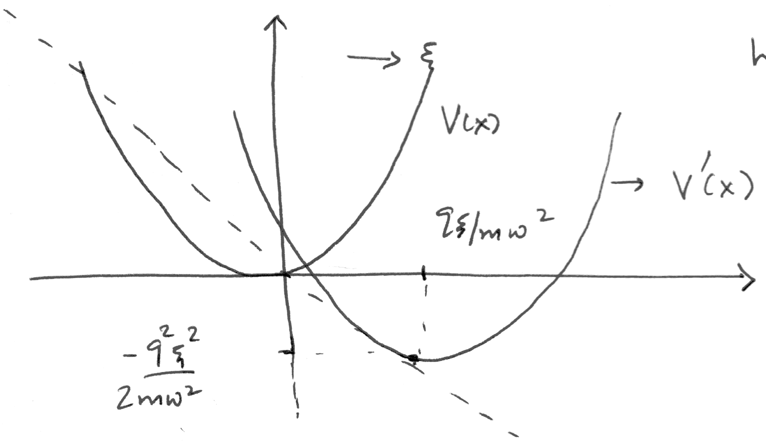
Application:
A charged Harmonic Oscillator in a Uniform Electric Field

1D Harmonic Oscillator has a particle of mass 'm' having a P.E:

$$V(x) = \frac{1}{2} m\omega^2 \hat{X}^2.$$

Let us assume that this particle has a charge q and it is placed in a uniform electric field (ξ) parallel to Ox



$W(\xi) = -q\xi x$

$\rightarrow V'(x) = V(x) + W(\xi)$

$$\hat{H}' = \frac{\hat{P}^2}{2m} + \frac{1}{2} m\omega^2 \hat{X}^2 - q\xi\hat{x} \qquad [\Rightarrow \quad H'|\varphi'\rangle = E'|\varphi'\rangle]$$

Choose $|x\rangle$ representation:

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2} m\omega^2 x^2 - q\xi x\right]\varphi'(x) = E'\varphi'(x)$$

$$\frac{1}{2} m\omega^2 x^2 - q\xi x = \frac{1}{2} m\omega^2 \left[x^2 - \frac{2q\xi}{m\omega^2} x\right]$$

$$= \frac{1}{2} m\omega^2 \left[\left(x - \frac{q\xi}{m\omega^2}\right)^2 - \frac{q^2\xi^2}{m^2\omega^4}\right] = \frac{1}{2} m\omega^2 \left(\left(x - \frac{q\xi}{m\omega^2}\right)^2\right) - \frac{q^2\xi^2}{2m\omega^2}$$

$$\Rightarrow \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2\left(x - \frac{q\xi}{m\omega^2}\right)^2 - \frac{q^2\xi^2}{2m\omega^2}\right]\varphi'(x) = E'\varphi'(x).$$

Define $u = x - \dfrac{q\xi}{m\omega^2}$

$$\Rightarrow \frac{d^2}{dx^2} = \frac{d^2}{du^2}.$$

$$\therefore \left[-\frac{\hbar^2}{2m}\frac{d^2}{du^2} + \frac{1}{2}m\omega^2 u^2\right]\varphi'(u) = \underbrace{\left(E' + \frac{q^2\xi^2}{2m\omega^2}\right)}_{E''}\varphi'(u)$$

We already know:

$$E'' = (n + 1/2)\hbar\omega$$

$$\Rightarrow E' + \frac{q^2\xi^2}{2m\omega^2} = (n+1/2)\hbar\omega$$

$$\therefore E' = E_n - \frac{q^2\xi^2}{2m\omega^2}.$$

and $\varphi_n'(x) = \varphi_n\left(x - \dfrac{q\xi}{m\omega^2}\right)$ $\Rightarrow$ Replace $x \to x - \dfrac{q\xi}{m\omega^2}$.

