DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH5100 Quantum Mechanics I Problem Set 2 24 August 2012
due: 31 Aug 2012

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1. Prove the following relation:

$$[AB, CD] = -AC\{D, B\} + A\{C, B\}D - C\{D, A\}B + \{C, A\}DB \tag{1}$$

where A, B, C, D are operators.

2. **Pauli-Matrices**: Pauli Matrices are defined as:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{3}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$

(a) Prove the following:

$$\sigma_i^2 = 1, \tag{5}$$

$$\det \sigma_i = -1, \tag{6}$$

and

$$\mathrm{Tr}(\sigma_i) = 0. \tag{7}$$

where $i = 1, 2, 3$.

(b) Using the following relations:

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \tag{8}$$

and

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} \tag{9}$$

show that:

$$\sigma_i\sigma_j = \delta_{ij}I + i\epsilon_{ijk}\sigma_k \tag{10}$$

where I is the identity and $\{A, B\} = AB + BA$ and is called the anti-commutator.

(c) If X is a 2×2 matrix, then it can be represented in the basis $\{I, \sigma_i\}$ as follows:

$$X = a_0 I + \sum_i a_i\sigma_i \tag{11}$$

Obtain the $\mathrm{Tr}(X)$ and $\mathrm{Tr}(\sigma_k X)$ in terms of a_0 and a_i. How does the matrix X looks like?

3. Suppose $|i\rangle$ and $|j\rangle$ are eigenkets of some Hermitian operator $\widehat{A}$. Under what condition can we conclude that $|i\rangle + |j\rangle$ is also an eigenket of $\widehat{A}$? Justify your answer.

4. If $\widehat{A}$ and $\widehat{B}$ are two hermitian operators such that $[\widehat{A}, \widehat{B}] = \widehat{C} \neq 0$. Show that:

   (a) $\widehat{C}$ is anti-hermitian, i.e.
   $$\widehat{C} = -\widehat{C}^\dagger \tag{12}$$

   (b) The expectation value of $\widehat{C}$ is pure imaginary in any arbitrary state $|\Psi\rangle$. (Hint: Expand $|\Psi\rangle$ in the basis of $\widehat{A}$ and calculate $\langle\Psi|\widehat{C}|\Psi\rangle$ and $\langle\Psi|\widehat{C}^\dagger|\Psi\rangle$.)

   (c) Can both $\widehat{A}$ and $\widehat{B}$ have sharp values simultaneously in any state $|\Psi\rangle$?

5. **Gaussian Wave Packets:** If the wave function (in position space) is given by:
   $$\langle x|\alpha\rangle = \frac{1}{\pi^{1/4}\sqrt{d}} \exp\left[ikx - \frac{x^2}{2d^2}\right], \tag{13}$$

   (a) Show that the probability density defined by $|\langle x|\alpha\rangle|^2$ is a Gaussian of width d.

   (b) Show that:
   $$\langle x\rangle = 0, \tag{14}$$
   $$\langle x^2\rangle = \frac{d^2}{2}, \tag{15}$$
   $$\langle p\rangle = \hbar k, \tag{16}$$
   and
   $$\langle p^2\rangle = \hbar^2 k^2 + \frac{\hbar^2}{2d^2}. \tag{17}$$

   (c) Determine $\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle$ where $\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2$ and $\langle(\Delta p)^2\rangle = \langle p^2\rangle - \langle p\rangle^2$. Justify why a Gaussian wave packet is called a minimum uncertainty wave packet.

   (d) Obtain the momentum space wave function $\langle p|\alpha\rangle$.