Position and Momentum Representations

Observables can either have continuous spectra or a discrete one (Spectra: set of eigenvalues). So far we had seen ~~continuous~~ discrete ones. In quantum mechanics, there are observables with continuous eigenvalues.

What are the modifications we must consider?

If $\hat{O}$ has a continuous spectra

$$\Rightarrow \hat{O}|\,\rangle = O|\,\rangle.$$

and the set of $\{O\}$, and $\{|\,\rangle\}$, which are the eigenkets are infinite in number. Therefore, the space has infinite dimensions. Fortunately, many of the results of finite dimensional VS with discrete eigenvalues can be generalized. There are of course some catches, which we will see as we move along.

Start with the eigenvalue eqn:

$$\hat{\xi}|\xi'\rangle = \xi'|\xi'\rangle$$

$\hat{\xi}$: operator

$|\xi'\rangle$: eigenkets

$\{\xi'\}$: eigenvalues

Orthonormality:

$$\langle a''|a'\rangle = \delta_{a''a'} \longrightarrow \langle \xi''|\xi'\rangle = \delta(\xi' - \xi'')$$

Completeness:

$$\sum_{a'} |a'\rangle\langle a'| = 1 \longrightarrow \int d\xi' |\xi'\rangle\langle\xi'| = \mathbb{1}$$

Expansion of an arbitrary ~~operator~~ ket:

$$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle \longrightarrow |\alpha\rangle = \int d\xi' |\xi'\rangle\langle\xi'|\alpha\rangle$$

where: $\sum_{a'} |\langle a'|\alpha\rangle|^2 = 1 \longrightarrow \int d\xi' \, |\langle \xi'|\alpha\rangle|^2 = 1.$

Inner pdts of arbitrary vectors:

$$\langle\beta|\alpha\rangle = \int d\xi' \langle\beta|\xi'\rangle\langle\xi'|\alpha\rangle$$

Representation of operators:

$$\langle\xi''|\hat{\xi}|\xi'\rangle = \xi'\delta(\xi''-\xi')$$

~~For Arbitrary operators:~~

$$\cancel{\langle\xi''|\hat{O}|\xi'\rangle =} \int \cancel{d\xi_1} \int d\xi_2 \cancel{\langle\xi''|\xi_1\rangle\langle\xi_1|\hat{O}|}$$

Position eigenkets and position measurements

Define position operator $\hat{x}$,

then:

$$\hat{x}|x\rangle = x|x\rangle$$

$|x\rangle$: position eigenkets

x : eigenvalues [dimensions of length]

An arbitrary vector:

$$|\alpha\rangle = \int_{-\infty}^{\infty} dx'' |x'\rangle\langle x''|\alpha\rangle$$

$\langle x'|\alpha\rangle$: expansion coeffns.

When a measurement is made, the ket $|\alpha\rangle$ collapses into one of the eigenvalues $\{x'\}$. Let us assume that the detector detects the particle within a small window Δ about x'.

Then:

$$|\alpha\rangle = \int_{-\infty}^{\infty} dx'' \, |x''\rangle\langle x''|\alpha\rangle \longrightarrow \int_{x'-\Delta/2}^{\Delta/2+x'} dx'' \, |x'\rangle\langle x''|\alpha\rangle$$

Prob of finding the particle within the interval dx'

$$|\langle x'|\alpha\rangle|^2 dx'$$

$\therefore$ Prob. of finding the particle somewhere between $-\infty$ and $+\infty$:

$$\int_{-\infty}^{\infty} dx' \, |\langle x'|\alpha\rangle|^2$$

which is normalized to unity if $|\alpha\rangle$ is normalized:

$$\langle\alpha|\alpha\rangle = \int_{-\infty}^{\infty} dx' \, |\langle x'|\alpha\rangle|^2 = 1$$

Incidentally: $\langle x'|\alpha\rangle$ is the representation of the vector $|\alpha\rangle$ in the basis $\{|x\rangle\}$.

We were dealing with 1D. But the ideas can be generalized to 3D:

$$|\alpha\rangle = \int d^3x' \, |\vec{x}'\rangle\langle\vec{x}'|\alpha\rangle$$

$\vec{x}'$: stands for ~~$|x'\rangle$~~, x', y', z'

i.e:

$|\vec{x}'\rangle = |x' \, y' \, z'\rangle$ simultaneous eigenkets of x', y', z'.

