

Primary and scalar-induced, secondary gravitational waves from the early universe

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Plan of the talk

- 1 Inflation and reheating
- 2 Tensor power spectrum in slow roll inflation
- 3 Constraints on inflation from the CMB data
- 4 GWs provide a new window to the universe
- 5 Reheating can boost the strengths of primary GWs
- 6 Generation of GWs by enhanced scalar perturbations on small scales
- 7 NANOGrav 15-year data and the stochastic GW background
- 8 Outlook
- 9 References



A few words on the conventions and notations

- ◆ We shall work in units such that $c = \hbar = 1$, and define the Planck mass to be $M_{\text{Pl}} = (8\pi G)^{-1/2}$.
- ◆ As is often done, particularly in the context of inflation, we shall assume the background universe to be described by the following spatially flat, Friedmann-Lemaître-Robertson-Walker (FLRW) line-element:

$$ds^2 = -dt^2 + a^2(t) d\mathbf{x}^2 = a^2(\eta) (-d\eta^2 + d\mathbf{x}^2),$$

where t is the cosmic time, $a(t)$ is the scale factor and $\eta = \int dt/a(t)$ denotes the conformal time coordinate.

- ◆ We shall denote differentiation with respect to the cosmic and the conformal times t and η by an overdot and an overprime, respectively.
- ◆ Moreover, as usual, $H = \dot{a}/a$ shall denote the Hubble parameter associated with the FLRW universe, and N shall denote the number of e -folds.

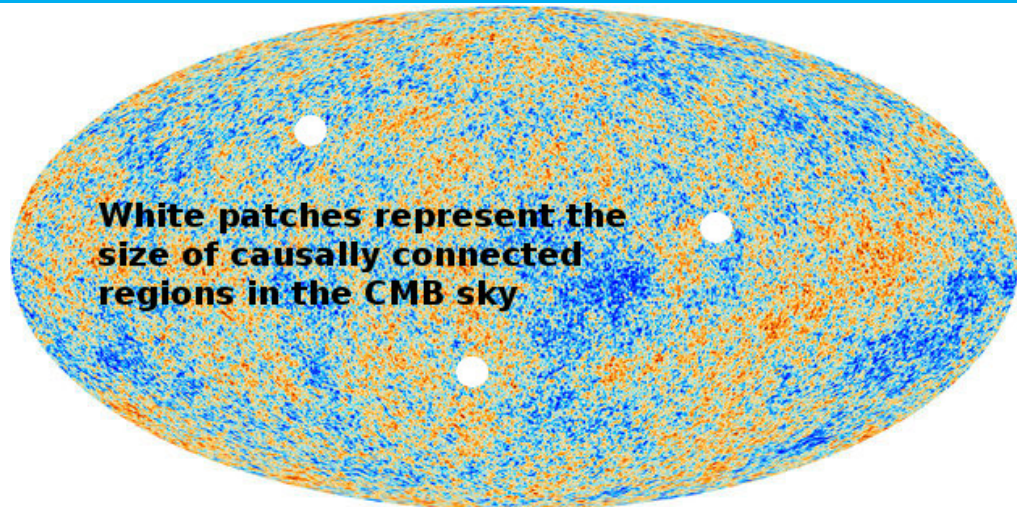


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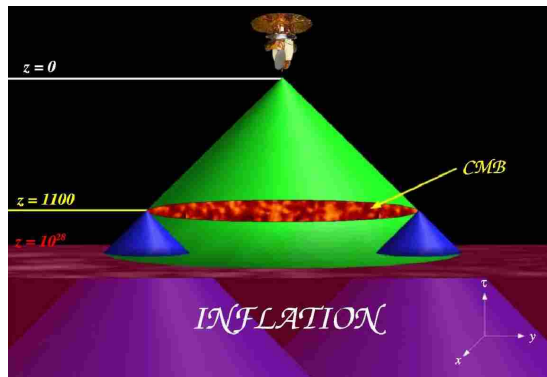
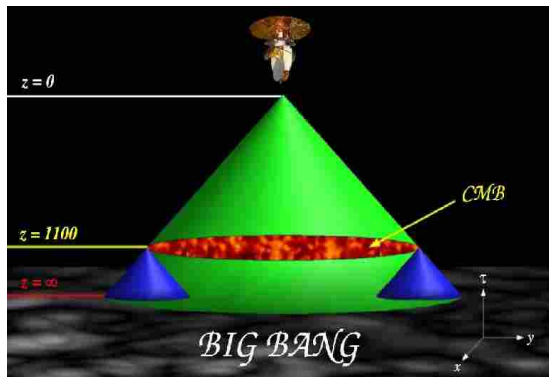
The horizon problem



The radiation from the CMB arriving at us from regions separated by more than the Hubble radius at the surface of last scattering, which subtends an angle of about 1° today, could not have interacted before decoupling.



The resolution of the horizon problem in the inflationary scenario

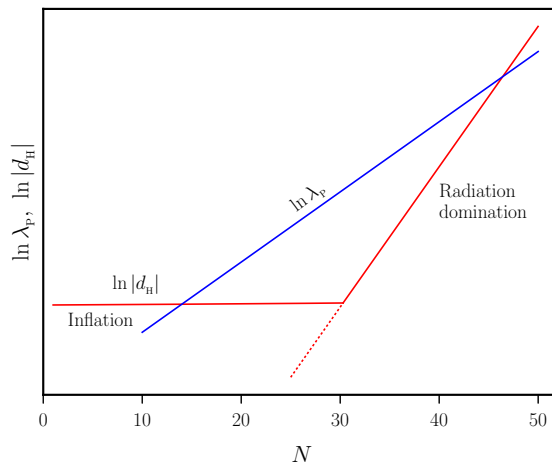


Another illustration of the horizon problem (on the left), and an illustration of its resolution (on the right) through an early and sufficiently long epoch of inflation¹.

¹Images from [W. Kinney, astro-ph/0301448](#).



Bringing the modes inside the Hubble radius



► Evolution of comoving lengths

The physical wavelength $\lambda_p \propto a$ (in blue) and the Hubble radius $d_H = H^{-1}$ (in red) in the inflationary scenario². The scale factor is expressed in terms of e-folds N as $a(N) \propto e^N$.

²See, for example, E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley Publishing Company, New York, 1990), Fig. 8.4.



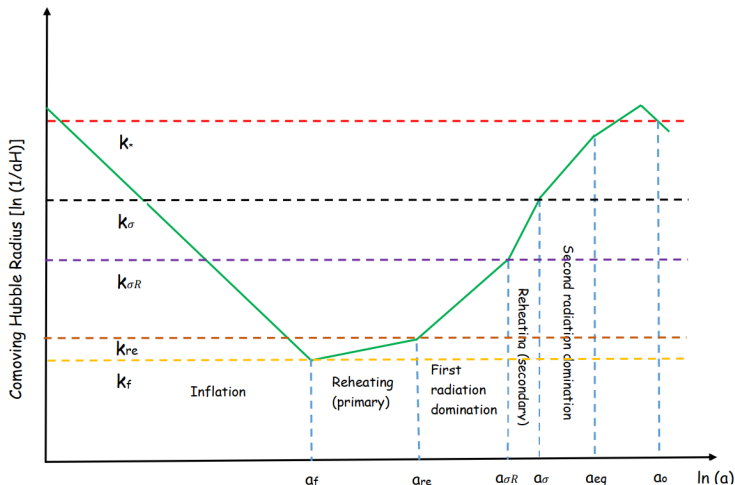
Condition for inflation

If we require that $\lambda_P < d_H$ at a sufficiently early time, then we need to have an epoch wherein λ_P decreases faster than the Hubble scale *as we go back in time*, i.e. a regime during which

$$-\frac{d}{dt} \left(\frac{\lambda_P}{d_H} \right) < 0 \quad \Rightarrow \quad \ddot{a} > 0.$$



Behavior of the comoving wave number and Hubble radius



Behavior of the comoving wave number k (horizontal lines in different colors) and the comoving Hubble radius $d_H/a = (aH)^{-1}$ (in green) across different epochs³.

³Md. R. Haque, D. Maity, T. Paul and L. Sriramkumar, Phys. Rev. D **104**, 063513 (2021).



Scalar fields in a FLRW universe

The equations that govern the dynamics of the spatially flat, FLRW universe are given by

$$H^2 = \frac{8\pi G}{3} \rho, \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p),$$

where ρ and p denote the energy density and pressure associated with the source(s) that is (are) driving the expansion.

The energy density and pressure associated with a homogeneous scalar field in a FLRW universe are given by

$$\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi),$$

where $V(\phi)$ is the potential describing the scalar field. From the equation governing the conservation of energy, viz.

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = 0,$$

the equation of motion governing the scalar field can be obtained to be

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = 0,$$

where $V_\phi = dV/d\phi$.



Driving inflation with scalar fields

From the Friedmann equations, for $\ddot{a} > 0$, we require that

$$(\rho + 3p) < 0.$$

In the case of a canonical scalar field, this condition simplifies to

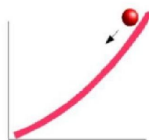
$$\dot{\phi}^2 < V(\phi).$$

This condition can be achieved if the scalar field ϕ is initially displaced from a minima of the potential, and inflation will end when the field approaches a minima with zero or negligible potential energy⁴.

⁴See, for instance, B. A. Bassett, S. Tsujikawa and D. Wands, *Rev. Mod. Phys.* **78**, 537 (2006).



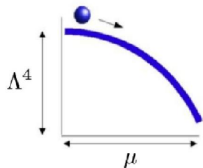
A variety of potentials to choose from



Large field

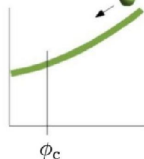
$$V(\phi) = \Lambda^4 (\phi/\mu)^p$$

$$V(\phi) = \Lambda^4 e^{\phi/\mu}$$



Small field

$$V(\phi) = \Lambda^4 [1 - (\phi/\mu)^p]$$



Hybrid

$$V(\phi) = \Lambda^4 [1 + (\phi/\mu)^p]$$

A variety of scalar field potentials have been considered to drive inflation⁵. Often, these potentials are classified as small field, large field and hybrid models.

⁵Image from [W. Kinney, astro-ph/0301448](https://arxiv.org/abs/astro-ph/0301448).



Slow roll approximation and the potential slow roll parameters

In the slow roll approximation, we can write the first Friedmann equation as

$$H^2 \simeq \frac{V(\phi)}{3 M_{\text{Pl}}^2}.$$

Also, if we assume that the acceleration of the field is small, we have

$$3 H \dot{\phi} \simeq -V_{\phi}.$$

Given a potential $V(\phi)$, the slow roll approximation requires that the following potential slow roll parameters be small when compared to unity⁶:

$$\epsilon_{\text{v}} = \frac{M_{\text{Pl}}^2}{2} \left(\frac{V_{\phi}}{V} \right)^2, \quad \eta_{\text{v}} = M_{\text{Pl}}^2 \left(\frac{V_{\phi\phi}}{V} \right), \quad \xi^2 = M_{\text{Pl}}^4 \left(\frac{V_{\phi} V_{\phi\phi\phi}}{V^2} \right),$$

where $V_{\phi\phi} = d^2V/d\phi^2$ and $V_{\phi\phi\phi} = d^3V/d\phi^3$.

⁶See, for instance, A. R. Liddle, P. Parsons and J. D. Barrow, *Phys. Rev. D* **50**, 7222 (1994).



Hierarchy of slow roll parameters

Nowadays, it is more common to use the following hierarchy of slow roll parameters⁷:

$$\epsilon_1 = -\frac{d \ln H}{dN}, \quad \epsilon_{i+1} = \frac{d \ln \epsilon_i}{dN} \quad (i \geq 0).$$

We can express the first Friedmann equation and the equation of motion for the scalar field in terms of the first two slow roll parameters ϵ_1 and ϵ_2 as

$$H^2 = \frac{V(\phi)}{M_{\text{Pl}}^2 (3 - \epsilon_1)}, \quad \left(3 - \epsilon_1 + \frac{\epsilon_2}{2}\right) H \dot{\phi} + V_\phi = 0.$$

In the slow roll approximation wherein $(\epsilon_1, \epsilon_2) \ll 1$, these two equations can be combined to arrive at

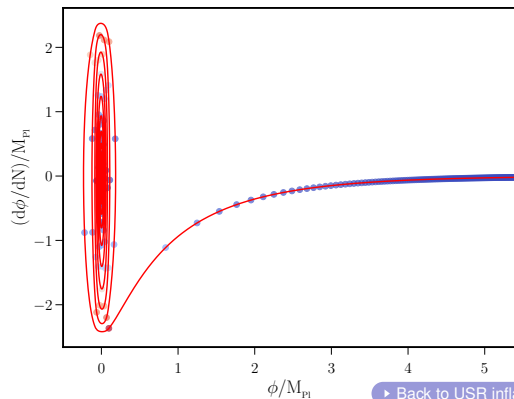
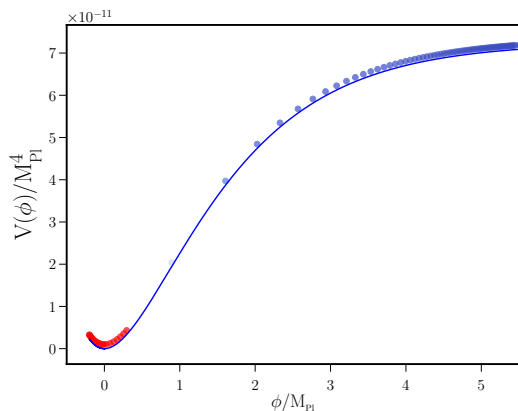
$$N - N_i \simeq -\frac{1}{M_{\text{Pl}}^2} \int_{\phi_i}^{\phi} d\phi \frac{V}{V_\phi}.$$

Given a potential $V(\phi)$, this equation can be integrated to obtain $\phi(N)$.

⁷D. J. Schwarz, C. A. Terrero-Escalante, A. A. Garcia, Phys. Lett. B **517**, 243 (2001);
S. M. Leach, A. R. Liddle, J. Martin and D. J. Schwarz, Phys. Rev. D **66**, 023515 (2002).



The inflationary attractor



[▶ Back to USR inflation](#)

Evolution of the scalar field in the popular Starobinsky model, which leads to slow roll inflation, is indicated (as circles, in blue and red) at regular intervals of time (on the left). Illustration of the behavior of the scalar field in phase space (on the right)⁸.

⁸Figure from H. V. Ragavendra, *Observational imprints of non-trivial inflationary dynamics over large and small scales*, Ph.D. Thesis, Indian Institute of Technology Madras, Chennai, India (2022).



Averaging over the oscillations after the end of inflation

Note that, the equation of motion of the inflaton can be written as

$$\dot{\rho}_\phi = -3H \dot{\phi}^2 = -6H [\rho_\phi - V(\phi)].$$

Upon averaging this equation over a time period of oscillation of the inflaton at the bottom of the potential, we obtain that

$$\langle \dot{\rho}_\phi \rangle = -\langle 6H [\rho_\phi - V(\phi)] \rangle \simeq -6H \langle \rho_\phi - V(\phi) \rangle,$$

where the angular brackets denote the time averages.

Since $dt = d\phi / \sqrt{2 [\rho_\phi - V(\phi)]}$, we can write

$$\langle \rho_\phi - V(\phi) \rangle \equiv \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} dt [\rho_\phi - V(\phi)] = \left[\int_{-\phi_m}^{\phi_m} d\phi \sqrt{\rho_\phi - V(\phi)} \right] \left[\int_{-\phi_m}^{\phi_m} \frac{d\phi}{\sqrt{\rho_\phi - V(\phi)}} \right]^{-1},$$

where \mathcal{T} and ϕ_m are the time period and the value of the scalar field when it has reached the maximum amplitude during the oscillations.



Equation-of-state (EoS) during preheating

For $V(\phi) \propto \phi^n$ and even n , if we assume that, over one period, $\rho_\phi \simeq V(\phi_m) = V_m$ is a constant, then we can write

$$\langle \rho_\phi - V(\phi) \rangle \simeq \alpha \rho_\phi,$$

where α is a number that is given by

$$\alpha = \left[\int_{-\phi_m}^{\phi_m} d\phi \sqrt{1 - [V(\phi)/V_m]} \right] \left[\int_{-\phi_m}^{\phi_m} \frac{d\phi}{\sqrt{1 - [V(\phi)/V_m]}} \right]^{-1} = \frac{n}{n+2}.$$

The equation governing the average value of ρ_ϕ simplifies to

$$\dot{\rho}_\phi \simeq -6H\alpha\rho_\phi,$$

which can be easily integrated to arrive at the result⁹

$$\rho_\phi = \rho_e (a/a_e)^{-6\alpha} = \rho_e (a/a_e)^{-6n/(n+2)} = \rho_e (a/a_e)^{-3(1+w)},$$

where ρ_e is the energy density when the scale factor was a_e and $w = (n-2)/(n+2)$.

⁹M. S. Turner, Phys. Rev. D **28**, 1243 (1983);

A. Albrecht, P. J. Steinhardt, M. S. Turner and F. Wilczek, Phys. Rev. Lett. **48**, 1437 (1982).



Coupling the inflaton to radiation

The inflaton can be coupled to radiation by modifying the equations that govern the energy densities of the inflation and radiation as follows:

$$\dot{\rho}_\phi = -3H(\rho_\phi + p_\phi) + Q_\phi, \quad \dot{\rho}_\gamma = -3H(\rho_\gamma + p_\gamma) + Q_\gamma,$$

where ρ_γ and p_γ denote the energy density and pressure associated with radiation.

