Harmonic Oscillator in 1 dimensions

A simplest example is a particle attached at one end by a spring ~~string~~. If the spring is stretched, the particle experiences a restoring force $-kx$ where k is the spring constant. The potential in which the particle moves is

$$V = \frac{1}{2} k x^2$$

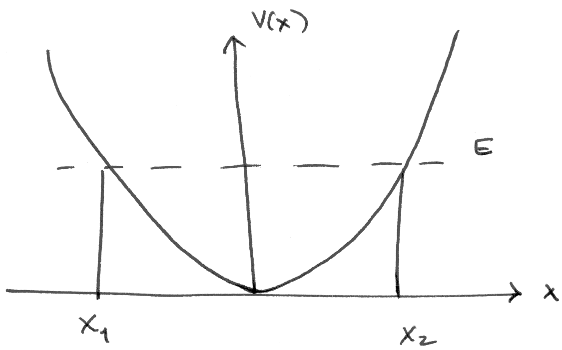
From classical mechanics, the particle has oscillatory motion about $x=0$ with angular frequency

$$\omega = \sqrt{\frac{k}{m}}$$

In fact even for a generic potential $V(x)$ that has local minima, if one studies the behavior of a physical system in the neighborhood of a stable equilibrium position, one arrives at equations which, in the limit of small oscillations are those of a harmonic oscillator. The results we obtain here are therefor applicable to a whole lot of physical phenomena - Vibrations of atoms about equilibrium inside a molecule, oscillations about equilibrium in crystal lattices etc.

It also turns out that the operators we define here creation and destruction operators are used throughout quantum statistical mechanics and field theory.

Oscillator in classical Mechanics



Classically a particle of mass m is governed by the following eqn. of motion:

$$m \frac{d^2x}{dt^2} = -kx$$

The general soln.

$$x = x_M \cos(\omega t - \phi)$$

$\omega = \sqrt{\frac{k}{m}}$, x_M, ϕ are the constants of integration fixed through initial conditions.

The K.E. of the particle $T = \frac{p^2}{2m}$

The total energy $E = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$

Insert the solution $x = x_M \cos(\omega t - \phi)$, it is easy to see that $E = \frac{1}{2} m\omega^2 x_M^2$ and is independent of time.
Also $E \geq 0$

Quantum Mechanical Hamiltonian

$x, p \longrightarrow \hat{x}, \hat{p}$ such that $[\hat{x}, \hat{p}] = i\hbar$

Then: $\hat{H} = \frac{\hat{P}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$.

H is time independent (conservative system), therefore we can look at the solutions to the eigenvalue eqn:

$$H|\varphi\rangle = E|\varphi\rangle$$

Written in the $\{|x\rangle\}$ basis:

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2\right)\varphi(x) = E\varphi(x)$$

Note that the eigenfunctions of H have a definite parity since

$V(-x) = V(x)$ (even function) $\Rightarrow$ We will have even and odd wave fns.

<u>Eigenvalues of the Hamiltonian</u>

$\hat{X}$ and $\hat{P}$ have dimensions of length and momentum respectively.

$\omega \rightarrow$ has dimensions of 1/time

$h \rightarrow$ dimensions of action [Energy x time]

$\frac{M^2L^2T^{-2}\,ML}{MLT^{-1}}$

Define $\hat{X}' = \sqrt{\frac{m\omega}{\hbar}}\,\hat{X}$ and $\hat{P}' = \frac{\hat{P}}{\sqrt{m\hbar\omega}}$

$\sqrt{\frac{mc^2\,\omega/c}{\hbar c}}$ $\quad \frac{m\omega}{\hbar}$

Then $\hat{X}'$ and $\hat{P}'$ are dimensionless.

$\hbar = 2\pi mvr$ $\quad \frac{MT^{-1}}{MLT^{-1}L} = \frac{1}{L^2}$

$= MLT^{-1} \times L$

$= ML^2T^{-1}$

$$\therefore\ H = \frac{1}{2}\frac{P^2}{\cancel{2}m} + \frac{1}{2}m\omega^2x^2$$

Can now be written as: $H = \frac{1}{2m}(m\hbar\omega)\hat{P}' + \frac{1}{2}m\omega^2\frac{\hbar}{m\omega}\hat{X}'$

$$= \frac{\hbar\omega}{2}\hat{X}'^2 + \frac{\hbar\omega}{2}\hat{P}'^2$$

$$= \hbar\omega\left[\frac{\hat{X}'^2 + \hat{P}'^2}{2}\right]$$

Define $\hat{H}' = \dfrac{\hat{X}'^2 + \hat{P}'^2}{2}$

Then: $\hat{H} = \hbar\omega\,\hat{H}'$.

Therefore we can seek solutions of $\hat{H}'$ and obtain dimensionless eigenvalues, which can be later scaled by $\hbar\omega$ to obtain the eigenvalues of $\hat{H}$.

If $\hat{X}', \hat{P}'$ were numbers, then we could have written them out as

$$\frac{1}{2}\left(X'^2 + P'^2\right) = \frac{1}{2}\left(\hat{X}' - i\hat{P}'\right)\left(X' + iP'\right)$$

Since $\hat{X}', \hat{P}'$ are non-commuting operators the product ~~Is not~~ $(\hat{X}' - i\hat{P}')(\hat{X}' + i\hat{P}') \neq \hat{X}'^2 + \hat{P}'^2$.

However it turns out that factorizing the Hamiltonian in the above form does simplify the calculations a whole lot!

