[NOTE: There could be sign errors and factors of m that could be missing!]

Charged Particle in a Magnetic field (Ref: Cohen Tannoudji Vol 1)

So far we have solved examples where a particle has been confined to a scalar potential. Here we will discuss the case of a vector potential – a charged particle in a Magnetic field. We begin by reviewing the problem classically.

Classical problem:

a. When a particle of position $\vec{r}$, charge q is subjected to a magnetic field $\vec{B}(\vec{r})$, the force $\vec{f}$ is given by:

$$\vec{f} = q\,\vec{v} \times \vec{B}(\vec{r})$$

$$\vec{v} = \frac{d\vec{r}}{dt}.$$

To describe the magnetic field $\vec{B}(\vec{r})$, one uses a vector potential $\vec{A}(\vec{r})$

so that $\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}$

$$\Rightarrow B_i = \partial_j A_k\, \epsilon_{ijk}$$

In order to get a uniform field B along one direction, one can choose $\vec{A}$

Ex: $A_x = 0$ $A_y = xB$, $A_z = 0$

$$B_z = \epsilon_{12}\, \partial_1 A_2 + \epsilon_{21}\, \partial_2 A_1$$

$$= \frac{\partial}{\partial x} A_{y} - \frac{\partial}{\partial y} A_y = B$$

The potential is not unique: We could have chosen:

$$A_x = +yB/2 \qquad A_y = -xB/2 \qquad A_z = 0$$

Then: $B_z = +\frac{\partial}{\partial y} A_x + \frac{\partial}{\partial x} A_y$

$$= +B/2 + B/2 = B.$$

Any other component: (for both choices).

$$B_x = \frac{\partial}{\partial y} A_z - \frac{\partial}{\partial z} A_y = 0$$

$$B_y = \frac{\partial}{\partial x} A_z - \frac{\partial}{\partial z} A_y = 0$$

In addition $\vec{A} \to \vec{A} + \vec{\nabla}\varphi$

since $\vec{\nabla} \times \vec{\nabla}\varphi = 0$ $\quad\varphi$: scalar.

The classical Lagrangian for a particle of charge q in a magnetic field is given by:

$$\mathcal{L}(\vec{r}, \vec{v}) = \frac{1}{2}\mu v^2 + q\,\vec{v}\cdot\vec{A}(\vec{r})$$

We can see from the Lagrange's Eqn. of motion that we do get the Lorentz Force! (Check this).

The Conjugate momenta:

$$\vec{p} = \nabla_{\vec{v}}\,\mathcal{L}(\vec{r}, \vec{v}) = \mu\vec{v} + q\vec{A}(\vec{r})$$

∴ the classical Hamiltonian:

$$\mathcal{H}(\vec{r},\vec{p}) = \frac{1}{2\mu}\left[\vec{p} - q\vec{A}(\vec{r})\right]^2$$

Reminders:

Maxwell's Eqn.

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$$
$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{B} = \mu_0\vec{j} + \epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}$$

Since $\vec{\nabla}\cdot\vec{B} = 0$, $\vec{B} = \vec{\nabla}\times\vec{A}$.

$$\Rightarrow \vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\vec{\nabla}\times\vec{A}$$

or: $\vec{\nabla}\times\left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right) = 0$.

or: $\vec{E} + \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}U(\vec{r},t)$ $U(\vec{r},t)$: Scalar fn.

$(\vec{A}, U)$ constitute a gauge for describing EM field.

$$\vec{B}(\vec{r},t) = \vec{\nabla}\times\vec{A}(\vec{r},t)$$
$$\vec{E}(\vec{r},t) = -\vec{\nabla}U(\vec{r},t) - \frac{\partial}{\partial t}\vec{A}(\vec{r},t)$$

if $\vec{A}'(\vec{r},t) = \vec{A}(\vec{r},t) + \vec{\nabla}\chi(\vec{r},t)$ and $U'(\vec{r},t) = U(\vec{r},t) - \frac{\partial}{\partial t}\chi(\vec{r},t)$

A' U' produce the same $\vec{E}$ and $\vec{B}$ fields.

Quantum Mechanical Hamiltonian

$$H = \frac{(\vec{P} - e\vec{A})^2}{2m}$$

If $\vec{B}$ is a field along z axis and $\vec{B}$ is uniform, then

$$\vec{A} = (A_x, A_y, 0)$$

$$\therefore \hat{H} = \frac{(P_x - eA_x)^2}{2m} + \frac{(P_y - eA_y)^2}{2m} + \frac{P_z^2}{2m}.$$

Define $\left.\begin{aligned} \mathcal{P}_x &= P_x - eA_x \\ \mathcal{P}_y &= P_y - eA_y \end{aligned}\right\}$ different from the mechanical Momenta.