Let the total energy density and pressure be $\rho = \rho_\phi + \rho_\gamma$ and $p = p_\phi + p_\gamma$. If the total energy density is to be conserved, we need to set $Q_\phi = -Q_\gamma$, so that we have

$$\dot{\rho} = -3H(\rho + p).$$

Let us now choose $Q_\phi = -\Gamma \dot{\phi}^2$, where Γ is a constant decay rate. In such a case, the conservation equation for ρ_ϕ is given by

$$\dot{\rho}_\phi = -(3H + \Gamma) \dot{\phi}^2 = -2(3H + \Gamma) [\rho_\phi - V(\phi)] = -2(3H + \Gamma) \alpha \rho_\phi,$$

which can be integrated to obtain that

$$\rho_\phi = \rho_e (a/a_e)^{-6\alpha} \exp[-2\Gamma\alpha(t - t_e)],$$

where $\rho_\phi = \rho_e$ when $a = a_e$ at $t = t_e$.



Transferring the energy from inflaton to radiation

Since $p_\gamma = \rho_\gamma/3$, the energy density of radiation is governed by the equation

$$\dot{\rho}_\gamma = -4H\rho_\gamma + 2\Gamma\alpha\rho_\phi.$$

Upon substituting the above solution for ρ_ϕ , the solution for ρ_γ can be expressed as

$$\rho_\gamma(t) = \rho_e (2\alpha\Gamma) (a/a_e)^{-4} \int_{t_e}^t d\tilde{t} [a(\tilde{t})/a_e]^{(4-6\alpha)} \exp - [2\alpha\Gamma(\tilde{t} - t_e)],$$

where we have assumed that $\rho_\gamma(t_e) = 0$.

If we assume that the scalar field still dominates the energy density, the evolution of the scale factor during this domain can be obtained to be: $a(t)/a_e = (t/t_e)^\beta$, where $\beta = 1/(3\alpha)$.

Under this condition, one can carry out the integral in the above expression to obtain that¹⁰

$$\rho_\gamma(t) = \rho_e (2\alpha\Gamma t_e)^{-(4\beta-2)} e^{2\alpha\Gamma t_e} (a/a_e)^{-4} [\gamma(4\beta-1, 2\alpha\Gamma t) - \gamma(4\beta-1, 2\alpha\Gamma t_e)],$$

where $\gamma(m, x)$ denotes the incomplete Gamma function.

¹⁰See, for instance, J. Martin, arXiv:hep-th/0406011.



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Character of the perturbations

In a Friedmann universe, the perturbations in the metric and the matter can be classified according to their behavior with respect to local rotation of the spatial coordinates on hypersurfaces of constant time as follows¹¹:

- ◆ Scalar perturbations – Density and pressure perturbations,
- ◆ Vector perturbations – Rotational velocity fields,
- ◆ Tensor perturbations – Gravitational waves.

The metric perturbations are related to the matter perturbations through the first order Einstein's equations.

Inflation does not produce any vector perturbations, while the tensor perturbations can be generated even in the absence of sources.

It is the fluctuations in the inflaton field ϕ that act as the seeds for the scalar perturbations that are primarily responsible for the anisotropies in the CMB and, eventually, the present day inhomogeneities.

¹¹See, for instance, [L. Sriramkumar, Curr. Sci. 97, 868 \(2009\)](#).



Perturbations in the metric tensor

The perturbations in the FLRW metric, say, $\delta g_{\mu\nu}$, can be split as¹²

$$\delta g_{\mu\nu} = (\delta g_{00}, \delta g_{0i}, \delta g_{ij}).$$

As it contains no running index, evidently, the perturbation $\delta g_{00} = \mathcal{A}$, say, is a scalar. Since, one can always decompose a vector into a gradient of a scalar and a vector that is divergence free, we can write δg_{0i} as

$$\delta g_{0i} = \nabla_i \mathcal{B} + \mathcal{S}_i,$$

where \mathcal{B} is a scalar, while \mathcal{S}_i is a vector that satisfies the condition $\nabla_i \mathcal{S}^i = 0$.

A similar decomposition can be carried for the quantity δg_{ij} by repeating the above analysis on each of the two indices so that we have

$$\delta g_{ij} = \psi \delta_{ij} + \frac{1}{2} (\nabla_i \mathcal{F}_j + \nabla_j \mathcal{F}_i) + \left[\frac{1}{2} (\nabla_i \nabla_j + \nabla_j \nabla_i) - \frac{1}{3} \delta_{ij} \nabla^2 \right] \mathcal{E} + \mathcal{H}_{ij},$$

where ψ and \mathcal{E} are scalars, \mathcal{F}_i (like \mathcal{S}_i) is a divergence free vector, and \mathcal{H}_{ij} is a symmetric, traceless and transverse tensor that satisfies the conditions $\mathcal{H}_i^i = 0$ and $\nabla_i \mathcal{H}^{ij} = 0$.

¹²See, for instance, L. Sriramkumar, *Curr. Sci.* **97**, 868 (2009).



Counting the number of degrees of freedom

To describe $\delta g_{\mu\nu}$, we require the scalars \mathcal{A} , \mathcal{B} , ψ and \mathcal{E} , amounting to four degrees of freedom.

In $(D+1)$ -spacetime dimensions, we also need the two divergence free spatial vectors \mathcal{S}_i and \mathcal{F}_i that add up to $2(D-1)$ degrees of freedom.

Moreover, after we impose the traceless and transverse conditions, the tensor \mathcal{H}_{ij} has

$$\frac{1}{2} D (D + 1) - (D + 1) = \frac{1}{2} (D + 1) (D - 2)$$

degrees of freedom.

Upon adding all these scalar, vector and tensor degrees of freedom, we obtain

$$4 + 2(D-1) + \frac{1}{2}(D+1)(D-2) = \frac{1}{2}(D+1)(D+2),$$

which are the total number of degrees of freedom associated with the perturbed metric in $(D+1)$ -spacetime dimensions.



Gauges or the coordinate degrees of freedom

In the FLRW background, the $(D + 1)$ coordinate transformations that relate different coordinate systems that describe the perturbed metric can be expressed in terms of scalars, say, δt and δx , as follows:

$$t \rightarrow t + \delta t, \quad x^i \rightarrow x^i + \nabla^i (\delta x).$$

Similarly, we can construct coordinate transformations in terms of a divergence free vector, say, δx^i , as

$$t \rightarrow t, \quad x^i \rightarrow x^i + \delta x^i.$$

There are no coordinate degrees of freedom associated with the tensor perturbations. The two scalar quantities δt and δx , and the divergence free vector δx^i , constitute the

$$2 + (D - 1) = (D + 1)$$

degrees of freedom associated with the coordinate transformations in $(D + 1)$ -spacetime dimensions.



Independent degrees of freedom

The scalar functions $(\mathcal{A}, \mathcal{B}, \psi, \mathcal{E})$ were required to describe the perturbed metric tensor. But, there were two scalar degrees of freedom, viz. δt and δx , associated with the choice of coordinates. Hence, the actual number of independent scalar degrees of freedom is $(4 - 2) = 2$, which is true in any spacetime dimension.

We needed two divergence free vectors, \mathcal{S}_i and \mathcal{F}_i , amounting to a total of $2(D - 1)$ degrees of freedom, to describe the perturbed metric tensor. If we subtract the $(D - 1)$ vector degrees of freedom corresponding to the coordinate transformations that can be achieved through δx^i from this number, we are left with $2(D - 1) - (D - 1) = (D - 1)$ true vector degrees of freedom.

The tensor perturbations contained $(D + 1)(D - 2)/2$ independent degrees of freedom.

Upon adding these, we obtain that

$$2 + (D - 1) + \frac{1}{2} (D + 1)(D - 2) = \frac{1}{2} D(D + 1) = \frac{1}{2} (D + 1)(D + 2) - (D + 1),$$

which is the total number of *independent* degrees of freedom describing the perturbed metric tensor.



Decomposition theorem

The perturbations in the stress-energy tensor of matter fields, say, δT_{ν}^{μ} , can also be decomposed as scalars, vectors and tensors, just as in the same manner as the perturbations in the metric tensor.

At the *linear* order on perturbation theory, it can be shown that the scalar, vector and tensor perturbations evolve independently¹³.

¹³See, for example, [S. Weinberg, Cosmology \(Oxford University Press, Oxford, England, 2008\)](#).



Tensor perturbations in a FLRW universe

On including the tensor perturbations, the FLRW line-element can be expressed as

$$ds^2 = -dt^2 + a^2(t) [\delta_{ij} + \gamma_{ij}(t, \mathbf{x})] dx^i dx^j,$$

where γ_{ij} is a symmetric, transverse and traceless tensor.

[▶ Back to spectral energy density of GWs](#)

The transverse and traceless conditions reduce the number of independent degrees of freedom of γ_{ij} to two. These two degrees of freedom correspond to the two types of polarization associated with the gravitational waves (GWs).

On imposing the transverse and the traceless conditions, the components of the perturbed Einstein tensor, say, δG_{ν}^{μ} , corresponding to the above line element simplify to be

$$\delta G_0^0 = \delta G_i^i = 0, \quad \delta G_j^i = -\frac{1}{2} \left(\ddot{\gamma}_{ij} + 3H \dot{\gamma}_{ij} - \frac{1}{a^2} \nabla^2 \gamma_{ij} \right).$$

Therefore, in the absence of sources, the tensor perturbations in a FLRW universe satisfy the equation¹⁴

$$\ddot{\gamma}_{ij} + 3H \dot{\gamma}_{ij} - \frac{1}{a^2} \nabla^2 \gamma_{ij} = 0.$$

¹⁴See, for example, S. Weinberg, *Cosmology* (Oxford University Press, Oxford, England 2008).



Quantization of the tensor perturbations

On quantization, the operator $\hat{\gamma}_{ij}(\eta, \mathbf{x})$ representing the tensor perturbations can be expressed in terms of the corresponding Fourier modes, say, g_k , as¹⁵

$$\begin{aligned}\hat{\gamma}_{ij}(\eta, \mathbf{x}) &= \int \frac{d^3 \mathbf{k}}{(2\pi)^{3/2}} \hat{\gamma}_{ij}^{\mathbf{k}}(\eta) e^{i \mathbf{k} \cdot \mathbf{x}} \\ &= \sum_{\lambda} \int \frac{d^3 \mathbf{k}}{(2\pi)^{3/2}} \varepsilon_{ij}^{\lambda}(\mathbf{k}) \left[\hat{b}_{\mathbf{k}}^{\lambda} g_k(\eta) e^{i \mathbf{k} \cdot \mathbf{x}} + \hat{b}_{\mathbf{k}}^{\lambda \dagger} g_k^*(\eta) e^{-i \mathbf{k} \cdot \mathbf{x}} \right],\end{aligned}$$

where $\lambda = (+, \times)$ denotes the two states of polarizations of the GWs.

In this decomposition, the operators $(\hat{b}_{\mathbf{k}}^{\lambda}, \hat{b}_{\mathbf{k}}^{\lambda \dagger})$ satisfy the standard commutation relations, while the quantity $\varepsilon_{ij}^{\lambda}(\mathbf{k})$, which is symmetric in the spatial indices, represents the transverse and traceless polarization tensor describing the GWs.

[▶ Back to spectral energy density of GWs](#)

The Fourier mode functions g_k satisfy the following equation of motion:

$$g_k'' + 2 \mathcal{H} g_k' + k^2 g_k = 0,$$

where $\mathcal{H} = a H = a'/a$ is the so-called conformal Hubble parameter.

¹⁵See, for instance, L. Sriramkumar, *Curr. Sci.* **97**, 868 (2009).



Polarization tensor

The transverse and traceless nature of GWs leads to the conditions

$$\delta^{ij} k_i \varepsilon_{jl}^\lambda(\mathbf{k}) = \delta^{ij} \varepsilon_{ij}^\lambda(\mathbf{k}) = 0.$$

Note that the polarization tensors $\varepsilon_{ij}^+(\mathbf{k})$ and $\varepsilon_{ij}^\times(\mathbf{k})$ can be expressed in terms of the set of orthonormal unit vectors $(\hat{e}(\mathbf{k}), \hat{\hat{e}}(\mathbf{k}), \hat{\hat{k}}(\mathbf{k}))$ in the following manner¹⁶:

$$\varepsilon_{ij}^+(\mathbf{k}) = \frac{1}{\sqrt{2}} [\hat{e}_i(\mathbf{k}) \hat{e}_j(\mathbf{k}) - \hat{\hat{e}}_i(\mathbf{k}) \hat{\hat{e}}_j(\mathbf{k})], \quad \varepsilon_{ij}^\times(\mathbf{k}) = \frac{1}{\sqrt{2}} [\hat{e}_i(\mathbf{k}) \hat{\hat{e}}_j(\mathbf{k}) + \hat{\hat{e}}_i(\mathbf{k}) \hat{e}_j(\mathbf{k})].$$

The orthonormal nature of the vectors $\hat{e}(\mathbf{k})$ and $\hat{\hat{e}}(\mathbf{k})$ lead to the normalization condition:

$$\delta^{il} \delta^{jm} \varepsilon_{ij}^\lambda(\mathbf{k}) \varepsilon_{lm}^{\lambda'}(\mathbf{k}) = \delta^{\lambda\lambda'},$$

where λ and λ' can represent either of the two states of polarization $+$ or \times .

¹⁶M. Maggiore, Phys. Rep. **331**, 283 (2000).



Primary tensor power spectrum

The dimensionless tensor power spectrum, say, $\mathcal{P}_T(k)$, is defined in terms of the correlation functions of the Fourier modes $\hat{\gamma}_{ij}^{\mathbf{k}}$ as follows:

$$\langle \hat{\gamma}_{ij}^{\mathbf{k}}(\eta) \hat{\gamma}_{mn}^{\mathbf{k}'}(\eta) \rangle = \frac{(2\pi)^2}{8k^3} \Pi_{ij,mn}^{\mathbf{k}} \mathcal{P}_T(k) \delta^3(\mathbf{k} + \mathbf{k}'),$$

where

$$\Pi_{ij,mn}^{\mathbf{k}} = \sum_{\lambda} \varepsilon_{ij}^{\lambda}(\mathbf{k}) \varepsilon_{mn}^{\lambda}(\mathbf{k}).$$

In the Bunch-Davies vacuum, say, $|0\rangle$, which is defined as $\hat{b}_{\mathbf{k}}^{\lambda}|0\rangle = 0 \forall \mathbf{k}$ and λ , we can express the tensor power spectrum in terms of the mode functions $g_{\mathbf{k}}$ as

$$\mathcal{P}_T(k) = 4 \frac{k^3}{2\pi^2} |g_{\mathbf{k}}|^2.$$

With the initial conditions imposed in the sub-Hubble domain, viz. when $k/(aH) \gg 1$ these spectra are to be evaluated on super-Hubble scales, i.e. as $k/(aH) \ll 1$.



Mukhanov-Sasaki variable and Bunch-Davies initial condition

If we write $g_k = (\sqrt{2}/M_{\text{Pl}}) (u_k/a)$, then the equation describing the tensor mode functions reduces to the form

▶ Back to solution to during radiation domination

$$u_k'' + \left(k^2 - \frac{a''}{a} \right) u_k = 0.$$

The tensor power spectrum $\mathcal{P}_T(k)$ can then be expressed in terms of the modes functions g_k and the Mukhanov-Sasaki variable u_k as follows:

$$\mathcal{P}_T(k) = 4 \frac{k^3}{2\pi^2} |g_k|^2 = \frac{8}{M_{\text{Pl}}^2} \frac{k^3}{2\pi^2} \left(\frac{|u_k|}{a} \right)^2.$$

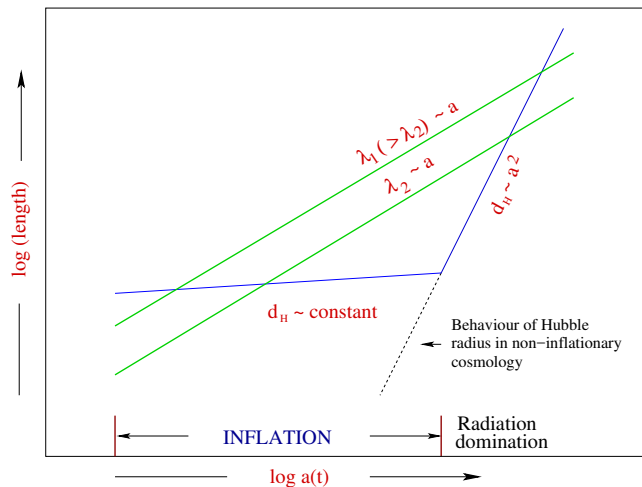
The assumption that the tensor perturbations are in the vacuum state then requires that the Mukhanov-Sasaki variable u_k behave as positive frequency modes at sub-Hubble scales, i.e. they have the asymptotic form¹⁷

$$\lim_{[k/(aH)] \rightarrow \infty} u_k(\eta) \rightarrow \frac{1}{\sqrt{2k}} e^{-ik\eta}.$$

¹⁷T. S. Bunch and P. C. W. Davies, Proc. Roy. Soc. Lond. A **360**, 117 (1978).