$\Rightarrow [x_i, x_j] = 0 \qquad i, j = 1, 2, 3$

$\qquad x_1 = x, \; x_2 = y, \; x_3 = z.$

- Translation:

Start with a state well localized around $\vec{x}'$. Consider an operation that changes this state into another well localized state around $\vec{x}' + d\vec{x}'$. Everything else is unchanged.

Such an operation is called an <u>infinitesimal translation</u> by $d\vec{x}'$.
Let $\mathcal{J}(d\vec{x}')$: denote the infinitesimal translation operator:

$$\mathcal{J}(d\vec{x}')\,|\vec{x}'\rangle = |\vec{x}' + d\vec{x}'\rangle$$

The RHS is also a position eigenstate, but with eigenvalue $\vec{x}' + d\vec{x}'$. Obviously $|\vec{x}'\rangle$ is not an eigenket of $\mathcal{J}(d\vec{x}')$.

Let us examine the effect of the infinitesimal translation operator on $|\alpha\rangle$.

$$|\alpha\rangle = \int d^3x'\, |\vec{x}'\rangle\langle\vec{x}'|\alpha\rangle$$

$$\therefore\ \mathcal{J}(d\vec{x}')|\alpha\rangle = \int d^3x'\, \mathcal{J}(d\vec{x}')\,|\vec{x}'\rangle\langle\vec{x}'|\alpha\rangle$$
$$= \int d^3x'\, |\vec{x}' + d\vec{x}'\rangle\langle\vec{x}'|\alpha\rangle$$

We can redefine: $\vec{x}'' = \vec{x}' + d\vec{x}'$

$$\therefore\ \int d^3x'\, |\vec{x}' + d\vec{x}'\rangle\langle\vec{x}'|\alpha\rangle = \int d^3x''\, |\vec{x}''\rangle\langle\vec{x}'' - dx'|\alpha\rangle$$

(since x' is a dummy variable)

Therefore the wave fn for a translated state is obtained by replacing $\vec{x} \to \vec{x} - d\vec{x}$. in $\langle\vec{x}|\alpha\rangle$

Here we considered a translation on the system. We could equivalently consider translation of the co-ordinates, where the origin is shifted in the opposite direction $(-d\vec{x}')$. That is we are asking how an observer in the new co-ordinate system would perceive $|\alpha\rangle$. We will stick to the first approach here.

<u>Properties of the translation operator</u>:

- Unitarity: If $|\alpha\rangle$ is normalized to unity, then the translated ket should also be normalized to 1.

$\Rightarrow \langle\alpha|\alpha\rangle = \langle\alpha|\mathcal{J}^{\dagger}(d\vec{x}')\ \mathcal{J}(d\vec{x})|\alpha\rangle = 1$

$\Rightarrow \mathcal{J}^{\dagger}(d\vec{x})\ \mathcal{J}(d\vec{x}) = \underline{1}$ (Norm of the ket is preserved under unitary transf).

- Suppose we consider two successive infinitesimal translations, first by $d\vec{x}'$ and subsequently by $d\vec{x}''$, where $d\vec{x}'$, $d\vec{x}''$ need not be in the same directions, we expect the net result to be a single translation by $d\vec{x}' + d\vec{x}''$. Therefore:

$$\mathcal{J}(d\vec{x}'')\ \mathcal{J}(d\vec{x}') = \mathcal{J}(d\vec{x}'' + d\vec{x}')$$

- If we first translate by $d\vec{x}'$ and then subsequently by $-d\vec{x}'$, then we expect to come back to the original value of $\vec{x}'$.

$$\Rightarrow \mathcal{J}(-d\vec{x}') = \mathcal{J}^{-1}(d\vec{x}')$$

- $\lim_{d\vec{x}' \to 0} \mathcal{J}(d\vec{x}') = 1$ (i.e, $|\vec{x}'\rangle$ remains the same).

Let us now choose our infinitesimal operator to be:

$$\mathcal{J}(d\vec{x}') = 1 - i\vec{K}\cdot d\vec{x}'$$

where $\vec{K}$ is a vector with components K_x, K_y, K_z. and the components are Hermitian operators, one can check that all the properties of the translation operator are satisfied.

