Mathematical Foundations

In these lectures, we go over formal mathematics that will allow us to "work" with the theory.

Linear Vector Spaces: (Refs: Dennery & Krzywicki, Arfken, Sakurai)

• Defn of a LVS:

Consider a set S of certain abstract objects: $|\alpha\rangle, |3\rangle, |\alpha\beta\rangle$
We must define rules to manipulate them. For instance we have a set of real numbers. Unless we define operations on them, they are just labels.

Rules of manipulation: Define rules analogous to real numbers:

(i) Addition:

$$|c\rangle = |a\rangle + |b\rangle, \qquad |a\rangle, |b\rangle, |c\rangle \in S$$

(ii) Multiplication by a complex scalar

$$|c\rangle = \alpha |b\rangle$$

Properties that elements of S should satisfy:

a) If $|a\rangle, |b\rangle \in S$, then $|c\rangle = |a\rangle + |b\rangle$ is also $\in S$

b) If $|a\rangle \in S$, α: complex number, then $\alpha |a\rangle = |c\rangle \in S$.

c) There exists a null element $|0\rangle$ such that $|a\rangle + |0\rangle = |a\rangle$

d) For any $|a\rangle \in S$, there exists an element $|a'\rangle$ such that $|a\rangle + |a'\rangle = |0\rangle$

The following rules make addition and multiplication well defined.

a) Commutative law: $|a\rangle + |b\rangle = |b\rangle + |a\rangle$

b) Associative law:

$$(|a\rangle + |b\rangle) + |c\rangle = |a\rangle + (|b\rangle + |c\rangle)$$

c) Identity (Multiplication)

$$1 \cdot |a\rangle = |a\rangle$$

d) $\alpha \cdot (\beta \cdot |a\rangle) = (\alpha \cdot \beta) \cdot |a\rangle$ (associative law of multiplication)

e) Distributive law: (w.r.to addition of complex numbers)

$$(\alpha + \beta)|a\rangle = \alpha|a\rangle + \beta|a\rangle$$

Distributive law w.r.to addition of $|\,\rangle$:

$$\alpha(|a\rangle + |b\rangle) = \alpha|a\rangle + \alpha|b\rangle$$

If the elements of set S has all the 2 sets of properties, then the set is called a LVS, and its elements $|\,\rangle$: are called Vectors

Check: if S is a set of all complex numbers, then we can show that it forms a LVS.

Scalar Product:

For $|b\rangle \in S$, $|a\rangle \in S$, there is a rule to define a complex number $\langle b|a\rangle$, then this ~~mu~~ complex number is called a Scalar product.

- Properties:

(i) $\langle b|a\rangle = \overline{\langle a|b\rangle}$ (complex conjugate)

(ii) If $|d\rangle = \alpha|a\rangle + \beta|b\rangle$

then

$$\langle c|d\rangle = \alpha\langle c|a\rangle + \beta\langle c|b\rangle$$

(iii) $\langle a|a\rangle \geq 0$ equality when $|a\rangle = 0$

$\langle c|a\rangle$: real number : $\langle a|a\rangle = \overline{\langle a|a\rangle}$

- $|a\rangle, |b\rangle$ are orthogonal if $\langle a|b\rangle = 0 = \langle b|a\rangle$

• If $|a\rangle \in S$ is orthogonal to every vector of S, then $|a\rangle = 0$

Dual Vectors ~~and the Cauchy-Schwarz inequality~~

If we start with

$$|d\rangle = \alpha|a\rangle + \beta|b\rangle$$

then:

$$\langle c|d\rangle = \alpha\langle c|a\rangle + \beta\langle c|b\rangle$$ is a linear fn of α, β

If:

$$|c\rangle = \alpha|a\rangle + \beta|b\rangle$$

Then:

$$\langle c|d\rangle = \bar{\alpha}\langle a|d\rangle + \bar{\beta}\langle a|d\rangle = \overline{\langle d|c\rangle}$$

$\Rightarrow$ $\langle c|d\rangle$ is linear w.r.t. $|d\rangle$ but antilinear w.r.t. $|c\rangle$.

