$\Rightarrow \sqrt{\langle 3|3\rangle} \leq \sqrt{\langle 1|1\rangle} + \sqrt{\langle 2|2\rangle}$

Inner pdts need not be defined in a LVS. If defined, the LVS is a normed space and triangle inequality helps define distances.

Cauchy - Schwarz inequality:

$$\sqrt{\langle a|a\rangle} \cdot \sqrt{\langle b|b\rangle} \geq |\langle b|a\rangle|$$

Linear Operators:

~~Consider a space of~~

A function $f(x)$ is a map which takes a value x and gives a value y. Analogously, we can define a fn of a vector argument $|x\rangle$. In this case:

$$f(|x\rangle) \equiv c|x\rangle$$

where the function multiplies the vector $|x\rangle$ by a scalar.

Let us denote $f(|x\rangle) \equiv \hat{F}|x\rangle$ $\hat{F}$: Operator.

The space of all functions (Operators) forms a LVS.
We will consider a subset of operators which are linear i.e.

$$\hat{A}(\alpha|a\rangle + \beta|b\rangle) = \alpha\hat{A}|a\rangle + \beta\hat{A}|b\rangle$$

i.e. preserves the linear VS of $|\ \rangle$.

Space F	Space V	Space W
$\hat{A}$	$\|\ \rangle$	$\|\ \rangle'$

We usually consider $V \equiv W$

Algebra of linear operators:

Let A and B be two linear operators defined in a linear space S of vectors $|\ \rangle$.

- $A = B \Rightarrow A|\ \rangle = B|\ \rangle$ for any $|\ \rangle \in S$

Define addition and multiplication of linear operators as:
$C = A+B$ and $D = A \cdot B$ if for any $| \rangle \in S$:

$$C| \rangle = (A+B)| \rangle = A| \rangle + B| \rangle$$
$$D| \rangle = (A \circ B)| \rangle = A \circ (B| \rangle)$$

$A+B$ & $A \cdot B$ are themselves linear operators. Addition and multiplication of operators satisfy all laws of addition and multiplication with the exception of commutation for multiplication:

$$AB \neq BA$$

Define Commutator: $AB - BA \equiv [A, B]$

If $[A, B] = 0$ then A, B are called commuting operators

If E is a unit operator: i.e. $E| \rangle = | \rangle$
then, ~~all~~ any op. $\{A\}$ commutes with E,
$$[A, E] = 0$$

- Multiplication of operators by numbers:

$B = \alpha A = A\alpha$ $\quad \alpha$: scalar

$\Rightarrow$

$B| \rangle = \alpha(A| \rangle)$ for any $| \rangle$

- If $A| \rangle = \alpha| \rangle$, then the RHS is interpreted as:
$\alpha | \rangle \equiv \alpha E| \rangle$ $\quad E$: unit operator.

and $A = \alpha E$.

- $A^m \Rightarrow \underbrace{A \cdot A \cdot A \cdots A}_{m \text{ times}}$

One can define functions of operators by their formal power series expansion: for ex:

$$e^A = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

If $A|a\rangle$ is defined, then one can define the action of A on ~~of~~ a dual $\langle b|$ as:

$$(\langle b|A)\cdot|a\rangle = \langle b|\{A|a\rangle\} = \langle b|A|a\rangle$$

In general $\langle|A$ is not the dual of $A|\rangle$.

For example:

$$\left.\begin{array}{l} E|b\rangle = |b\rangle \\ \langle b|E = \langle b| \end{array}\right\} \quad \langle b|E \text{ is the dual of } E|b\rangle$$

But: $\qquad \alpha$: Complex scalar.

$$\alpha E|b\rangle = \alpha|b\rangle$$

and its dual is $\bar{\alpha}\langle b| \neq \langle b|\alpha = \langle b|\alpha E$

~~Since~~

Special operators: Some operators have special properties that are very useful for theory calculations.

- Inverse operator:

 Define A_ℓ^{-1} and A_r^{-1} such that

$$A_\ell^{-1} A = E$$

and $A A_r^{-1} = E$

Usually $A_\ell^{-1} \neq A_r^{-1}$.

