-1}\, K^{-4}} \quad \text{and} \quad b = 2.9 \times 10^{-3}\, \mathrm{m\, K}.$$

   Given that the value of the speed of light is $c = 2.998 \times 10^8$ m s^{-1}, determine the values of the Planck's constant h and the Boltzmann's constant k_{B} from these values.

2. *Photoelectrons from a zinc plate:* Consider the emission of electrons due to photoelectric effect from a zinc plate. The work function of zinc is known to be 3.6 eV. What is the maximum energy of the electrons ejected when ultra-violet light of wavelength 3000 Å is incident on the zinc plate?

3. *Radiation emitted in the Frank and Hertz experiment:* Recall that, in the Frank and Hertz experiment, the emission line from the mercury vapor was at the wavelength of 2536 Å. The spectrum of mercury has a strong second line at the wavelength of 1849 Å. What will be the voltage corresponding to this line at which we can expect the current in the Frank and Hertz experiment to drop?

4. *Value of the Rydberg's constant:* Evaluate the numerical value of the Rydberg's constant R_{H} and compare with its experimental value of 109677.58 cm^{-1}. How does the numerical value change if the finite mass of the nucleus is taken into account?

5. *Mean values and fluctuations:* The expectation values $\langle \hat{A} \rangle$ and $\langle \hat{A}^2 \rangle$ associated with the operator $\hat{A}$ and the wavefunction Ψ are defined as

$$\langle \hat{A} \rangle = \int \mathrm{d}x\ \Psi^* \hat{A}\, \Psi \quad \text{and} \quad \langle \hat{A}^2 \rangle = \int \mathrm{d}x\ \Psi^* \hat{A}^2\, \Psi,$$

where the integrals are to be carried out over the domain of interest. Recall that the momentum operator is given by

$$\hat{p}_x = -i\, \hbar\, \frac{\partial}{\partial x}.$$

Given the wavefunction

$$\Psi(x) = \left(\frac{\pi}{\alpha}\right)^{-1/4}\ \mathrm{e}^{-\alpha\, x^2/2},$$

calculate the following quantities: (i) $\langle \hat{x} \rangle$, (ii) $\langle \hat{x}^2 \rangle$, (iii) $\Delta x = \left[\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2\right]^{1/2}$, (iv) $\langle \hat{p}_x \rangle$, (v) $\langle \hat{p}_x^2 \rangle$, (vi) $\Delta p_x = \left[\langle \hat{p}_x^2 \rangle - \langle \hat{p}_x \rangle^2\right]^{1/2}$, and (vii) $\Delta x\, \Delta p_x$.

Exercise sheet 2

The postulates of quantum mechanics and the Schrodinger equation

1. *Hermitian operators:* Recall that the expectation value of an operator $\hat{A}$ is defined as

$$\langle \hat{A} \rangle = \int \mathrm{d}x \; \Psi^* \hat{A}\, \Psi.$$

An operator $\hat{A}$ is said to be hermitian if

$$\langle \hat{A} \rangle = \langle \hat{A} \rangle^*.$$

Show that the position, the momentum and the Hamiltonian operators are hermitian.

2. *Motivating the momentum operator:* Using the time-dependent Schrodinger equation, show that

$$\frac{\mathrm{d}\langle x \rangle}{\mathrm{d}t} = -\frac{i\,\hbar}{m} \int \mathrm{d}x \; \Psi^* \frac{\partial \Psi}{\partial x} = \langle \hat{p}_x \rangle,$$

a relation which can be said to motivate the expression for the momentum operator, viz. that $\hat{p}_x = -i\,\hbar\,\partial/\partial x$.