Define:

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\hat{X}' + i\hat{P}'\right)$$

$$\hat{a}^{+} = \frac{1}{\sqrt{2}}\left(\hat{X}' - i\hat{P}'\right)$$

We can immediately obtain

$$\hat{X}' = \frac{1}{\sqrt{2}} (\hat{a}^+ + \hat{a})$$

$$\hat{P}' = \frac{i}{\sqrt{2}} (a^+ - a)$$

~~Check~~: $\hat{X}'$, $\hat{P}'$ are Hermitian, while $\hat{a}, \hat{a}^+$ are not, but are adjoints of each other.

Determine

$$[\hat{a}, \hat{a}^+] = \frac{1}{2} [\hat{X}' + i\hat{P}', \hat{X}' - i\hat{P}']$$

$$= -\frac{i}{2} [\hat{X}', \hat{P}'] + \frac{i}{2} [\hat{P}', \hat{X}']$$

$$[\hat{X}', \hat{P}'] = \sqrt{\frac{m\omega}{\hbar}} \frac{1}{\sqrt{m\hbar\omega}} [\hat{X}, \hat{P}] = \frac{i\hbar}{\hbar} = i$$

$$\therefore [\hat{a}, \hat{a}^+] = -\frac{i}{2}(i) + \frac{i}{2}(-i) = 1$$

$$\boxed{[\hat{a}, \hat{a}^+] = 1}$$ $\rightarrow$ Completely equivalent to the canonical commutation relation.

Now: $\cancel{\hat{a}\hat{a}^+ = \frac{1}{2} (\hat{X}' - i\hat{P}')(\hat{X}' + i\hat{P}')}$

$$\hat{a}^+\hat{a} = \frac{1}{2} (\hat{X}' - i\hat{P}')(\hat{X}' + i\hat{P}')$$

$$= \frac{1}{2} \left(\hat{X}'^2 + \hat{P}'^2 + i\hat{X}'\hat{P}' - i\hat{P}'\hat{X}' \right)$$

$$= \frac{1}{2} \left(\hat{X}' + \hat{P}'^2 + i[\hat{X}', \hat{P}'] \right) = \frac{1}{2} \left(\hat{X}'^2 + \hat{P}'^2 - 1 \right)$$

Therefore:

$$\hat{H}' = \frac{1}{2}\left(\hat{x}'^2 + \hat{p}'^2\right)$$

$$= \hat{a}^+\hat{a} + 1/2$$

Introduce an operator $\hat{N} = \hat{a}^+\hat{a}$

Now: $\hat{N}^+ = (\hat{a}^+\hat{a})^+ = \hat{a}^+\hat{a} = \hat{N} \Rightarrow \hat{N}$ is Hermitian.

$\therefore \hat{H} = \hat{N} + 1/2\,\hat{I}$

$\Rightarrow$ eigenvectors of $\hat{H}$ are the eigenvectors of $\hat{N}$ and vice versa.

Calculate:

$$[\hat{N}, \hat{a}] = [\hat{a}^+\hat{a}, \hat{a}] = \hat{a}^+\underbrace{[\hat{a}, \hat{a}]}_{0} + [\hat{a}^+, \hat{a}]\hat{a}$$

$$= -\hat{a}$$

$$[\hat{N}, \hat{a}^+] = [\hat{a}^+\hat{a}, \hat{a}^+] = \hat{a}^+[\hat{a}, \hat{a}^+] = \hat{a}^+$$

If we can find: $\hat{N}|\varphi_\nu\rangle = \nu|\varphi_\nu\rangle$

Then: $H|\varphi_\nu\rangle = (\nu + 1/2)|\varphi_\nu\rangle$

Let us prove some properties of the eigenvalue eqn of $\hat{N}$.

- Let $|\varphi_\nu\rangle$ be an eigenvector of $\hat{N}$. Now consider $a|\varphi_\nu\rangle = |\varphi_\nu'\rangle$.

$$\langle \varphi'_\nu | \varphi'_\nu \rangle \geq 0$$

$$\Rightarrow \langle \varphi_\nu | \hat{a}^+ \hat{a} | \varphi_\nu \rangle \geq 0$$

But $\hat{a}^+ \hat{a} = \hat{N}$

$$\therefore \langle \varphi_\nu | \hat{N} | \varphi_\nu \rangle \geq 0 \Rightarrow \nu \langle \varphi_\nu | \varphi_\nu \rangle \geq 0$$

$$\Rightarrow \nu \geq 0 \quad (\text{since } \langle \varphi_\nu | \varphi_\nu \rangle > 0)$$

$\Rightarrow$ $\boxed{\text{Eigenvalues of the operator } \hat{N} \text{ are positive}}$

- Properties of the eigenvector $|\varphi_\nu\rangle$.

Let $|\varphi_\nu\rangle$ be the eigenvector of $\hat{N}$ with eigenvalue ν

We will prove that:

(i) If $\nu = 0$, the ket $a|\varphi_{\nu=0}\rangle = 0$

(ii) If $\nu > 0$, the ket $a|\varphi_\nu\rangle$ is a non-zero eigenvector of $\hat{N}$ with eigenvalue $(\nu - 1)$.