→ <u>Prove this!</u>

Let us now derive a fundamental relation between the $\vec{K}$ operator and $\vec{x}$ operator.

Note that:

$$\hat{\vec{x}}\,\hat{\mathcal{J}}(d\vec{x}')|\vec{x}'\rangle = \hat{\vec{x}}\,|\vec{x}'+d\vec{x}'\rangle = (\vec{x}'+d\vec{x}')|\vec{x}'+d\vec{x}'\rangle$$
$$\hat{\mathcal{J}}(d\vec{x}')\,\hat{\vec{x}}\,|\vec{x}'\rangle = \vec{x}'|\vec{x}'+d\vec{x}'\rangle$$

$$\therefore\ [\vec{x}, \mathcal{J}(d\vec{x}')]|\vec{x}'\rangle = d\vec{x}'|\vec{x}'+d\vec{x}'\rangle \simeq d\vec{x}'|\vec{x}'\rangle$$

$|\vec{x}'\rangle$ is arbitrary and also $|\vec{x}'\rangle$ forms a complete set (basis). Therefore: we can only have an operator identity:

$$[\hat{\vec{x}}, \hat{\mathcal{J}}(d\vec{x}')] = d\hat{\vec{x}}$$

Using the form of $\mathcal{J}(d\vec{x}')$:

$$[\hat{x}_i, \hat{k}_j] = i\delta_{ij} \qquad \text{[Fundamental commutation relation]}$$

· What is the physical significance of $\vec{k}$?

→ In classical Mechanics, momentum is a generator of translation. Since $\vec{k}$ is an infinitesimal translation op., it should be related to momentum.

In classical Mechanics (Goldstein, 395, 411), an infinitesimal translation can be regarded as a canonical transformation,

$$\vec{X}_{new} = \vec{X} = \vec{x} + d\vec{x} \quad \text{and} \quad \vec{P}_{new} \equiv \vec{P} = \vec{p}$$

and can be obtained from the generating fn:

$$F(\vec{x}, \vec{P}) = \vec{x}\cdot\vec{P} + \vec{p}\cdot d\vec{x}$$

Comparing with the form of the quantum mechanical infinitesimal transl. op:

$$\mathcal{J}(d\vec{x}) = 1 - i\vec{k}\cdot d\vec{x}$$

$\vec{x}\cdot\vec{P}$ in the classical equivalent is the generating fn for the identity transf: $(\vec{P} = \vec{p}\,.\ \vec{X} = \vec{x})$. Comparing the second term, we see that $\vec{K}$ is related to the momentum operator in Quantum Mechanics.

We will get the exact relation with a bit of dimension analysis:

$$[\vec{k} \cdot d\vec{x}]_{\text{dim}} \equiv 0 \quad \text{(dimensionless)}$$

$$[\vec{p}] = MLT^{-1}$$
$$[\vec{k} \cdot d\vec{x}] = 0$$
$$MLT^{-1}(\alpha)\, L = 0$$
$$ML^2T^{-1}(\alpha) = 0$$
$$\Rightarrow \alpha = [ML^2T^{-1}]^{-1}$$
$$\sim [\text{ang. momentum}]^{-1}$$
$$[k] \sim (1/L)$$

$$\therefore [k] \propto \frac{p}{\text{const.}}$$

Const: dim of angular momentum: ML^2T^{-1}

$$[p] : MLT^{-1}$$
$$\therefore [k] = \left(\frac{1}{L}\right).$$

Constant (Universal const) $\equiv h$ (Planck's const.)

$$\hat{\vec{k}} \equiv \frac{\hat{\vec{p}}}{\hbar} \quad \text{(wave number operator)}$$

$$\therefore \cancel{\hbar}[X_i, P_j/\hbar] = i\,\delta_{ij}$$

$$\therefore [X_i, P_j] = i\hbar\,\delta_{ij}$$

$\hat{x}, \vec{p}$ cannot be simultaneously determined!