From sub-Hubble to super-Hubble scales



The initial conditions are imposed in the sub-Hubble regime when the modes are well inside the Hubble radius [viz. when $k/(aH) \gg 1$] and the power spectrum is evaluated when they are sufficiently outside [i.e. when $k/(aH) \ll 1$].



Solution to the tensor Mukhanov-Sasaki variable in slow roll inflation

From the definition of ϵ_1 , it can be established that

$$\frac{a''}{a} = \mathcal{H}^2 (2 - \epsilon_1).$$

Also, we can write

$$\eta = -\frac{1}{(1 - \epsilon_1) \mathcal{H}} - \int \frac{\epsilon_1 \epsilon_2}{(1 - \epsilon_1)^3} d\left(\frac{1}{\mathcal{H}}\right).$$

At the leading order in the slow roll approximation, the second term can be ignored and, at the same order, one can assume ϵ_1 to be a constant. Therefore, we have

$$\mathcal{H} \simeq -\frac{1}{(1 - \epsilon_1) \eta}, \quad \frac{a''}{a} \simeq (2 + 3\epsilon_1) \frac{1}{\eta^2}$$

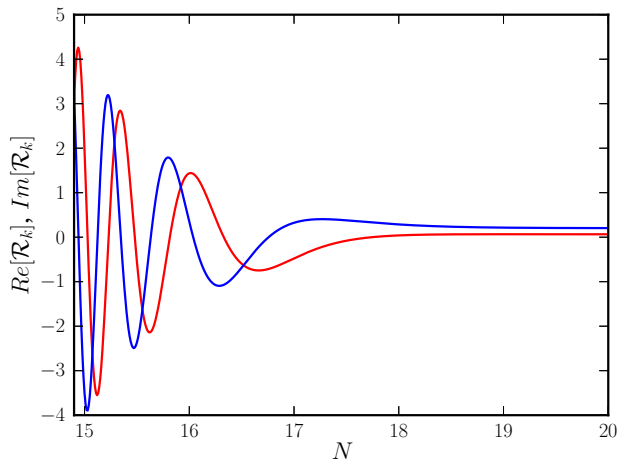
and the solution which satisfies the Bunch-Davies initial condition is given by

$$u_k(\eta) = \sqrt{\frac{-\pi \eta}{4}} e^{i[\nu_T + 1/2] \pi/2} H_{\nu_T}^{(1)}(-k \eta),$$

where $H_{\nu}^{(1)}(z)$ denotes Hankel function of the first kind and $\nu_T = (3/2) + \epsilon_1$.



Typical evolution of the perturbations



Typical evolution of the real and the imaginary parts of the scalar modes during slow roll inflation. The mode considered here leaves the Hubble radius at about $18 e$ -folds¹⁸.

¹⁸Figure from V. Sreenath, *Computation and characteristics of inflationary three-point functions*, Ph.D. Thesis, Indian Institute of Technology Madras, Chennai, India (2015).



Tensor power spectrum in slow roll inflation

The tensor power spectrum evaluated in the super-Hubble limit, i.e. when $(-k\eta) \ll 1$, can be obtained to be¹⁹

► [Back to scalar power spectrum in slow roll inflation](#)

$$\mathcal{P}_T(k) = \frac{2 H_I^2}{\pi^2 M_{\text{Pl}}^2} [1 - 2(C + 1)\epsilon_1 - 2\epsilon_1 \ln(-k\eta)],$$

where $C = (\gamma_E - 2 + \ln 2)$ with γ_E being the Euler constant.

The tensor spectral index n_T is defined as

$$n_T = \frac{d \ln \mathcal{P}_T}{d \ln k}$$

and, in slow roll inflation, we have

$$n_T = -2\epsilon_1.$$

¹⁹See, for instance, J. Martin and R. Brandenberger, Phys. Rev. D **68**, 063513 (2003) [hep-th/0305161].



Tensor power spectrum in de Sitter inflation

In the case of de Sitter inflation, which corresponds to $\epsilon_1 = 0$, the scale factor is given by

$$a(\eta) = -\frac{1}{H_1 \eta},$$

where H_1 is a constant.

In such a case, we have

$$\frac{a''}{a} = \frac{2}{\eta^2}$$

and the solution to the Mukhanov-Sasaki variable u_k has the following simple form:

$$u_k(\eta) = \frac{1}{\sqrt{k} \eta} \left(1 - \frac{i}{k \eta} \right) e^{-i k \eta}.$$

The tensor power spectrum at late times, i.e. as $(-k \eta) \ll 1$, can be easily evaluated to be

$$\mathcal{P}_T(k) = \frac{2 H_1^2}{\pi^2 M_{\text{Pl}}^2},$$

which is strictly scale invariant.

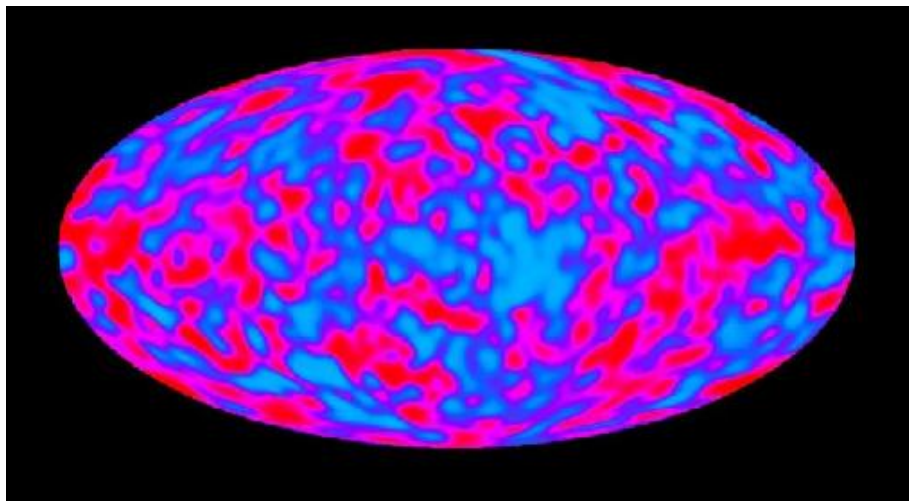


Plan of the talk

- 1 Inflation and reheating
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- 8 Outlook
- 9 References



Anisotropies in the CMB as observed by COBE

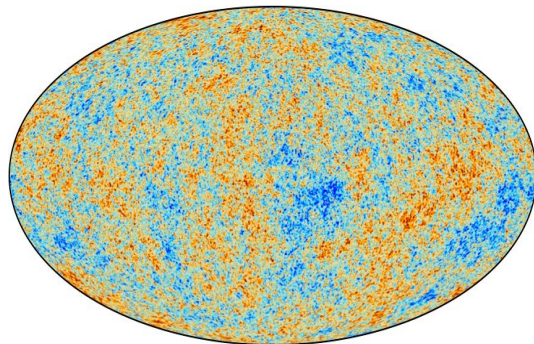
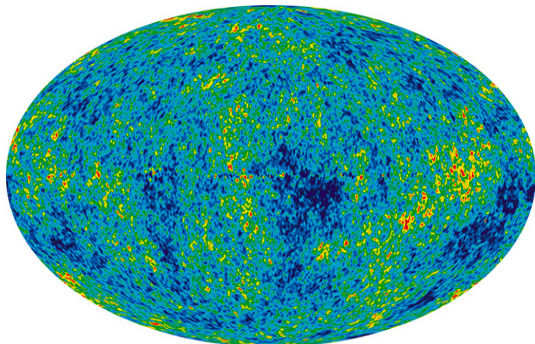


The fluctuations in the temperature of the CMB as observed by COBE²⁰. The amplitude of the fluctuations are of the order of one part in 10^5 .

²⁰Image from http://aether.lbl.gov/www/projects/cobe/COBE_Home/DMR_Images.html.



Anisotropies in the CMB as observed by WMAP and Planck



Left: All-sky map of the anisotropies in the CMB created from nine years of Wilkinson Microwave Anisotropy Probe (WMAP) data²¹.

Right: CMB intensity map derived from the joint analysis of Planck, WMAP, and 408 MHz observations²². The above images show temperature variations (as color differences) of the order of $200 \mu\text{K}$.

²¹ Image from <http://wmap.gsfc.nasa.gov/media/121238/index.html>.

²² P. A. R. Ade *et al.*, arXiv:1502.01582 [astro-ph.CO].



Spectral indices and tensor-to-scalar ratio

While comparing with the observations, for convenience, one often uses the following power law template for the primordial scalar and the tensor spectra:

$$\mathcal{P}_S(k) = A_S \left(\frac{k}{k_*} \right)^{n_S - 1}, \quad \mathcal{P}_T(k) = A_T \left(\frac{k}{k_*} \right)^{n_T},$$

with the spectral indices n_S and n_T assumed to be constant.

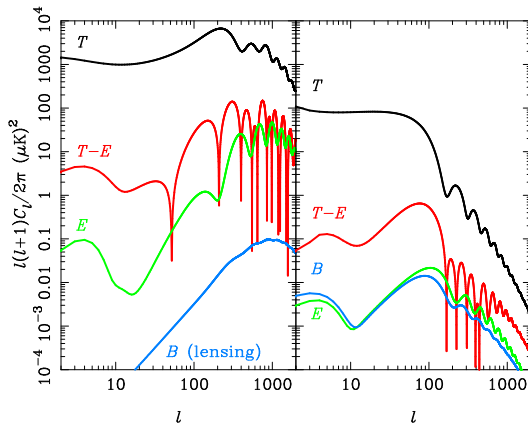
The tensor-to-scalar ratio r is defined as

$$r(k) = \frac{\mathcal{P}_T(k)}{\mathcal{P}_S(k)}$$

and it is usual to further set $r = -8n_T$, viz. the so-called consistency relation, which is valid during slow roll inflation.



Theoretical angular power spectra²³

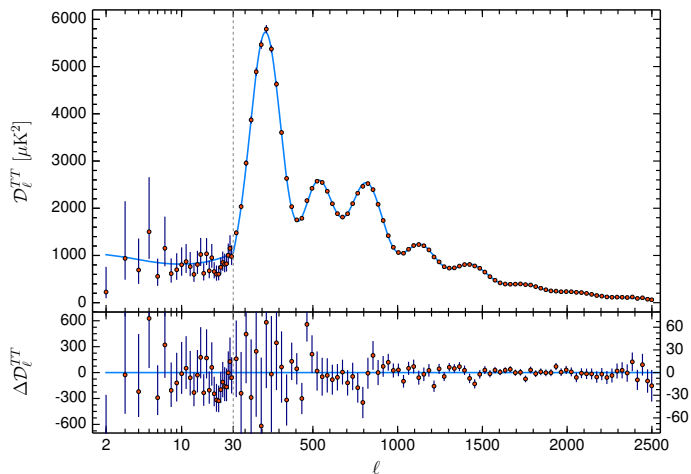


The *theoretically* computed, angular power and cross-correlation spectra of the CMB arising due to scalars (on the left) and tensors (on the right) corresponding to a tensor-to-scalar ratio of $r = 0.24$. The B-mode spectrum induced by weak gravitational lensing has also been shown (in blue) in the panel on the left.

²³Figure from, A. Challinor, arXiv:1210.6008 [astro-ph.CO].



CMB angular power spectrum from Planck

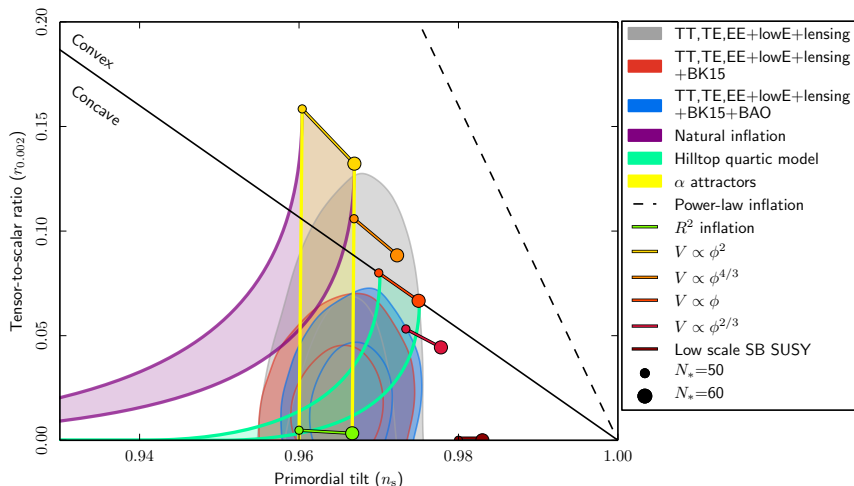


The CMB TT angular power spectrum from the Planck 2018 data (red dots with error bars) and the best fit Λ CDM model with a power law primordial spectrum (solid blue curve)²⁴

²⁴Planck Collaboration (N. Aghanim *et al.*), *Astron. Astrophys.* **641**, A6 (2020).



Performance of inflationary models in the n_s - r plane

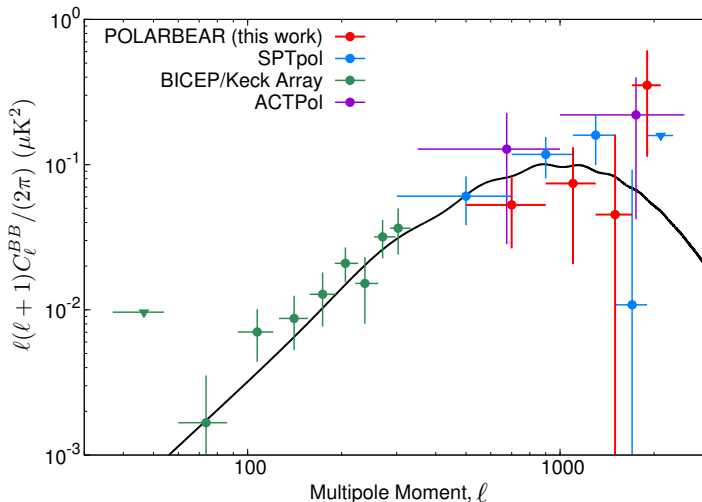


Joint constraints on n_s and $r_{0.002}$ from Planck in combination with other data sets, compared to the theoretical predictions of some of the popular inflationary models²⁵.

²⁵Planck Collaboration (Y. Akrami *et al.*), *Astron. Astrophys.* **641**, A10 (2020).



Constraints on the B-mode polarization of the CMB

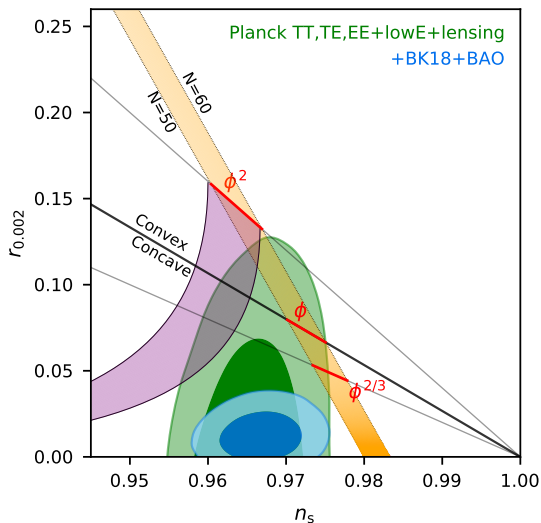


Constraints on the B-mode, angular power spectrum of the CMB from two years of POLARBEAR data²⁶.

²⁶POLARBEAR Collaboration (P. A. R. Ade *et al.*), *Ap. J.* **848**, 141, (2017).



Latest constraints on the tensor-to-scalar ratio r

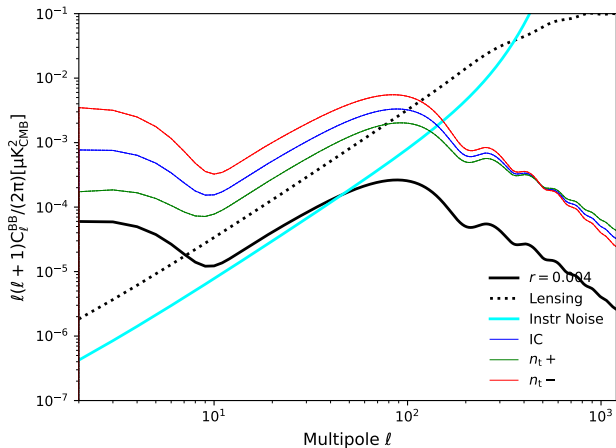


Latest constraints on the tensor-to-scalar ratio r from the BICEP/Keck telescopes²⁷.

²⁷ BICEP/Keck Collaboration (P. A. R. Ade *et al.*), arXiv:2203.16556 [astro-ph.CO].



Prospects of observing the imprints of the tensor perturbations



The B-mode angular power spectra of the CMB resulting from the primordial tensor perturbations for three models with $r_{0.05} = 0.05$ have been plotted, along with the CMB lensing signal and the instrumental noise of a LiteBIRD-like configuration²⁸.

²⁸ D. Paoletti, F. Finelli, J. Valiviita and M. Hazumi, *Phys. Rev. D* **106**, 083528 (2022).