We have seen that:

$$\langle \varphi_\nu | a^+ a | \varphi_\nu \rangle = \nu \langle \varphi_\nu | \varphi_\nu \rangle \geq 0$$

The equality is established when $\nu = 0$, the norm $\langle \varphi_\nu | \varphi_\nu \rangle \to 0$ iff $|\varphi_\nu\rangle = 0$ (Null vector). If $\nu = 0$ is an eigenvalue of $\hat{N}$, and $|\varphi_0\rangle$ the corresponding eigenvector, then:

$$a|\varphi_0\rangle = 0 \quad (\because \text{ We have } \langle \varphi_0 | a^+ a | \varphi_0 \rangle = 0 \text{ since } \nu = 0. \Rightarrow \text{the vector itself i.e. } a|\varphi_0\rangle = 0)$$

Multiply by a^+ $\Rightarrow$ $a^+ a | \varphi_0 \rangle = \hat{N} | \varphi_0 \rangle = 0$

$\Rightarrow$ any vector $|\varphi_0\rangle$ that satisfies $a|\varphi_v\rangle = 0$ is an eigenvector of $\hat{N}$ with eigenvalue '0.'

(ii) if $v > 0$, then: $\langle\varphi_v|a^{+}a|\varphi_v\rangle > 0$
$\Rightarrow a|\varphi_v\rangle \neq 0$

Now: Consider: $[N, a]|\varphi_v\rangle = -a|\varphi_v\rangle$

$(Na - aN)|\varphi_v\rangle = -a|\varphi_v\rangle$
$Na|\varphi_v\rangle = aN|\varphi_v\rangle - a|\varphi_v\rangle = v\,a|\varphi_v\rangle - a|\varphi_v\rangle$
$= (v-1)\,a|\varphi_v\rangle$

$\Rightarrow N|\varphi'\rangle = (v-1)|\varphi'\rangle$ where $|\varphi'\rangle = a|\varphi_v\rangle$
$\therefore a|\varphi_v\rangle$ is an eigenvector of $\hat{N}$ with eigenvalue $(v-1)$.

Next let us check what happens to $a^{+}|\varphi_v\rangle$.

If $|\varphi_v\rangle$ is a non-zero eigenvector of $\hat{N}$ with eigenvalue v, then: $a^{+}|\varphi_v\rangle > 0$
and $a^{+}|\varphi_v\rangle$ is an eigenvector of $\hat{N}$ with eigenvalue $(v+1)$.

Norm of $a^{+}|\varphi_v\rangle = \langle\varphi_v|aa^{+}|\varphi_v\rangle$

Use: $[a, a^{+}] = 1 \Rightarrow aa^{+} = 1 + a^{+}a$
$\Rightarrow \langle\varphi_v|aa^{+}|\varphi_v\rangle = \langle\varphi_v|1 + a^{+}a|\varphi_v\rangle = \langle\varphi_v|\varphi_v\rangle + v\langle\varphi_v|\varphi_v\rangle$
$= (v+1)\langle\varphi_v|\varphi_v\rangle$.

Even if $\nu = 0$, $\| a^{+} |\varphi_\nu\rangle \| \neq 0$
$\Rightarrow$ $a^{+} |\varphi_\nu\rangle \neq 0$ (since it is not a null vector).

Consider:

$$[N, a^{+}] |\varphi_\nu\rangle = a^{+} |\varphi_\nu\rangle$$

$$N a^{+} |\varphi_\nu\rangle - \underbrace{a^{+} N |\varphi_\nu\rangle}_{\nu\, a^{+} |\varphi_\nu\rangle} = a^{+} |\varphi_\nu\rangle$$

$$\Rightarrow \; N a^{+} |\varphi_\nu\rangle = (\nu + 1)\, a^{+} |\varphi_\nu\rangle$$

$\Rightarrow$ $a^{+} |\varphi_\nu\rangle$ is an eigenvector of $\hat{N}$ with eigenvalue $(\nu+1)$.

Now we are in a position to fix the spectrum of $\hat{N}$.

Let $|\varphi_\nu\rangle$ be an eigenket of $\hat{N}$, then $a|\varphi_\nu\rangle$ is an eigenket of $\hat{N}$ with eigenvalue $(\nu-1)$.

Now: $a^2 |\varphi_\nu\rangle = N a^2 |\varphi_\nu\rangle =$

$$[N, a] = Na - aN = -a \quad \Rightarrow \quad Na = -a + aN$$

$$\begin{aligned} \therefore \; N a a |\varphi_\nu\rangle &= (-a + aN)\, a |\varphi_\nu\rangle = (-a^2 + aNa)\, |\varphi_\nu\rangle \\ &= -a^2 |\varphi_\nu\rangle + a(-a + aN) |\varphi_\nu\rangle \\ &= -a^2 |\varphi_\nu\rangle - a^2 |\varphi_\nu\rangle + a^2 \nu |\varphi_\nu\rangle \\ &= (-2 + \nu)\, [a^2 |\varphi_\nu\rangle] \end{aligned}$$

$\therefore$ $a^3 |\varphi_\nu\rangle$ is an eigenstate of $\hat{N}$ with eigenvalue: $(\nu - 3)$.

We know that $\nu \geq 0$. Let us assume that ν is non-integer.

Then we can always find an integer n such that

$n < \nu < n+1$.