Grnd state : $E_0' = E_0 - \dfrac{q^2\xi^2}{2m\omega^2}$

<u>Discuss</u>: Think of $\bar{e}$ that are elastically bound to atoms. Then this system behaves like an Oscillator. In the absence of a field the electric dipole moment $D = q\hat{X}$ has a mean value $\langle D \rangle = 0$. If an ξ field is applied the the dipole orients itself in the field direction and

$$\langle D \rangle' = q \langle \varphi_n | \hat{X} | \varphi_n \rangle' = q \int_{-\infty}^{\infty} dx \; x \, |\varphi_n'(x)|^2 = \frac{q^2 \mathcal{E}}{m\omega^2}$$

$$\therefore \text{ the susceptibility } \chi = \frac{\langle D \rangle'}{\mathcal{E}} = \frac{q^2}{m\omega^2} > 0.$$

Energetically favorable for the dipole to orient along the field
$\Rightarrow$ G. state E is lowered and indeed it is!

Coherent Quasi - classical states

In the limit of large n, we expect the quantum mechanical description to go over to the classical result. The question we will address is: Is it possible to construct quantum mechanical states, which results in predictions which are almost identical to the classical ones? The answer is, as we shall see, such a state exists and it turns out to be a coherent linear superposition of all states $|\varphi_n\rangle$. They are referred to as "Coherent states or quasi-classical states".

This problem is very important because many physical systems can be likened to Harmonic Oscillators to a first approximation. This question is analogous to asking in what limits do we have classical behavior to be valid and what are the limits where this breaks down.

Position, momentum and energy are given by operators that do not commute. Therefore we can at best look for a limit where the expectation values of $\hat{X}$, $\hat{P}$ and $\hat{H}$ are close to their classical limit. (in other words the variance in each is negligible).

Quasi - classical states :

classical Eqns of motion of a 1d HO of mass m and angular frequency ω:

$$\frac{d}{dt} x(t) = \frac{p(t)}{m}$$

$$\frac{d}{dt} p(t) = -m\omega^2 x(t)$$

Let us first introduce dimensionless quantities:

$$\tilde{x}(t) = \beta\, x(t)$$

$$\tilde{p}(t) = \frac{1}{\hbar \beta} p(t)$$

where $\beta = \sqrt{\dfrac{m\omega}{\hbar}}$

$\therefore$ We can re-write eqns of motion

$$\frac{d}{dt} \tilde{x}(t) = \omega\, \tilde{p}(t)$$

$$\frac{d}{dt} \tilde{p}(t) = -\omega\, \tilde{x}(t)$$

Usually the classical state of the system is known if we know $\tilde{x}(t)$ and $\tilde{p}(t)$. Let us combine these two into one complex number:

$$\alpha(t) = \frac{1}{\sqrt{2}} \left[\tilde{x}(t) + i\, \tilde{p}(t) \right]$$

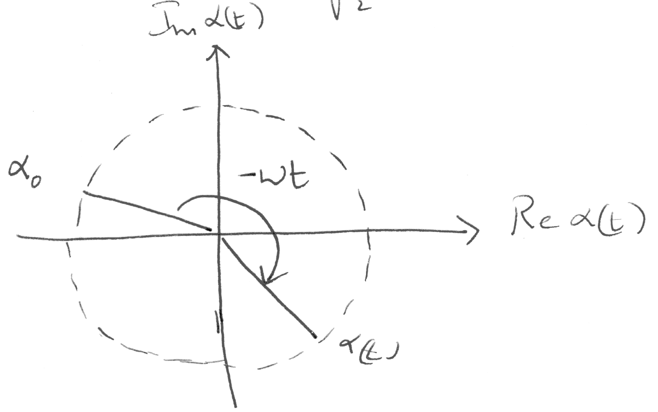
$$\frac{d\alpha(t)}{dt} = \frac{1}{\sqrt{2}} \left[\frac{d\tilde{x}(t)}{dt} + i\, \frac{d\tilde{p}(t)}{dt} \right]$$