3. *The conserved current:* Consider a quantum mechanical particle propagating in a given potential and described by the wave function $\Psi(x,t)$. The probability $P(x,t)$ of finding the particle at the position x and the time t is given by

$$P(x,t) = |\Psi(x,t)|^2.$$

Using the one-dimensional Schrodinger equation, show that the probability $P(x,t)$ satisfies the conservation law

$$\frac{\partial P(x,t)}{\partial t} + \frac{\partial j(x,t)}{\partial x} = 0,$$

where the quantity $j(x,t)$ represents the conserved current given by

$$j(x,t) = \frac{\hbar}{2\,i\,m} \left[\Psi^*(x,t) \left(\frac{\partial \Psi(x,t)}{\partial x}\right) - \Psi(x,t) \left(\frac{\partial \Psi^*(x,t)}{\partial x}\right)\right].$$

4. *Ehrenfest's theorem:* Show that

$$\frac{\mathrm{d}\langle \hat{p}_x \rangle}{\mathrm{d}t} = -\left\langle \frac{\partial V}{\partial x} \right\rangle,$$

a relation that is often referred to as the Ehrenfest's theorem.

5. *Conservation of the scalar product:* The scalar product between two normalizable wavefunctions, say, Ψ_1 and Ψ_2, which describe a one-dimensional system is defined as

$$\langle \Psi_2 | \Psi_1 \rangle \equiv \int \mathrm{d}x \; \Psi_2^*(x,t)\, \Psi_1(x,t),$$

where the integral is to be carried out over the domain of interest. Show that

$$\frac{\mathrm{d}\langle \Psi_2 | \Psi_1 \rangle}{\mathrm{d}t} = 0.$$

Exercise sheet 3

The time-independent Schrodinger equation in one dimension I

1. *Superposition of energy eigen states:* Consider a particle in the infinite square well. Let the initial wave function of the particle be given by

$$\Psi(x,0) = A\,[\psi_1(x) + \psi_2(x)]\,,$$

where $\psi_1(x)$ and $\psi_2(x)$ denote the ground and the first excited states of the particle.

   (a) Normalize the wave function $\Psi(x,0)$.

   (b) Obtain the wave function at a later time t, viz. $\Psi(x,t)$, and show that the probability $|\Psi(x,t)|^2$ is an oscillating function of time.

   (c) Evaluate the expectation value of the position in the state $\Psi(x,t)$ and show that it oscillates. What are the angular frequency and the amplitude of the oscillation?

   (d) What will be the values that you will obtain if you measure the energy of the particle? What are the probabilities for obtaining these values?

   (e) Evaluate the expectation value of the Hamiltonian operator corresponding to the particle in the state $\Psi(x,t)$. How does it compare with the energy eigen values of the ground and the first excited states?

2. *Spreading of wave packets:* A free particle has the initial wave function

$$\Psi(x,0) = A\,\mathrm{e}^{-a\,x^2},$$

where A and a are constants, with a being real and positive.

   (a) Normalize $\Psi(x,0)$.

   (b) Find $\Psi(x,t)$.

   (c) Plot $\Psi(x,t)$ at $t=0$ and for large t. Determine qualitatively what happens as time goes on?

   (d) Find $\langle\hat{x}\rangle$, $\langle\hat{x}^2\rangle$, $\langle\hat{p}\rangle$, $\langle\hat{p}^2\rangle$, Δx and Δp.

   (e) Does the uncertainty principle hold? At what time does the system have the minimum uncertainty?

3. *Properties of Hermite polynomials:* In this problem, we shall explore a few useful relations involving the Hermite polynomials.

   (a) According to the so-called Rodrigues's formula

$$H_n(x) = (-1)^n\,\mathrm{e}^{x^2}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\left(\mathrm{e}^{-x^2}\right).$$

   Use this relation to obtain $H_3(x)$ and $H_4(x)$.

   (b) Utilize the following recursion relation:

$$H_{n+1}(x) = 2\,x\,H_n(x) - 2\,n\,H_{n-1}(x),$$

   and the results of the above problem to arrive at $H_5(x)$ and $H_6(x)$.

