DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH5100 Quantum Mechanics I **Problem Set 3** **17 Sept 2012**
due: 24 Sept 2012

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1. Let $\widehat{K}$ be an operator defined by: $\widehat{K} = |\phi\rangle\langle\psi|$ where $|\phi\rangle$ and $|\psi\rangle$ are two vectors of the state space.
   (a) Under what condition in $\widehat{K}$ Hermitian?
   (b) Calculate $\widehat{K}^2$. Under what condition is $\widehat{K}$ a projector?
   (c) Show that $\widehat{K}$ can always be written in the form $\widehat{K} = \lambda \widehat{P}_1 \widehat{P}_2$ where λ is a constant to be calculated and $\widehat{P}_1$ and $\widehat{P}_2$ are projectors.

2. Consider the Hamiltonian of a particle in a one-dimensional problem defined by:
$$\widehat{H} = \frac{1}{2m}\widehat{P}^2 + V(\widehat{X})$$
   where $\widehat{X}$ and $\widehat{P}$ are operators that satisfy $[\widehat{X}, \widehat{P}] = i\hbar$. The eigenvectors of $\widehat{H}$ are denoted by $|\phi_n\rangle$ and the eigenvalues are E_n where n is a discrete index.
   (a) Show that:
$$\langle\phi_n|\widehat{P}|\phi_{n'}\rangle = \alpha\langle\phi_n|\widehat{X}|\phi_{n'}\rangle$$
   where α is a coefficient that depends on the difference between E_n and $E_{n'}$. Calculate α. (Hint: consider the commutator: $[\widehat{X}, \widehat{H}]$)
   (b) From the above exercise, deduce the following using closure relation:
$$\sum_n (E_n - E_{n'})^2 |\langle\phi_n|\widehat{X}|\phi_{n'}\rangle|^2 = \frac{\hbar^2}{m^2}\langle\phi_n|\widehat{P}|\phi_n\rangle.$$

3. Consider the three-dimensional wave function
$$\psi(x, y, z) = N e^{-\left[\frac{|x|}{2a} + \frac{|y|}{2b} + \frac{|z|}{2c}\right]}$$
   where a, b, c are three positive lengths.
   (a) Calculate N such that ψ is normalized.
   (b) Calculate the probability that a measurement of X will yeild a result included between 0 and a.

(c) Calculate the probability that simultaneous measurements of Y and Z will yield results included repectively between $-b$ and b, and $-c$ and c.

(d) Calculate the probability that a measurement of the momentum will yield a result included in the element $dp_x dp_y dp_z$ centered at the point $p_x = p_y = 0$ and $p_z = \hbar/c$.

4. Consider a particle of mass m in a potential:

$$V(x) = 0 \qquad \text{if } 0 \leq x \leq a$$

$$V(x) = \infty \qquad \text{otherwise}$$

Let $|\phi_n\rangle$ be the eigenstates of the Hamiltonian and the eigenvalues are given by $E_n = \dfrac{n^2\pi^2\hbar^2}{2ma^2}$. The state of the particle at the instant $t = 0$ is:

$$|\psi(0)\rangle = a_1|\phi_1\rangle + a_2|\phi_2\rangle + a_3|\phi_3\rangle + a_4|\phi_4\rangle$$

(a) What is the probability, when the energy of the particle in the state $|\psi(0)\rangle$ is measured, of finding a value smaller than $\dfrac{3\pi^2\hbar^2}{ma^2}$?

(b) What is the mean value of the energy in the state $|\psi(0)\rangle$?

(c) Calculate the state $|\psi(t)\rangle$ at a later time t. How does it compare to the results calculated in the first two parts of this problem?

(d) When the energy is measured, the result $\dfrac{8\pi^2\hbar^2}{ma^2}$ is found. After this measurement, what is the state of the system and what happens if the energy is measured again?