Now consider the series of vectors
$|\varphi_\nu\rangle$, $a|\varphi_\nu\rangle$, $a^2|\varphi_\nu\rangle \dots a^n|\varphi_\nu\rangle$

each of the vectors ~~are~~ is non-zero eigenvector of $\hat{N}$ and the eigenvalue is $(\nu - p)$ for the p^{th} vector $a^p|\varphi_\nu\rangle$.
→ The vectors are non-zero since ν cannot be zero if ν is assumed to be non-integer. Now: $0 \leq p \leq n$

If $\hat{a}$ acts on $\cancel{a^p|\varphi_\nu\rangle}$ $a^n|\varphi_\nu\rangle$, then it is also an eigenket of $\hat{N}$ with eigenvalue $(\nu - n - 1)$. But $\nu < n+1$
$\therefore [\nu - (n+1)] < 0$. But we have already established that the eigenvalues of $\hat{N}$ are positive or zero.

If we assume that ν is an integer, then: (say $\nu = n$)
$\cancel{a^{n+1}|\varphi_\nu\rangle = \nu - (n+1)}$
then in the series of vectors, $a^n|\varphi_n\rangle$ has an eigenvalue $(n-n) = 0$.

2) ~~$a\, a^n|\varphi_n\rangle$ is an eigenvector of N with eigenvalue 0~~

~~Then: $a^{n+1}|\varphi_n\rangle = ?$~~
~~$N a^{n+1}|\varphi_n\rangle = N a[a^n|\varphi_n\rangle] = [-a + aN]\, a^n|\varphi_n\rangle$~~
~~$= -a^{n+1}|\varphi_n\rangle + a[0]$~~

But we have seen that if $\nu = 0$, then ~~the corresp~~ $\hat{a}$ acting on the corresp eigenket is a Null ket.

i.e. $\Rightarrow N[a^n|\varphi_n\rangle] = 0$ $\quad$ ($a^n|\varphi_n\rangle$ is a non-zero ket of $\hat{N}$)

$\Rightarrow a^{n+1}|\varphi_n\rangle = 0$ etc. $[\langle\varphi_n|a^{+n}]\, a^+ a\, [a^n|\varphi_n\rangle] \geq 0$

$\therefore$ It is not possible to get a non-zero eigenvector of $\hat{N}$ that has negative eigenvalues.

$\Rightarrow$ $\boxed{\nu \geq 0 \text{ and } \nu \text{ takes integer values.}}$

$\therefore$ The energy eigenvalues:

$$H'|\varphi_n\rangle = (n + 1/2)|\varphi_n\rangle$$

$$\Rightarrow H|\varphi_n\rangle = (n + 1/2)\hbar\omega|\varphi_n\rangle$$

$$\Rightarrow E_n = (n + 1/2)\hbar\omega \qquad n = 0, 1, 2 \ldots$$

Once again we see that E cannot take arbitrary values and is <u>quantized</u>.

<u>Interpretation of the operators $\hat{a}$ and $\hat{a}^+$:</u>

Let $|\varphi_n\rangle$ be an eigenvector of $\hat{H}$ with eigenvalues $(n + 1/2)\hbar\omega$

$$\Rightarrow H|\varphi_n\rangle = (n + 1/2)\hbar\omega|\varphi_n\rangle$$

Now: $\hat{H}[\hat{a}|\varphi_n\rangle] = \hbar\omega(\hat{N} + 1/2)[\hat{a}|\varphi_n\rangle] = [\hat{N}\hat{a}|\varphi_n\rangle + \hat{a}|\varphi_n\rangle]\hbar\omega$

Using $\hat{N}\hat{a} - \hat{a}\hat{N} = -\hat{a}$

We can write:

$$\hat{H}[\hat{a}|\varphi_n\rangle] = \hbar\omega\left([-\hat{a} + \hat{a}\hat{N}]|\varphi_n\rangle + \tfrac{1}{2}\hat{a}|\varphi_n\rangle\right)$$

$$= \left(-\hat{a}|\varphi_n\rangle + n\hat{a}|\varphi_n\rangle + \tfrac{1}{2}\hat{a}|\varphi_n\rangle\right)\hbar\omega$$

$$= \left((E_n - 1)\,\hat{a}|\varphi_n\rangle\right)\hbar\omega$$

$|||^{rly}$ $$\hat{H}[a^+|\varphi_n\rangle] = \left(\hat{N}\hat{a}^+|\varphi_n\rangle + \frac{\hat{a}^+}{2}|\varphi_n\rangle\right)\hbar\omega$$

$$= \left(\hat{a}^+|\varphi_n\rangle + \cancel{\hat{N}a}\ \hat{a}^+\hat{N}|\varphi_n\rangle + \frac{\hat{a}^+}{2}|\varphi_n\rangle\right)\hbar\omega$$

$$= \left[(E_n + 1)\hat{a}^+|\varphi_n\rangle\right]\hbar\omega$$

$\therefore$ $a|\varphi_n\rangle$ is an eigenstate of $\hat{H}$ with eigenvalues $(E_n - 1)\hbar\omega$ and $a^+|\varphi_n\rangle$ is an eigenstate of $\hat{H}$ with eigenvalue $(E_n + 1)\hbar\omega$

$\therefore$ $\hat{a}$: destruction operator (destroys an energy quantum $\hbar\omega$)

$\hat{a}^+$: Creation operator (creates an energy quantum $\hbar\omega$).