   (c) Using the expressions for $H_5(x)$ and $H_6(x)$ that you have obtained, check that the following relation is satisfied:

$$\frac{\mathrm{d}H_n}{\mathrm{d}x} = 2\,n\,H_{n-1}(x).$$

(d) Obtain $H_0(x)$, $H_1(x)$ and $H_2(x)$ from the following generating function for the Hermite polynomials:

$$e^{-\left(z^2-2\,z\,x\right)} = \sum_{n=0}^{\infty} \frac{z^n}{n!}\, H_n(x).$$

4. *Wagging the dog:* Recall that the time-independent Schrodinger equation satisfied by a simple harmonic oscillator of mass m and frequency ω is given by

$$-\frac{\hbar^2}{2\,m}\,\frac{\mathrm{d}^2\psi_{_E}}{\mathrm{d}x^2} + \frac{1}{2}\,m\,\omega^2\,x^2\,\psi_{_E} = E\,\psi_{_E}.$$

In terms of the dimensionless variable

$$\xi = \sqrt{\frac{m\,\omega}{\hbar}}\,x,$$

the above time-independent Schrodinger equation reduces to

$$\frac{\mathrm{d}^2\psi_{_E}}{\mathrm{d}\xi^2} + \left(\mathcal{E} - \xi^2\right)\,\psi_{_E} = 0,$$

where $\mathcal{E}$ is the energy expressed in units of $(\hbar\,\omega/2)$, and is given by

$$\mathcal{E} = \frac{2\,E}{\hbar\,\omega}.$$

According to the 'wag-the-dog' method, one solves the above differential equation numerically, say, using Mathematica, varying $\mathcal{E}$ until a wave function that goes to zero at large ξ is obtained.

Find the ground state energy and the energies of the first two excited states of the harmonic oscillator to five significant digits by the 'wag-the-dog' method.

5. *The case of the particle in an infinite square well:* Find the first three allowed energies to, say, five significant digits, of a particle in the infinite square well, by 'wagging the dog'.

Quiz I

From the origins of quantum theory and the wave aspects of matter to the time independent Schrodinger equation in one dimension

1. *Semi-classical quantization precedure:* Consider a particle moving in one-dimension. Let the particle be described by the generalized coordinate q, and let the corresponding conjugate momentum be p. According to the semi-classical quantization rule, the so-called action I satisfies the following relation:

$$I \equiv \int dq\, p = n\, h,$$

where n is an integer, while h is the Planck constant. Using the above quantization procedure, determine the energy levels of the following systems:

   (a) A particle in an infinite square well. [4 marks]

   (b) A rigid rotator. [6 marks]

   Note: A rigid rotator is a particle which rotates about an axis and is located at a fixed length from the axis. Also, the particle moves *only* along the azimuthal direction.

2. *The Klein-Gordon equation:* Consider the following Klein-Gordon equation governing a wavefunction $\Psi(x,t)$:

$$\frac{1}{c^2}\,\frac{\partial^2\Psi(x,t)}{\partial t^2} - \frac{\partial^2\Psi(x,t)}{\partial x^2} + \left(\frac{\mu\, c}{\hbar}\right)^2 \Psi(x,t) = 0,$$

where c and μ are constants. Show that there exists a corresponding 'probability' conservation law of the form

$$\frac{\partial P(x,t)}{\partial t} + \frac{\partial j(x,t)}{\partial x} = 0,$$

where the quantity $j(x,t)$ represents the conserved current given by

$$j(x,t) = \frac{\hbar}{2\, i\, \mu}\left[\Psi^*(x,t)\left(\frac{\partial\Psi(x,t)}{\partial x}\right) - \Psi(x,t)\left(\frac{\partial\Psi^*(x,t)}{\partial x}\right)\right].$$

   (a) Express the 'probability' $P(x,t)$ in terms of the wavefunction $\Psi(x,t)$. [6 marks]

   (b) Can you identify any issue with interpreting $P(x,t)$ as the probability? [4 marks]

3. *Particle in an infinite square well:* A particle in an infinite square well, with its walls located at $x = 0$ and $x = a$, is described by the following initial wave function at the time, say, $t = 0$:

$$\Psi(x,0) = A\,\sin^3(\pi\, x/a).$$

   (a) Determine A. [2 marks]

   (b) Find the wavefunction $\Psi(x,t)$ at a time $t > 0$. [4 marks]

   (c) Calculate $\langle\hat{x}\rangle$ as a function of time in the state $\Psi(x,t)$. [4 marks]

4. *Particle in an attractive delta function potential:* Consider a particle moving in one-dimension in the following delta function potential:

$$V(x) = -a\,\delta(x),$$

where $a > 0$.

