Pictures in Quantum Mechanics

Schrödinger, Heinsenberg and Interaction pictures

So far all the time dependence for example on position, momentum, K.E. etc came from the time evolution of the state vector, which is a solution of the TDSE. The observables which are operators acting on the state vectors are independent of time. This approach is called the S. picture.

We usually calculate expectation values (inner prods)

$$\langle \Psi(t) | \hat{O} | \Psi(t) \rangle$$

We will introduce a transformation U which is unitary such that

$$U | \Psi(t) \rangle = | \Psi \rangle \quad (\text{Time independent}).$$

Then the time dependence gets transfered to the operator O. That picture where the operators time evolve is refered to as the Heisenberg picture.

Let $|\Psi_s(t_0)\rangle$ be the ket at some initial time t_0 in the S. picture.

Now:

$$\underbrace{|\Psi_s(t, t_0)\rangle}_{\text{Schrödinger picture}} = U(t, t_0) | \Psi_s(t_0) \rangle$$

$\hat{U}(t,t_0)$: Time evolution operator.

Define : $|\Psi_H\rangle = U^+(t,t_0)|\Psi_S(t)\rangle = U^+(t,t_0)\,U(t,t_0)\,|\Psi_S(t_0)\rangle$
$= |\Psi_S(t_0)\rangle$

$\therefore \underbrace{|\Psi_H\rangle = |\Psi_S(t_0)\rangle}$: starting point.

Heisenberg picture, the wave fn is equal to its initial value at $t=t_0$ and does not time evolve.

$\therefore \langle \Psi_S(t)|\,\hat{O}_S(t)\,|\Psi_S(t)\rangle = \langle \Psi_S(t_0)|\,U^+(t,t_0)\,\hat{O}_S(t)\,U(t,t_0)\,|\Psi_S(t_0)\rangle$

$= \langle \Psi_H|\,U^+\hat{O}_S U\,|\Psi_H\rangle$

Define $\hat{O}_H(t,t_0) = U^+(t,t_0)\,\hat{O}_S\,U(t,t_0)$

The operator $\hat{O}_S(t)$ can depend on time, but in the Schrödinger picture, the operator does not time evolve.

If $\hat{O}_S(t)$ is independent of time, then

$\hat{O}_H = U^+(t,t_0)\,O_S\,U(t,t_0)$ is also time-independent.
If $\hat{H}$ is independent of time, and $[\hat{H}, O_S]=0$,

then $U(t,t_0) = e^{-iH_S(t-t_0)/\hbar}$
$\therefore$ ~~$A_H(t$~~ $O_H = e^{-iH_S(t-t_0)/\hbar}\,e^{iH_S(t-t_0)/\hbar}\,O_S = O_S$.

For an arbitrary $\hat{O}_s(t)$:

$$\hat{O}_H(t) = U^\dagger(t,t_0)\, O_s(t)\, U(t,t_0)$$

$$\frac{d}{dt} O_H(t) = \frac{\partial}{\partial t} U^\dagger(t,t_0)\, O_s(t)\, U(t,t_0) + U^\dagger(t,t_0)\, \frac{d O_s(t)}{dt}\, U(t,t_0)$$

$$+ U^\dagger(t,t_0)\, O_s(t)\, \frac{\partial}{\partial t} U(t,t_0)$$

The time evolution op. satisfies the following eqn:

$$i\hbar \frac{\partial}{\partial t} U(t,t_0) = \hat{H}_s(t)\, U(t,t_0)$$

$$\therefore \quad \frac{\partial U(t,t_0)}{\partial t} = \frac{H_s(t)}{i\hbar}\, U(t,t_0)$$

$$\frac{\partial U^\dagger(t,t_0)}{\partial t} = -\cancel{\frac{H_s(t)\, U^\dagger(t,t_0)}{i\hbar}} \; -\frac{1}{i\hbar}\left[U^\dagger(t,t_0)\, H_s(t)\right]$$