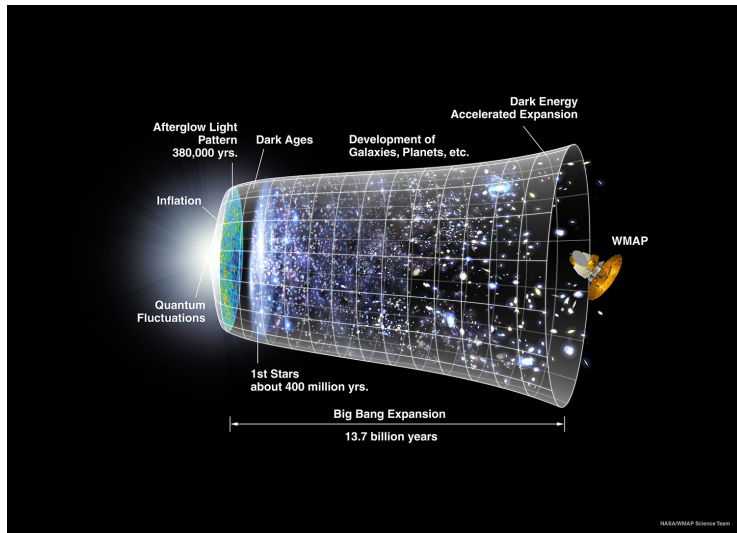


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- 9 References



Timeline of the universe

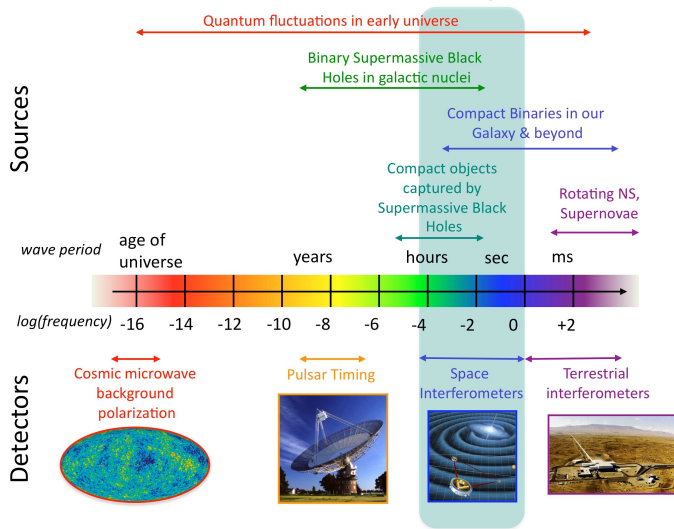


A pictorial timeline of the universe²⁹.

²⁹See http://wmap.gsfc.nasa.gov/media/060915/060915_CMB_Timeline150.jpg.



The spectrum of GWs

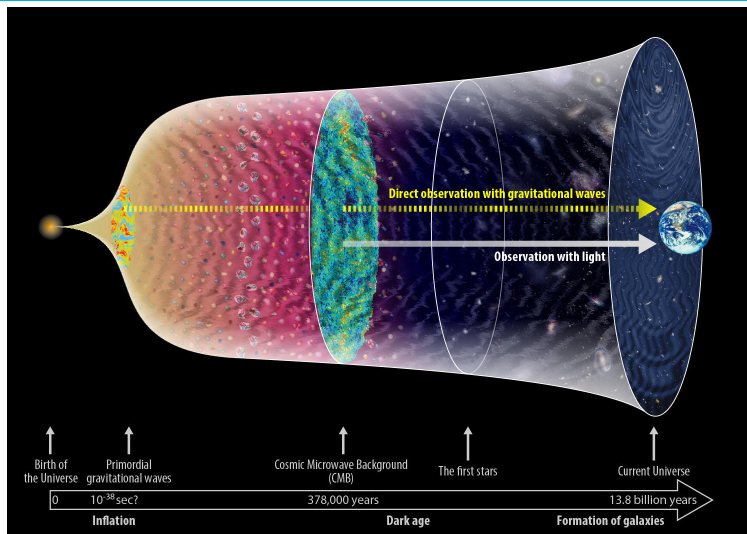


Different sources of GWs and corresponding detectors³⁰.

³⁰ J. B. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (Pearson Education, Delhi, 2003).



Probing the primordial universe through GWs



GWs provide a unique window to probe the primordial universe³¹.

³¹ Image from <https://gwpo.nao.ac.jp/en/gallery/000061.html>.

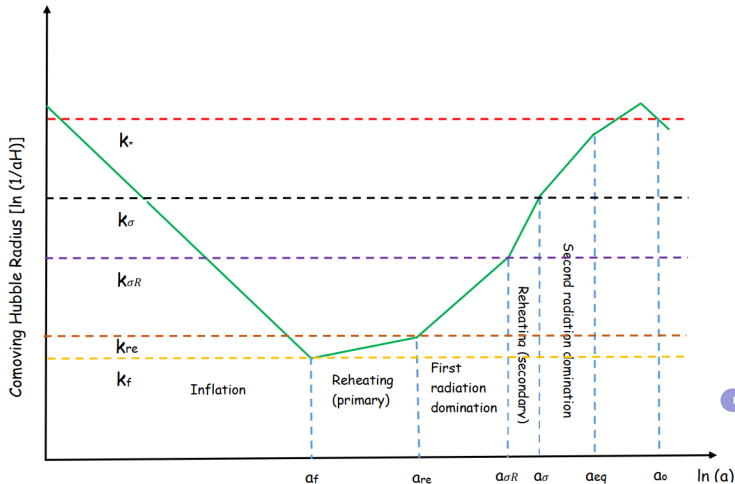


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- 9 References



Behavior of the comoving wave number and Hubble radius



Behavior of the comoving wave number k (horizontal lines in different colors) and the comoving Hubble radius $d_H/a = (aH)^{-1}$ (in green) across different epochs³².

³²Md. R. Haque, D. Maity, T. Paul and L. Sriramkumar, Phys. Rev. D **104**, 063513 (2021).



Reheating temperature

The reheating temperature T_{re} can be expressed in terms of the EoS parameter during reheating w_{re} , the duration of reheating N_{re} and the inflationary parameters as follows³³:

$$T_{\text{re}} = \left(\frac{43}{11 g_{s,\text{re}}} \right)^{1/3} \left(\frac{a_0 H_I}{k_*} \right) e^{-(N_* + N_{\text{re}})} T_0,$$

where H_I is the Hubble parameter during inflation and $T_0 = 2.725$ K is the temperature of the CMB today, while $g_{s,\text{re}}$ denotes the effective number of relativistic degrees of freedom that contribute to the entropy of radiation at the time of reheating.

Note that $k_*/a_0 \simeq 0.05 \text{ Mpc}^{-1}$ represents the CMB pivot scale and N_* denotes the number of e -folds *prior to the end of inflation* when the pivot scale leaves the Hubble radius.

³³ J. Martin and C. Ringeval, Phys. Rev. D **82**, 023511 (2010);
 L. Dai, M. Kamionkowski and J. Wang, Phys. Rev. Lett. **113**, 041302 (2014);
 J. L. Cook, E. Dimastrogiovanni, D. A. Easson and L. M. Krauss, JCAP **04**, 047 (2015).



Duration of reheating

The duration of the epoch of reheating in terms of e -folds, say, N_{re} , can be expressed in terms of the EoS parameter during reheating w_{re} and the inflationary parameters as follows³⁴:

$$N_{\text{re}} = \frac{4}{(3w_{\text{re}} - 1)} \left[N_* + \frac{1}{4} \ln \left(\frac{30}{\pi^2 g_{\text{re}}} \right) + \frac{1}{3} \ln \left(\frac{11 g_{s,\text{re}}}{43} \right) + \ln \left(\frac{k_*}{a_0 T_0} \right) + \ln \left(\frac{\rho_e^{1/4}}{H_1} \right) \right],$$

where g_{re} denotes the effective number of relativistic degrees of freedom that contribute to the energy density of radiation at the time of reheating.

Also, recall that ρ_e denotes the energy density at the end of inflation.

³⁴ J. Martin and C. Ringeval, Phys. Rev. D **82**, 023511 (2010);
 L. Dai, M. Kamionkowski and J. Wang, Phys. Rev. Lett. **113**, 041302 (2014);
 J. L. Cook, E. Dimastrogiovanni, D. A. Easson and L. M. Krauss, JCAP **04**, 047 (2015).



Stress-energy tensor associated with GWs

Consider a line-element which includes the tensor perturbations at the linear order, say, γ_{ij} , in a given background.

The stress-energy tensor associated with the GWs can be calculated using the Einstein's equation, viz.

$$\delta T_{\mu\nu} = -M_{\text{Pl}}^2 \delta G_{\mu\nu}^{(2)},$$

where the superscript (2) on the right hand side indicates that the quantity has to be evaluated at the quadratic order in γ_{ij} .

It can be shown that, in a spatially flat FLRW background, the energy density of primary GWs is given by³⁵

► FLRW line-element with tensor perturbations

$$\rho = \delta G_{00}^{(2)} = -\frac{1}{8} \dot{\gamma}^{ij} \dot{\gamma}_{ij} + \frac{H}{2} \dot{\gamma}^{ij} \gamma_{ij} + \frac{1}{2} \gamma^{ij} \ddot{\gamma}_{ij} + \frac{3}{8} (\partial^l \gamma^{ij}) (\partial_l \gamma_{ij}) - \frac{1}{4} (\partial^l \gamma^{ij}) (\partial_j \gamma_{il}).$$

³⁵See, for instance, F. Finelli, G. Marozzi, G. P. Vacca and G. Venturi, Phys. Rev. D **71**, 023522 (2005).



Spectral energy density of GWs

If we now substitute the decomposition of $\hat{\gamma}_{ij}$ in the above expression for the energy density of GWs, we obtain that

► Fourier decomposition of the tensor perturbations in a FLRW universe

$$\rho_{\text{GW}}(\eta) = M_{\text{Pl}}^2 \sum_{\lambda} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \left[\frac{1}{4} |\dot{g}_{\mathbf{k}}|^2 + \frac{k^2}{4a^2} |g_{\mathbf{k}}|^2 + H (\dot{g}_{\mathbf{k}} g_{\mathbf{k}}^* + g_{\mathbf{k}} \dot{g}_{\mathbf{k}}^*) \right].$$

The spectral energy density of GWs, say, $\rho_{\text{GW}}(k, \eta)$, is defined through the relation

$$\rho_{\text{GW}}(\eta) = \int_0^{\infty} d \ln k \rho_{\text{GW}}(k, \eta).$$

From the above expression for $\rho_{\text{GW}}(\eta)$, we obtain that

$$\rho_{\text{GW}}(k, \eta) = \frac{M_{\text{Pl}}^2}{a^2} \frac{k^3}{2\pi^2} \left[\frac{1}{2} |g'_{\mathbf{k}}|^2 + \frac{k^2}{2} |g_{\mathbf{k}}|^2 + 2\mathcal{H} (g'_{\mathbf{k}} g_{\mathbf{k}}^* + g_{\mathbf{k}}^* g'_{\mathbf{k}}) \right].$$



Dimensionless spectral energy density of GWs today

The dimensionless spectral energy density of GWs, say, $\Omega_{\text{GW}}(k, \eta)$, is defined as

$$\Omega_{\text{GW}}(k, \eta) = \frac{\rho_{\text{GW}}(k, \eta)}{\rho_{\text{c}}(\eta)} = \frac{\rho_{\text{GW}}(k, \eta)}{3 H^2 M_{\text{Pl}}^2},$$

where $\rho_{\text{c}} = 3 H^2 M_{\text{Pl}}^2$ is the critical density at time η .

During the late stages of radiation domination, the wave numbers such that $k \gg k_{\text{eq}}$ are well inside the Hubble radius. Therefore, the spectral energy density of GWs will fall in the same manner as the energy density of radiation (i.e. as a^{-4}).

Utilizing this behavior, we can express the dimensionless spectral energy density of GWs *today*, i.e. $\Omega_{\text{GW}}(k)$, in terms of $\Omega_{\text{GW}}(k, \eta)$ above in the following manner:

$$\begin{aligned} \Omega_{\text{GW}}(k) h^2 &= \frac{g_k}{g_0} \left(\frac{g_{s,0}}{g_{s,k}} \right)^{4/3} \Omega_{\text{r}} h^2 \Omega_{\text{GW}}(k, \eta) \simeq \left(\frac{g_0}{g_k} \right)^{1/3} \Omega_{\text{r}} h^2 \Omega_{\text{GW}}(k, \eta) \\ &\simeq 1.38 \times 10^{-5} \left(\frac{g_{*,k}}{106.75} \right)^{-1/3} \left(\frac{\Omega_{\text{r}} h^2}{4.16 \times 10^{-5}} \right) \Omega_{\text{GW}}(k, \eta), \end{aligned}$$

where $\Omega_{\text{r}} h^2$ denotes the present day dimensionless energy density of radiation.



Transition from de Sitter inflation to radiation domination

During radiation domination, the scale factor can be expressed as

$$a(\eta) = a_r (\eta - \eta_r),$$

where $a_r = -a_e/\eta_e$ and $\eta_r = 2\eta_e$, which are arrived at by matching the scale factor and its time derivative at the end of inflation corresponding to the time η_e and scale factor a_e .

Since $a'' = 0$ during radiation domination, it is straightforward to see that the general solution to the Mukhanov-Sasaki variable can be written as

► Mukhanov-Sasaki equation for tensors

$$u_k(\eta) = \frac{1}{\sqrt{2k}} \left[A_r(k) e^{-ik(\eta-\eta_r)} + B_r(k) e^{ik(\eta-\eta_r)} \right].$$

On matching the Mukhanov-Sasaki variable u_k and its time derivative at η_e , we obtain the coefficients $A_r(k)$ and $B_r(k)$ to be

► Solution in de Sitter

$$A_r(k) = \frac{1}{2k^2\eta_e^2} (-1 - 2ik\eta_e + 2k^2\eta_e^2) e^{-2ik\eta_e}, \quad B_r(k) = \frac{1}{2k^2\eta_e^2}.$$

It is useful to note that $|A_r(k)|^2 - |B_r(k)|^2 = 1$, which is the Wronskian condition.



Wave numbers k_{re} and k_e

Let k_{re} and k_e be the wave numbers that re-enter the Hubble radius at the end of reheating and at the end of inflation, respectively.

The wave number k_e can be approximately expressed in terms of the inflationary energy scale H_I and the reheating parameters T_{re} and w_{re} as follows:

$$k_e = a_e H_I \simeq H_I \frac{T_0}{T_{\text{re}}} \left(\frac{43}{11 g_{\text{s, re}}} \right)^{1/3} \left(\frac{\pi^2 g_{\text{re}} T_{\text{re}}^4}{90 M_{\text{Pl}}^2 H_I^2} \right)^{1/[3(1+w_{\text{re}})]}.$$

Also, note that the ratio between k_e and k_{re} can be expressed as³⁶

$$\frac{k_e}{k_{\text{re}}} = \left(\frac{90 H_I^2 M_{\text{Pl}}^2}{\pi^2 g_{\text{re}}} \right)^{(1+3w_{\text{re}})/[6(1+w_{\text{re}})]} T_{\text{re}}^{-2(1+3w_{\text{re}})/[3(1+w_{\text{re}})]}.$$

³⁶S. Maity, N. Bhaumik, Md. R. Haque, D. Maity and L. Sriramkumar, arXiv:2403.16963 [astro-ph.CO], accepted in JCAP.



$\Omega_{\text{GW}}(k)$ over $k_{\text{re}} < k < k_e$

Over the wave numbers $k_{\text{re}} < k < k_e$, the dimensionless spectral energy density of primary GWs today can be expressed as

$$\begin{aligned}\Omega_{\text{GW}}(k) h^2 &\simeq \Omega_r h^2 \frac{H_I^2}{12 \pi^2 M_{\text{Pl}}^2} \frac{\mu(w_{\text{re}})}{\pi} \left(\frac{k}{k_{\text{re}}}\right)^{n_{w_{\text{re}}}} \\ &\simeq 3.5 \times 10^{-17} \left(\frac{H_I}{10^{-5} M_{\text{Pl}}}\right)^2 \frac{\mu(w_{\text{re}})}{\pi} \left(\frac{k}{k_{\text{re}}}\right)^{n_{w_{\text{re}}}},\end{aligned}$$

where the quantity $\mu(w_{\text{re}})$ and the index $n_{w_{\text{re}}}$ are given by³⁷

$$\mu(w_{\text{re}}) = (1 + 3 w_{\text{re}})^{4/(1+3 w_{\text{re}})} \Gamma^2\left(\frac{5 + 3 w_{\text{re}}}{2 + 6 w_{\text{re}}}\right), \quad n_{w_{\text{re}}} = -\frac{2(1 - 3 w_{\text{re}})}{1 + 3 w_{\text{re}}}.$$

Note that $n_{w_{\text{re}}} = (0, -2, 1)$ when $w_{\text{re}} = (1/3, 0, 1)$.

³⁷S. Maity, N. Bhaumik, Md. R. Haque, D. Maity and L. Sriramkumar, arXiv:2403.16963 [astro-ph.CO], accepted in JCAP.



Bound from ΔN_{eff}

Since, at suitably late times when the wave numbers of interest are inside the Hubble radius, the energy density of GWs behaves in the same manner as that of radiation, they can contribute additional relativistic degrees of freedom to radiation.

These additional number of relativistic degrees of freedom, which is usually denoted as ΔN_{eff} , is well constrained by the CMB observations³⁸.

This constraint becomes important when $w_{\text{re}} > 1/3$ because, in such cases, the spectral energy density of primary GWs has a positive spectral index over $k_{\text{re}} < k < k_{\text{e}}$.

