Schwinger's Oscillator Model : Angular Momentum

Consider two simple harmonic oscillators. Let us call one "plus type" and the other "minus type". Then we have for the plus type oscillator: a_+ and $a_+^\dagger$ annihilation and creation operators and for the minus type oscillator a_- and $a_-^\dagger$. We can define two number operators

$$\hat{N}_+ = \hat{a}_+^\dagger \hat{a}_+ \qquad \hat{N}_- = \hat{a}_-^\dagger \hat{a}_-$$

We have:

$$[a_+, a_+^\dagger] = 1 \qquad [a_-, a_-^\dagger] = 1$$

$$[N_+, a_+] = -a_+ \qquad [N_-, a_-] = -a_-$$

$$[N_+, a_+^\dagger] = a_+^\dagger \qquad [N_-, a_-^\dagger] = a_-^\dagger$$

The two oscillators are uncoupled. This implies:

$$[a_+, a_-^\dagger] = [a_-, a_+^\dagger] = 0 \text{ etc.}$$

$$\Rightarrow [N_+, N_-] = 0$$

Since they commute, we can build simultaneous eigenkets of $\hat{N}_+$, $\hat{N}_-$. Let us denote them: $|n_+, n_-\rangle$

$$N_+ |n_+, n_-\rangle = n_+ |n_+, n_-\rangle$$

$$\hat{N}_- |n_+, n_-\rangle = n_- |n_+, n_-\rangle$$

We already know the action of the various raising and lowering operators: i.e

$$\cancel{\hat{N}_+ |n_+, n_-\rangle}$$

$$a_+^\dagger |n_+, n_-\rangle = \sqrt{n_+ + 1}\, |n_+ + 1, n_-\rangle$$

$$a_-^\dagger |n_+, n_-\rangle = \sqrt{n_- + 1}\, |n_+, n_- + 1\rangle$$

$$a_+ |n_+, n_-\rangle = \sqrt{n_+}\, |n_+ - 1, n_-\rangle$$

$$a_- |n_+, n_-\rangle = \sqrt{n_-}\, |n_+, n_- - 1\rangle$$

In general:

$$|n_+, n_-\rangle = \frac{(a_+^\dagger)^{n_+} \, (a_-^\dagger)^{n_-}}{\sqrt{n_+!}\,\sqrt{n_-!}} \underbrace{|0, 0\rangle}_{\text{grnd state}}$$

Let us re-define some operators:

$$J_+ = \hbar\, a_+^\dagger a_- \qquad J_- = \hbar\, a_-^\dagger a_+$$

$$J_z = \frac{\hbar}{2}\left(a_+^\dagger a_+ - a_-^\dagger a_-\right) = \frac{\hbar}{2}\left(N_+ - N_-\right)$$

Then:

$$[J_+, J_-] = \left[\hbar\, a_+^\dagger a_-,\ \hbar\, a_-^\dagger a_+\right]$$

$$= \hbar^2 \left[a_+^\dagger a_-,\ a_-^\dagger a_+\right]$$

$$= \cancel{\hbar^2 \Big[\ \hbar^2 \left(a_+^\dagger [a_-, a_-^\dagger] a_+ + a_-^\dagger [a_+^\dagger, a_+] a_- \right)}$$

$$= \hbar^2 \cancel{\left[a_+^\dagger (1) a_+ + a_-^\dagger (-1) a_- \right]} = \hbar^2 \cancel{\left[N_+ - N_- \right]} = \hbar \cancel{J_z}$$

$$= \hbar^2 \left(a_+^\dagger [a_-, a_-^\dagger a_+] + [a_+^\dagger, a_-^\dagger a_+] a_- \right)$$

$$= \hbar^2 \left(a_+^\dagger [a_-, a_-^\dagger] a_+ + a_-^\dagger [a_+^\dagger, a_+] a_- \right)$$

$$= \hbar^2 \left(a_+^\dagger a_+ - a_-^\dagger a_- \right)$$

$$= 2\hbar J_z$$

$$\therefore [J_+, J_-] = 2\hbar J_z \quad \text{——— (1)}$$

$$[J_z, J_+] = \left[\frac{\hbar}{2}(a_+^\dagger a_+ - a_-^\dagger a_-),\ \hbar a_+^\dagger a_-\right]$$

$$= \left[\frac{\hbar}{2}(N_+ - N_-),\ \hbar a_+^\dagger a_-\right]$$

$$= \frac{\hbar^2}{2}[N_+, a_+^\dagger a_-] - \frac{\hbar^2}{2}[N_-, \cancel{\hbar}\, a_+^\dagger a_-]$$

$$= \frac{\hbar^2}{2}[N_+, a_+^\dagger] a_- - \frac{\hbar^2}{2} a_+^\dagger [N_-, a_-]$$

$$= \frac{\hbar^2}{2}\left(a_+^\dagger a_- + a_+^\dagger a_- \right)$$

$$= \hbar^2 (a_+^\dagger a_-) = \hbar J_+$$

$$[J_z, J_+] = \hbar J_+$$

III^{ly} $[J_z, J_-] = -\hbar J_-$

Let us define a total $\hat{N}$ as:

$$\hat{N} = \hat{N}_+ + \hat{N}_- = a_+^\dagger a_+ + a_-^\dagger a_-$$

Define:

$$|\vec{J}|^2 = J_z^2 + \frac{1}{2}\left(J_+ J_- + J_- J_+\right)$$

$$= \frac{\hbar^2}{2}\left(N_+ \mp N_-\right)^2 + \frac{\hbar^2}{2}\left(a_+^\dagger a_- a_-^\dagger a_+ + a_-^\dagger a_+ a_+^\dagger a_-\right)$$

$$= \frac{\hbar^2}{2}\Big(N_+^2 + N_-^2 + N_+ N_- - N_- N_+$$
$$+ a_+ a_+^\dagger a_- a_-^\dagger + a_- a_-^\dagger a_+ a_+^\dagger\Big)$$

$$= \frac{\hbar^2}{2}\Big(N_+^2 + N_-^2 + N_+ N_- - N_- N_+ + (N_+ + 1)(N_- + 1)$$
$$+ (N_- + 1)(N_+ + 1)\Big]$$

$$= \frac{\hbar^2}{2}\Big(N_+^2 + N_-^2 + N_+ N_- - \cancel{N_- N_+} + N_+ N_- + N_+ + N_- + 1$$
$$+ \cancel{N_- N_+} + N_- + N_+ + 1\Big)$$

left as an exercise! $\Big\{$ $\vdots$

$$= \frac{\hbar^2}{2}(N)\left(\frac{N}{2} + 1\right)$$

$\longrightarrow$ (obtained using the defn. for $\hat{N}$ and re-grouping terms).

We have 2 types of oscillators. Let the plus oscillator represent the spin up state $|\uparrow\rangle$, and the minus oscillator the spin down state $|\downarrow\rangle$.

$$|\uparrow\rangle = |\tfrac{1}{2}\ m_s = \tfrac{1}{2}\rangle$$

$$|\downarrow\rangle = |\tfrac{1}{2}\ m_s = -\tfrac{1}{2}\rangle$$

Now: $\hat{n}_+$ ~~creates a unit of~~ counts the number of spin '+'

$\hat{n}_-$: counts the number of spin downs.

J_+ : destroys a spin down and creates a spin up.

J_-: destroys a spin up and creates a spin minus.

J_z: counts the diff: (no. of spin up − no. of spin downs).

$$J_+|n_+, n_-\rangle = \hbar\, a_+^{\dagger} a_- |n_+, n_-\rangle = \hbar\sqrt{(n_+ + 1)\, n_-}\ |n_+ + 1, n_- - 1\rangle$$

$$J_-|n_+, n_-\rangle = \hbar\, a_-^{\dagger} a_+ |n_+, n_-\rangle = \hbar\sqrt{(n_- + 1)\, n_+}\ |n_+ - 1, n_+ + 1\rangle$$

$$J_z|n_+, n_-\rangle = \frac{\hbar}{2}(\hat{N}_+ - \hat{N}_-)|n_+, n_-\rangle = \frac{\hbar}{2}(n_+ - n_-)|n_+, n_-\rangle.$$

Notice that the total no. of spins remain constant.
⇒ Total no. of spin ½ particles remains constant.

Now the energy of the oscillator:

$$E = E_+ + E_- = \cancel{\tfrac{1}{2}}\, \hbar\omega(\cancel{N_+}\, n_+ + 1/2) + \hbar\omega(n_- + 1/2)$$
$$= \hbar\omega(n_+ + n_- + 1)$$
$$= \hbar\omega(N+1) \qquad N = 0, 1, 2 \ldots$$

$$E_0 = \hbar\omega$$
$$E_1 = 2\hbar\omega$$
$$E_2 = 3\hbar\omega$$
$$\vdots$$
$$E_n = (n+1)\hbar\omega$$

$E_0 : \hbar\omega \Rightarrow n_+ = n_- = 0 \longrightarrow$ Non-degenerate

$E_1 = 2\hbar\omega \Rightarrow n_+, n_- = ((0,1)\ (1,0)) \Rightarrow$ 2 solutions $\Big\}$ $g=2$

$|\Psi\rangle = |\varphi_1\, \varphi_0\rangle$ or $|\varphi_0, \varphi_1\rangle$

$E_2 = 3\hbar\omega \Rightarrow n_+, n_- = ((2,0), (0,2), (1,1))\Big\}$ $g=3$

$\therefore$ N^{th} level is $(N+1)$ fold degenerate.

If $S = 1/2$: $1/2$, $-1/2$ $\Big\}$ 2 fold degeneracy.

$S = 1$: 1, 0, -1 $\Big\}$ 3 fold degeneracy.

$S = 3/2$: $-3/2$, $-1/2$, $1/2$, $3/2$ $\Big\}$ 4 fold degeneracy

etc.

