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PH5100 Quantum Mechanics I   Problem Set 1   10 August 2012

Problem Set 1

1. Check whether the following sets of elements form an LVS. If they do, find the dimensionality of the LVS.

   (a) The set of all polynomials (of order $\leq n$) of a complex variable z.

   (b) The set of all $n \times n$ matrices with complex entries.

   (c) The set of all 2×2 matrices whose trace is zero.

   (d) The set of all solutions of the differential equation $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 0\,.$

2. **The Cauchy-Schwarz inequality** is of fundamental importance. It says that $|\langle u|v\rangle| \leq ||u||\,||v||$, for any two vectors $u,\, v \in \mathbb{V}$, the equality holding iff u and v are linearly dependent. In terms of ordinary vectors in Euclidean space, it amounts to saying that the cosine of the angle between two vectors has a magnitude between 0 and 1, the limiting value of unity occurring iff the vectors are collinear. Establish the Cauchy-Schwarz inequality. *Hint*: Consider the inner product $\langle u + a\,v\,|\,u + a\,v\rangle$ where a is an arbitrary complex number. Choosing a appropriately leads to the desired inequality.

3. The set of all $n \times n$ matrices (with complex entries) forms an LVS. The inner product of two elements in this space may be defined as $\langle A\,B\rangle = \mathrm{Tr}\,(A^\dagger B)\,$, where $A^\dagger$ denotes the hermitian conjugate of A. If A is an arbitrary $n \times n$ matrix, and U is an unitary $n \times n$ matrix, show that

$$\langle A\,A\rangle \;\geq\; \frac{1}{n}\,|\langle U^\dagger\,A\rangle|^2\,.$$

4. Let $|\phi_0\rangle, |\phi_1\rangle, \cdots$ be an orthonormal basis in an infinite dimensional linear space. Let $\widehat{A}$ be an operator acting on the vectors in this space such that

$$\widehat{A}|\phi_0\rangle \;=\; 0 \qquad (1)$$

$$\widehat{A}|\phi_n\rangle \;=\; \sqrt{n}|\phi_{n-1}\rangle \qquad \text{for } n \geq 1 \qquad (2)$$

   Write down the matrix representations of the operators $\widehat{A}$ and $\widehat{A^2}$ in the basis provided by the set of vectors $|\phi_0\rangle, |\phi_1\rangle, \cdots$, i.e. write down the matrix whose ij^{th} element is $\langle\phi_i|A|\phi_j\rangle$ for the first case and $\langle\phi_i|A^2|\phi_j\rangle$ for the second case.

5. Find the eigenvalues and Eigenvectors (normalized) of the following 3×3 matrices

(a)

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(b)

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Determine degeneracies if any and explain your results for the eigenvectors in the degenerate case.