Can obtain: Show this!

$$\langle (\Delta x)^2 \rangle \langle (\Delta p_x)^2 \rangle \geq \hbar^2/4$$

So far we have seen translation by infinitesimal amounts. A finite translation can be obtained by successive infinite translations:

$$\mathcal{J}\left(\frac{\Delta\vec{x}'}{N},\hat{x}\right)|\vec{x}'\rangle = |\vec{x}' + \frac{\Delta\vec{x}'}{N}, \hat{x}\rangle$$

$$\mathcal{J}\left(\frac{\Delta\vec{x}'}{N},\hat{x}\right)|\vec{x}' + \frac{\Delta\vec{x}}{N}\rangle = |\vec{x}' + 2\frac{\Delta\vec{x}'}{N}, \hat{x}\rangle \text{ etc}$$

$\vdots$

Thus after N such operations, the total translation is $\Delta\vec{x}'$.

$$\therefore \mathcal{J}(\Delta\vec{x}', \hat{x}) = \lim_{N\to\infty}\left(1 - \frac{iP_x\Delta x'}{N\hbar}\right)^N$$

$$= e^{-iP_x\Delta x/\hbar}$$

Exp. of an operator:

$$e^X = 1 + X + \frac{X^2}{2!} + \cdots$$

Fundamental prop. of translation:

- Translations in different directions commute.

i.e. $\mathcal{J}(\Delta y', \hat{y})\,\mathcal{J}(\Delta x', \hat{x}) = \mathcal{J}(\Delta x'\hat{x} + \Delta y'\hat{y})$

$\mathcal{J}(\Delta x', \hat{x})\,\mathcal{J}(\Delta y', \hat{y}) = \mathcal{J}(\Delta x'\hat{x} + \Delta y'\hat{y})$

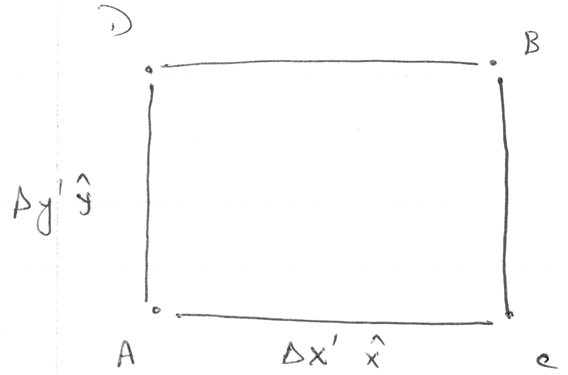
Therefore:

$$[\mathcal{J}(\Delta y', \hat{y}), \mathcal{J}(\Delta x', \hat{x})] = \left[\left(1 - \frac{iP_y\Delta y'}{\hbar} + O(\Delta y'^2)\right), \left(1 - \frac{iP_x\Delta x'}{\hbar} + O(\Delta x')^2\right)\right]$$

$$= -\frac{\Delta x'\Delta y'}{\hbar^2}[P_x, P_y]$$

Since: $[\mathcal{J}(\Delta y', \hat{y}), \mathcal{J}(\Delta x', \hat{x})] = 0$

$\Rightarrow [P_x, P_y] = 0$ i.e: $[P_i, P_j] = 0$



$$\boxed{ACB = ADB}$$

This property is not true for rotations!

Whenever the generators of a transformation commute, they from an abelian group. Translation group in 3D is <u>Abelian</u>. Therefore: P_x, P_y, P_z can have simultaneous eigenkets.

$$\Rightarrow |\vec{P}\rangle = |P_x, P_y, P_z\rangle$$

$$\hat{P}_x |\vec{P}\rangle = P_x |\vec{P}\rangle \quad \text{etc.}$$

Now: $$\mathcal{J}(d\vec{x}')\,|\vec{P}\rangle = \underbrace{\left[1 - \frac{i\vec{p}\cdot d\vec{x}'}{\hbar}\right]}_{\text{Complex eigenvalue.}} |\vec{P}\rangle$$

$$\Rightarrow \left[\vec{P}, \mathcal{J}(d\vec{x}')\right] = 0$$

Eigenvalues are complex. Reason: $\mathcal{J}(dx')$ is unitary, but not <u>Hermitian</u>.

<u>Canonical Commutation relation:</u>

$$\underbrace{[x_i, x_j] = 0, \quad [P_i, P_j] = 0, \quad [x_i, P_j] = i\hbar\,\delta_{ij}}_{\text{fundamental commutation relation}}$$

Wave functions in position and momentum space

We start with 1D case and later generalize to 3D.