In some special cases $A_\ell^{-1} = A_r^{-1}$ [When A_ℓ^{-1} and A_r^{-1} exist, then they are equal and if A_ℓ^{-1} is unique, then A_r^{-1} is unique and vice versa)

→ Product of operators: $(AB)^{-1} = B^{-1}A^{-1}$ (if B^{-1}, A^{-1} exist)

- Adjoint operator:

If a scalar pdt is defined in S, then the operator X satisfying:

$$\langle a|X|b\rangle = \overline{\langle b|A|a\rangle}$$

for $|a\rangle, |b\rangle \in S$ is called the Adjoint operator of A and is denoted by A^+:

$$\langle a|A^+|b\rangle \underset{\text{def}}{\equiv} \overline{\langle b|A|a\rangle} \quad \text{for } |a\rangle, |b\rangle \in S$$

We see that $\langle a|A^+$ is the dual of $A|a\rangle$

and

$$\langle b|(A^+)^+|a\rangle = \langle b|A|a\rangle$$

$$\text{ie: } (A^+)^+ = A$$

Since for any $|a\rangle, |b\rangle$, $B|b\rangle \underset{q}{-} \langle b|B^+$ are dual vector pairs, $A|a\rangle - \langle a|A^+$ are dual vector pairs.

$$\langle b|B^+A^+|a\rangle = [\langle b|B^+][A^+|a\rangle]$$

$$= [\overline{\langle a|A}][B|b\rangle] \quad \left(\text{Since: } \langle b|B^+ \text{ is obtained by } \overline{B|b\rangle}\right)$$

$$= \overline{\langle a|AB|b\rangle}$$

$$= \langle b|(AB)^+|a\rangle$$

$$\Rightarrow (AB)^+ = B^+A^+$$

- If an operator H is equal to its adjoint, then

$$H = H^+$$

Then H is called Hermitian operator.

- An operator U that satisfies the condition:

$$U^+ = U^{-1}$$

is called Unitary

Unitary operators have the special property that their action on vectors preserves its norm.

ie: $\langle a|a\rangle \Rightarrow \langle a|u^{\dagger}u|a\rangle = [\langle a|u^{\dagger}][u|a\rangle]$
$= \langle a|u^{-1}u|a\rangle = \langle a|a\rangle.$

ie. we act on $|a\rangle$ with operator u and form its ~~adj~~ dual. $\langle a|u^{\dagger}$. Then the scalar pdt: $\langle a|u^{\dagger}u|a\rangle = \langle a|u^{-1}u|a\rangle = \langle a|a\rangle$ which is the norm of $|a\rangle$.

- Define an operator $|a\rangle\langle b|$:

$|a\rangle\langle b|c\rangle = \langle b|c\rangle|a\rangle$ i.e. another vector $|a\rangle$

→ Can check the linear properties of $|a\rangle\langle b|$

also:

$\{|a\rangle\langle b|\}^{\dagger} = |b\rangle\langle a|$

Proof:

$\langle x|\{|b\rangle\langle a|\}|y\rangle = \langle x|b\rangle\langle a|y\rangle$
$= \overline{\langle y|a\rangle\langle b|x\rangle}$
$= \langle x|\overline{\{|a\rangle\langle b|\}}|y\rangle.$

~~Now Consider a set of all operators that can~~ be ~~obtained~~

→ Define $P_e = |e\rangle\langle e|$

$P_e|a\rangle = |e\rangle\langle e|a\rangle = \langle e|a\rangle|e\rangle$

and $P_e^2|a\rangle = \langle e|a\rangle|e\rangle\underbrace{\langle e|e\rangle}_{1} = \langle e|a\rangle|e\rangle = P_e|a\rangle.$

ie: $P_e^2 = P_e$

P_e: has the property that when acting on any $|a\rangle \in S$, it gives the component of $|a\rangle$ along $|e\rangle$ and $P_e^2 = P_e$.

<u>P_e is Hermitian.</u>

if ~~P_1+P_2~~ P_1, P_2 are projection operators, then $(P_1+P_2)^2$ is also a projection operator iff $P_1 P_2 = P_2 P_1 = 0$

If an operator satisfies: $AB = BA = 0$, then A, B are orthogonal operators. Therefore $P_1 + P_2$ can be projection operators if they are orthogonal operators.

In general if $\{P_i\}$ $i = 1, 2, \dots N$, then $P = \sum_i P_i$ is also a projection operator iff:

$$P_i P_j = \begin{cases} P_i & i = j \\ 0 & i \neq j \end{cases}$$

Example: Unit vectors in ~~real~~ 2 dimensions.