The constraint is given by

$$\int_{k_{\text{re}}}^{k_{\text{e}}} \frac{dk}{k} \Omega_{\text{GW}}(k) h^2 \leq \frac{7}{8} \left(\frac{4}{11} \right)^{4/3} \Omega_{\gamma} h^2 \Delta N_{\text{eff}},$$

where $\Omega_{\gamma} h^2 \simeq 2.47 \times 10^{-5}$ is the dimensionless energy density of photons today.

According to Planck, at $2\text{-}\sigma$, $\Delta N_{\text{eff}} = 0.284$.

³⁸Planck Collaboration (N. Aghanim *et al.*), *Astron. Astrophys.* **641**, A6 (2020)
[Erratum: *Astron. Astrophys.* **652**, C4 (2021)].



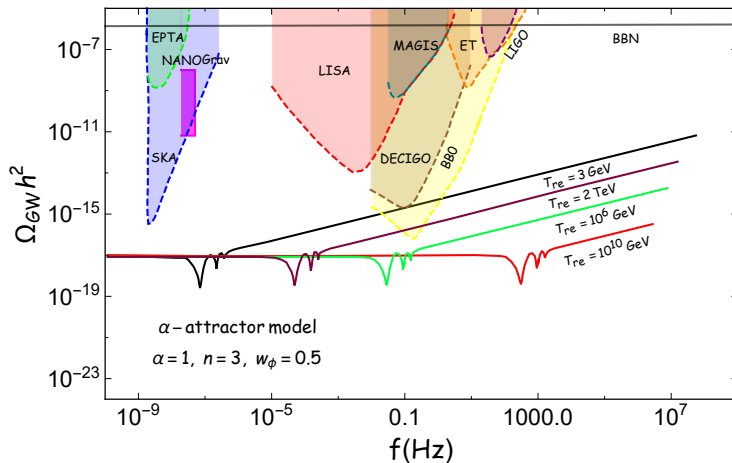
Relation between wave number and frequency

Note that the frequency f and the wave number k are related as follows:

$$f = \frac{k}{2\pi} = 1.55 \times 10^{-15} \left(\frac{k}{1 \text{ Mpc}^{-1}} \right) \text{ Hz.}$$

For instance, $k \simeq 10^6 \text{ Mpc}^{-1}$ corresponds to $f \simeq 10^{-9} \text{ Hz} = 1 \text{ nHz}$.

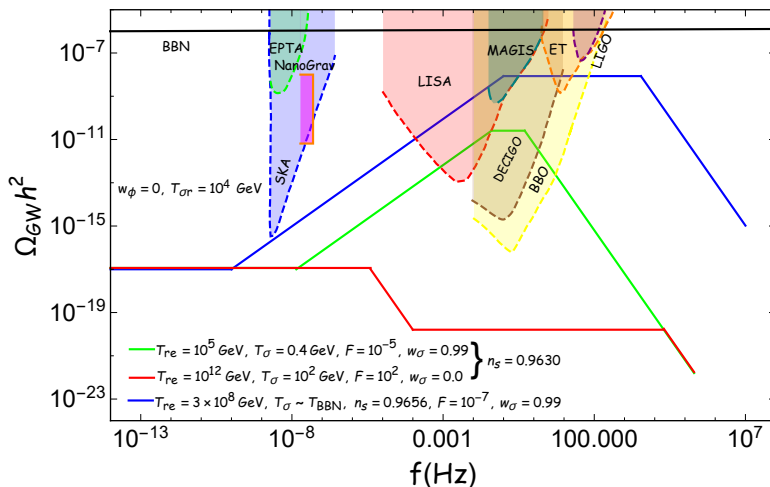


Effects on $\Omega_{\text{GW}}(f)$ due to reheating

The behavior of the dimensionless spectral energy density of primary GWs today, viz. Ω_{GW} , has been plotted, over a wide range of frequency f , for different reheating temperatures (in red, green, brown and black)³⁹.

³⁹Md. R. Haque, D. Maity, T. Paul and L. Sriramkumar, Phys. Rev. D **104**, 063513 (2021).



Effects on $\Omega_{\text{GW}}(f)$ due to late time entropy production

The dimensionless spectral energy density of primary GWs observed today $\Omega_{\text{GW}}(f)$ has been plotted in a scenario involving late time production of entropy⁴⁰.

⁴⁰Md. R. Haque, D. Maity, T. Paul and L. Sriramkumar, Phys. Rev. D **104**, 063513 (2021).



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- 9 References



FLRW metric with scalar perturbations

Recall that, in a FLRW universe, there are two independent degrees of freedom associated with the scalar perturbations. In the so-called longitudinal gauge, the FLRW line-element is given by

$$ds^2 = -(1 + 2\Phi) dt^2 + a^2(t) (1 - 2\Psi) \delta_{ij} dx^i dx^j,$$

where Φ and Ψ are often referred to as the Bardeen potentials.

If we write the perturbations in the stress-energy tensor in terms of scalars as $\delta T^\mu_\nu = (-\delta\rho, \partial_i \delta q, \delta p \delta^i_j)$, then the Einstein's equations at the first order lead to

$$\begin{aligned} 3H \left(H\Phi + \dot{\Psi} \right) - \frac{1}{a^2} \nabla^2 \Psi &= - (4\pi G) \delta\rho, \\ \partial_i \left(H\Phi + \dot{\Psi} \right) &= (4\pi G) \partial_i(\delta q), \\ \ddot{\Psi} + H \left(\dot{\Phi} + 3\dot{\Psi} \right) + \left(2\dot{H} + 3H^2 \right) \Phi &= (4\pi G) \delta p, \\ \Phi - \Psi &= 0. \end{aligned}$$

Note that, in the absence of anisotropic stress, $\Phi = \Psi$.



Bardeen equation

We can combine the first and third of the above equations to arrive at the following differential equation that governs the Bardeen potential Φ :

$$\Phi'' + 3\mathcal{H} (1 + c_A^2) \Phi' - c_A^2 \nabla^2 \Phi + [2\mathcal{H}' + (1 + 3c_A^2) \mathcal{H}^2] \Phi = (4\pi G a^2) \delta p^{\text{NA}},$$

where $\delta p = c_A^2 \delta \rho + \delta p^{\text{NA}}$ and $c_A^2 = p'/\rho'$ denotes the adiabatic speed of the perturbations, while δp^{NA} represents the non-adiabatic pressure perturbation.

[▶ Back to Bardeen equation in the case of scalar field](#)



Curvature perturbation

Consider the following combination of the Bardeen potential Φ and its time derivative

$$\mathcal{R} = \Phi + \frac{2\rho}{3\mathcal{H}} \left(\frac{\Phi' + \mathcal{H}\Phi}{\rho + p} \right),$$

a quantity that is referred to as the curvature perturbation⁴¹.

Upon substituting this expression in the Bardeen equation and making use of the background equations, we obtain that, in Fourier space,

$$\mathcal{R}'_k = \frac{\mathcal{H}}{\mathcal{H}^2 - \mathcal{H}'} \left[(4\pi G) a^2 \delta p_k^{\text{NA}} - c_A^2 k^2 \Phi_k \right].$$

Note that, on super-Hubble scales wherein $(k/aH) \ll 1$, the term $(c_A^2 k^2 \Phi_k)$ can be neglected. If one further assumes that no non-adiabatic pressure perturbations are present (i.e. $\delta p^{\text{NA}} = 0$), say, as in the case of ideal fluids, then the above equation implies that $\mathcal{R}'_k \simeq 0$ on super-Hubble scales.

▶ Back to equation for \mathcal{R}_k in the case of scalar field



⁴¹It is called so since it is proportional to the local three curvature on the spatial hypersurface.

Evolution of the Bardeen potential on super-Hubble scales

If we now define

$$\Phi = \frac{\mathcal{H}}{a^2 \theta} \mathcal{U}, \quad \theta = \left[\frac{\mathcal{H}^2}{(\mathcal{H}^2 - \mathcal{H}') a^2} \right]^{1/2},$$

then, in Fourier space, the equation that governs the Bardeen potential reduces to

$$\mathcal{U}_k'' + \left(c_A^2 k^2 - \frac{\theta''}{\theta} \right) \mathcal{U}_k = \left(\frac{4 \pi G a^4 \theta}{\mathcal{H}} \right) \delta p_k^{\text{NA}}.$$

When $\delta p_k^{\text{NA}} = 0$, on super-Hubble scales, the Bardeen potential Φ_k can be expressed as

$$\Phi_k(\eta) \simeq C_G(k) \frac{\mathcal{H}}{a^2(\eta)} \int^\eta \frac{d\tilde{\eta}}{\theta^2(\tilde{\eta})} + C_D(k) \frac{\mathcal{H}}{a^2(\eta)}.$$

In power law expansion wherein $a(\eta) = (-\bar{\mathcal{H}} \eta)^{\gamma+1}$, we find that

$$\Phi_k(\eta) \simeq C_G(k) \frac{3(w+1)}{3w+5} + C_D(k) \frac{2}{(3w+1) \bar{\mathcal{H}}^{2(\gamma+1)} \eta^{2\gamma+3}},$$

where $w = (1 - \gamma)/[3(1 + \gamma)]$.



Relation between Bardeen potential and curvature perturbation

Since Φ_k is a constant on super-Hubble scales, in this limit, the curvature perturbation \mathcal{R}_k is given by

$$\mathcal{R}_k \simeq \frac{3w + 5}{3(w + 1)} \Phi_k \simeq C_G(k).$$

Since \mathcal{R}_k is conserved and Φ_k is a constant at super-Hubble scales in power law expansion, when the modes enter the Hubble radius during the radiation or the matter dominated epochs, the Bardeen potentials at entry are given by

$$\Phi_k^r \simeq \frac{3(w_r + 1)}{3w_r + 5} \mathcal{R}_k = \frac{2}{3} C_G(k), \quad \Phi_k^m \simeq \frac{3(w_m + 1)}{3w_m + 5} \mathcal{R}_k = \frac{3}{5} C_G(k),$$

where we have made use of the fact that $w_r = 1/3$ and $w_m = 0$. Note that $\Phi_k^m = (9/10) \Phi_k^r$.



Perturbations in the stress-energy tensor of a scalar field

Let $\delta\phi$ denote the perturbations in the scalar field.

In the longitudinal gauge, the perturbations in the stress-energy tensor of a canonical scalar field are given by

$$\delta T_0^0 = -\delta\rho_\phi = -\dot{\phi} \delta\dot{\phi} + \Phi \dot{\phi}^2 - V_\phi \delta\phi,$$

$$\delta T_i^0 = -\partial_i \delta q_\phi = -\partial_i (\dot{\phi} \delta\phi),$$

$$\delta T_j^i = \delta p_\phi \delta_j^i = (\dot{\phi} \delta\dot{\phi} - \Phi \dot{\phi}^2 - V_\phi \delta\phi) \delta_j^i.$$

These expressions for the perturbed stress-energy tensor lead to the following equation for the Bardeen potential:

$$\Phi'' + 3\mathcal{H} (1 + c_A^2) \Phi' - c_A^2 \nabla^2 \Phi + [2\mathcal{H}' + (1 + 3c_A^2) \mathcal{H}^2] \Phi = (1 - c_A^2) \nabla^2 \Phi$$

so that

► Bardeen equation in the case of fluid

$$\delta p^{\text{NA}} = \left(\frac{1 - c_A^2}{4\pi G a^2} \right) \nabla^2 \Phi.$$



Equation of motion governing the curvature perturbation

For such a δp^{NA} , the equation that describes the evolution of the curvature perturbation simplifies to

► Equation for \mathcal{R}_k in the case of fluid

$$\mathcal{R}'_k = - \left(\frac{\mathcal{H}}{\mathcal{H}^2 - \mathcal{H}'} \right) k^2 \Phi_k.$$

On differentiating this equation with respect to time, and on using the background equations as well as the Bardeen equation, we obtain the following equation of motion governing the Fourier modes of the curvature perturbation induced by the scalar field

$$\mathcal{R}''_k + 2 \frac{z'}{z} \mathcal{R}'_k + k^2 \mathcal{R}_k = 0,$$

where $z = a \dot{\phi} / H$.



Quantization of the scalar perturbations

On quantization, the operator $\hat{\mathcal{R}}(\eta, \mathbf{x})$ representing the scalar perturbations can be expressed in terms of the corresponding Fourier modes, say, f_k , as⁴²

$$\hat{\mathcal{R}}(\eta, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \hat{\mathcal{R}}_{\mathbf{k}}(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left[\hat{a}_{\mathbf{k}} f_k(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a}_{\mathbf{k}}^\dagger f_k^*(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}} \right],$$

where the operators $(\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}}^\dagger)$ satisfy the standard commutation relations.

The Fourier mode functions f_k satisfy the following equation of motion:

$$f_k'' + 2 \frac{z'}{z} f_k' + k^2 f_k = 0.$$

⁴²See, for instance, L. Sriramkumar, *Curr. Sci.* **97**, 868 (2009).



Scalar power spectrum

The dimensionless scalar power spectrum, say, $\mathcal{P}_s(k)$, is defined in terms of the correlation functions of the Fourier modes $\hat{\gamma}_{ij}^{\mathbf{k}}$ as follows:

$$\langle \hat{\mathcal{R}}_{\mathbf{k}}(\eta) \hat{\mathcal{R}}_{\mathbf{k}'}(\eta) \rangle = \frac{(2\pi)^2}{8k^3} \mathcal{P}_s(k) \delta^3(\mathbf{k} + \mathbf{k}').$$

In the Bunch-Davies vacuum, say, $|0\rangle$, which is defined as $\hat{a}_{\mathbf{k}}|0\rangle = 0 \forall \mathbf{k}$, we can express the scalar power spectrum in terms of the mode functions $f_{\mathbf{k}}$ as

$$\mathcal{P}_s(k) = \frac{k^3}{2\pi^2} |f_{\mathbf{k}}|^2.$$

With the initial conditions imposed in the sub-Hubble domain, viz. when $k/(aH) \gg 1$, these spectra are to be evaluated on super-Hubble scales, i.e. as $k/(aH) \ll 1$.



Scalar Mukhanov-Sasaki variable and Bunch-Davies initial condition

If we write $f_k = v_k/z$, then the equation describing the scalar mode functions reduces to the form

$$v_k'' + \left(k^2 - \frac{z''}{z} \right) v_k = 0.$$

The scalar power spectrum $\mathcal{P}_s(k)$ can then be expressed in terms of the modes functions f_k and the corresponding Mukhanov-Sasaki variable v_k as follows:

$$\mathcal{P}_s(k) = \frac{k^3}{2\pi^2} |f_k|^2 = \frac{k^3}{2\pi^2} \left(\frac{|v_k|}{z} \right)^2.$$

The assumption that the scalar perturbations are in the vacuum state then requires that the corresponding Mukhanov-Sasaki variable v_k behave as positive frequency modes at sub-Hubble scales, i.e. they have the asymptotic form⁴³

$$\lim_{[k/(aH)] \rightarrow \infty} v_k(\eta) \rightarrow \frac{1}{\sqrt{2k}} e^{-ik\eta}.$$

⁴³T. S. Bunch and P. C. W. Davies, Proc. Roy. Soc. Lond. A **360**, 117 (1978).



Solution to the scalar Mukhanov-Sasaki variable in slow roll inflation

It can be shown that

$$\frac{z''}{z} = \mathcal{H}^2 \left(2 - \epsilon_1 + \frac{3\epsilon_2}{2} - \frac{\epsilon_1 \epsilon_2}{2} + \frac{\epsilon_2^2}{4} + \frac{\epsilon_2 \epsilon_3}{2} \right),$$

which is an exact relation.

[▶ Back to ultra slow roll inflation](#)

At the leading order in the slow roll parameters, we have

$$\mathcal{H} \simeq -\frac{1}{(1 - \epsilon_1)\eta}, \quad \frac{z''}{z} = \left(2 + 3\epsilon_1 + \frac{3\epsilon_2}{2} \right) \frac{1}{\eta^2}.$$

The solution to the scalar Mukhanov-Sasaki variable v_k , which satisfies the Bunch-Davies initial condition is given by

$$v_k(\eta) = \sqrt{\frac{-\pi\eta}{4}} e^{i[\nu_T + 1/2]\pi/2} H_{\nu_S}^{(1)}(-k\eta),$$

where $H_{\nu}^{(1)}(z)$ denotes Hankel function of the first kind and $\nu_S = (3/2) + \epsilon_1 + (\epsilon_2/2)$.



Scalar power spectrum in slow roll inflation

The scalar power spectrum evaluated in the super-Hubble limit, i.e. when $(-k\eta) \ll 1$, can be obtained to be⁴⁴

► Tensor power spectrum in slow roll inflation

$$\mathcal{P}_s(k) = \frac{H_I^2}{8\pi^2 M_{\text{Pl}}^2 \epsilon_1} [1 - 2(C + 1)\epsilon_1 - C\epsilon_2 - (2\epsilon_1 + \epsilon_2)\ln(-k\eta)],$$

where $C = (\gamma_E - 2 + \ln 2)$ with γ_E being the Euler constant.