$$= \frac{1}{\sqrt{2}} \left[\omega\, \tilde{p}(t) - i\, \omega\, \tilde{x}(t) \right] = -i\omega \left[\frac{\tilde{x}(t) + i\, \tilde{p}(t)}{\sqrt{2}} \right]$$

$$= -i\omega\, \alpha(t)$$

$\Rightarrow \alpha(t) = \alpha_0 e^{-i\omega t}$

where $\alpha_0 = \frac{1}{\sqrt{2}}\left[\tilde{x}(0) + i\,\tilde{p}(0)\right]$



Amplitude given by $|\alpha(t)|$ and phase by phase of $\alpha(t)$.

We can invert the relation between $\tilde{x}(t)$, $\tilde{p}(t)$ and $\alpha(t)$ and obtain $\tilde{x}(t)$ and $\tilde{p}(t)$

$$\tilde{x}(t) = \frac{1}{\sqrt{2}}\left[\alpha_0 e^{-i\omega t} + \alpha_0^* e^{i\omega t}\right]$$

$$\tilde{p}(t) = -\frac{i}{\sqrt{2}}\left[\alpha_0 e^{-i\omega t} - \alpha_0^* e^{i\omega t}\right]$$

The classical energy of the system:

$$\mathcal{H} = \frac{1}{2m}\left[p(0)\right]^2 + \frac{1}{2} m\omega^2 \left[x(0)\right]^2$$

$$= \frac{\hbar\omega}{2}\left[(\tilde{x}(0))^2 + (\tilde{p}(0))^2\right] = \hbar\omega|\alpha_0|^2$$

For a macroscopic oscillator the energy is much greater than $\hbar\omega$

$\Rightarrow |\alpha_0| \gg 1$.

Conditions defining quasi-classical states

We want to find a quantum mechanical state for which at every instant the mean values $\langle X \rangle$ $\langle P \rangle$ and $\langle H \rangle$ are practically equal to the values x, p and $\mathcal{H}$ which correspond to a given classical motion.

$$\hat{X}' = \beta \hat{X} = \frac{1}{\sqrt{2}} (\hat{a} + \hat{a}^+)$$

$$\hat{P}' = \frac{1}{\beta \hbar} \hat{P} = -\frac{i}{\sqrt{2}} (a - a^+)$$

$$\Rightarrow \hat{H} = \hbar \omega \left(a^+ a + 1/2 \right)$$

If $|\Psi(t)\rangle$ is an arbitrary state, the time evolution of the matrix element

$$\langle a(t) \rangle = \langle \Psi(t) | a | \Psi(t) \rangle \quad \text{is given by:}$$

$$\frac{d}{dt} \langle a(t) \rangle = \left[\frac{d}{dt} \langle \Psi(t) | \right] a | \Psi(t) \rangle + \langle \Psi(t) | a \left(\frac{d}{dt} | \Psi(t) \rangle \right)$$

Multiply by $i\hbar$:

$$i\hbar \frac{d}{dt} \langle a \rangle(t) = \left[i\hbar \frac{d}{dt} \langle \Psi(t) | \right] a | \Psi(t) \rangle + \langle \Psi(t) | a \, i\hbar \frac{d}{dt} | \Psi(t) \rangle$$

Use: $i\hbar \frac{d}{dt} | \Psi(t) \rangle = H | \Psi(t) \rangle$

$$- i\hbar \frac{d}{dt} \langle \Psi(t) | = \langle \Psi(t) | H$$

$$= - \langle \Psi(t) | H a | \Psi(t) \rangle + \langle \Psi(t) | a H | \Psi(t) \rangle$$

$$= \langle [a, H] \rangle (t)$$

Now let us calculate $[a, H]$:

$$[a, \hbar\omega(a^+a + 1/2)] = \hbar\omega[a, a^+a] = \hbar\omega[a, a^+]a = a\hbar\omega$$

$$\therefore \; i\hbar \frac{\partial}{\partial t} \langle a \rangle (t) = a \hbar\omega$$

$$\Rightarrow \langle a \rangle (t) = \langle a \rangle (0) \, e^{-i\omega t}$$

$$\therefore \langle a^+ \rangle (t) = \langle a^+ \rangle^* (0) \, e^{i\omega t}$$

$$= \langle a \rangle^* (0) \, e^{i\omega t}$$

analogous to the classical eqn.