$\therefore$ We see that $N=0 \rightarrow$ spinless

$N=1 \rightarrow$ spin $1/2$

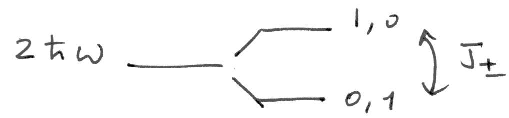
$N=2 \rightarrow$ spin 1

etc.

J_+, J_- moves up and down within a spin multiplet!

Ex:

$N=1 = (0,1),\ (1,0).$



$$J_+ |0,1\rangle = \hbar a_+^\dagger a_- |0,1\rangle = \hbar |1,0\rangle$$

$$J_- |1,0\rangle = \hbar a_-^\dagger a_+ |1,0\rangle = \hbar |0,1\rangle$$

$$\boxed{\begin{aligned} J_- |0,1\rangle &= \hbar a_-^\dagger a_+ |0,1\rangle = 0 \\ J_+ |1,0\rangle &= \hbar a_+^\dagger a_- |1,0\rangle = 0 \end{aligned}}$$

$(a_+ |0\rangle = 0)$

$(a_- |0\rangle = 0)$

$\downarrow$

Cannot move past the top most level, or the bottommost level!

<u>What have we seen:</u>

- J : Represents a generic angular momentum (Angular, Spin, total $= l+s$, Isospin)
- $J \rightarrow J_x, J_y, J_z \longrightarrow J_+, J_-, J_z$ and J_+, J_- are linear combinations of J_x and J_y. and are raising and lowering operators.
- $\{J_\pm, J_z\}$ or $\{J_x, J_y, J_z\}$ satisfy commutation relations.
- Any angular momentum J has $(2J+1)$ degeneracies.

Theory of Angular momentum :

An angular momentum J_i is a generator of rotations and the origin of the commutation relations lies in the geometric properties of rotations. ~~in 3 dim~~ Given a generic angular momentum vector $\vec{J}$, it has three components J_x, J_y, J_z. Quantum mechanically, these are the three operators $\hat{J}_x, \hat{J}_y, \hat{J}_z$ (related to observables).
and they satisfy:

$$\left.\begin{aligned} [J_x, J_y] &= i\hbar J_z \\ [J_y, J_z] &= i\hbar J_x \\ [J_z, J_x] &= i\hbar J_y \end{aligned}\right\} \quad [J_i, J_j] = i\epsilon_{ijk} J_k \hbar$$

Define an operator $J^2 = J_x^2 + J_y^2 + J_z^2$.

- J_x, J_y, J_z, J^2 are all Hermitian

Calculate: $[J^2, J_x] = [J_x^2 + J_y^2 + J_z^2, J_x]$

$$= [J_y^2, J_x] + [J_z^2, J_x]$$

$$= J_y [J_y, J_x] + [J_y, J_x] J_y + J_z [J_z, J_x] + [J_z, J_x] J_z$$

$$= \cancel{J_y (-i\hbar J_z)} - i\hbar \cancel{J_z J_y} + \cancel{J_z [i\hbar J_y]} + i\hbar \cancel{J_y J_z} = 0$$

Same is true for the other two components $\Rightarrow$ $[J^2, J_i] = 0$

- Since the different components do not commute, it is not possible to measure all three components simultaneously.

- Define $J_+ = J_x + i J_y$
$J_- = J_x - i J_y$

J_+, J_- are not Hermitian. They are adjoints of each other.

We have already established:

$$\left.\begin{aligned} &[J_z, J_+] = \hbar J_+ \\ &[J_z, J_-] = -\hbar J_- \\ &[J_+, J_-] = 2\hbar J_z \\ &[J^2, J_z] = [J^2, J_\pm] = 0 \end{aligned}\right\}$$

You can show this using $[J_i, J_j] = i \epsilon_{ijk} J_k$

$$J_+ J_- = (J_x + iJ_y)(J_x - iJ_y)$$

$$\begin{aligned} &= J_x^2 + J_y^2 + i J_y J_x - i J_x J_y \\ &= J_x^2 + J_y^2 - i [J_x, J_y] \\ &= J_x^2 + J_y^2 - i (i\hbar J_z) \\ &= J_x^2 + J_y^2 + \hbar J_z \end{aligned}$$

lllly

$$J_- J_+ = J_x^2 + J_y^2 - \hbar J_z$$

$$\therefore \left.\begin{aligned} J_+ J_- &= J^2 - J_z^2 + \hbar J_z \\ J_- J_+ &= J^2 - J_z^2 - \hbar J_z \end{aligned}\right\} \Rightarrow \boxed{J^2 = J_z^2 + \frac{1}{2}(J_+ J_- + J_- J_+)}$$

Notice that $[J^2, J_i] = 0$. Our <u>Goal</u>: is to find the common eigenstates of J^2 and a component of $\vec{J}$, say the z-component J_z.

<u>Eigenvalues of J^2 and J_z:</u>

$$J^2 = J_x^2 + J_y^2 + J_z^2$$

$$\Rightarrow \langle\psi|J^2|\psi\rangle = \langle\psi|J_x^2|\psi\rangle + \langle\psi|J_y^2|\psi\rangle + \langle\psi|J_z^2|\psi\rangle \geq 0$$

– Expected, since J^2 is the length of a vector.