$$\therefore \quad \frac{d}{dt} O_H(t) = -\frac{1}{i\hbar}\, U^\dagger(t,t_0)\, H_s(t)\, O_s(t)\, U(t,t_0)$$

$$+ U^\dagger(t,t_0)\, \frac{d O_s(t)}{dt}\, U(t,t_0) + U^\dagger(t,t_0)\, O_s(t) \left(\frac{1}{i\hbar}\, H_s(t)\, U(t,t_0)\right)$$

$$= \Big[\underbrace{U^\dagger(t,t_0)\, H_s(t)\, U(t,t_0)}_{H_H(t)}\, \underbrace{\;U^\dagger(t,t_0)}_{1}\, O_s(t)\, U(t,t_0)$$

$$- U^\dagger(t,t_0)\, O_s(t)\, U(t,t_0)\, U^\dagger(t,t_0)\, H_s(t)\, U(t,t_0)\Big]\left(-\frac{1}{i\hbar}\right)$$

$$+ U^\dagger(t,t_0)\, \frac{d O_s(t)}{dt}\, U(t,t_0)$$

$$= -\frac{i}{i\hbar}\left(H_H(t)\, \theta_H(t) - \theta_H(t)\, H_H(t) \right) + \underbrace{u^{+}(t,t_0)\, \frac{d\theta_S(t)}{dt}\, u(t,t_0)}_{\cancel{\frac{d}{dt}\left[\theta_H(t)\right]}\ \cdot\ \left[\frac{d}{dt}\theta_S(t)\right]_H}$$

$$2)\quad \boxed{i\hbar \frac{d}{dt}\theta_H(t) = \left[\theta_H(t),\ H_H(t)\right] + i\hbar\left[\frac{d}{dt}\theta_S(t)\right]_H}$$

- Historically: Schrödinger developed Wave Mechanics leading to S. eqn. Heisenberg developed Matrix Mechanics and the connection came later that the 2 approaches are equivalent.

In the S. picture the time evolution of expectation value:

$$\langle \theta \rangle(t) = \langle \Psi_S(t) | \theta_S | \Psi_S(t) \rangle . = \langle \Psi_H | \theta_H(t) | \Psi_H \rangle$$

We can S-T.

$$\frac{d}{dt}\langle \theta \rangle = \frac{1}{i\hbar}\left\langle [A, H(t)] \right\rangle + \left\langle \frac{\partial A}{\partial t} \right\rangle$$ can be obtained

no matter what picture we use!

Example: Consider a system composed of a particle of mass 'm' under the influence of a potential.

Say: $H_S(t) = \frac{P_S^2}{2m} + V(X_S, t)$

$$\cancel{i\hbar}\frac{d}{dt}P_H(t) = [P_H, H_H]$$

$$= \cancel{-\partial V}\,[P_H, V(x_H, t)]$$

$$= -\frac{\partial V}{\partial x}(x_H, t)\,\cancel{i\hbar}$$

IIIly

$$i\hbar\frac{d}{dt}X_H(t) = [X_H, H_H]$$

$$= \left[X_H, \frac{P_H^2}{2m}\right] = \frac{1}{2m}\left(P_H[X_H, P_H] + [X_H, P_H]P_H\right)$$

$$= \frac{P_H}{m}\,i\hbar$$

$$\boxed{\begin{aligned} \frac{d}{dt}P_H(t) &= -\frac{\partial V(x_H,t)}{\partial x} \\ \frac{d}{dt}X_H(t) &= \frac{P_H}{m} \end{aligned}}$$

~~Very similar~~

Heisenberg's Eqn. of Motion

$$\text{or } \underbrace{m\frac{d^2}{dt^2}\langle x\rangle = -\frac{\partial\langle V\rangle}{\partial x}}_{\text{Ehrenfest thm.}}$$

Very similar to the classical eqn.