Position space: $|x'\rangle$

- Eigenkets of $\hat{x}$

$$\hat{x}|x'\rangle = x'|x'\rangle$$

- ~~Orgo~~ Orthogonality:

$$\langle x''|x'\rangle = \delta(x''-x')$$

- General state

$$|\alpha\rangle = \int dx'\, |x'\rangle\langle x'|\alpha\rangle$$

and: $|\langle x'|\alpha\rangle|^2 dx'$: prob. of finding the particle $|\alpha\rangle$ in a state $|x'\rangle$ within an interval dx'.

Wave fn: $\psi_\alpha(x') \underset{\text{defn}}{\equiv} \langle x'|\alpha\rangle$

$\psi_\alpha(x')$: expansion coeffn like c_α

Consider the following inner pdt:

$$\langle\beta|\alpha\rangle = \int dx'\, \langle\beta|x'\rangle\langle x'|\alpha\rangle$$

$$= \int dx'\, \psi^*_\beta(x')\, \psi_\alpha(x')$$

$\langle\beta|\alpha\rangle \Rightarrow$ Overlap between 2 wave fns. (prob. amplitude for a state $|\alpha\rangle$ to be found in state $|\beta\rangle$.

Consider:

$$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle$$

$$\Rightarrow \langle x'|\alpha\rangle = \sum_{a'}{}' \langle x'|a'\rangle \langle a'|\alpha\rangle$$

$$\psi_\alpha(x') = \sum_{a'}{}' c_{a'} \underbrace{u_{a'}(x')}_{\text{eigenfns of an operator } \hat{A}}$$

- Expectation values:

$$\langle\beta|A|\alpha\rangle = \int dx' \int dx'' \langle\beta|x'\rangle\langle x'|A|x''\rangle\langle x''|\alpha\rangle$$

$$= \int dx' \int dx'' \psi_\beta^*(x') \langle x'|A|x''\rangle \psi_\alpha(x'')$$

If $A \equiv x^2$ or $f(x)$ in general.

$$\langle x'|A|x''\rangle = \langle x'|x^2|x''\rangle = x''^2 \delta(x'-x'')$$

$$\therefore \langle\beta|x^2|\alpha\rangle = \int dx' \int dx'' \psi_\beta^*(x') x''^2 \delta(x'-x'') \psi_\alpha(x'')$$

$$= \int dx' x'^2 \psi_\beta^*(x') \psi_\alpha(x')$$

$$\text{and } \langle\beta|\underbrace{\hat{f}(x)}_{\text{op}}|\alpha\rangle = \int dx' \underbrace{f(x')}_{\text{number}} \psi_\beta^*(x') \psi_\alpha(x')$$

<u>Representation of momentum op. in the position basis</u>:

Start with the infinitesimal translation operator:

$$\left[1 - \frac{ip\Delta x'}{\hbar}\right]|\alpha\rangle = \int dx'' \left[1 - \frac{ip\Delta x'}{\hbar}\right]|x''\rangle\langle x''|\alpha\rangle$$

$$= \int dx'' \left(\frac{1 - ip\Delta x''}{\hbar}\right)$$

$$\mathcal{J}(\Delta x')\,|\alpha\rangle = \int dx'\; \mathcal{J}(\Delta x')\,|x'\rangle\,\langle x'|\alpha\rangle$$

$$= \int dx'\; |x'+\Delta x'\rangle\,\langle x'|\alpha\rangle$$

$$= \int dx'\,|x'\rangle\langle x'-\Delta x'|\alpha\rangle$$

$$= \int dx'\,|x'\rangle\left[\langle x'|\alpha\rangle - \Delta x'\,\frac{\partial}{\partial x'}\langle x'|\alpha\rangle + \cdots\right]$$

But:

$$\mathcal{J}(\Delta x')\,|\alpha\rangle = 1 - \frac{i p\,\Delta x'}{\hbar}$$

$\therefore$ Comparing $\Delta x'$ terms:

$$\frac{i p|\alpha\rangle}{\hbar} = \int \frac{\partial}{\partial x'}\langle x'|\alpha\rangle\; dx'\,|x'\rangle$$

$$\text{or}\quad p|\alpha\rangle = \int -i\hbar\,\frac{\partial}{\partial x'}\langle x'|\alpha\rangle\; dx'\,|x'\rangle$$

$$\Rightarrow\quad \langle x'|p|\alpha\rangle = -i\hbar\,\frac{\partial}{\partial x'}\langle x'|\alpha\rangle$$