Now:

$$\langle x \rangle (t) = \frac{1}{\sqrt{2}} \left[\langle a \rangle (t) + \langle a^+ \rangle (t) \right]$$

and

$$\langle p \rangle (t) = -\frac{i}{\sqrt{2}} \left[\langle a \rangle (t) - \langle a^+ \rangle (t) \right]$$

$$\Rightarrow \langle \hat{x} \rangle (t) = \frac{1}{\sqrt{2}} \left[\langle a \rangle (0) \, e^{-i\omega t} + \langle a \rangle^* (0) \, e^{i\omega t} \right]$$

$$\langle \hat{p} \rangle (t) = \frac{-i}{\sqrt{2}} \left[\langle a \rangle (0) \, e^{-i\omega t} - \langle a \rangle^* (0) \, e^{i\omega t} \right]$$

comparing with the classical vectors $\bar{x}(t)$ and $\bar{p}(t)$ we see that if:

$\langle \hat{X} \rangle (t) = \bar{x}(t)$

and $\langle \hat{p} \rangle (t) = \bar{p}(t)$

then: $\langle a \rangle (0) = \alpha_0$

$$\Rightarrow \boxed{\langle \Psi(0) | \hat{a} | \Psi(0) \rangle = \alpha_0} \text{ ——— I}$$

Next:

$$\langle \hat{H} \rangle = \hbar\omega \langle a^{+} a \rangle (0) + \frac{\hbar\omega}{2}$$

if this has to be equal to the classical energy ~~then~~:

$\mathcal{H} = \hbar\omega |\alpha_0|^2$

Then (i) $\frac{\hbar\omega}{2}$ should be negligible (makes sense in the limit of a macroscopic oscillator)

and $|\alpha_0|^2 = \langle a^{+} a \rangle (0)$.

$$\Rightarrow \boxed{\langle \Psi(0) | a^{+} a | \Psi(0) \rangle = |\alpha_0|^2} \text{ ——— II}$$

Using the 2 conditions I, II, we will determine the state vector $|\Psi(0)\rangle$.

Define an operator $\hat{b}(\alpha_0)$:

$$\hat{b}(\alpha_0) = \hat{a} - \alpha_0$$

then:

$$b^{+}(\alpha_0)\, b(\alpha_0) = (a^{+} - \alpha_0^{*})(\hat{a} - \alpha_0)$$

$$= \hat{a}^{+}\hat{a} - \alpha_0 \hat{a}^{+} - \alpha_0^{*} \hat{a} + |\alpha_0|^2$$

Now:

$$\langle \Psi(0) | b^{+}(\alpha_0)\, b(\alpha_0) | \Psi(0) \rangle = \langle \Psi(0) | \hat{a}^{+} \hat{a} | \Psi(0) \rangle - \alpha_0 \langle \Psi(0) | \hat{a}^{+} | \Psi(0) \rangle - \alpha_0^{*} \langle \Psi(0) | \hat{a} | \Psi(0) \rangle + \alpha_0^{*} \alpha_0$$

$$= \cancel{|\alpha_0|^2} - \cancel{\alpha_0}\,\alpha_0^{*} - \alpha_0^{*}\cancel{\alpha_0} + \alpha_0^{*}\cancel{\alpha_0} = 0 .$$

$$\therefore \quad b(\alpha_0) | \Psi(0) \rangle = 0 \qquad \text{(A null vector)}$$

$$\Rightarrow (\hat{a} - \alpha_0) | \Psi(0) \rangle = 0$$

or: $\quad \hat{a} | \Psi(0) \rangle = \alpha | \Psi(0) \rangle$

Therefore: the quasi-classical state $|\Psi(0)\rangle$ associated with classical motion, parameterized by α_0 is such that $|\Psi(0)\rangle$ is an eigenvector of $\hat{a}$ with eigenvalue α_0.