The scalar spectral index n_s is defined as

$$n_s - 1 = \frac{d \ln \mathcal{P}_s}{d \ln k}$$

and, in slow roll inflation, we have

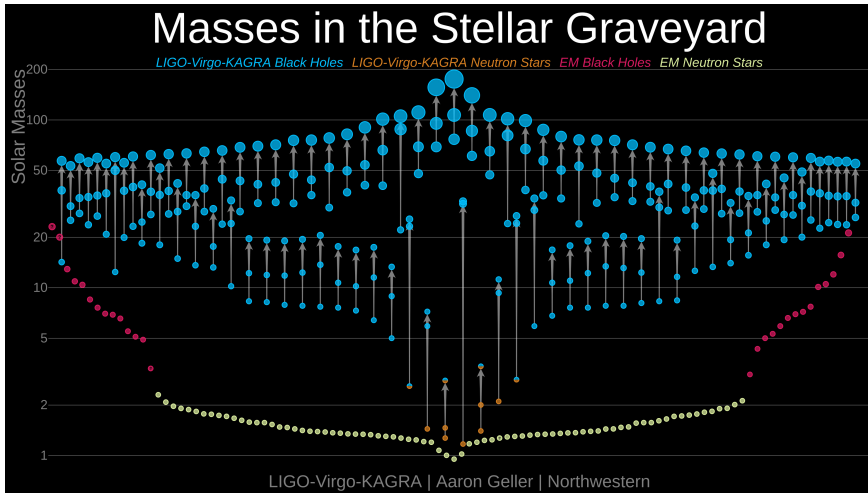
$$n_s = 1 - 2\epsilon_1 + \epsilon_2.$$

Also, in slow roll inflation, the tensor-to-scalar ratio is given by $r = 16\epsilon_1 = -8n_T$.

⁴⁴See, for instance, J. Martin and R. Brandenberger, Phys. Rev. D **68**, 063513 (2003) [hep-th/0305161].



Coalescence of compact binaries observed by LIGO

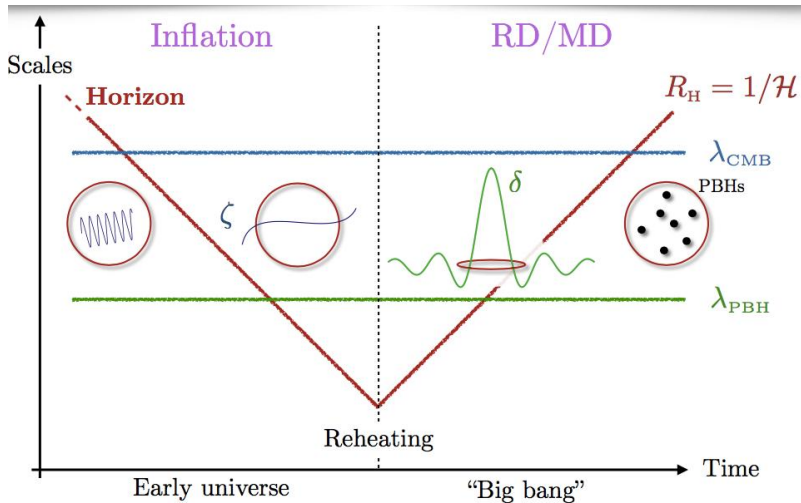


The third GW Transient Catalog of mergers involving black holes and neutron stars observed by the LIGO-Virgo-KAGRA collaboration⁴⁵.

⁴⁵Image from <https://www.ligo.caltech.edu/LA/image/ligo20211107a>.



Formation of primordial black holes (PBHs)

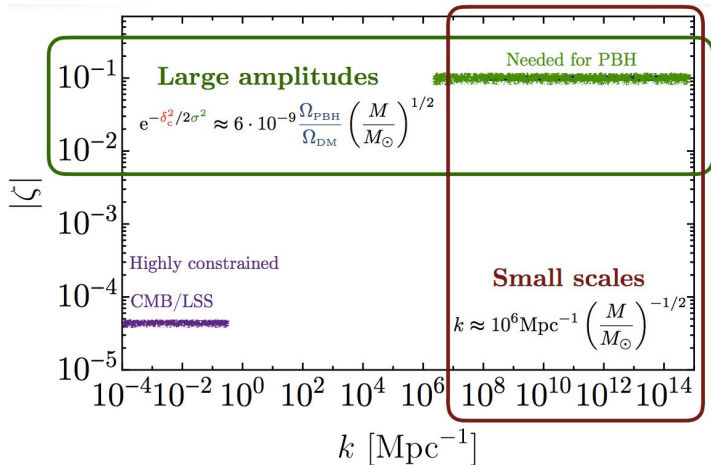


BHs can form in the primordial universe when perturbations with significant amplitudes on small scales reenter the Hubble radius during the radiation dominated epoch⁴⁶.

⁴⁶Figure from G. Franciolini, [arXiv:2110.06815 \[astro-ph.CO\]](https://arxiv.org/abs/2110.06815).



Amplitude required to form significant number of PBHs



In order to form significant number of black holes, the amplitude of the perturbations on small scales has to be large enough such that the dimensionless amplitude of the scalar perturbation is close to unity⁴⁷.

⁴⁷Figure credit G. Franciolini.



Point of inflection in the potential and ultra slow roll inflation

Recall that the canonical scalar field ϕ satisfies the following equation in a FLRW universe:

$$\ddot{\phi} + 3H\dot{\phi} + V_{\phi} = 0.$$

Consider a potential $V(\phi)$ that contains a point of inflection. Since $V_{\phi} \simeq 0$ near the point, the above equation can be integrated immediately to arrive at

$$\dot{\phi} \propto \frac{1}{a^3}.$$

In such a situation, as we can assume H to be constant, we have

$$\epsilon_1 = \frac{\dot{\phi}^2}{2M_{\text{Pl}}^2 H^2} \propto \frac{1}{a^6}.$$

Also, in such a situation

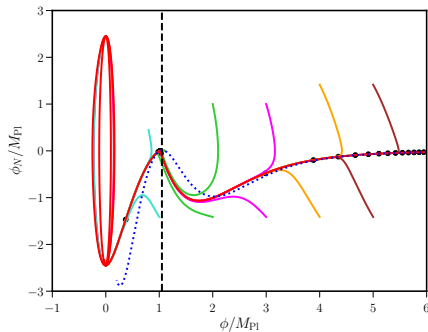
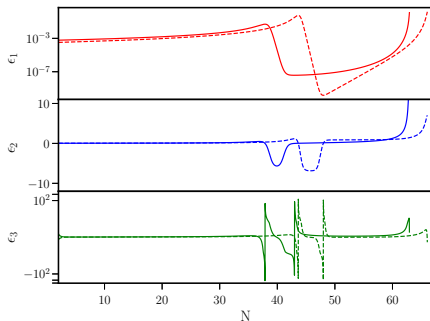
$$\epsilon_2 = \frac{d \ln \epsilon_1}{dN} = \frac{\dot{\epsilon}_1}{H \epsilon_1} = -6.$$

Such a period of inflation is referred to as ultra slow roll inflation⁴⁸

⁴⁸N. C. Tsamis and R. P. Woodard, Phys. Rev. D **69**, 084005 (2004) [astro-ph/0307463];
W. H. Kinney, Phys. Rev. D **72**, 023515 (2005) [gr-qc/0503017].



Single-field models admitting ultra slow roll inflation



Potentials leading to ultra slow roll inflation (with $x = \phi/v$, v being a constant)⁴⁹:

► Inflationary attractor

$$\text{USR1} : V(\phi) = V_0 \frac{6x^2 - 4\alpha x^3 + 3x^4}{(1 + \beta x^2)^2},$$

$$\text{USR2} : V(\phi) = V_0 \left\{ \tanh\left(\frac{\phi}{\sqrt{6} M_{Pl}}\right) + A \sin\left[\frac{\tanh[\phi/(\sqrt{6} M_{Pl})]}{f_\phi}\right] \right\}^2.$$

⁴⁹ J. Garcia-Bellido and E. R. Morales, Phys. Dark Univ. **18**, 47 (2017);
I. Dalianis, A. Kehagias and G. Tringas, JCAP **01**, 037 (2019).



Analytical modelling of a brief phase of ultra slow roll inflation

Let us assume the following behavior of the first slow roll parameter:

$$\epsilon_1(a) = \begin{cases} \epsilon_{1i}, & \text{for } \eta_i < \eta < \eta_1, \\ \epsilon_{1i} (a_1/a)^{-6}, & \text{for } \eta_1 < \eta < \eta_2, \\ \epsilon_{1a} (a/a_2)^2, & \text{for } \eta_2 < \eta < \eta_e, \end{cases}$$

The corresponding second slow roll parameter behaves as

$$\epsilon_2 \simeq \begin{cases} 0, & \text{for } \eta_i < \eta < \eta_1, \\ -6, & \text{for } \eta_1 < \eta < \eta_2, \\ 2, & \text{for } \eta_2 < \eta < \eta_e. \end{cases}$$

Note that a period of ultra slow roll inflation follows an epoch of slow roll inflation before ϵ_1 rises leading to the end of inflation.

It can be shown that, in the three phases, the quantity z''/z behaves as

z''/z

$$\frac{z''}{z} \simeq \frac{2}{\eta^2}, \quad \frac{z''}{z} \simeq \frac{2}{\eta^2}, \quad \frac{z''}{z} \simeq \frac{6}{\eta^2}.$$



Solutions to the Mukhanov-Sasaki variable in the three phases

The solutions to the Mukhanov-Sasaki variable in the three phases can be expressed as

$$v_k^{\text{I}}(\eta) = \frac{1}{\sqrt{2k}} \left(1 - \frac{i}{k\eta}\right) e^{-ik\eta},$$

$$v_k^{\text{II}}(\eta) = \frac{\gamma_k}{\sqrt{2k}} \left(1 - \frac{i}{k\eta}\right) e^{-ik\eta} + \frac{\delta_k}{\sqrt{2k}} \left(1 + \frac{i}{k\eta}\right) e^{ik\eta},$$

$$v_k^{\text{III}}(\eta) = \frac{\alpha_k}{\sqrt{2k}} \left(1 - \frac{3i}{k\eta} - \frac{3}{k^2\eta^2}\right) e^{-ik\eta} + \frac{\beta_k}{\sqrt{2k}} \left(1 + \frac{3i}{k\eta} - \frac{3}{k^2\eta^2}\right) e^{ik\eta}.$$

The coefficients $(\alpha_k, \beta_k, \gamma_k, \delta_k)$ can be obtained to be

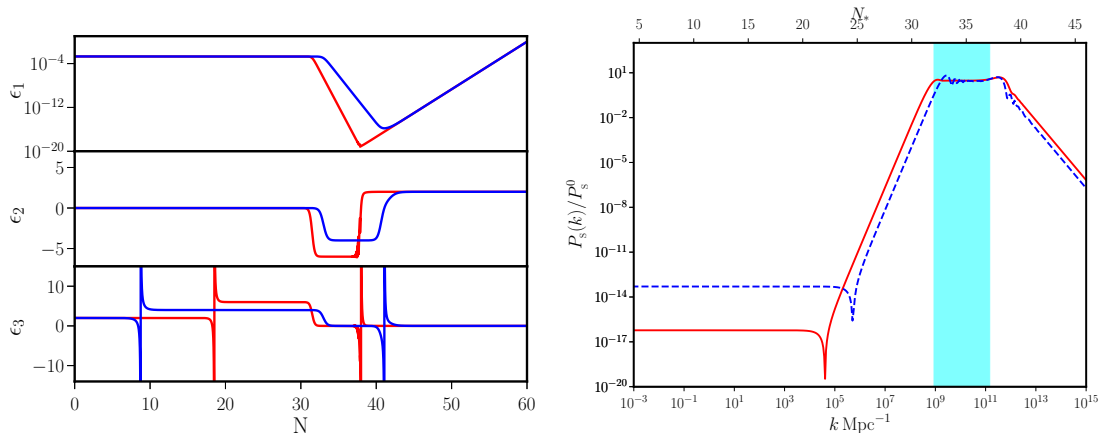
$$\gamma_k = 1 + \frac{3i}{2k\eta_1} + \frac{3i}{2k^3\eta_1^3}, \quad \delta_k = \left(-\frac{3i}{2k\eta_1} - \frac{3}{k^2\eta_1^2} + \frac{3i}{2k^3\eta_1^3}\right) e^{-2ik\eta_1},$$

$$\alpha_k = \left(1 + \frac{2}{k^2\eta_2^2} + \frac{9}{2k^4\eta_2^4}\right) \gamma_k - \left(\frac{2i}{k\eta_2} - \frac{7}{k^2\eta_2^2} - \frac{9i}{k^3\eta_2^3} + \frac{9}{2k^4\eta_2^4}\right) \delta_k e^{2ik\eta_2},$$

$$\beta_k = \left(1 + \frac{2}{k^2\eta_2^2} + \frac{9}{2k^4\eta_2^4}\right) \delta_k + \left(\frac{2i}{k\eta_2} + \frac{7}{k^2\eta_2^2} - \frac{9i}{k^3\eta_2^3} - \frac{9}{2k^4\eta_2^4}\right) \gamma_k e^{-2ik\eta_2}.$$



Behavior of the slow roll parameters and the scalar power spectrum

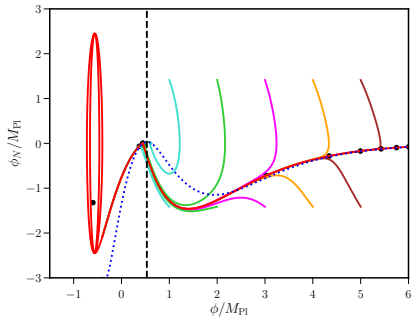
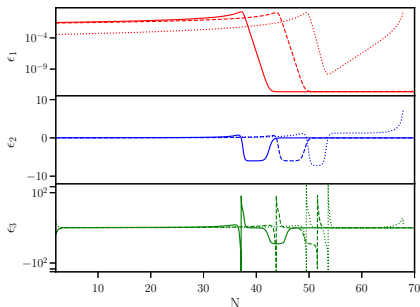


The behavior of the first three slow roll parameters ($\epsilon_1, \epsilon_2, \epsilon_3$) (on the left) and the resulting scalar power spectrum $\mathcal{P}_s(k)$ (on the right). In addition to the analytically evaluated quantities (plotted in blue), we have also plotted the numerical results (in red). On small scales, the power spectrum rises as k^4 and falls as k^{-2} after the peak⁵⁰.

⁵⁰C. T. Byrnes, P. S. Cole, and S. P. Patil, JCAP **06**, 028 (2019).



Single-field models permitting punctuated inflation



Potentials admitting punctuated inflation⁵¹:

$$\text{PI1} : V(\phi) = V_0 (1 + B \phi^4), \quad \text{PI2} : V(\phi) = \frac{m^2}{2} \phi^2 - \frac{2m^2}{3\phi_0} \phi^3 + \frac{m^2}{4\phi_0^2} \phi^4,$$

$$\text{PI3} : V(\phi) = V_0 \left[c_0 + c_1 \tanh \left(\frac{\phi}{\sqrt{6\alpha} M_{\text{Pl}}} \right) + c_2 \tanh^2 \left(\frac{\phi}{\sqrt{6\alpha} M_{\text{Pl}}} \right) + c_3 \tanh^3 \left(\frac{\phi}{\sqrt{6\alpha} M_{\text{Pl}}} \right) \right]^2.$$

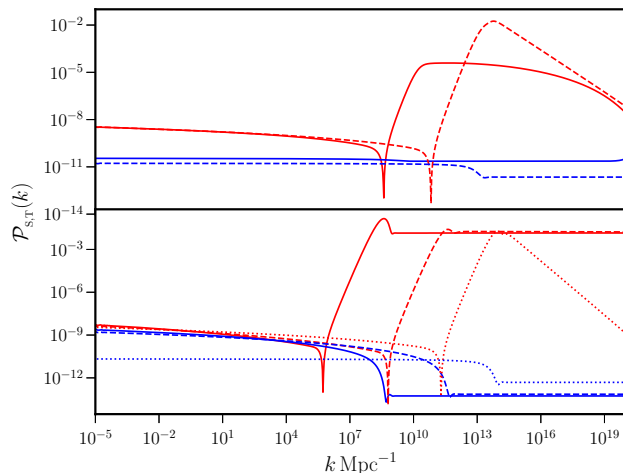
⁵¹D. Roberts, A. R. Liddle and D. H. Lyth, Phys. Rev. D **51**, 4122 (1995);

R. K. Jain, P. Chingangbam, J.-O. Gong, L. Sriramkumar and T. Souradeep, JCAP **01**, 009 (2009);

I. Dalianis, A. Kehagias and G. Tringas, JCAP **01**, 037 (2019).



Power spectra in ultra slow roll and punctuated inflation



Scalar (in red) and the tensor (in blue) power spectra arising in various single field models that permit a period of ultra slow roll (on top) and punctuated⁵².

⁵²H. V. Ragavendra, P. Saha, L. Sriramkumar and J. Silk, *Phys. Rev. D* **103**, 083510 (2021);
Also see H. V. Ragavendra and L. Sriramkumar, *Galaxies* **11**, 34 (2023).



Tensor perturbations at the second order

In order to compute the second-order tensor perturbations induced by the first order scalar perturbations, we can start with the following line-element:

$$ds^2 = - \left(1 + 2 \Phi^{(1)} + 2 \Phi^{(2)} \right) dt^2 + 2 V_i^{(2)} a(t) dt dx^i + a^2(t) \left[\left(1 - 2 \Psi^{(1)} - 2 \Psi^{(2)} \right) \delta_{ij} + \frac{1}{2} h_{ij}^{(2)} \right] dx^i dx^j,$$

where the superscripts (1) and (2) denote the perturbations at the first and the second order perturbations.