$\therefore$ In the x basis:

$$\langle x'|p|x''\rangle = -i\hbar\,\frac{\partial}{\partial x'}\,\delta(x'-x'')$$

and:

$$\langle\beta|\,p\,|\alpha\rangle = \int dx'dx''\langle\beta|x'\rangle\,\langle x'|p|x''\rangle\,\langle x''|\alpha\rangle$$

$$= \int dx'\;\psi^*_\beta(x')\left(-i\hbar\,\frac{\partial}{\partial x'}\right)\psi_\alpha(x')$$

Momentum - Space wave fn

The base kets in momentum space: $\{|p'\rangle\}$ such that

$$\hat{p}\,|p'\rangle = p'\,|p'\rangle$$

$$\langle p'|p''\rangle = \delta(p'-p'')$$

An arbitrary ket $|\alpha\rangle$ can be expanded in $\{|p'\rangle\}$:

$$|\alpha\rangle = \int dp'\; |p'\rangle\langle p'|\alpha\rangle$$

$\langle p'|\alpha\rangle$: Momentum space wave fn.

if $\langle\alpha|\alpha\rangle = 1$ (Normalized) then:

$$\int dp'\; |\langle p'|\alpha\rangle|^2 = 1$$

· Connection between the x - representation and the p-representation
We need elements of $\langle x'|p'\rangle$ analogous to the elements of a transformation matrix for discrete spectra ($\cancel{\langle\alpha|\beta\rangle}$) ($\langle a'|b'\rangle$)
We know that:

$$\langle x'|p|\alpha\rangle = -i\hbar\frac{\partial}{\partial x'}\langle x'|\alpha\rangle$$

Let $|\alpha\rangle = p'$

Then $$\langle x'|p|p'\rangle = -i\hbar\frac{\partial}{\partial x'}\langle x'|p'\rangle$$

or: $$p'\langle x'|p'\rangle = -i\hbar\frac{\partial}{\partial x'}\langle x'|p'\rangle$$

$$\Rightarrow \langle x'|p'\rangle = N\exp\left(\frac{ip'x'}{\hbar}\right).$$

N: Normalization constant to be determined

While $\langle x'|p'\rangle$ are the matrix elements of the transformation matrix that takes us from $|p'\rangle \to |x'\rangle$, for a fixed value of p', it can be regarded as a fn of x', hence the prob. amplitude of the overlap between states $|p'\rangle$ and $\langle x'|$. This is also referred to as the momentum eigenfunction (working in the $\{|x'\rangle\}$ basis). We see that the momentum eigenstate is a plane wave.

To fix the normalization, let us consider:

$$\langle x'|x''\rangle = \int dp' \, \langle x'|p'\rangle\langle p'|x''\rangle$$

$$\delta(x'-x'') = N^2 \int dp' \, e^{ip'(x'-x'')/\hbar}$$

$$= 2\pi\hbar N^2 \, \delta(x'-x'').$$

$$\therefore N = \frac{1}{\sqrt{2\pi\hbar}}$$

$$\Rightarrow \langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ip'x'}{\hbar}\right)$$

Now we will see the relation between the position and momentum space wave fns:

$$\psi_\alpha(x') \equiv \langle x'|\alpha\rangle = \int dp' \, \langle x'|p'\rangle\langle p'|\alpha\rangle$$

and

$$\phi_\alpha(p') \equiv \langle p'|\alpha\rangle = \int dx' \, \langle p'|x'\rangle\langle x'|\alpha\rangle$$

$$\left.\begin{aligned} \Rightarrow \psi_\alpha(x') &= \frac{1}{\sqrt{2\pi\hbar}} \int dp' \, \cancel{\langle x'|p'\rangle} \, e^{ip'x'/\hbar} \phi_\alpha(p') \\ \phi_\alpha(p') &= \frac{1}{\sqrt{2\pi\hbar}} \int dx' \, e^{-ip'x'/\hbar} \psi_\alpha(x') \end{aligned}\right\} \text{\underline{F.T pairs}}$$

Fourier transforms, Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}{}' a_n \underbrace{\cos(nx)} + \sum_{n=1}^{\infty}{}' b_n \underbrace{\sin(nx)}$$

periodic fns.