Proof: Let $|\psi\rangle$ be an eigenvector of J^2

$$\Rightarrow \quad J^2|\psi\rangle = G|\psi\rangle$$

$$\therefore \quad \langle\psi|J^2|\psi\rangle = G\langle\psi|\psi\rangle \geq 0$$

Since we want to find the simultaneous eigenkets of J^2 and J_z, let us denote this ket as $|a, b\rangle$, where a is the eigenvalue of J^2 and b is the eigenvalue of J_z. i.e:

$$J^2|a, b\rangle = a|a, b\rangle$$

$$J_z|a, b\rangle = b|a, b\rangle.$$

In order to figure out the allowed values of a, b. we will work with $J_\pm$, the non-Hermitian operators instead of J_x, J_y.

We already know: $[J_+, J_-] = 2\hbar J_z$, $\quad [J^2, J_\pm] = 0$

$$[J_z, J_\pm] = \pm\hbar J_\pm$$

Consider:

~~Let~~ $J_+|a,b\rangle = |\ \rangle'$

Apply $J_z|\ \rangle' = J_z J_+ |a,b\rangle$

Using the relation: $[J_z, J_+] = \hbar J_+$

$$J_z J_+ - J_+ J_z = \hbar J_+$$

Then: $$J_z J_+ |a,b\rangle = (\hbar J_+ + J_+ J_z)|a,b\rangle$$
$$= \hbar J_+ |a,b\rangle + b J_+ |a,b\rangle$$
$$= (\hbar + b) J_+ |a,b\rangle$$

IIIly:

$$J_z J_- |a,b\rangle = (b - \hbar) J_- |a,b\rangle.$$

$\therefore$ ~~J_+ raises~~ (i) $J_\pm|a,b\rangle$ is an eigenket of J_z

(ii) $J_+|a,b\rangle$ has an eigenvalue $b+\hbar$

(iii) $J_-|a,b\rangle$ has an eigenvalue $b-\hbar$

$J_\pm \rightarrow$ Ladder operators. They increase/decrease the eigenvalue of J_z i.e. b by one unit of $\hbar$. (Reminder: the commutation relations seen here are very similar to the ones seen with the oscillator).

Now let us see ~~the~~ what happens when we apply J^2 to the ket $J_\pm|a,b\rangle$:

$$J^2 J_\pm |a,b\rangle = J_\pm J^2 |a,b\rangle \quad (\text{Since } [J^2, J_\pm] = 0)$$
$$= J_\pm a |a,b\rangle$$
$$= a\,[J_\pm |a,b\rangle]$$

$J_\pm|a,b\rangle$ is an eigenket of J^2 with the same eigenvalue.

Once again very similar to the oscillator case:
if $|a,b\rangle$ is a simultaneous eigenket of J^2 and J_z, then $J_\pm |a,b\rangle$ is also a simultaneous eigenket of J^2 and J_z.
But:

$$J^2 |a,b\rangle = a|a,b\rangle \qquad J_z|a,b\rangle = b|a,b\rangle$$
$$J^2 [J_\pm |a,b\rangle] = a[J_\pm |a,b\rangle] \qquad J_z [J_\pm |a,b\rangle] = (b\pm\hbar)[J_\pm|a,b\rangle]$$

∴ We can conclude:
$$J_\pm |a,b\rangle = C_\pm |a, b\pm\hbar\rangle$$

so that $J_z [J_\pm |a,b\rangle] = C_\pm (b\pm\hbar) |\underbrace{a, b\pm\hbar}_{\text{Labelled by the eigenvalue of } J_z}\rangle$

$C_\pm$ will be determined later using the normalization of the eigenkets.

- If we apply J_+ n times to the ket $|a,b\rangle$, then the value of J_z eigenvalue becomes $(b+n\hbar)$; while the eigenvalue of J^2 is unchanged. But this process has an upper limit. It turns out that for a given 'a' which is the eigenvalue of J^2,
$$a \geq b^2$$

<u>Proof:</u>
$$J^2 - J_z^2 = \frac{1}{2}(J_+ J_- + J_- J_+)$$

We know that $J_- = J_+^+$
$$\Rightarrow J^2 - J_z = \frac{1}{2}(J_+ J_+^+ + J_+^+ J_+)$$

Now:

$$\langle a, b | J^2 - J_z^2 | a, b\rangle = \frac{1}{2} \langle a, b | J_+ J_+^+ + J_+^+ J_+ | a, b\rangle$$

$$= \frac{1}{2} \langle a^* b^* | J_+ J_+^+ | a, b\rangle + \langle a' b' | J_+^+ J_+ | a, b\rangle$$

Now: $J_+ | a, b\rangle = | \ \rangle'$

then its dual $'\langle \ | = \langle a, b | J_+^+$

$|||^{ly}$ $J_+^+ | a, b\rangle$ has $\langle a, b | J_+$ for its dual.