Degeneracy of the eigenvalues:

Ground state: $n = 0 \Rightarrow E_0 = \frac{\hbar\omega}{2}$

Since $|\varphi_0\rangle$ is an eigenstate of $\hat{N}$ with an eigenvalue '0', $a|\varphi_0\rangle = 0$

In terms of $\hat{x}$, $\hat{p}$:

$$a = \frac{1}{\sqrt{2}}[\hat{x}' + i\hat{p}'] = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{m\omega}{\hbar}}\,\hat{x} + \frac{i}{\sqrt{m\hbar\omega}}\,\hat{p}\right]$$

$\Rightarrow$ in the $\{|x\rangle\}$ basis:

$$a|\varphi_0\rangle \equiv \frac{1}{\sqrt{2}}\left[\sqrt{\frac{m\omega}{\hbar}}\,\hat{X} + \frac{i}{\sqrt{m\hbar\omega}}(-i\hbar)\frac{d}{dx}\right]\varphi_0(x) = 0$$

$$\Rightarrow \left(\sqrt{\frac{m\omega}{\hbar}}\,x + \sqrt{\frac{\hbar}{m\omega}}\frac{d}{dx}\right)\varphi_0(x) = 0$$

$$\left(\frac{m\omega}{\hbar}x + \frac{d}{dx}\right)\varphi_0(x) = 0$$

$$\Rightarrow \varphi_0(x) = c\, e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2}$$

$\Rightarrow$ Only one solution to the ground state. Hence the ground state is degenerate.

In order to check that the other kets are also ~~deg~~ non-degenerate, we need to establish that if the level $E_n = (n+\frac{1}{2})\hbar\omega$ is non-degenerate, then E_{n+1} is also non-degenerate.

Let $|\varphi_n\rangle$ be a non-degenerate eigenvector of $\hat{N}$.

$$\Rightarrow \hat{N}|\varphi_n\rangle = n|\varphi_n\rangle$$

Now: $\hat{N}|\varphi_{n+1}\rangle = (n+1)|\varphi_{n+1}\rangle$

Now: $a|\varphi_{n+1}\rangle$ is an eigenket of $\hat{N}$ with eigenvalue n. Since $|\varphi_n\rangle$ is non-degenerate,

$$a|\varphi_{n+1}\rangle = c\,|\varphi_n\rangle \qquad c: \text{ complex number.}$$

$$a^\dagger a|\varphi_{n+1}\rangle = c\, a^\dagger|\varphi_n\rangle$$
$$(n+1)|\varphi_{n+1}\rangle = c\, a^\dagger|\varphi_n\rangle$$

$$\therefore |\varphi_{n+1}\rangle = \frac{c}{n+1}\, a^\dagger|\varphi_n\rangle$$

$a^\dagger|\varphi_n\rangle$ is an eigenket of $\hat{N}$ with eigenvalue $(n+1)$. Now we see that all such eigenkets are proportional to each other. Hence $|\varphi_{n+1}\rangle$ has to be non-degenerate.
Since $n=0$ eigenket is non-degenerate, any other $n\neq 0$ ket is also non-degenerate.

Properties of the Eigenstates of the Hamiltonian:

The eigenkets of $\hat{N}$ (or $\hat{H}$) are non-degenerate. ~~and hence~~
These eigenkets constitute a basis in the space ξ_x (space of a particle in a 1d problem).

$|\varphi_0\rangle$ satisfies: $\quad a|\varphi_0\rangle = 0$
$\quad\langle\varphi_0|\varphi_0\rangle = 1 \quad$ (Normalized).

$|\varphi_1\rangle$ is an eigenvector of $\hat{N}$ with value 1.
$$\hat{N}|\varphi_1\rangle = |\varphi_1\rangle$$
But $|\varphi_1\rangle = c_1\, a^\dagger|\varphi_0\rangle$
We determine C_1 by requiring $|\varphi_1\rangle$ to be normalized and choosing the phase of $|\varphi_1\rangle$ such that C_1 is real and positive.

$$\langle \varphi_1 | \varphi_1 \rangle = |c_1|^2 \langle \varphi_0 | a a^{+} | \varphi_0 \rangle$$

Use: $[a, a^{+}] = 1 \quad \Rightarrow \quad \hat{a}\hat{a}^{+} = 1 + \hat{a}^{+}\hat{a}$

$$\Rightarrow \langle \varphi_1 | \varphi_1 \rangle = |c_1|^2 \langle \varphi_0 | (1 + \hat{a}^{+}\hat{a}) | \varphi_0 \rangle$$
$$= |c_1|^2 \underbrace{\langle \varphi_0 | \varphi_0 \rangle}_{1} + |c_1|^2 \underbrace{\langle \varphi_0 | \hat{a}^{+}\hat{a} | \varphi_0 \rangle}_{\overset{!!}{0}}$$

$$\Rightarrow |c_1|^2 = \underline{1}$$

$$\therefore |\varphi_1\rangle = a^{+} |\varphi_0\rangle$$

Now: $|\varphi_2\rangle = c_2 a^{+} |\varphi_1\rangle$

Again:

$$\langle \varphi_2 | \varphi_2 \rangle = 1 = |c_2|^2 \langle \varphi_1 | \hat{a}\hat{a}^{+} | \varphi_1 \rangle$$
$$= |c_2|^2 \langle \varphi_1 | \hat{a}^{+}\hat{a} + 1 | \varphi_1 \rangle$$
$$= |c_2|^2 \langle \varphi_1 | \varphi_1 \rangle + |c_2|^2 \langle \varphi_1 | a^{+} a | \varphi_1 \rangle$$

$$= 2 |c_2|^2$$

$$\therefore c_2 = \frac{1}{\sqrt{2}}$$

$$\therefore |\varphi_2\rangle = \frac{a^{+}}{\sqrt{2}} |\varphi_1\rangle = \frac{(a^{+})^2}{\sqrt{2}} |\varphi_0\rangle$$