$|e_1\rangle, |e_2\rangle$

$\langle e_1|e_1\rangle = 1 = \langle e_2|e_2\rangle$

$\langle e_2|e_1\rangle = \langle e_1|e_2\rangle = 0$

Then:

$$P_1 = |e_1\rangle\langle e_1| \qquad P_2 = |e_2\rangle\langle e_2|$$

$$P_1^2 = |e_1\rangle\langle e_1|e_1\rangle\langle e_1| = 1\,|e_1\rangle\langle e_1| = P_1$$

$|||^{ly}$ $\quad P_2^2 = P_2$

$$P_1 P_2 = |e_1\rangle\langle e_1|e_2\rangle\langle e_2| = 0.$$

If $|a\rangle$ is an arbitrary 2D vector:

$$P_1|a\rangle = \langle e_1|a\rangle\,|e_1\rangle = |\vec{a}|\cos\theta_{a,e_1}\,|e_1\rangle$$

- <u>Linear Independence of Vectors:</u>

The vectors $|1\rangle \dots |N\rangle$ are said to be linearly independent if the relation:

$$\sum_i a^i |i\rangle = 0 \qquad (|i\rangle \neq 0)$$

$\Rightarrow a^i = 0$ for all i.

If any of the $z\, a^i$ do not vanish, then the vectors $|1\rangle \cdots |N\rangle$ are linearly dependent.

- The maximum no. of linearly independent vectors in a space (assuming it is finite) is called the dimension of this space. If the no. of linearly independent vectors is not bounded, then the space has infinite dimension.

The set of linearly independent vectors $|i\rangle$ that have the property that any $|a\rangle \in S$ can be expressed as a linear combination of $|i\rangle$:

$$|a\rangle = \sum_i{}' a_i |i\rangle$$

is called a basis of the space S. i.e. $\{|i\rangle\}$ spans the space S.

a_i: are the components of $|a\rangle$ w.r.t. $|i\rangle$ and the components are unique.

i.e. $|a\rangle = \sum_i{}' a^i |i\rangle$

$$|a\rangle = \sum_i{}' a'^i |i\rangle$$

$$\therefore |a\rangle - |a\rangle = 0 = \sum_i{}' (a^i - a'^i)|i\rangle = 0$$

Unless $a^i = a'^i$, we have a relation between the $|i\rangle$, ~~which~~ which is not possible since $\{|i\rangle\}$ is a linearly independent set.

Eigenvalues and Eigenvectors

If an operator B acting on ~~a~~ $|b\rangle$ changes the length of the vector, while preserving its original direction:

$$B|b\rangle = b|b\rangle$$

b: complex no. (in general)

Then the eqn is an Eigenvalue eqn, $|b\rangle$ is the eigenvector of the operator and the number b is an eigenvalue of that operator.

Ex: (i) any $|\;\rangle \in S$ is an eigenvector of E, the identity operator.

$$E|\;\rangle = 1.|\;\rangle$$

(ii) $|a\rangle$ is an eigenvector of $|a\rangle\langle b|$

since: $|a\rangle\langle b|a\rangle = \underbrace{\langle b|a\rangle}_{\text{number}}|a\rangle$

<u>Hermitian operators</u>: These operators are self adjoint

$$H = H^{+}$$

and play very important role in Physics (Quantum Mechanics)

<u>Prop. of Hermitian Operators:</u>

(i) Eigenvalues of a Hermitian operator are all real

Let $H|h_1\rangle = h_1|h_1\rangle$

$\langle h_1|H|h_1\rangle = h_1$

Now: $\langle h_1|H|h_1\rangle = \langle h_1|H^{+}|h_1\rangle$

$$\Rightarrow \boxed{h_1 = h_1^{*}}$$

(ii) If $|h_1\rangle$, $|h_2\rangle$ are 2 eigenvectors of H, then they are orthogonal.

$$H|h_1\rangle = h_1|h_1\rangle$$
$$H|h_2\rangle = h_2|h_2\rangle$$

Now: $\langle h_2|H|h_1\rangle = h_1 \langle h_2|h_1\rangle$ and
$\langle h_1|H|h_2\rangle = h_2 \langle h_1|h_2\rangle$

or: $\langle h_2|H|h_1\rangle = h_2 \langle h_2|h_1\rangle \longrightarrow$ Complex Conjugate of Second eqn.

$$\therefore \; (h_1 - h_2)\langle h_2|h_1\rangle = 0$$

If there are 2 eigenvalues, then $h_1 \neq h_2$
$\Rightarrow \langle h_2|h_1\rangle = 0$.