$\therefore$ we have sum of 2 norms and this has to be ≥ 0

$\Rightarrow (a - b^2) \geq 0$ or $\boxed{a \geq b^2}$

$\Rightarrow$ There is a b_{max} such that:

$J_+ | a, b_{max}\rangle = 0$

$\Rightarrow J_- J_+ | a, b_{max}\rangle = 0 \quad (J_+ | a, b_{max}\rangle = \text{NULL})$.

$$J_- J_+ = (J_x - i J_y)(J_x + i J_y)$$
$$= J_x^2 + J_y^2 - i (J_y J_x - J_x J_y)$$
$$= J_x^2 + J_y^2 - i \underbrace{[J_y, J_x]}_{-J_z \hbar i}$$
$$= J_x^2 + J_y^2 - \hbar J_z$$

$$\Rightarrow (\underbrace{J_x^2 + J_y^2}- \hbar J_z) | a, b_{max}\rangle = 0$$
$$(J^2 - J_z^2 - \hbar J_z) | a, b_{max}\rangle = 0.$$

$$a - b_{max}^2 - b_{max}\hbar = 0$$

$$\boxed{a = b_{max}(b_{max} + \hbar)}$$

There should also exist a b_{min} such that: $\left(\begin{array}{l} b^2 \leq a \\ b_{min} \leq b \leq b_{max} \end{array} \right)$

$$J_- |a, b_{min}\rangle = 0$$

$$\Rightarrow J_+ J_- |a, b_{min}\rangle = 0$$

$$J_+ J_- = J^2 - J_z^2 + \hbar J_z .$$

$$\Rightarrow \quad J_+ J_- |a, b_{min}\rangle = (J^2 - J_z^2 + \hbar J_z) |a, b_{min}\rangle = 0$$

or: $\boxed{a = b_{min}(b_{min} - \hbar)}$

$\Rightarrow b_{max} = -b_{min}$ or: $-b_{max} \leq b \leq b_{max}$.

If we apply J_+ successively to $|a, b_{min}\rangle$, then we should have $|a, b_{max}\rangle$

$$\Rightarrow b_{max} = b_{min} + n\hbar$$

$$= -b_{max} + n\hbar$$

$$\Rightarrow \boxed{b_{max} = \cancel{\frac{m\hbar}{2}} \quad \frac{n\hbar}{2}}$$

Define $\dfrac{b_{max}}{\hbar} = j = \dfrac{n}{2}$ (convention).

$$\Rightarrow a = \hbar^2 \left(\frac{b_{max}}{\hbar}\right)\left(\frac{b_{max}}{\hbar} + 1\right)$$

$$= \hbar^2 j (j+1)$$

and let $b = m\hbar$

If j = integer

then b_{max} = integer b_{min} = − integer.

⇒ m can take values in the range $-j$ to j. with increments of 1 (action of J_+).

$$\boxed{m = -j,\ -j+1,\ -j+2 \ldots\ +j-1,\ j}$$

label the ket $|a,b\rangle$ with $|j,m\rangle$

Then: $\boxed{\begin{array}{l} J^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle \\ J_z|j,m\rangle = m\hbar\,|j,m\rangle \end{array}}$ → Comes from commutation rules. → Comes from angular momentum being a generator of rotations

If j = ½ integer, then m has to take ½ integer values.

Therefore in the $|j,m\rangle$ basis:

$$\langle j'm'|J^2|j,m\rangle = \hbar^2 j(j+1)\,\delta_{jj'}\,\delta_{mm'}$$

and

$$\langle j'm'|J_z|j,m\rangle = m\hbar\,\delta_{jj'}\,\delta_{mm'}$$

$$\langle j,m|J_+^{+}J_+|j,m\rangle = \langle j,m|J^2 - J_z^2 - \hbar J_z|j,m\rangle$$
$$= \hbar^2\left[j(j+1) - m^2 - m\right]$$

$$\left.\begin{array}{l} J_+|j,m\rangle = c_{jm}^{+}|j,m+1\rangle \\ \langle j,m|J_+^{+} = c_{jm}^{*}\langle j,m+1| \end{array}\right\} \Rightarrow |c_{jm}^{+}|^2 = \hbar^2\left(j(j+1) - m^2 - m\right)$$

$$\Rightarrow \boxed{c_{jm} = \hbar\sqrt{(j-m)(j+m+1)}}$$

$|||^{ly}$

Using $J_+ J_- |jm\rangle = J_-^\dagger J_- |j,m\rangle$

One can obtain $c_{jm} = \sqrt{(j+m)(j-m+1)}\ \hbar$ ~~K_j~~

$$\therefore \boxed{\begin{array}{l} J_+ |j,m\rangle = \hbar\sqrt{(j-m)(j+m+1)}\ |j,m+1\rangle \\ J_- |j,m\rangle = \hbar\sqrt{(j+m)(j-m+1)}\ |j,m-1\rangle \end{array}}$$

$$\therefore \langle j'm' | J_\pm | j,m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\ \hbar\ \cancel{K_j}\ \langle j',m' | j, m\pm 1\rangle$$

$$= \sqrt{(j\mp m)(j\pm m+1)}\ \hbar\ \delta_{jj'}\,\delta_{m',m\pm 1}$$

<u>How do operators in this space look like?</u>

Let us look at the rotation operator: $\hat{R} = e^{-i\vec{J}\cdot\hat{n}\,\phi/\hbar}$