Interaction Picture:

Consider an arbitrary physical system: $H_0(t)$. If $U_0(t, t')$ is the time evolution operator, then:

$$i\hbar\frac{\partial}{\partial t}U(t, t_0) = H_0(t)\,U(t, t_0)$$

$$U(t_0, t_0) = \mathbb{1}$$

Let $H(t) = H_0(t) + \underbrace{W(t)}_{\text{perturbation}}$.

Define $|\Psi_I(t)\rangle = U_0^+(t, t_0)\,|\Psi_S(t)\rangle$ ie: $\cancel{|\Psi_S(t)\rangle}$

Then:

$$i\hbar \frac{\partial}{\partial t} |\Psi_I(t)\rangle = i\hbar \frac{\partial}{\partial t} U_0^+(t,t_0) |\Psi_s(t)\rangle + i\hbar\, U_0^+(t,t_0) \frac{\partial}{\partial t} |\Psi_s(t)\rangle .$$

Given a Hamiltonian $H(t) = H_0(t) + W(t)$.

$$i\hbar \frac{\partial}{\partial t} \Psi_s(t) = H(t)\, \Psi_s(t) \longrightarrow \text{S. eqn.}$$

and we already have: $i\hbar \frac{\partial}{\partial t} U_0(t,t_0) = H_0(t)\, U_0(t,t_0)$

and $-i\hbar \frac{\partial}{\partial t} U_0^+(t,t_0) = U_0^+(t,t_0)\, H_0(t)$.

$$\therefore\ i\hbar \frac{\partial}{\partial t} \Psi_I(t) = -U_0^+(t,t_0) H_0(t) |\Psi_s(t)\rangle + \cancel{i\hbar}\, U^+(t,t_0) H(t) |\Psi_s(t)\rangle$$

$$= U_0^+(t,t_0) \underbrace{\left[-H_0(t) + H(t)\right]}_{W(t)} |\Psi_s(t)\rangle$$

$$= \underbrace{U_0^+(t,t_0)\, W(t)\, U_0(t,t_0)} |\Psi_I(t)\rangle .$$

$$= W_I(t)\, |\Psi_I(t)\rangle .$$

$\therefore$ $|\Psi_I(t)\rangle$ evolves according to:

$$i\hbar \frac{\partial}{\partial t} |\Psi_I(t)\rangle = W_I(t) |\Psi_I(t)\rangle .$$

and $W_I(t)$ evolves according to:

$i\hbar \frac{d}{dt} W_I(t) = [W_I(t), H_0]$ if H_0 is independent of time.

$\cancel{|\Psi_I(t)\rangle = |\Psi_S(t_0)\rangle}$

Integrating eqn for $|\Psi_I(t)\rangle$

$$|\Psi_I(t)\rangle - |\Psi_I(t_0)\rangle = -\frac{i}{\hbar}\int_{t_0}^{t} dt'\, W(t')\, |\Psi_I(t')\rangle$$

$$\therefore\ |\Psi_I(t)\rangle = |\Psi_I(t_0)\rangle \mp \frac{i}{\hbar}\int_{t_0}^{t} dt'\, W(t')\, |\Psi_I(t')\rangle$$

$$= |\Psi_I(t_0)\rangle \mp \frac{i}{\hbar}\int_{t_0}^{t} dt'\, W(t')\, |\Psi_I(t_0)\rangle \mp \left(\frac{-i}{\hbar}\right)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''$$

$$W(t')\, W(t'') + \cdots$$

$$= \left[\mathbb{1} - \frac{i}{\hbar}\int_{t_0}^{t} dt'\, W(t') + \left(\frac{-i}{\hbar}\right)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, W(t')\, W(t'')\right.$$

$$\left. + \cdots \right] |\Psi_I(t_0)\rangle .$$

NOTE: at $t = t_0$, all three pictures coincide.

i.e:

$$\boxed{|\Psi_S(t_0)\rangle = |\Psi_H(t_0)\rangle = |\Psi_I(t_0)\rangle}$$

Interaction picture is very useful in perturbation theory calculation.