Generalize:

$$|\varphi_n\rangle = a_n a^{+} |\varphi_{n-1}\rangle$$

Normalization: $\langle \varphi_n | \varphi_n \rangle = 1 = |c_n|^2 \langle \varphi_{n-1} | a a^{+} | \varphi_{n-1} \rangle$
$$= |c_n|^2 \langle \varphi_{n-1} | 1 + a^{+} a | \varphi_{n-1} \rangle$$
$$= n |c_n|^2$$

$\Rightarrow \quad C_n = \frac{1}{\sqrt{n}}$

or: $|\varphi_n\rangle = \frac{a^+}{\sqrt{n}} |\varphi_{n-1}\rangle = \left(\frac{1}{\sqrt{n}} \frac{1}{\sqrt{n-1}} \frac{1}{\sqrt{n-2}} \cdots 1 \right) (a^+)^n |\varphi_0\rangle$

$\therefore \quad |\varphi_n\rangle = \frac{(a^+)^n}{\sqrt{n!}} |\varphi_0\rangle$

Orthonormalization and Closure:

$\hat{H}$ is Hermitian, $\Rightarrow \langle \varphi_n | \varphi_{n'} \rangle = \delta_{nn'}$.

$|\varphi_n\rangle$ constitutes a basis

$\Rightarrow \quad \sum_n{}' |\varphi_n\rangle \langle \varphi_n| = 1$

Action of various operators:

$\hat{X}, \hat{P}$ are linear combinations of $\hat{a}, \hat{a}^+$. We can express other observables in this basis. In order to do this, we need to determine the action of $\hat{a}, \hat{a}^+$ on $|\varphi_n\rangle$.

We have already seen:

$|\varphi_{n+1}\rangle = \frac{1}{\sqrt{n+1}} a^+ |\varphi_n\rangle$

or: $a^+ |\varphi_n\rangle = \sqrt{n+1} |\varphi_{n+1}\rangle$.

Next we need to determine: $a|\varphi_n\rangle = a \frac{a^+}{\sqrt{n}} |\varphi_{n-1}\rangle$

Using $[a, a^+] = 1$: $\quad a a^+ = 1 + a^+ a$

$$\Rightarrow a|\varphi_n\rangle = \frac{a a^+}{\sqrt{n}}|\varphi_{n-1}\rangle = \frac{(1 + a^+ a)}{\sqrt{n}}|\varphi_{n-1}\rangle$$

$$= \frac{\cancel{1} + n - \cancel{1}}{\sqrt{n}}|\varphi_{n-1}\rangle = \sqrt{n}|\varphi_{n-1}\rangle$$

$$\therefore \boxed{\begin{aligned} a|\varphi_n\rangle &= \sqrt{n}|\varphi_{n-1}\rangle \\ a^+|\varphi_n\rangle &= \sqrt{n+1}\,|\varphi_{n+1}\rangle \end{aligned}}$$

its adjoint:

$$\boxed{\begin{aligned} \langle\varphi_n|a &= \sqrt{n+1}\,\langle\varphi_{n+1}| \\ \langle\varphi_n|a^+ &= \sqrt{n}\,\langle\varphi_{n-1}| \end{aligned}}$$

We now can obtain matrix representations for the ~~kets~~ operators $\hat{x}$, $\hat{p}$:

$$\hat{x}|\varphi_n\rangle = \sqrt{\frac{\hbar}{m\omega}}\,\frac{(a^+ + a)}{\sqrt{2}}|\varphi_n\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n+1}\,|\varphi_{n+1}\rangle + \sqrt{n}\,|\varphi_{n-1}\rangle\right]$$

$$\Rightarrow \langle\varphi_{n'}|\hat{x}|\varphi_n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n+1}\,\langle\varphi_{n'}|\varphi_{n+1}\rangle + \sqrt{n}\,\langle\varphi_{n'}|\varphi_{n-1}\rangle\right]$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n+1}\,\delta_{n', n+1} + \sqrt{n}\,\delta_{n', n-1}\right]$$

IIIrly

$$\hat{p}|\varphi_n\rangle = \frac{i\sqrt{m\hbar\omega}}{\sqrt{2}}\,(\hat{a}^+ - \hat{a})\,|\varphi_n\rangle$$

$$= i\sqrt{\frac{m\hbar\omega}{2}}\left[\sqrt{n+1}\,|\varphi_{n+1}\rangle - \sqrt{n}\,|\varphi_{n-1}\rangle\right]$$

$$\therefore \langle\varphi_{n'}|\hat{p}|\varphi_n\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left[\sqrt{n+1}\,\delta_{n',n+1} - \sqrt{n}\,\delta_{n',n-1}\right]$$

The matrices representing $\hat{a}$, $\hat{a}^+$ are:

$$\langle\varphi_{n'}|\hat{a}|\varphi_n\rangle = \sqrt{n}\,\cancel{\varphi_{n'}}\,\delta_{n',n-1}$$

$$\langle\varphi_{n'}|a^+|\varphi_n\rangle = \sqrt{n+1}\,\delta_{n',n+1}$$

$$\therefore)\quad a \equiv \begin{pmatrix} 0 & \sqrt{1} & 0 & \cdots & & 0 \\ 0 & 0 & \sqrt{2} & 0 & \cdots & \\ 0 & 0 & 0 & \sqrt{3} & 0 & \cdots \\ - & - & - & & & \\ & & - & & & \\ & & & & 0 & \sqrt{n}\ \cdots \end{pmatrix}$$