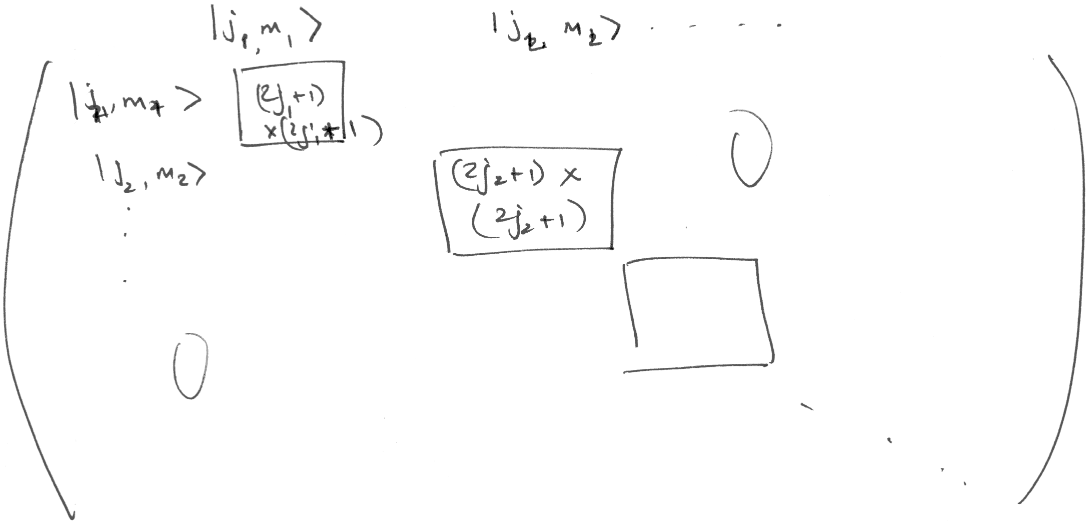
$$J^2 \hat{R}|j,m\rangle = \hat{R} J^2 |j,m\rangle = j(j+1)\hbar^2\, \hat{R}\,|j,m\rangle.$$

$\Rightarrow \hat{R}|j,m\rangle$ is an eigenket of J^2. $\because [J^2, J_i] = 0$

Unless $\hat{n} = \hat{z}$, $R_z = e^{-iJ_z\phi/\hbar}$

$[J_z, R_z] = 0$

But for arbitrary $\hat{n}$, $[J_z, \hat{R}] \neq 0 \Rightarrow$ a general rotation mixes m values but not j.



→ Best possible decomposition
IRREPS

True of any operator that commutes with J^2. (Rotationally invariant).

Orbital Angular Momentum:

When a particle does not have spin ($s=0$), then the theory we discussed so far applies to orbital angular momentum. For the orbital angular momentum (that has a classical analogue) we can set up the angular momentum algebra in a completely different way, by working in the $|\vec{r}\rangle$ basis.

We will look for eigenfunctions of L^2 and L_z.

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\therefore L_x = -i\hbar \left(\vec{r} \times \nabla_{\vec{x}} \right)_x = -i\hbar \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right)$$

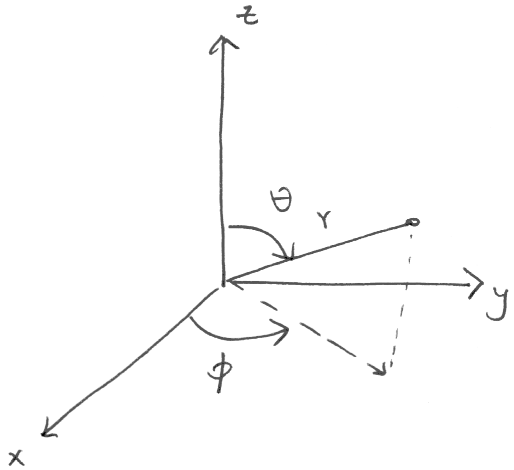
$$L_y = -i\hbar \left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right)$$

$$L_z = -i\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)$$

$$\cancel{L_y = -i\hbar \left(x \frac{\partial}{\partial z} - \right.}$$

It turns out that spherical co-ordinates are more useful here.

$$\left.\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta \end{aligned}\right\} \quad \begin{aligned} & r \geq 0 \\ & 0 \leq \theta \leq \pi \\ & 0 \leq \phi \leq 2\pi \end{aligned}$$