   (a) Determine the bound state energy eigen functions. [4 marks]

   (b) Plot the energy eigen functions. [2 marks]

(c) How many bound states exist? What are the corresponding energy eigen values? 4 marks

5. *Relations involving operators:* Consider a one-dimensional particle described by the wavefunction $\psi(x)$. Establish the following operator relations: 2+4+4 marks

$$\begin{aligned}
[\hat{x}, \hat{p}_x]\ \psi(x) \equiv (\hat{x}\,\hat{p}_x - \hat{p}_x\,\hat{x})\ \psi &= i\,\hbar\,\psi(x), \\
\exp\ (i\ \hat{p}_x\,a/\hbar)\ \psi(x) &= \psi(x+a), \\
\exp\ (i\ \hat{p}_x\,a/\hbar)\ \hat{x}\ \exp\ -(i\ \hat{p}_x\,a/\hbar) &= \hat{x} + a,
\end{aligned}$$

where a is a constant, while, recall that, $\hat{p} = -i\,\hbar\,\mathrm{d}/\mathrm{d}x$.

Exercise sheet 4

Essential mathematical formalism I

1. *Eigen values and eigen functions of the momentum operator:* Determine the eigen values and the eigen functions of the momentum operator. Establish the completeness of the momentum eigen functions.

2. *The angular momentum operator:* Consider the operator

$$L_\phi = -i\,\hbar \frac{\mathrm{d}}{\mathrm{d}\,\phi},$$

where ϕ is an angular variable. Is the operator hermitian? Determine its eigenfunctions and eigenvalues.

Note: The operator L_ϕ, for instance, could describe the conjugate momentum of a bead that is constrained to move on a circle of a fixed radius.

3. *Probabilities in momentum space:* A particle of mass m is bound in the delta function well $V(x) = -a\,\delta(x)$, where $a > 0$. What is the probability that a measurement of the particle's momentum would yield a value greater than $p_0 = m\,a/\hbar$?

4. *The energy-time uncertainty principle:* Consider a system that is described by the Hamiltonian operator $\hat{H}$.

   (a) Given an operator, say, $\hat{Q}$, establish the following relation:

$$\frac{\mathrm{d}\langle\hat{Q}\rangle}{\mathrm{d}t} = \frac{i}{\hbar}\,\langle[\hat{H},\hat{Q}]\rangle + \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle,$$

   where the expectation values are evaluated in a specific state.

   (b) When $\hat{Q}$ does not explicitly depend on time, using the generalized uncertainty principle, show that

$$\Delta H\,\Delta Q \geq \frac{\hbar}{2}\left|\frac{\mathrm{d}\langle\hat{Q}\rangle}{\mathrm{d}t}\right|.$$

   (c) Defining

$$\Delta t \equiv \frac{\Delta Q}{|\mathrm{d}\langle\hat{Q}\rangle/\mathrm{d}t|},$$

   establish that

$$\Delta E\,\Delta t \geq \frac{\hbar}{2},$$

   and interpret this result.

5. *Two-dimensional Hilbert space:* Imagine a system in which there are only two linearly independent states, viz.

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The most general state would then be a normalized linear combination, i.e.

$$|\psi\rangle = \alpha\,|1\rangle + \beta\,|2\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

with $|\alpha|^2+|\beta|^2=1$. The Hamiltonian of the system can, evidently, be expressed as a 2×2 hermitian matrix. Suppose it has the following form:

$$\mathsf{H} = \begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

where a and b are *real* constants. If the system starts in the state $|1\rangle$ at an initial time, say, $t=0$, determine the state of the system at a later time t.