Note that $(\Phi^{(1)}, \Psi^{(1)}, \Phi^{(2)}, \Psi^{(2)})$ denote the scalar perturbations, $V_i^{(2)}$ is the vector perturbation and $h_{ij}^{(2)}$ represents the tensor perturbation of our interest⁵³.

⁵³D. Baumann, P. J. Steinhardt, K. Takahashi and K. Ichiki, Phys. Rev. D **76**, 084019 (2007).



GWs sourced by second order scalar perturbations I

At the second order in the perturbations, one finds that the equation governing the tensor modes, say, $h_{\mathbf{k}}$, can be written as⁵⁴

$$h_{\mathbf{k}}^{\lambda''} + 2\mathcal{H} h_{\mathbf{k}}^{\lambda'} + k^2 h_{\mathbf{k}}^{\lambda} = S_{\mathbf{k}}^{\lambda}$$

with the source term $S_{\mathbf{k}}^{\lambda}$ being given by

$$S_{\mathbf{k}}^{\lambda}(\eta) = 4 \int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}} e^{\lambda}(\mathbf{k}, \mathbf{p}) \left\{ 2 \Phi_{\mathbf{p}}(\eta) \Phi_{\mathbf{k}-\mathbf{p}}(\eta) + \frac{4}{3(1+w)\mathcal{H}^2} [\Phi'_{\mathbf{p}}(\eta) + \mathcal{H} \Phi_{\mathbf{p}}(\eta)] [\Phi'_{\mathbf{k}-\mathbf{p}}(\eta) + \mathcal{H} \Phi_{\mathbf{k}-\mathbf{p}}(\eta)] \right\},$$

where $\Phi_{\mathbf{k}}$ represents the Fourier modes of the Bardeen potential, while \mathcal{H} and w denote the conformal Hubble parameter and the EoS parameter describing the universe at the conformal time η . Also, $e^{\lambda}(\mathbf{k}, \mathbf{p}) = e_{ij}^{\lambda}(\mathbf{k}) p^i p^j$, with $e_{ij}^{\lambda}(\mathbf{k})$ representing the polarization of the tensor perturbations.

⁵⁴K. N. Ananda, C. Clarkson and D. Wands, Phys. Rev. D **75**, 123518 (2007);
D. Baumann, P. J. Steinhardt, K. Takahashi and K. Ichiki, Phys. Rev. D **76**, 084019 (2007).



GWs sourced by second order scalar perturbations II

During radiation domination, we can express the Fourier modes $\Phi_{\mathbf{k}}$ of the Bardeen potential in terms of the inflationary Fourier modes $\mathcal{R}_{\mathbf{k}}$ of the curvature perturbations generated during inflation through the relation

$$\Phi_{\mathbf{k}}(\eta) = \frac{2}{3} \mathcal{T}(k\eta) \mathcal{R}_{\mathbf{k}},$$

where $\mathcal{T}(k\eta)$ is the transfer function given by

$$\mathcal{T}(k\eta) = \frac{9}{(k\eta)^2} \left[\frac{\sin(k\eta/\sqrt{3})}{k\eta/\sqrt{3}} - \cos(k\eta/\sqrt{3}) \right].$$



Particular solution to the inhomogeneous equation

If we make use of the Green's function corresponding to the tensor modes during radiation domination, we find that we can express the inhomogeneous contribution to $h_{\mathbf{k}}^\lambda$ as⁵⁵

$$h_{\mathbf{k}}^\lambda(\eta) = \frac{4}{9 k^3 \eta} \int \frac{d^3 \mathbf{p}}{(2\pi)^{3/2}} e^{i\lambda(\mathbf{k}, \mathbf{p})} \mathcal{R}_{\mathbf{k}} \mathcal{R}_{\mathbf{k}-\mathbf{p}} \\ \times [\mathcal{I}_c(p/k, |\mathbf{k} - \mathbf{p}|/k) \cos(k\eta) + \mathcal{I}_s(p/k, |\mathbf{k} - \mathbf{p}|/k) \sin(k\eta)],$$

where the kernels $\mathcal{I}_c(v, u)$ and $\mathcal{I}_s(v, u)$ are described by the integrals (with $\mathcal{T}_z = d\mathcal{T}/dz$)

$$\mathcal{I}_c(v, u) = -4 \int_0^\infty d\tau \tau \sin \tau \left\{ 2 \mathcal{T}(v\tau) \mathcal{T}(u\tau) \right. \\ \left. + [\mathcal{T}(v\tau) + v\tau \mathcal{T}_{v\tau}(v\tau)] [\mathcal{T}(u\tau) + u\tau \mathcal{T}_{u\tau}(u\tau)] \right\}, \\ \mathcal{I}_s(v, u) = 4 \int_0^\infty d\tau \tau \cos \tau \left\{ 2 \mathcal{T}(v\tau) \mathcal{T}(u\tau) \right. \\ \left. + [\mathcal{T}(v\tau) + v\tau \mathcal{T}_{v\tau}(v\tau)] [\mathcal{T}(u\tau) + u\tau \mathcal{T}_{u\tau}(u\tau)] \right\}.$$

⁵⁵J. R. Espinosa, D. Racco and A. Riotto, JCAP, **1809**, 012 (2018).



Forms of the kernels

Upon utilizing the transfer function $\mathcal{T}(k\eta)$ the kernels $\mathcal{I}_c(v, u)$ and $\mathcal{I}_s(v, u)$ can be calculated analytically to obtain that⁵⁶

$$\mathcal{I}_c(v, u) = -\frac{27\pi}{4v^3u^3} \Theta\left(v + u - \sqrt{3}\right) (v^2 + u^2 - 3)^2,$$

$$\mathcal{I}_s(v, u) = -\frac{27}{4v^3u^3} (v^2 + u^2 - 3) \left[4vu + (v^2 + u^2 - 3) \log \left| \frac{3 - (v - u)^2}{3 - (v + u)^2} \right| \right],$$

where $\Theta(z)$ denotes the theta function. Note that $\mathcal{I}_{c,s}(v, u) = \mathcal{I}_{c,s}(u, v)$.

⁵⁶K. Kohri and T. Terada, Phys. Rev. D **97**, 123532 (2018);
J. R. Espinosa, D. Racco and A. Riotto, JCAP, **1809**, 012 (2018).



The secondary tensor power spectrum

The power spectrum of the secondary GWs, say, $\mathcal{P}_h(k, \eta)$, generated due to the second order scalar perturbations can be defined through the relation

$$\langle h_{\mathbf{k}}^\lambda(\eta) h_{\mathbf{k}'}^{\lambda'}(\eta) \rangle = \frac{2\pi^2}{k^3} \mathcal{P}_h(k, \eta) \delta^{(3)}(\mathbf{k} + \mathbf{k}') \delta^{\lambda\lambda'}.$$

If we assume that the Fourier modes of the curvature perturbations are Gaussian random variables, using Wick's theorem, we can express the four-point function involving $\mathcal{R}_{\mathbf{k}}$ in terms of the inflationary scalar power spectrum $\mathcal{P}_s(k)$ to arrive at the following expression for $\mathcal{P}_h(k, \eta)$:

$$\begin{aligned} \mathcal{P}_h(k, \eta) &= 481 k^2 \eta^2 \int_0^\infty dv \int_{|1-v|}^{1+v} du \left[\frac{4v^2 - (1 + v^2 - u^2)^2}{4uv} \right]^2 \mathcal{P}_s(kv) \mathcal{P}_s(ku) \\ &\times [\mathcal{I}_c(u, v) \cos(k\eta) + \mathcal{I}_s(u, v) \sin(k\eta)]^2. \end{aligned}$$

The trigonometric functions in this expression arise because of the form of the transfer function $\mathcal{T}(k\eta)$.



Averaging over the oscillations

On averaging the secondary tensor power spectrum $\mathcal{P}_h(k, \eta)$ over small time scales, we can replace the trigonometric functions in the above expression by their average over a time period. This leads to

$$\begin{aligned} \overline{\mathcal{P}_h(k, \eta)} &= \frac{2}{81 k^2 \eta^2} \int_0^\infty dv \int_{|1-v|}^{1+v} du \left[\frac{4v^2 - (1 + v^2 - u^2)^2}{4uv} \right]^2 \mathcal{P}_s(kv) \mathcal{P}_s(ku) \\ &\quad \times [\mathcal{I}_c^2(u, v) + \mathcal{I}_s^2(u, v)], \end{aligned}$$

where the line over $\mathcal{P}_h(k, \eta)$ implies the average over small time scales.

The spectral energy density of GWs at a time η is given by

$$\rho_{\text{GW}}(k, \eta) = \frac{M_{\text{Pl}}^2}{8} \left(\frac{k}{a} \right)^2 \overline{\mathcal{P}_h(k, \eta)}.$$

We can define the corresponding dimensionless density parameter $\Omega_{\text{GW}}(k, \eta)$ in terms of the critical density $\rho_{\text{cr}}(\eta)$ as

$$\Omega_{\text{GW}}(k, \eta) = \frac{\rho_{\text{GW}}(k, \eta)}{\rho_{\text{cr}}(\eta)} = \frac{1}{24} \left(\frac{k}{\mathcal{H}} \right)^2 \overline{\mathcal{P}_h(k, \eta)} = \frac{k^2 \eta^2}{24} \overline{\mathcal{P}_h(k, \eta)}.$$



Spectral energy density of secondary GWs today

The dimensionless spectral energy density of GWs $\Omega_{\text{GW}}(k, \eta)$, when evaluated at late times during the radiation dominated epoch, can be expressed as⁵⁷

$$\Omega_{\text{GW}}(k, \eta) = \frac{\rho_{\text{GW}}(k, \eta)}{\rho_{\text{cr}}(\eta)} = \frac{1}{972} \int_0^\infty dv \int_{|1-v|}^{1+v} du \left[\frac{4v^2 - (1 + v^2 - u^2)^2}{4uv} \right]^2 \mathcal{P}_s(kv) \mathcal{P}_s(ku) \\ \times [\mathcal{I}_c^2(u, v) + \mathcal{I}_s^2(u, v)],$$

where the quantities $\mathcal{I}_c(u, v)$ and $\mathcal{I}_s(u, v)$ are determined by the transfer function $\mathcal{T}(k, \eta)$ for the scalar perturbations.

Recall that, we can express $\Omega_{\text{GW}}(k)$ today in terms of the above $\Omega_{\text{GW}}(k, \eta)$ as follows:

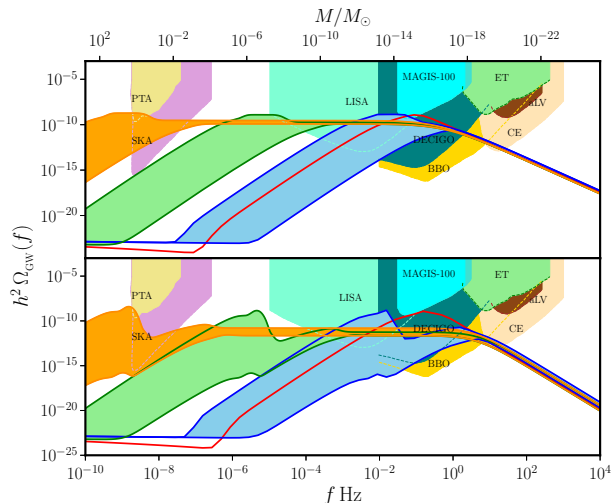
$$h^2 \Omega_{\text{GW}}(k) \simeq 1.38 \times 10^{-5} \left(\frac{g_k}{106.75} \right)^{-1/3} \left(\frac{\Omega_r h^2}{4.16 \times 10^{-5}} \right) \Omega_{\text{GW}}(k, \eta),$$

where Ω_r denotes the dimensionless energy density of radiation today, while g_k represents the number of relativistic degrees of freedom at reentry.

⁵⁷ K. Kohri and T. Terada, Phys. Rev. D **97**, 123532 (2018);
J. R. Espinosa, D. Racco and A. Riotto, JCAP **09**, 012 (2018).



$\Omega_{\text{GW}}(f)$ in ultra slow roll inflation



The dimensionless spectral density of GWs today $\Omega_{\text{GW}}(f)$ arising in single field models that permit a brief epoch of ultra slow roll inflation⁵⁸.

⁵⁸H. V. Ragavendra, P. Saha, L. Sriramkumar and J. Silk, Phys. Rev. D **103**, 083510 (2021).

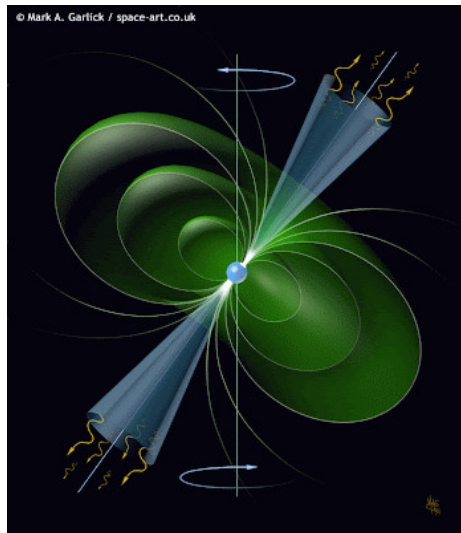


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- 2 Tensor power spectrum in slow roll inflation
- 3 Constraints on inflation from the CMB data
- 4 GWs provide a new window to the universe
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- 6 Generation of GWs by enhanced scalar perturbations on small scales
- 7 NANOGrav 15-year data and the stochastic GW background**
- 8 Outlook
- 9 References



Pulsars

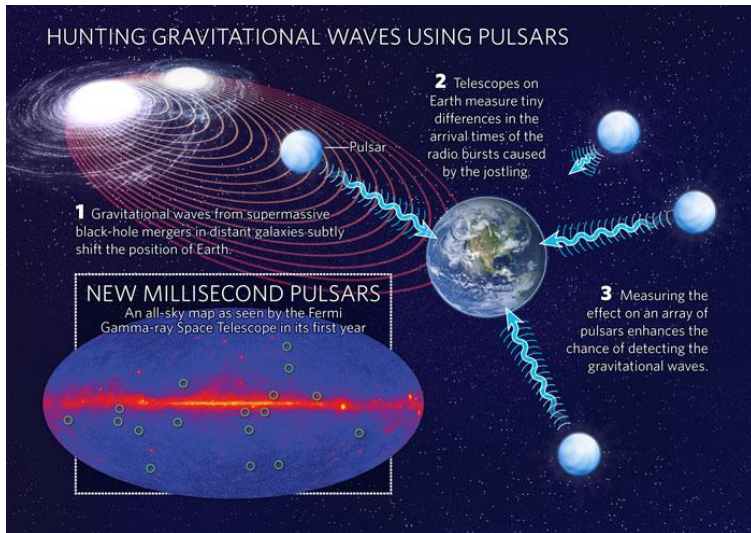


Pulsars are dense and rotating neutron stars that emit regular beams of light⁵⁹.

⁵⁹Image from <https://dmr-astronomersclub.blogspot.com/2012/07/what-is-pulsar.html>.



Pulsar timing arrays (PTAs)

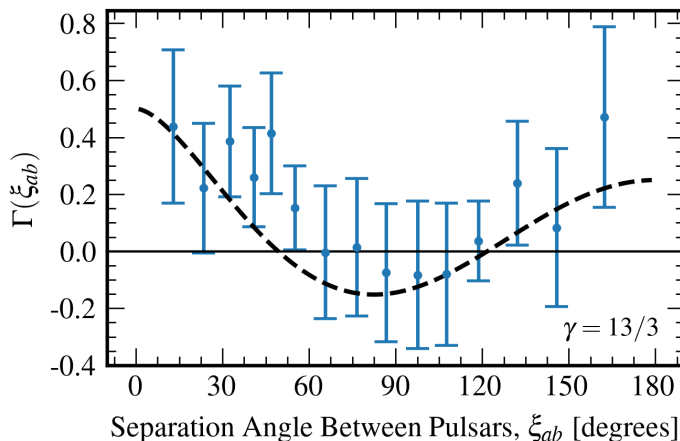


The PTAs monitor an array of millisecond pulsars⁶⁰.

⁶⁰See https://ipta.github.io/mock_data_challenge/.



Hellings-Downs curve

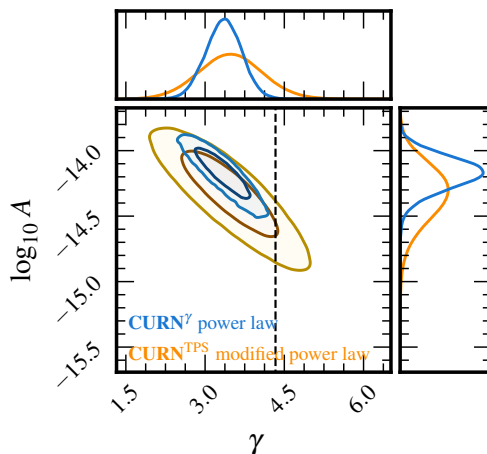


The inter-pulsar correlations measured from 2,211 distinct pairings in the 67-pulsar array of the NANOGrav 15-year data. The dashed black line shows the Hellings-Downs correlation pattern⁶¹.

⁶¹ NANOGrav Collaboration (G. Agazie *et al.*), *Astrophys. J. Lett.* **951**, 1 (2023).