IIIrly

$$a^+ \equiv \begin{pmatrix} 0 & 0 & 0 & \cdots & & \\ \sqrt{1} & 0 & 0 & \cdots & & \\ 0 & \sqrt{2} & 0 & \cdots & & \\ 0 & 0 & \sqrt{3} & 0 & \cdots & \\ \vdots & \vdots & \vdots & & & \\ & & & \sqrt{n+1} & 0 & \cdots 0 \end{pmatrix}$$

Wave fns associated with the stationary states

We are going to figure out the quantity $\langle x|\varphi_n\rangle$ (i.e. obtain an $|x\rangle$ representation for the $|\varphi_n\rangle$)

We have already obtained $\varphi_0(x)$:
Recall:

$$a|\varphi_0\rangle = 0$$

$$\Rightarrow \left[\sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} + \frac{i}{\sqrt{2m\hbar\omega}}\,\hat{p}\right]|\varphi_0\rangle = 0$$

$$\langle x|\sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} + \frac{i}{\sqrt{2m\hbar\omega}}\,\hat{p}\,|\varphi_0\rangle = 0$$

$$\Rightarrow \left(\frac{m\omega}{\hbar}x + \frac{d}{dx}\right)\varphi_0(x) = 0$$

$$\therefore \varphi_0(x) = c\,e^{-\frac{m\omega}{2\hbar}x^2}$$

Obtain 'c' through normalization $= \sqrt{\frac{m\omega}{\pi\hbar}} = c$

$$\therefore \varphi_0(x) = \sqrt{\frac{m\omega}{\pi\hbar}}\,e^{-(m\omega/2\hbar)x^2}$$

To obtain the other eigenkets in the x representation use:

$$|\varphi_n\rangle = \frac{(a^+)^n}{\sqrt{n!}}|\varphi_0\rangle$$ and use the expression for $\hat{a}^+$ in terms of $\hat{x}$ and $\hat{p}$.

$$\cancel{\varphi_1(x)} \quad |\varphi_1\rangle = \frac{a^{+}}{\sqrt{1}} |\varphi_0\rangle$$

$$\langle x | \varphi_1 \rangle = \langle x | a^{+} | \varphi_0 \rangle$$

$$= \langle x | \sqrt{\frac{m\omega}{2\hbar}}\, x - \frac{i}{\sqrt{2m\omega\hbar}}\, \hat{p} \,|\varphi_0\rangle$$

$$\varphi_1(x) = \sqrt{\frac{\hbar}{2m\omega}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \left[\frac{m\omega}{\hbar} x - \frac{d}{dx}\right] e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2}$$

n^{th} degree polynomial

$\downarrow$

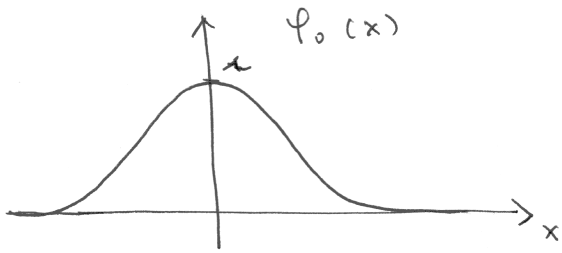
Hermite Polynomial

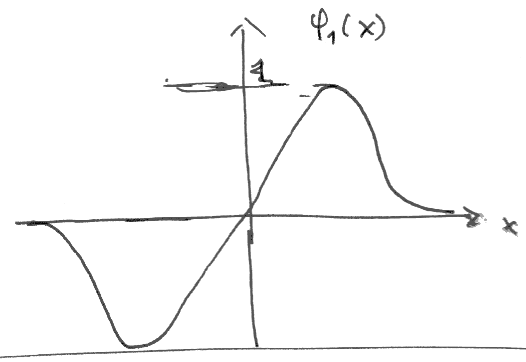
$$\Rightarrow \varphi_1(x) = \left[\frac{4}{\pi}\left(\frac{m\omega}{\pi}\right)^3\right]^{1/4} x\, e^{-(m\omega/2\hbar)x^2}$$

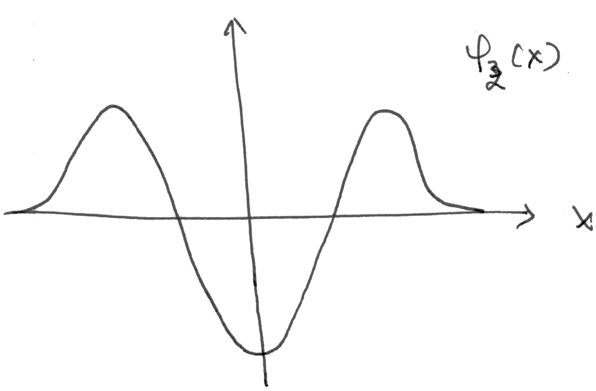
lll^{ly}

$$\varphi_2(x) = \left(\frac{m\omega}{4\pi\hbar}\right)^{1/4} \left[2\frac{m\omega}{\hbar}x^2 - 1\right] e^{-(m\omega/2\hbar)x^2}$$

$$\Rightarrow \varphi_n(x) = \left[\frac{1}{2^n n!}\left(\frac{\hbar}{m\omega}\right)^n\right]^{1/2} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \cdot \left[\frac{m\omega}{\hbar}x - \frac{d}{dx}\right]^n e^{-(m\omega/2\hbar)x^2}$$