Let $|\alpha\rangle \equiv |\Psi(0)\rangle$

Then: $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. [It turns out that this solution is unique to within a constant factor].

<u>Properties of the state $|\alpha\rangle$:</u>

Let us expand $|\alpha\rangle$ in the basis $|\varphi_n\rangle$.

$$|\alpha\rangle = \sum_n{}' c_n(\alpha) \, |\varphi_n\rangle$$

Then: $\quad a|\alpha\rangle = \sum_n{}' c_n(\alpha)\, a|\varphi_n\rangle = \sum_n c_n(\alpha) \sqrt{n}\, |\varphi_{n-1}\rangle$

But $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$

$\Rightarrow \quad \alpha|\alpha\rangle = \sum_n' C_n(\alpha)\sqrt{n}\,|\varphi_{n-1}\rangle$

~~or: $\sum_n' C_n(\alpha)\,\alpha\,|\varphi_n\rangle = \sum_n' C_n(\alpha)\sqrt{n}\,|\varphi_{n-1}\rangle$.~~

$|\varphi_{n-1}\rangle =$

$C_n(\alpha) = \langle\varphi_n|\alpha\rangle$

$\langle\varphi_n|a|\alpha\rangle = \alpha\langle\varphi_n|\alpha\rangle$ (form inner pdt w.r.t. $\langle\varphi_n|$.)

$a|\varphi_n\rangle = \sqrt{n}\,|\varphi_{n-1}\rangle$

$\Rightarrow \langle\varphi_n|a^+ = \langle\varphi_{n-1}|\sqrt{n}$.

$\therefore \quad a^+|\varphi_n\rangle = \sqrt{n+1}\,|\varphi_{n+1}\rangle$

and $\langle\varphi_n|a = \langle\varphi_{n+1}|\sqrt{n+1}$

$\therefore \langle\varphi_n|a|\alpha\rangle = \langle\varphi_{n+1}|\sqrt{n+1}\,|\alpha\rangle = \alpha\langle\varphi_n|\alpha\rangle$

$\Rightarrow \sqrt{n+1}\,C_{n+1}(\alpha) = \alpha\,C_n(\alpha)$

$$\Rightarrow \boxed{C_{n+1}(\alpha) = \frac{\alpha}{\sqrt{n+1}}\,C_n(\alpha)}$$

$$\Rightarrow C_n(\alpha) = \frac{\alpha^n}{\sqrt{n!}}\,C_0(\alpha)$$

$\Rightarrow$ if $C_0(\alpha)$ is fixed then all $C_n(\alpha)$ are also fixed.

Let us choose $c_0(\alpha)$ to be real and positive, and normalize the ket $|\alpha\rangle$ ⇒

$$\langle\alpha|\alpha\rangle = 1 \Rightarrow \sum_n' |c_n(\alpha)|^2 = 1 .$$

$$\text{or: } |c_0(\alpha)|^2 \underbrace{\sum_n' \frac{|\alpha|^{2n}}{n!}}_{e^{|\alpha|^2}} = 1$$

$$\therefore |c_0(\alpha)|^2 = e^{-|\alpha|^2}$$

$$\Rightarrow c_0(\alpha) = e^{-|\alpha|^2/2}$$

$$\therefore \boxed{|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n' \frac{\alpha^n}{\sqrt{n!}} |\varphi_n\rangle}$$