Constraints on the spectral amplitude and index of GWs

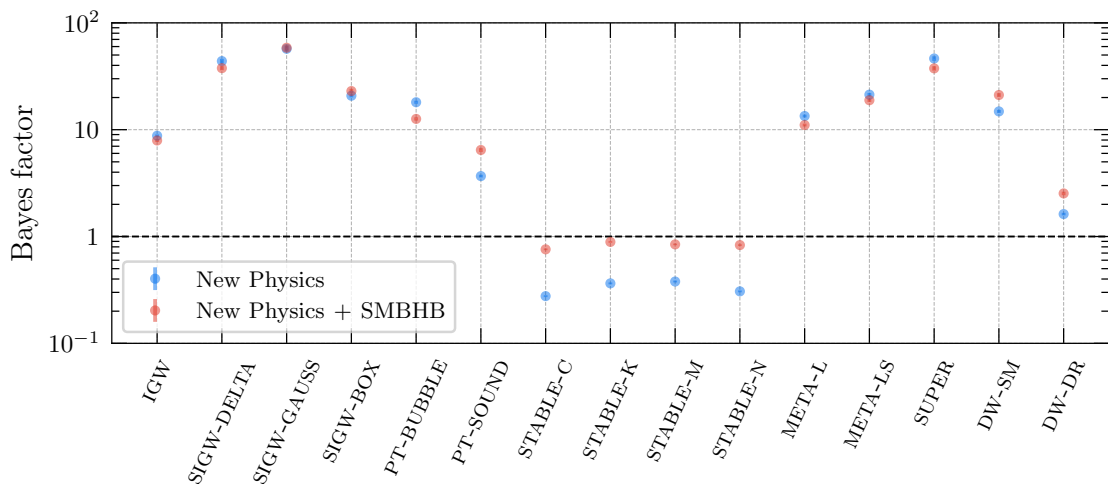


Constraints on the amplitude A and the index γ of the stochastic background of GWs from the NANOGrav 15-year data⁶².

⁶²NANOGrav Collaboration (G. Agazie *et al.*), *Astrophys. J. Lett.* **951**, 1 (2023).



Stochastic GW background observed by the PTAs



The Bayesian evidence for a variety of astrophysical and cosmological sources for the stochastic GW background suggested by the observations of the PTAs⁶³.

⁶³ NANOGrav Collaboration (G. Agazie *et al.*), *Astrophys. J. Lett.* **951**, 1 (2023).



Form of the inflationary scalar power spectrum

We assume that the inflationary scalar power spectrum is given by⁶⁴

$$\mathcal{P}_{\mathcal{R}}(k) = A_S \left(\frac{k}{k_*} \right)^{n_S - 1} + A_0 \begin{cases} \left(\frac{k}{k_{\text{peak}}} \right)^4 & k \leq k_{\text{peak}} \\ \left(\frac{k}{k_{\text{peak}}} \right)^{n_0} & k \geq k_{\text{peak}} \end{cases},$$

where A_S and n_S are the amplitude and spectral index of the power spectrum at the CMB pivot scale of $k_* = 0.05 \text{ Mpc}^{-1}$.

We set the reheating temperature to the rather low value of $T_{\text{re}} = 50 \text{ MeV}$.

We shall assume that the threshold value of the density contrast for the formation of PBHs is given by⁶⁵

$$\delta_c^{\text{an}} = \frac{3(1 + w_{\text{re}})}{5 + 3w_{\text{re}}} \sin^2 \left(\frac{\pi \sqrt{w_{\text{re}}}}{1 + 3w_{\text{re}}} \right).$$

⁶⁴For other forms of spectra, see [G. Domènech, S. Pi, A. Wang and J. Wang, arXiv:2402.18965 \[astro-ph.CO\]](#).

⁶⁵In this context, see [T. Harada, C.-M. Yoo, and K. Kohri, Phys. Rev. D **88**, 084051 \(2013\)](#).



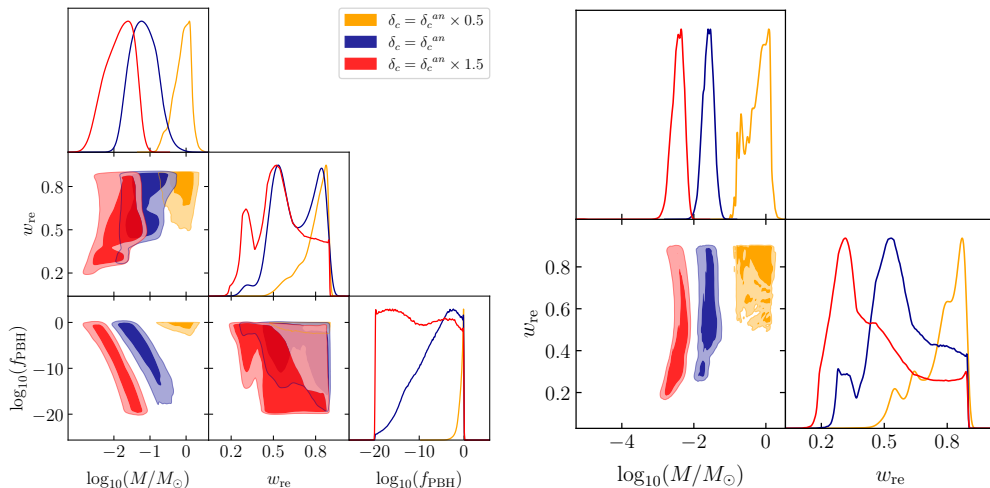
Best-fit values

Model	Parameter	Prior	Mean value		
R4pF	$\log_{10} \left(\frac{k_{\text{peak}}}{\text{Mpc}^{-1}} \right)$	[6, 9]	$7.62^{+0.35}_{-0.41}$		
	$\log_{10}(A_0)$	[-3, 0]	$-1.23^{+0.38}_{-0.66}$		
	w_{re}	[0.1, 0.9]	0.52 ± 0.23		
	n_0	[-3.0, -1.5]	-2.26 ± 0.43		
R3pF	$\log_{10} \left(\frac{k_{\text{peak}}}{\text{Mpc}^{-1}} \right)$	[6, 9]	$7.54^{+0.36}_{-0.44}$		
	$\log_{10}(A_0)$	[-3, 0]	$-1.26^{+0.26}_{-0.64}$		
	w_{re}	[0.1, 0.9]	$0.55^{+0.39}_{-0.14}$		
			$0.5 \delta_c^{\text{an}}$	δ_c^{an}	$1.5 \delta_c^{\text{an}}$
R3pB	$\log_{10} \left(\frac{M}{M_{\odot}} \right)$	[-6, 3.5]	$-0.12^{+0.28}_{-0.15}$	$-1.18^{+0.35}_{-0.39}$	$-1.85^{+0.49}_{-0.30}$
	$\log_{10}(f_{\text{PBH}})$	[-20, 0]	$-0.67^{+0.68}_{-0.16}$	$-6.6^{+6.5}_{-1.9}$	$-10.2^{+8.2}_{-9.6}$
	w_{re}	[0.1, 0.9]	$0.78^{+0.11}_{-0.030}$	$0.66^{+0.23}_{-0.19}$	0.55 ± 0.17
R2pB	$\log_{10} \left(\frac{M}{M_{\odot}} \right)$	[-6, 3.5]	$-0.24^{+0.38}_{-0.45}$	$-1.60^{+0.16}_{-0.14}$	$-2.45^{+0.20}_{-0.13}$
	w_{re}	[0.1, 0.9]	$0.77^{+0.13}_{-0.038}$	0.59 ± 0.16	$0.464^{+0.095}_{-0.25}$

The best-fit values arrived at upon comparison with the NANOGrav 15-year data.



Constraints on the epoch of reheating

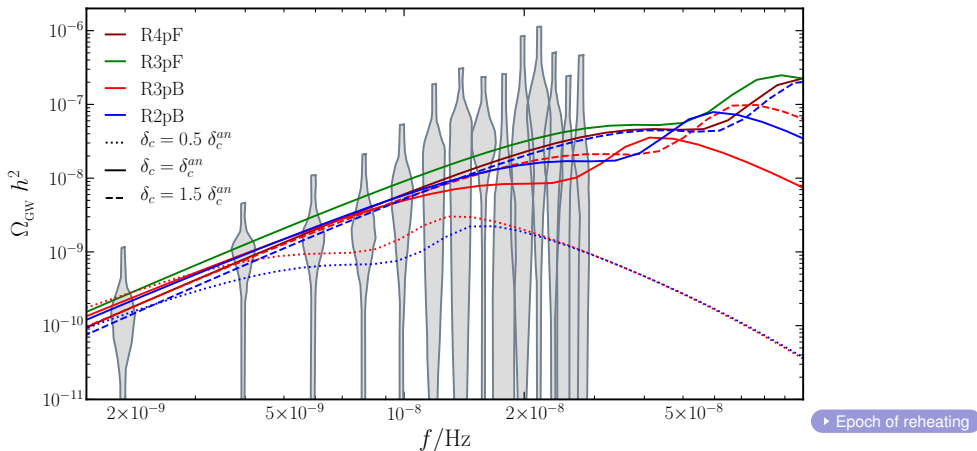


Constraints on the parameters in the models R3pB (on the left) and R2pB (on the right), arrived at upon comparison with the NANOGrav 15-year data⁶⁶.

⁶⁶S. Maity, N. Bhaumik, Md. R. Haque, D. Maity and L. Sriramkumar, arXiv:2403.16963 [astro-ph.CO], accepted in JCAP.



Generation of secondary GWs during the epoch of reheating

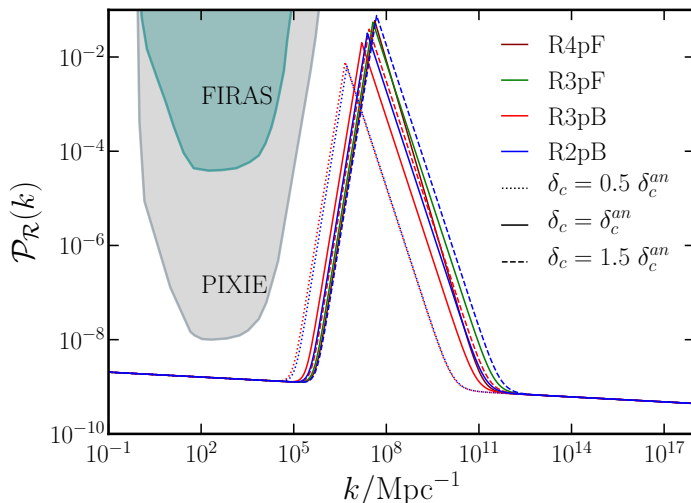


The dimensionless spectral energy density of the secondary GWs today $\Omega_{\text{GW}}(f)$ is plotted for a given reheating temperature and the best-fit values of the parameters in the different models⁶⁷.

⁶⁷S. Maity, N. Bhaumik, Md. R. Haque, D. Maity and L. Sriramkumar, arXiv:2403.16963 [astro-ph.CO], accepted in JCAP.



Constraints from spectral distortions

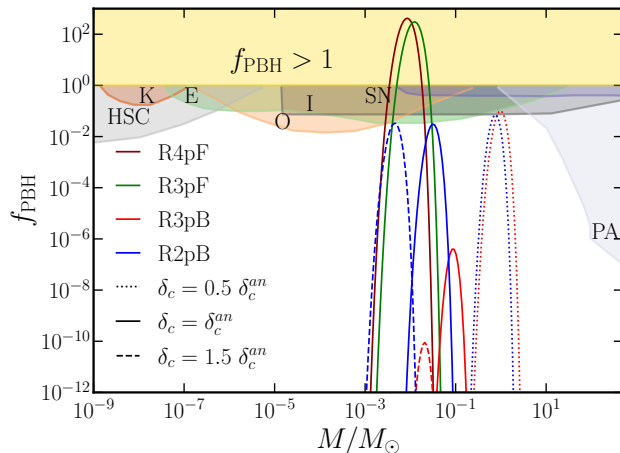


Constraints on the scalar power spectrum from spectral distortions in the CMB⁶⁸.

⁶⁸S. Maity, N. Bhaumik, Md. R. Haque, D. Maity and L. Sriramkumar, arXiv:2403.16963 [astro-ph.CO], accepted in JCAP.



Formation of PBHs during the epoch of reheating



The fraction of PBHs that constitute the dark matter density today, viz. $f_{\text{PBH}}(M)$ is plotted for a given reheating temperature and the best-fit values of the parameters in the different models⁶⁹.

⁶⁹S. Maity, N. Bhaumik, Md. R. Haque, D. Maity and L. Sriramkumar, arXiv:2403.16963 [astro-ph.CO], accepted in JCAP.



Bayesian evidence

Model X	Model Y	$BF_{Y,X}$		
		$\delta_c = 0.5 \delta_c^{\text{an}}$	$\delta_c = \delta_c^{\text{an}}$	$\delta_c = 1.5 \delta_c^{\text{an}}$
SMBHB	R2pB	$1.7 \pm .06$	260.04 ± 19.21	350.61 ± 27.36

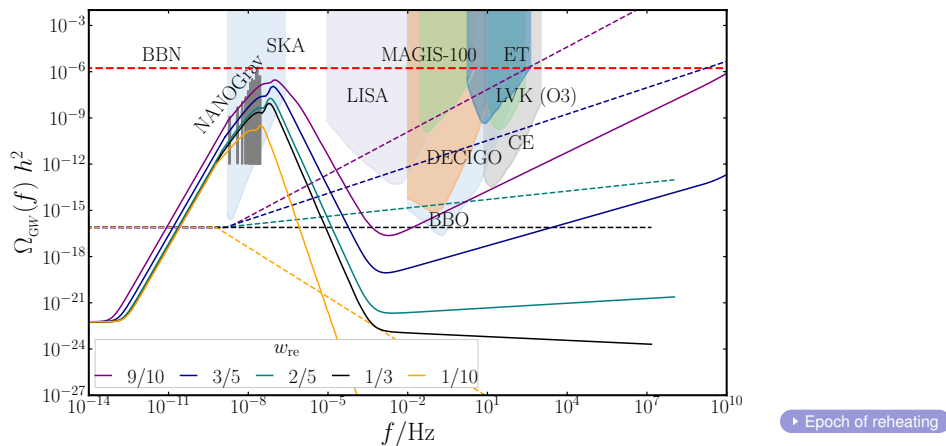
The Bayesian factors $BF_{Y,X}$ for the model R2pB that invokes primordial physics as the source of the stochastic GW background observed by the NANOGrav 15-year data, when compared to the astrophysical SMBHB model.

Bayesian factors $BF_{Y,X}$ that far exceed unity indicate strong evidence for the model Y with respect to the model X .

Clearly, when $\delta_c = \delta_c^{\text{an}}$ and $\delta_c = 1.5 \delta_c^{\text{an}}$, the NANOGrav 15-year data strongly favors the model R2pB when compared to the SMBHM model.



Generation of secondary GWs during the epoch of reheating



The dimensionless spectral energy density of primary and secondary GWs today $\Omega_{\text{GW}}(f)$ have been plotted for a given reheating temperature and different values of the parameter describing the EoS during reheating⁷⁰.

⁷⁰S. Maity, N. Bhaumik, Md. R. Haque, D. Maity and L. Sriramkumar, arXiv:2403.16963 [astro-ph.CO], accepted in JCAP.



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- 8 Outlook**
- 9 References



Outlook

- ◆ If one of the future CMB missions—such as LiteBIRD (Lite, Light satellite for the studies of B-mode polarization and Inflation from cosmic background Radiation Detection), Primordial Inflation Explorer (PIXIE) or Exploring Cosmic History and Origin (ECHO, a proposed Indian effort)—detect the signatures of the primordial GWs, it will help us arrive at strong constraints on the dynamics during inflation and reheating.
- ◆ The observations by LIGO are a culmination of almost fifty years of effort to detect GWs. They have opened up a new window to observe the universe.
- ◆ The observations by the PTAs and their possible implications for the stochastic GW background offer a wonderful opportunity to understand the physics operating over a wider range of scales in the early universe.
- ◆ During the coming decades, GW observatories such as the Laser Interferometer Space Antenna, Einstein Telescope and Cosmic Explorer, can be expected to provide us with an unhindered view of the primordial universe.



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- 8 Outlook
- 9 **References**



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- ◆ C. Caprini and D. G. Figueroa, *Cosmological backgrounds of gravitational waves*, Class. Quant. Grav. **35**, 163001 (2018) [arXiv:1801.04268 [astro-ph.CO]].
- ◆ G. Doménech, *Scalar induced gravitational waves review*, Universe **7**, 398 (2021) [arXiv:2109.01398 [astro-ph.CO]].
- ◆ R. Roshan and G. White, *Using gravitational waves to see the first second of the universe*, arXiv:2401.04388 [astro-ph.CO].



My work

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- ◆ H. V. Ragavendra and L. Sriramkumar, *Observational imprints of enhanced scalar power on small scales in ultra slow roll inflation and associated non-Gaussianities*, Galaxies **11**, 34 (2023) [arXiv:2301.08887 [astro-ph.CO]].
- ◆ Md. R. Haque, D. Maity, T. Paul and L. Sriramkumar, *Decoding the phases of early and late time reheating through imprints on primordial gravitational waves*, Phys. Rev. D **104**, 063513 (2021) [arXiv:2105.09242 [astro-ph.CO]].
- ◆ S. Maity, N. Bhaumik, Md. R. Haque, D. Maity and L. Sriramkumar, *Constraining the history of reheating with the NANOGrav 15-year data*, arXiv:2403.16963 [astro-ph.CO], accepted for publication in JCAP.



Thank you for your attention