$[X, \mathcal{P}_x] = i\hbar$ (since $[X, A_x] = 0$

$[Y, \mathcal{P}_y] = i\hbar$ and $[Y, A_y] = 0$)

But $[\mathcal{P}_x, \mathcal{P}_y] \neq 0$

$$\begin{aligned} [\mathcal{P}_x, \mathcal{P}_y] &= [P_x - eA_x, P_y - eA_y] \\ &= -e[P_x, A_y] - e[A_x, P_y] \\ &= -e[P_x, A_y] + e[P_y, A_x]. \end{aligned}$$

Consider: $[X, P] = i\hbar$

$$[x, p^2] = p[x, p] + [x, p]\,p$$
$$= 2i\hbar\, p$$

$$[x, p^3] = p[x, p^2] + [x, p]\,p^2$$
$$= 2i\hbar\, p^2 + i\hbar\, p^2$$
$$= 3i\hbar\, p^2$$

$\therefore$

$$[x, p^n] = n i\hbar\, p^{n-1}$$

$$\therefore [x, F(p)] = \sum_n [x, f_n p^n] = \sum_n{}' i\hbar\, n f_n p^{n-1}$$

$$= \cancel{\sum_n [x,} = i\hbar\, F'(p).$$

$$\text{ll}^{ly} \quad [p, G(x)] = -i\hbar\, G'(x) \longrightarrow \text{SHOW THIS!}$$

$$\Rightarrow [p_x, A_y] = -i\hbar \frac{\partial}{\partial x} A_y$$

$$[p_y, A_x] = -i\hbar \frac{\partial}{\partial y} A_x$$

$$\therefore -e[p_x, A_y] + e[p_y, A_x] = i\hbar e \underbrace{\left[\frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x\right]}_{B_z}$$

$$= i\hbar e B_z$$

$\therefore [\rho_i, \rho_j] = -i\hbar e\, \epsilon_{ijk} B_k$. (particle has a -ve charge)

Define: $\hat{Q} = \dfrac{\rho_y}{\sqrt{\hbar e B_z}} \qquad \hat{S} = \dfrac{\rho_x}{\sqrt{\hbar e B_z}}$

$$\Rightarrow [\hat{Q}, \hat{S}] = i$$

Reminds me of the commutation relation between x', p' for the SHO.

Define:

$$\hat{a} = \frac{\hat{Q} + i\hat{S}}{\sqrt{2}} \qquad \hat{a}^{+} = \frac{\hat{Q} - i\hat{S}}{\sqrt{2}}$$

Then: $[\hat{a}, \hat{a}^{+}] = 1$.

$$\therefore \; H = (n + 1/2)\hbar\omega_c + \frac{P_z^2}{2m}$$

where $\boxed{\omega_c = |e|\hbar B_z}$.

For a given P_z, the xy plane is quantized and the levels are equally spaced. (Landau Levels)

Choose a particular Gauge:

$$H = \frac{(P - eA)^2}{2m} = \frac{P^2}{2m} + \frac{e^2A^2}{2m} + -e\frac{\vec{P}\cdot\vec{A}}{2m} - e\frac{\vec{A}\cdot\vec{P}}{2m}.$$

Choose

$$\vec{A} = -\frac{1}{2}(\vec{r} \times \vec{B}) = \left(-y\frac{B}{2}, \; x\frac{B}{2}, 0\right).$$

Then: $\vec{p}\cdot\vec{A} = 0$

In position basis:

$$H = \frac{p^2}{2m} + \frac{e^2A^2}{2m} + \frac{e}{4m}(\vec{r} \times \vec{B})\cdot\vec{p} = \frac{p^2}{2m} + \frac{e^2A^2}{2m} + \frac{e}{4m}(\vec{r} \times \vec{p}).$$

$$= \frac{P^2}{2m} + \frac{e^2 A^2}{2m} - \frac{eB}{2m} L_z$$

$$= \frac{P_x^2 + P_y^2}{2m} + \frac{e^2 B^2}{\cancel{8}m}(x^2 + y^2) - \frac{eB}{2m} L_z + \frac{P_z^2}{2m}.$$

z- direction : plane wave.

xy is an oscillator.