Check:
$$a|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n\geq 1}' \frac{\alpha^n}{\sqrt{n!}} a|\varphi_n\rangle$$

$$= e^{-|\alpha|^2/2} \sum_{n\geq 1}' \frac{\alpha^n}{\sqrt{n!}} \sqrt{n}\,|\varphi_{n-1}\rangle$$

$$= \alpha \left[e^{-|\alpha|^2/2} \underbrace{\sum_n' \frac{\alpha^{n-1}}{\sqrt{(n-1)!}} |\varphi_{n-1}\rangle}_{\text{Define } n-1 = n'} \right]$$

$$= \alpha \left[e^{-|\alpha|^2/2} \sum_{n'}' \frac{\alpha^{n'}}{\sqrt{(n'!)}} |\varphi_{n'}\rangle \right]$$

$$= \alpha \left[|\alpha\rangle \right]$$

Value of Energy in the $|\alpha\rangle$ state :

Consider an Oscillator in the state $|\alpha\rangle$. Then:
a measurement of energy can yield a value $E_n = (n+1/2)\hbar\omega$ with a probability:

$$P_n(\alpha) = |c_n(\alpha)|^2 = \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2}$$

Now: the mean value of Energy:

$$\langle H \rangle = \sum_n{}' P_n(\alpha)\left[n + 1/2\right]\hbar\omega$$

Or:

$$\langle\alpha| H |\alpha\rangle = \langle\alpha|(a^{+}a + 1/2)\hbar\omega|\alpha\rangle$$

$$= \langle\alpha| a^{+}a|\alpha\rangle + 1/2\,\hbar\omega\,\langle\alpha|\alpha\rangle$$

$$= \hbar\omega\left[\,|\alpha|^2 + 1/2\right]$$

(Since $\langle\alpha|a^{+}a|\alpha\rangle = |\alpha|^2$ }. and $\langle\alpha|\alpha\rangle = \underline{1}$)

$\therefore$ if $|\alpha|^2 \gg 1$, then $\langle H\rangle$ is very much like the classical energy.

We can calculate $(\Delta H)^2$:

$$\langle H^2\rangle = (\hbar\omega)^2\langle\alpha|\,(a^{+}a + 1/2)^2\,|\alpha\rangle$$

$$= (\hbar\omega)^2\left[\,|\alpha|^2 + 2|\alpha|^2 + 1/4\right].$$

$\Rightarrow (\Delta H) = \hbar\omega|\alpha|$.

finally: $\dfrac{(\Delta H)}{\langle H\rangle} \approx \dfrac{1}{|\alpha|} \ll 1$. [interpret: $\dfrac{(\Delta H)}{\langle H\rangle} = \dfrac{-\langle H\rangle + \hbar}{\langle H\rangle} \approx \dfrac{1}{|\alpha|} \ll$

$\frac{(\Delta H)}{\langle H\rangle}$ the relative uncertainty in the state $|\alpha\rangle$ in limit $|\alpha| \gg 1$ is very small.

Therefore a quasi-classical state is obtained if we superpose a large number of states $|\varphi_n\rangle$.

Calculation of $\langle\hat{X}\rangle$, $\langle\hat{P}\rangle$ and (ΔX), (ΔP) in the state $|\alpha\rangle$:

$$\langle X\rangle_\alpha = \langle\alpha|\hat{X}|\alpha\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,\langle\alpha|a^\dagger + a|\alpha\rangle$$

Using $a|\alpha\rangle = \alpha|\alpha\rangle$
$\langle\alpha|a^\dagger = \langle\alpha|\alpha^*$

Then: $\langle\alpha|\hat{X}|\alpha\rangle = \sqrt{\dfrac{\hbar}{2m\omega}}\,(\alpha+\alpha^*) = \sqrt{\dfrac{2\hbar}{m\omega}}\,\mathrm{Re}(\alpha)$

$$\langle P\rangle_\alpha = \sqrt{2m\hbar\omega}\,\mathrm{Im}(\alpha)$$

$$\langle X^2\rangle = \frac{\hbar}{2m\alpha}\left[(\alpha+\alpha^*)^2 + 1\right]$$

$$\langle P^2\rangle = \frac{\hbar m\omega}{2}\left[1 - (\alpha-\alpha^*)^2\right]$$