Changing variables:

$$L_x = i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{\tan\theta}\frac{\partial}{\partial\varphi}\right)$$

$$L_y = i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \frac{\sin\varphi}{\tan\theta}\frac{\partial}{\partial\varphi}\right)$$

$$L_z = \cancel{\frac{\hbar}{i}}\ -i\hbar\frac{\partial}{\partial\phi}$$

$$\therefore L^2 = L_x^2 + L_y^2 + L_z^2$$

$$= -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \frac{1}{\tan\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

$$L_+ = L_x + iL_y = \hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$$

$$L_- = L_x - iL_y = \hbar e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$$

We are looking for the eigenfunctions:

$$L^2 \psi_{\ell m}(\theta,\varphi) = \ell(\ell+1)\hbar^2 \psi_{\ell m}(\theta,\varphi)$$

$$L_z \psi_{\ell m}(\theta,\varphi) = m\hbar\, \psi_{\ell m}(\theta,\varphi).$$

We already know that ℓ can take integral or half integral values and m takes $(2\ell+1)$ values $-\ell \leq m \leq \ell$ in steps of 1.

Values of ℓ and m:

Using:

$$L_z \psi_{\ell m}(\theta,\varphi) = m\hbar\, \psi_{\ell m}(\theta,\varphi)$$

$$-i\hbar \frac{\partial}{\partial\varphi} \psi_{\ell m}(\theta,\varphi) = m\hbar\, \psi_{\ell m}(\theta,\varphi)$$

~~Let us~~ We see that since this not depend on θ:

$$\psi_{\ell m}(\theta,\varphi) = F_{\ell m}(\theta)\, \cancel{e^{im\varphi}}\, \chi(\varphi)$$

$$\Rightarrow -i\cancel{\hbar} \frac{\partial}{\partial\varphi} \chi(\varphi) = m\cancel{\hbar}\, \chi(\varphi)$$

Or: $\chi(\varphi) = e^{im\varphi}$

$$\therefore \psi_{\ell m}(\theta,\varphi) = F_{\ell m}(\theta)\, e^{im\varphi}.$$

~~$\varphi = 2\pi$~~ From continuity: $\psi_{\ell m}(\theta, 0) = \psi_{\ell m}(\theta, 2\pi)$

$\Rightarrow e^{2im\pi} = 1$ $\quad\therefore$ m = integer values.

$\Rightarrow$ $\boxed{\ell = \text{integer values}}$

Now we choose $m = l$.

Then $\Psi_{ll}(\theta, \varphi) = F_{ll}(\theta)\, e^{il\varphi}$

Now: $L_+ \Psi_{ll} = 0$

$$\hbar e^{i\varphi} \left(\frac{\partial}{\partial \theta} + i \cot\theta \frac{\partial}{\partial \phi} \right) F_{ll}(\theta)\, e^{il\phi} = 0$$

$$\Rightarrow \left[\frac{\partial}{\partial \theta} F_{ll}(\theta) \right] e^{il\varphi} - (l \cot\theta) F_{ll}(\theta)\, e^{il\phi} = 0$$

$$\Rightarrow \frac{\partial}{\partial \theta} F_{ll}(\theta) - (l \cot\theta) F_{ll}(\theta) = 0$$

$$F_{ll}(\theta) = C_l (\sin\theta)^l$$

C_l : Normalization constant.

$$\therefore \Psi_{ll}(\theta, \varphi) = C_l (\sin\theta)^l e^{il\varphi}$$

Repeatedly applying $L_- \Psi_{ll}(\theta, \varphi) \rightarrow \Psi_{l, l-1}(\theta\, \varphi)$.
$\Psi_{l, l-2}(\theta, \varphi)$ . . . . $\Psi_{l\, -l}$

These eigenfunctions $\Psi_{ll}(\theta, \varphi)$ are the spherical Harmonics $\underline{\underline{Y_{lm}(\theta, \varphi)}}$

Therefore: $\cancel{L^2 \Psi}$ $\quad L^2 Y_{lm}(\theta, \varphi) = l(l+1)\hbar^2 Y_{lm}(\theta, \varphi)$

$$L_z Y_{lm}(\theta, \varphi) = m\hbar Y_{lm}(\theta, \varphi).$$

One can show:

$$L_{\pm} Y_{\ell m}(\theta,\varphi) = \hbar\sqrt{\ell(\ell+1) - m(m\pm 1)}\; Y_{\ell}^{m\pm 1}(\theta,\varphi)$$

We can choose the constant factor through orthonormalization

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta\; Y^*_{\ell' m'}(\theta,\varphi)\; Y_{\ell m}(\theta,\varphi) = \delta_{\ell\ell'}\,\delta_{mm'}$$

Any function $f(\theta,\varphi)$ can be expanded in this basis

$$f(\theta,\varphi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} c_{\ell m}\, Y_{\ell m}(\theta,\varphi)$$

where

$$c_{\ell m} = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\, \sin\theta\; Y^*_{\ell m}(\theta,\varphi)\, f(\theta,\varphi)$$

$Y_{\ell m} \to$ form an orthonormal basis in the space $\mathcal{E}_{\Omega}$ of functions of θ, φ.