N dimensional Vector space: S_N

An N dimensional vector space contains N linearly independent vectors. Any set $|1\rangle \cdots |N\rangle$ of N linearly independent vectors in an N dimensional Space S_N forms a basis for this space.

Once a basis is chosen, any vector can be represented by a set of Complex numbers (Unique) in this basis.

Representations: So far we have dealt with vectors using abstract notations. We can decompose a vector w.r.t. some basis:

$$|a\rangle = \sum_i{}' a_i |i\rangle$$

The set of number a_i represent the vector $|a\rangle$, since this decomposition is unique. Therefore all manipulation of vectors can be replaced by manipulation with the components (just numbers). Therefore it is important to find "Representation" of abstract qties in order to simplify subsequent algebra.

. There is a one-one correspondence between the vectors in an N dim. complex space and 2N dimensional real space.

Ex: $|a\rangle + |b\rangle = \sum_i' (a_i + b_i)|i\rangle$ a_i; b_i are the components of $|a\rangle$, $|b\rangle$.

Next we address the representation of a Linear Operator in an N dimensional space: this will lead us to the concept of a matrix.

Let $|i\rangle$ $(i=1,2,\cdots N)$ denote the basis of S_N. Consider a linear operator A such that $A|i\rangle \in S_N$, $\Rightarrow$

$$A|i\rangle = \sum_i A_i^j |j\rangle$$

where A_i^j: is component of $A|i\rangle$. It has 2 indices. The superscript 'j' identifies the component of the vector being decomposed. and the subscript denotes the vector being decomposed. That is A_i^j is the j^{th} component of the i^{th} vector $A|i\rangle$.

If $|a\rangle$ is not a basis vector: Then:

$|a'\rangle = A|a\rangle$

$|a\rangle$ can be decomposed into components in the basis $\{|i\rangle\}$

$$|a\rangle = \sum_{i=1}^{N}{}' a^i |i\rangle$$

and $|a'\rangle = \sum_{i=1}^{N}{}' (a')^i |i\rangle$

$$\therefore |a'\rangle = A|a\rangle = \sum_{i=1}^{N}{}' A\, a^i |i\rangle$$

But $A|i\rangle = \sum_{j=1}^{N}{}' A_i^j |j\rangle$

$$\Rightarrow |a'\rangle = \sum_{i,j=1}^{N} A_i^j a^i |j\rangle$$

Use: $|a'\rangle = \sum_{j=1}^{N}{}' a'^j |j\rangle$

$$\Rightarrow \sum_{j=1}^{N}{}' a'^j |j\rangle = \sum_{i,j}{}' A_i^j a^i |j\rangle$$

$$\Rightarrow \underbrace{a'^j = \sum_{i=1}^{N}{}' A_i^j a^i}$$

Effect of A on an arbitrary vector A.

<u>Aside</u>: Einstein convention: Any index appearing twice is summed over

$\Rightarrow a'^j = A_i^j a^i$ (Summation is implied and explicit $\sum_i{}'$ can be omitted).

The index summed over is a "dummy" index and can be replaced by anything else:

$$a'^j = A_m^j a^m$$

Once a basis is chosen; in the space S_N, the multiplication of a vector by a linear operator is REPRESENTED by a linear transformation of the components of this vector.

The numbers A_i^j can be arranged in a table:

$$\begin{pmatrix} A_1^1 & A_2^1 & A_3^1 & \cdots & A_N^1 \\ A_1^2 & A_2^2 & A_3^2 & \cdots & A_N^2 \\ \vdots & & & & \\ A_1^N & A_2^N & A_3^N & \cdots & A_N^N \end{pmatrix}$$

If we decide: lower index is a column index
upper index is a row index.