$\therefore\ \Delta X_\alpha = \sqrt{\dfrac{\hbar}{2m\omega}}$, $\Delta P_\alpha = \sqrt{\dfrac{m\hbar\omega}{2}}$ and $\underbrace{\Delta X_\alpha\,\Delta P_\alpha = \dfrac{\hbar}{2}}_{\text{Min uncertainty}}$

Time evolution of a quasi-classical state

Consider a Harmonic Oscillator which at $t=0$ is in the state $|\alpha\rangle$:

$$|\Psi(0)\rangle = |\alpha_0\rangle .$$

$$|\Psi(t)\rangle = e^{-i\hat{H}t/\hbar} |\Psi(0)\rangle \quad (\text{since } \hat{H} \text{ is time independent})$$

$$= e^{-i\hat{H}t/\hbar} |\alpha\rangle$$

$$= e^{-i\hat{H}t/\hbar} \sum_n{}' c_n(\alpha) |\varphi_n\rangle$$

$$= \sum_n{}' c_n(\alpha)\, e^{-i\hat{H}t/\hbar} |\varphi_n\rangle$$

$$= \sum_n{}' c_n(\alpha)\, e^{-iE_n t/\hbar} |\varphi_n\rangle$$

Use: $E_n = (n+1/2)\hbar\omega$

Then $$|\Psi(t)\rangle = \sum_n{}' c_n(\alpha)\, e^{-int\omega}\, e^{-i\omega t/2} |\varphi_n\rangle$$

$$= e^{-i\omega t/2} \sum_n{}' c_n(\alpha)\, e^{-i\omega t n} |\varphi_n\rangle$$

Use the expression we had for $c_n(\alpha)$:

$$|\Psi(t)\rangle = e^{-i\omega t/2}\, e^{-|\alpha_0|^2/2} \sum_n{}' \frac{\alpha_0^n\, e^{-in\omega t}}{\sqrt{n!}} |\varphi_n\rangle .$$

We know how α_0 time evolves (classically)

$$\alpha(t) = \alpha_0 \, e^{-i\omega t}$$

Therefore in our definition of $|\alpha\rangle$

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n{}' \frac{\alpha^n}{\sqrt{n!}} |\varphi_n\rangle$$

if we replace $\alpha \longrightarrow \alpha_0 e^{-i\omega t}$

then $$|\alpha(t)\rangle = e^{-|\alpha_0|^2/2} \sum_n{}' \frac{\alpha_0^n \, e^{-in\omega t}}{\sqrt{n!}} |\varphi_n\rangle$$

which differs from $|\Psi(t)\rangle$ by a phase $e^{-i\omega t/2}$ $\longrightarrow$ But a global phase factor does not matter!

$\therefore$ In order to time evolve, we need to replace $\alpha \longrightarrow \alpha_0 e^{-i\omega t}$

$$\therefore \quad \left.\begin{aligned} \langle X\rangle(t) &= \sqrt{\frac{2\hbar}{m\omega}} \, \mathrm{Re}\left[\alpha_0 \, e^{-i\omega t}\right] \\ \langle P\rangle(t) &= \sqrt{2m\hbar\omega} \, \mathrm{Im}\left[\alpha_0 \, e^{-i\omega t}\right] \end{aligned}\right\} \text{Analogous to classical result.}$$

$$\langle H\rangle = \hbar\omega\left[|\alpha_0|^2 + 1/2\right] \longrightarrow \text{Time independent.}$$

$$(\Delta X)_\alpha = \sqrt{\frac{\hbar}{2m\omega}} \quad \text{and} \quad (\Delta P)_\alpha = \sqrt{\frac{m\hbar\omega}{2}}$$

$$\Rightarrow (\Delta X)_\alpha (\Delta P)_\alpha = \frac{\hbar}{2} \quad \text{(independent of time!)}$$