Now introduce a space containing $\langle\,|$ which has a one-to-one correspondence with $|\,\rangle$. Space is called Dual space, $|\,\rangle$, $\langle\,|$ form dual vector pairs.

$\langle b|$ is the dual vector of $|b\rangle$

Define multiplication of vectors by vectors by requiring:

a) The pdt of $\langle b|$ with $|a\rangle$ is identified by the scalar pdt

$$\langle b|\cdot|a\rangle = \langle b|a\rangle$$

b) A scalar pdt of $\langle c|d\rangle$ depends linearly on $\langle c|$

$$\langle c|\cdot|d\rangle = \left[\bar{\alpha}\langle a| + \bar{\beta}\langle b|\right]\cdot|d\rangle$$
$$= \bar{\alpha}\langle a|d\rangle + \bar{\beta}\langle b|d\rangle.$$

$$\langle c|\cdot\left[\alpha|a\rangle + \beta|b\rangle\right] = \alpha\langle c|a\rangle + \beta\langle c|b\rangle$$

If $|c\rangle = \alpha|a\rangle + \beta|b\rangle$, then

$\langle c| = \bar{\alpha}\langle a| + \bar{\beta}\langle b|$

Then:

$$\langle c|d\rangle = \bar{\alpha}\langle a|d\rangle + \bar{\beta}\langle b|d\rangle$$

$\therefore$ Inner pdt is linear w.r.t. $|d\rangle$ and $\langle c|$

~~Real and Complex Vector Spaces:~~

With a scalar pdt defined on a LVS an important inequality gets satisfied: Triangle inequality:

$\cancel{\langle a|b\rangle}$ $\rho(a,b) + \rho(b,c) \geq \rho(a,c)$

ρ: distance between 2 points

Let us define distance between 2 vectors $|a\rangle, |b\rangle$ as norm of $|a\rangle - |b\rangle$.

Defn of norm: Norm or length of vector is defined as:

$$\sqrt{\langle a|a\rangle} = |\vec{a}|$$

Let $|1\rangle = |a\rangle - |b\rangle$

$|2\rangle = |b\rangle - |c\rangle$

$|3\rangle = |a\rangle - |c\rangle$

$|3\rangle = |1\rangle + |2\rangle$

$\therefore \langle 3|3\rangle = \cancel{\langle 1|1\rangle} \left[\langle 1| + \langle 2|\right] \cdot \left[|1\rangle + |2\rangle\right]$

$= \langle 1|1\rangle + \langle 2|2\rangle + 2\,\mathrm{Re}\,\langle 1|2\rangle$

$\leq \langle 1|1\rangle + \langle 2|2\rangle + 2\,|\langle 1|2\rangle|$

$\langle 1|1\rangle + \langle 2|2\rangle$

$+ \langle 1|2\rangle + \underbrace{\langle 2|1\rangle}_{\overline{\langle 1|2\rangle}}$

if $\langle 1|2\rangle = a+ib$

$\overline{\langle 1|2\rangle} = a-ib \Rightarrow 2\mathrm{Re}\,\langle 1|2\rangle$

$= \langle 1|2\rangle + \overline{\langle 1|2\rangle}$

Using Cauchy Schwarz inequality:

$$\langle 3|3\rangle \leq \langle 1|1\rangle + \langle 2|2\rangle + 2\sqrt{\langle 1|1\rangle\langle 2|2\rangle} = \left(\sqrt{\langle 1|1\rangle} + \sqrt{\langle 2|2\rangle}\right)^2$$

$$\Rightarrow \sqrt{\langle 3|3\rangle} \leq \sqrt{\langle 1|1\rangle} + \sqrt{\langle 2|2\rangle}$$

Inner pdts need not be defined in a LVS. If defined, the LVS is a normed space and triangle inequality helps define distances.

<u>Cauchy-Schwarz inequality</u>:

$$\sqrt{\langle a|a\rangle} \cdot \sqrt{\langle b|b\rangle} \geq |\langle b|a\rangle|$$