This table is a matrix

In an N dim space, there is a one-one correspondence between linear operators and matrices. ~~Eq~~

Equality of 2 matrices $A = A' \Rightarrow A_i^j = A'^j_i$

The components of a vector $|a\rangle$ i.e. a^i can also be arranged in a "table":

$$|a\rangle = \begin{pmatrix} a^1 \\ a^2 \\ \vdots \\ a^N \end{pmatrix}.$$

Therefore:

$$a'^i = A_j^i a^j$$

can be denoted as matrix multiplication

$$\begin{pmatrix} a'^1 \\ a'^2 \\ \vdots \\ a'^N \end{pmatrix} = \begin{pmatrix} A_1^1 & A_2^1 & \cdots & A_N^1 \\ A_1^2 & A_2^2 & \cdots & A_N^2 \\ \vdots & & & \\ A_1^N & A_2^N & \cdots & A_N^N \end{pmatrix} \begin{pmatrix} a^1 \\ a^2 \\ \vdots \\ a^N \end{pmatrix}$$

One can verify that ~~addition~~ matrices and linear operators follow the same algebra (i.e. they have the same properties). We know that in general operators do not commute. This then is translated to non-commutativity of matrices:

ie $\boxed{AB \neq BA}$

- Representation of unit op.

$E|\rangle = |\rangle$

$$\Rightarrow E = \begin{pmatrix} 1 & 0 & 0 & \cdots & \\ 0 & 1 & 0 & \cdots & \\ \vdots & & 1 & & 0 \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix} \equiv \text{Unit Matrix.}$$

- Multiplication by a scalar α:

$A'^{i}_{j} = \alpha A^{i}_{j} \Rightarrow$ all elements are augmented by α.

- <u>Inverse of a Matrix</u>

$A^{-1}A = E$

When A represents the op. $\hat{A}$, then A^{-1} is the left inv. $\hat{A}_e^{-1}$.

Inv $\Rightarrow$ det $A \neq 0$.

For finite dim space,

A^{-1} = unique left inv = right inv

Not true for infinite dim.

Change of basis in an N-dim. space

We have so far seen representations of vectors and linear operators w.r.t. a given basis in S_N. A basis is by no means unique. In fact there exists an infinite number of sets of N linearly independent vectors in S_N and each one of these sets can be equally well chosen as a basis of the space.

If $\hat{R}$ is some linear operator represented in the basis $|i\rangle$ by a matrix R and $\det R \neq 0$,

then. $|i'\rangle = \hat{R}|i\rangle = \sum_j' R_{ij}|j\rangle$

Since $\det R \neq 0$:

$$R_{ij} R^{-1}_{ik} = E_{jk}$$

$$\sum_i' R^{-1}_{ik} |i'\rangle = \sum_{ij}' R^{-1}_{ik} R_{ij} |j\rangle$$

$$= \sum_j' E_{kj} |j\rangle = |k\rangle$$

or: $|i\rangle = \sum_j' (R^{-1})_{ij} |j'\rangle$

$|i'\rangle$ is also linearly independent

i.e. $R_{e} \quad \sum_i' c_i |i'\rangle = 0$

or $\sum_{ij}' c_i R_i^j |j\rangle = 0$

$\Rightarrow \quad c_i R_i^j = 0$.

$|c\rangle$

Switch basis from $|i\rangle \to |i'\rangle$
Take an arbitrary vector $|a\rangle$

$$|a\rangle = \sum_i' a_i |i\rangle$$

and in the new basis:

$$|a'\rangle = \sum_j' a_i' |i'\rangle$$

$$|a\rangle = \sum_i' a_i (R_i^j)^{-1} |j'\rangle$$

$$\text{or}: \quad a_j' = a_i (R^{-1})_{ij}$$

$$\Rightarrow a' = R^{-1} a$$

Transformation of a linear op.

$$A|i'\rangle = A \sum_j R_i^j |j\rangle = \sum_{m'} A_j'^m |m'\rangle$$

$$= \sum_{jk}' \cancel{A_j^i} R_i^j A_j^k |k\rangle$$

Using $|k\rangle = R^{-1}|m'\rangle$

$$A|i'\rangle = \sum_{kjm}' (R^{-1})^m_k R_i^j A_j^k |m'\rangle$$

$$\Rightarrow (A')_i^m = (R^{-1})^m_k R_i^j A_j^k$$

$$\boxed{A' = R^{-1} A R}$$

~~When $|i'\rangle = R|i\rangle = \sum_j' R_i^j |j\rangle$, then we know the~~