I. EXERCISES - SESSION 9

A. Comments on Sessions:

The goals of this session are the following:

1. Solving ODE and PDE
2. Plot files

Everyone attempts all the problems!

B. Preparations:

Preparations: Download the session09.tar.gz file from the course website and untar it. There is a sample file for the Euler algorithm in it, which you could use for your reference. Please use makefiles to compile all your codes and any array in the code should be dynamically allocated and freed at the end of the code!

C. Problems:

1. Solving ODE: In this question, you will set up various ODE solvers (Euler, Predictor-Corrector, Runge Kutta second and fourth order solvers) as separate C codes. See the attached: ode_solver.c.

   (a) Set up separate functions for predictor-corrector, Runge-Kutta second and fourth order algorithms analogous to the Euler function seen in ode_solver.c. The derivatives are calculated through the function 'rhs' found in the main code. The functions that you set up, should be capable of handling N coupled first order ODEs, as seen in the example for Euler.

   (b) Solve the following differential equation that governs the decay of radioactive particles.

   $$\frac{dN(t)}{dt} = -N(t) \tag{1}$$

   The analytic solution is:

   $$N(t) = N(0)\exp(-t) \tag{2}$$

   using $N(0) = 2.0$. Track the solution from time $t = 0$ up to a time $t = 1$ which means you need to calculate the relative error between the analytic and the numerical solution for each time step and plot your results. Comment on your results.

   (c) Next we will use the codes that has been set up to solve a coupled first order ODE. We will solve Newton's equations for a harmonic potential. The Equation is:

   $$m\frac{d^2x(t)}{dt^2} = -kx(t) \tag{3}$$

   As a first step write out this equation as two coupled first order differential equations as seen in class (work this out below):

This system has an analytic solution:

$$x(t) = A\cos(\omega t + \phi) \tag{4}$$

where A is the amplitude, ϕ is the phase and ω is the frequency. Now

$$T = \frac{2\pi}{\omega}, \tag{5}$$

and

$$\omega = \sqrt{\frac{k}{m}} \tag{6}$$

Set $k/m = 1$ so that $T = 2\pi$. Choose the following initial conditions:

$$x(0) = 1 \tag{7}$$

$$v(0) = 0. \tag{8}$$

Use them to determine the constants A and ϕ for the analytic solution (show your work below):

Time evolve from $t_{\min} = 0$ to $t_{\max} = 4T$. Since this system conserves energy, the total energy:

$$E = \frac{1}{2}mv(t)^2 + \frac{1}{2}kx(t)^2 \tag{9}$$

should be a constant. The total energy at $T = 0$ is $E = \frac{1}{2}k$. Now check the energy conservation using the calculated results as you evolve in time for the various algorithms against the result at $t = 0$. Plot the errors together in a single output file.

2. In this exercise, you are going to solve the Diffusion Equation seen in class in 1-D. This is given by:

$$\frac{\partial^2 T(x,t)}{\partial x^2} = D\frac{\partial T(x,t)}{\partial t} \tag{10}$$

where $T(x,t)$ is the temperature gradient and $D = C\rho/\kappa$. C is the specific heat, ρ is the density of the material and κ is the thermal conductivity of the material. We can re-scale this equation by defining $x = \alpha\widehat{x}$ such that $\alpha^2 D = 1$. Then Eq. 10 becomes:

$$\frac{\partial^2 T(\widehat{x},\widehat{t})}{\partial \widehat{x}^2} = \frac{\partial T(\widehat{x},\widehat{t})}{\partial \widehat{t}} \tag{11}$$

Consider a 1D rod of length $L = 1$ and let its ends be dipped in a sink which is at $T = 0$. Let us assume that the rest of the rod except for the end points are at a constant temperature $T_0 = 100$. Note these are just numbers since the equation is now dimensionless. Your task is to find the solution $T(x,t)$, where x and t are still dimensionless but the hat has been dropped for notational simplicity, following the steps outlined below:

(a) Write a C function ONLY for the Implicit and Crank Nicolson schemes and solve for the diffusion equation numerically. Note that in both cases we have sparse matrices and we can cast the equation in terms of $AX = B$ which can be solved using a tridiagonal solver from the gsl library.

(b) Use the 3d plotter file and insert the data file that you have and make a 3D plot using the following command for both the Implicit and the Crank Nicolson schemes:

```
gnuplot 3d_plot.plt
```