Closure:

$$\sum_{\ell}\sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\varphi)\, Y^*_{\ell m}(\theta',\varphi') = \delta(\cos\theta - \cos\theta')\;\delta(\varphi-\varphi')$$

$$= \frac{1}{\sin\theta}\,\delta(\theta-\theta')\;\delta(\varphi-\varphi')$$

In fact it turns out that L^2, L_z alone do not form a complete set of commuting operators. In $\{\vec r\}$ space a particle of mass m, moving in a spherically symm. potential ~~is given~~ has a wave function:

$$\Psi(\vec r) = R_{\ell}(r)\, Y_{\ell m}(\theta,\varphi)$$

where $R_\ell(r)$ is the radial function. One can show that $R_\ell(r)$ is independent of m and depends on ℓ.

Spin:

- Experiments by Stern and Gerlach showed that angular momenta of silver atoms subjected to a varying field in the z direction ~~had~~ became quantized and the results could be explained only with half-integer angular momentum. Therefore, this could not be the orbital angular momentum. This angular momentum called Spin generates rotation in some internal space and is an intrinsic property of the particle.

- In order to accommodate spin, the postulates of quantum mechanics had to be modified.

- A particle has a spin $\vec{S}$ which is an angular momentum. This implies:

$$[S_x, S_y] = i\hbar S_z \quad \text{i.e: } [S_i, S_j] = i\,\epsilon_{ijk} S_k .$$

- Spin operators act in a new space "the spin state space" $\mathcal{E}_s$ and here S^2 and S_z form a complete set of commuting observables. Therefore $\mathcal{E}_s$ is spanned by $\{|s,m\rangle\}$.

where

$$S^2 |s,m\rangle = s(s+1)\hbar^2 |s,m\rangle$$
$$S_z |s,m\rangle = m\hbar |s,m\rangle$$

- s can take integral or half integral values.

- A given particle is characterized by a unique value of s.

- The spin space is always finite dimensional.

- The state space of a particle being considered is the tensor product of ξ_r and ξ_s, ξ_r : ~~state~~ orbital state space, ξ_s : spin state.

  i.e. $\xi = \xi_r \otimes \xi_s$

- All spin observables commute with all orbital observables.

Special properties of angular momentum $s = 1/2$

- ξ_s : 2 dimensional

- $|+\rangle, |-\rangle \rightarrow$ eigenkets $= |1/2 \; 1/2\rangle, \; |1/2 \; -1/2\rangle$

Most general state $|\chi\rangle = c_+|+\rangle + c_-|-\rangle$

- Any operator acting in this 2d space can be represented by a 2x2 Matrix. In particular:

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}$$

where $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

are the Pauli Matrices.

The Pauli Matrices have the following properties:

$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$

$$\left.\begin{aligned} &\sigma_x\sigma_y + \sigma_y\sigma_x = 0 \\ &[\sigma_x, \sigma_y] = 2i\sigma_z \\ &\sigma_x\sigma_y = i\sigma_z \end{aligned}\right\} + \text{cyclic permutations}$$

also:

$Tr(\sigma_i) = 0$

$Det\ (\sigma_i) = -1$

• Any 2x2 matrix (with complex coeffns) can be written as a linear combination of the 3 Pauli Matrices and the unit Matrix.

Consider an e^- of mass m. An e^- has $s = 1/2$. Therefore its state space can be described by the following q.nos. of the ops:

$\{x, y, z, S^2, S_z\}$ or $\{P_x, P_y, P_z, S^2, S_z\}$ or $\{H, L^2, L_z, S^2, S_z\}$

where H is Central Hamiltonian (Hamiltonian of with a central potential).

Denote: a basis $|\cancel{\ }\rangle = |\vec{r}, \chi\rangle$ where χ: denotes spin.

Then: Orthonormality:

$$\langle \vec{r}'\chi'|\vec{r}, \chi\rangle = \delta_{\chi'\chi}\ \delta(\vec{r} - \vec{r}')$$

and closure:

$$\sum_x{}' \int d^3r \, |\vec{r} x\rangle \langle \vec{r} x| = \int d^3r \, |\vec{r}+\rangle\langle +\vec{r}| + \int d^3r \, |\vec{r}-\rangle\langle -\vec{r}| = \mathbb{1} .$$

Any arbitrary state:

$$|\psi\rangle = \sum_x{}' \int d^3r \, |\vec{r} x\rangle\langle \vec{r} x|\psi\rangle$$

$\Rightarrow$ expansion coefn: $\langle \vec{r} x|\psi\rangle$.

Therefore we need to specify 2 fns: $\underbrace{\langle \vec{r}+|\psi\rangle}_{\Psi_+(r)}$ and $\underbrace{\langle \vec{r}-|\psi\rangle}_{\Psi_-(r)}$

These 2 numbers are usually written as:

$$[\Psi](\vec{r}) = \begin{pmatrix} \Psi_+(\vec{r}) \\ \Psi_-(\vec{r}) \end{pmatrix} \longrightarrow \text{2 component object}$$

and the bra:

$$[\Psi]^+(\vec{r}) = (\Psi_+^*(\vec{r}), \ \Psi_-^*(\vec{r}))$$