Exercise sheet 5

Essential mathematical formalism II

1. *A three-dimensional vector space:* Consider a three-dimensional vector space spanned by the orthonormal basis $|1\rangle$, $|2\rangle$ and $|3\rangle$. Let two kets, say, $|\alpha\rangle$ and $|\beta\rangle$ be given by

$$|\alpha\rangle = i\,|1\rangle - 2\,|2\rangle - i\,|3\rangle \quad \text{and} \quad |\beta\rangle = i\,|1\rangle + 2\,|3\rangle.$$

   (a) Construct $\langle\alpha|$ and $\langle\beta|$ in terms of the dual basis, i.e. $\langle 1|$, $\langle 2|$ and $\langle 3|$.

   (b) Find $\langle\alpha|\beta\rangle$ and $\langle\beta|\alpha\rangle$ and show that $\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$.

   (c) Determine all the matrix elements of the operator $\hat{A} = |\alpha\rangle\langle\beta|$ in this basis and construct the corresponding matrix. Is the matrix hermitian?

2. *A two level system:* The Hamiltonian operator of a certain two level system is given by

$$\hat{H} = E\left(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|\right),$$

   where $|1\rangle$ and $|2\rangle$ form an orthonormal basis, while E is a number with the dimensions of energy.

   (a) Find the eigen values and the normalized eigen vectors, i.e. as a linear combination of the basis vectors $|1\rangle$ and $|2\rangle$, of the above Hamiltonian operator.

   (b) What is the matrix that represents the operator $\hat{H}$ in this basis?

3. *Matrix elements for the harmonic oscillator:* Let $|n\rangle$ denote the orthonormal basis of energy eigen states of the harmonic oscillator. Determine the matrix elements $\langle n|\hat{x}|m\rangle$ and $\langle n|\hat{p}_x|m\rangle$ in this basis.

4. *Coherent states of the harmonic oscillator:* Consider states, say, $|\alpha\rangle$, which are eigen states of the annihilation (or, more precisely, the lowering) operator, i.e.

$$\hat{a}|\alpha\rangle = \alpha\,|\alpha\rangle,$$

   where α is a complex number.

   Note: The state $|\alpha\rangle$ is called the coherent state.

   (a) Calculate the quantities $\langle\hat{x}\rangle$, $\langle\hat{x}^2\rangle$, $\langle\hat{p}_x\rangle$ and $\langle\hat{p}_x^2\rangle$ in the coherent state.

   (b) Also, evaluate the quantities Δx and Δp_x in the state, and show that $\Delta x\,\Delta p_x = \hbar/2$.

   (c) Like any other general state, the coherent state can be expanded in terms of the energy eigen states $|n\rangle$ of the harmonic oscillator as follows:

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n\,|n\rangle.$$

   Show that the quantities c_n are given by

$$c_n = \frac{\alpha^n}{\sqrt{n!}}\,c_0.$$

   (d) Determine c_0 by normalizing $|\alpha\rangle$.

   (e) Upon including the time dependence, show that the coherent state continues to be an eigen state of the lowering operator $\hat{a}$ with the eigen value evolving in time as

$$\alpha(t) = e^{-i\,\omega\,t}\,\alpha.$$

   Note: Therefore, a coherent state *remains* coherent, and continues to minimize the uncertainty.

(f) Is the ground state $|0\rangle$ itself a coherent state? If so, what is the eigen value?

5. <u>*A three level system:*</u> The Hamiltonian for a three level system is represented by the matrix

$$\mathsf{H} = \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Two other observables, say, A and B, are represented by the matrices

$$\mathsf{A} = \lambda \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \text{and} \qquad \mathsf{B} = \mu \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where ω, λ and μ are positive real numbers.

(a) Find the eigen values and normalized eigen vectors of H, A, and B.

(b) Suppose the system starts in the generic state

$$|\psi(t=0)\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix},$$

with $|c_1|^2 + |c_2|^2 + |c_3|^2 = 1$. Find the expectation values of H, A and B in the state at $t = 0$.

(c) What is $|\psi(t)\rangle$ for $t > 0$? If you measure the energy of the state at a time t, what are the values of energies that you will get and what would be the probability for obtaining each of the values?

(d) Also, arrive at the corresponding answers for the quantities A and B.