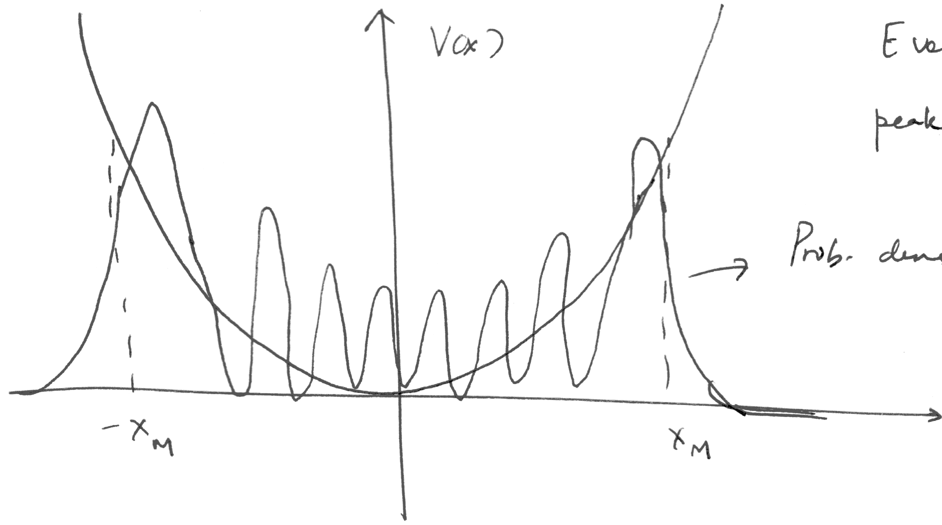


no. of zeros $= n$

n incr. - $\varphi_n(x)$ takes on non-negligible values for a larger region $\Rightarrow \langle V(x)\rangle$ incr.

n inc. no. of zeros incr $\Rightarrow$ Curvature inc.

$\langle P \rangle \sim \frac{d^2}{dx^2}\varphi_n(x) \rightarrow$ incr



Eventually prob. density peak around $\pm x_M$. [classically a particle spends a lot of time at the end points where k.E = 0]

Mean values and RMS value of $\hat{x}, \hat{p}$ in a state $|\varphi_n\rangle$.

$\hat{X}, \hat{P}$ do not commute with $\hat{H}$. Hence the eigenstates of $\hat{H}$ are not the eigenstates of $\hat{x}$ or $\hat{p}$.

Let us calculate:

$\langle \varphi_n | \hat{x} | \varphi_n \rangle$ and $\langle \varphi_n | \hat{p} | \varphi_n \rangle$

$\hat{X}$ in the $\hat{a}, \hat{a}^+$ basis:

$$\hat{X} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a}^+ + \hat{a})$$

$$\hat{P} = \cancel{\frac{i}{\sqrt{2\hbar m\omega}}} (\cancel{\hat{a}^+} - \cancel{\hat{a}}) \quad ; \quad \sqrt{\frac{m\hbar\omega}{2}} (\hat{a}^+ - \hat{a})$$

$$\Rightarrow \langle \varphi_n | \hat{X} | \varphi_n \rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle \varphi_n | \hat{a}^+ + \hat{a} | \varphi_n \rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left[\langle \varphi_n | \varphi_{n+1} \rangle \sqrt{n+1} + \langle \varphi_n | \varphi_{n-1} \rangle \sqrt{n} \right]$$

$$= 0$$

III^{ly}

$$\langle \varphi_n | \hat{p} | \varphi_n \rangle = 0$$

$$\hat{X}^2 = \frac{\hbar}{2m\omega}(a^+ + a)(a^+ + a) = \frac{\hbar}{2m\omega}\left(a^{+2} + aa^+ + a^+a + a^2\right)$$

$$\hat{p}^2 = -\frac{m\hbar\omega}{2}(a^+ - a)(a^+ - a)$$

$$= -\frac{m\hbar\omega}{2}\left(\hat{a}^{+2} - aa^+ - a^+a + a^2\right)$$

$$\therefore \langle \varphi_n | \hat{x}^2 | \varphi_n \rangle = \frac{\hbar}{2m\omega} \langle \varphi_n | aa^+ + a^+a | \varphi_n \rangle$$

But $aa^+ = 1 + a^+a$

$$\Rightarrow \langle \varphi_n | \hat{x}^2 | \varphi_n \rangle = \frac{\hbar}{2m\omega} \langle \varphi_n | 2a^+a + 1 | \varphi_n \rangle = \frac{\hbar}{2m\omega}(2n+1)$$

$$= \frac{\hbar}{m\omega}(n + 1/2)$$

$$\langle \varphi_n | \hat{p}^2 | \varphi_n \rangle = +\frac{m\hbar\omega}{2}\left(\langle \varphi_n | aa^+ + a^+a | \varphi_n \rangle\right)$$

$$= \frac{m\hbar\omega}{2}(2n+1) = m\hbar\omega(n + 1/2)$$

$$\therefore (\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{\hbar}{m\omega}(n + 1/2)$$

$$(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2 = m\hbar\omega(n + 1/2)$$

$$\therefore (\Delta x)^2 \cdot (\Delta p)^2 = (n + 1/2)^2 \hbar^2$$

$$\Rightarrow \boxed{(\Delta x)(\Delta p) = (n + 1/2)\hbar}$$

Note that we get minimum uncertainty when $n=0$. This is because the ground state wavefn is a Gaussian.