DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH5100 Quantum Mechanics I **Problem Set 4** **2 Oct 2012**
due: 9 Oct 2012

---

1. A very useful variant of the Baker-Campbell-Hausdorff formula is the following operator identity. Let A and B be two operators, and λ a scalar. Then

$$e^{\lambda A}\, B\, e^{-\lambda A} = B + \lambda\,[A\,,\,B] + \frac{\lambda^2}{2!}\,\big[A\,,\,[A\,,\,B]\big] + \frac{\lambda^3}{3!}\,\Big[A\,,\,\big[A\,,\,[A\,,\,B]\big]\Big] + \cdots$$

   (a) Verify this identity up to order λ^3 by explicitly working out the LHS up to this order in λ.

   (b) Using the formula, show that

$$e^{\lambda A}\, e^{B}\, e^{-\lambda A} = \exp\left\{B + \lambda\,[A\,,\,B] + \frac{\lambda^2}{2!}\,\big[A\,,\,[A\,,\,B]\big] + \frac{\lambda^3}{3!}\,\Big[A\,,\,\big[A\,,\,[A\,,\,B]\big]\Big] + \cdots\right\}.$$

2. Consider a one dimensional harmonic oscillator where $|\phi_n\rangle$ are the energy eigenstates. The operator $U(k)$ is defined by:

$$\widehat{U}(k) = e^{ik\widehat{X}}$$

   where k is real.

   (a) Is $\widehat{U}(k)$ unitary?

   (b) Show that for all n the matrix elements of $\widehat{U}$ satisfy:

$$\sum_{n'} |\langle\phi_n|U(k)|\phi_{n'}\rangle|^2 = 1$$

   (c) Express $U(k)$ in terms of $\widehat{a}$ and $\widehat{a}^\dagger$. Using the following relation:

$$e^A e^B = e^{A+B} e^{[A,B]/2}$$

   where A and B are operators obtain $\widehat{U}(k)$ in the form of a product of exponential operators.

   (d) Establish the relations:

$$e^{\lambda\widehat{a}}|\phi_0\rangle = |\phi_0\rangle$$
$$\langle\phi_n|e^{\lambda\widehat{a}^\dagger}|\phi_0\rangle = \frac{\lambda^n}{\sqrt{n!}}$$

   where λ is an arbitrary complex parameter.

(e) Find: $\langle \phi_0 | \widehat{U}(k) | \phi_n \rangle$

3. The evolution operator $\widehat{U}(t,0)$ of a one dimensional harmonic oscillator is written as:

$$\widehat{U}(t,0) = e^{-i\widehat{H}t/\hbar}$$

where $\widehat{H} = \hbar\omega \left(\widehat{a}^{\dagger}\widehat{a} + \frac{1}{2} \right)$.

   (a) Consider the operators:

$$\tilde{a}(t) = U^{\dagger}(t,0) a U(t,0)$$

$$\tilde{a}^{\dagger}(t) = U^{\dagger}(t,0) a^{\dagger} U(t,0)$$

   By calculating their action on the eigenkets $|\phi_n\rangle$ of $\widehat{H}$, find the expression for $\tilde{a}(t)$ and $\tilde{a}^{\dagger}(t)$ in terms of a and $a^{\dagger}$.

   (b) Calculate the operators $\tilde{X}(t)$ and $\tilde{P}(t)$ obtained from X and P by the unitary transformation:

$$\tilde{X}(t) = U^{\dagger}(t,0) X U(t,0)$$

$$\tilde{P}(t) = U^{\dagger}(t,0) P U(t,0)$$

   (c) Show that $U(\frac{\pi}{2\omega}, 0)|x\rangle$ is an eigenvector of P and determine its eigenvalue. Similarly show that $U^{\dagger}(\frac{\pi}{2\omega}, 0)|p\rangle$ is an eigenvector of X.

4. **Anisotropic HO**: In a three dimensional problem, consider a particle of mass m and of potential energy:

$$V(X,Y,Z) = \frac{m\omega^2}{2} \left[\left(1 + \frac{2\lambda}{3}\right)(X^2 + Y^2) + \left(1 - \frac{4\lambda}{3}\right) Z^2 \right]$$

where ω and λ are constants which satisfy: $\omega \geq 0$ and $0 \leq \lambda < 3/4$.

   (a) What are the eigenstates of the Hamiltonian and the corresponding energies?

   (b) Discuss for $\lambda = 0$, $\lambda = 1/2$, the energy eigenvalues and the degeneracies of the ground state and the first two excited states. Is Parity a good quantum number? If so, what happens to Parity as a function of λ?