$\Rightarrow$ $f(x)$ has to be periodic.

Integral transforms:

$$g(\alpha) = \int_a^b f(t)\, K(\alpha, t)\, dt$$

$g(\alpha)$: integral transform of $f(t)$.

Operation maps a function from t space $\rightarrow$ α space. Becomes significant when the spaces are position - momentum, frequency - time etc.

Fourier Transform:

$$g(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\alpha t}\, dt$$

$K(\alpha, t) = e^{i\alpha t}$ $\qquad a, b : [-\infty, \infty]$

Laplace transforms

$a, b : [0, \infty]$ $\qquad K(\alpha, t) = e^{-\alpha t}$

· These transformations are linear $\Rightarrow$ $c_1 f_1(t) + c_2 f_2(t) = f(t)$

Then: $\displaystyle\int_a^b [c_1 f_1(t) + c_2 f_2(t)]\, K(\alpha, t)\, dt = c_1 \int_a^b f_1(t)\, K(\alpha, t)\, dt$

$\displaystyle + c_2 \int_a^b f_2(t)\, K(\alpha, t)\, dt.$

In other words:

$$g(\alpha) = \mathcal{L}\, f(t)$$

and therefore we can define an inverse:

$$\mathcal{L}^{-1}\, g(\alpha) = f(t)$$

$$\mathcal{L}^{-1} g(\alpha) = \int_{-\infty}^{\infty} \frac{d\alpha}{\sqrt{2\pi}}\, e^{-i\alpha t}\, g(\alpha) = f(t)$$

$g(\alpha)$, $f(t)$: Fourier transform pairs.

Now consider the following transformation

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\alpha\; e^{i\alpha t}\, g(\alpha)$$

$$g(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\; e^{-i\alpha t}\, f(t)$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\alpha\; e^{i\alpha t} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt'\, e^{-i\alpha t'}\, f(t') \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt'\, f(t') \underbrace{\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\alpha\; e^{i\alpha(t-t')} \right]}_{(\sqrt{2\pi})^2\, \delta(t-t')}$$

$$= \frac{1}{(2\pi)} \int_{-\infty}^{\infty} dt'\, f(t')\, (2\pi)\, \delta(t-t') = f(t) \quad \therefore$$

$$\boxed{(2\pi)\,\delta(t-t') = \int_{-\infty}^{\infty} d\alpha \; e^{i\alpha(t-t')}}$$

Representation of a delta fn.

Going back

if $\varphi_\alpha(x) = \delta(x) = \frac{1}{(2\pi\hbar)}\int dp \; e^{ipx/\hbar}$

Now: $\varphi_\alpha(p) = \frac{1}{\sqrt{2\pi\hbar}} \int dx' \; e^{-ip'x'/\hbar} \, \Phi_\alpha(x)$

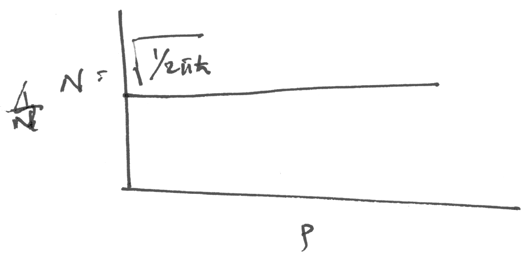
$$= \frac{1}{\sqrt{2\pi\hbar}} \int dx' \; e^{-ip'x'/\hbar} \int \frac{dp}{2\pi\hbar} \; e^{ipx/\hbar}$$

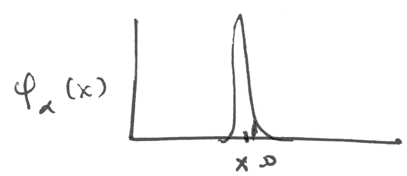
$$\cancel{= \frac{1}{\sqrt{2\pi\hbar}} \int \frac{dp}{2\pi\hbar} \underbrace{\int dx' \; e^{-ip'(x'-x)/\hbar}}_{\delta}}$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int \frac{dp}{2\pi\hbar} \underbrace{\int dx' \; e^{-ix(p'-p)/\hbar}}_{2\pi\hbar \, \delta(p'-p)}$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int dp \; \delta(p'-p) = \frac{1}{\sqrt{2\pi\hbar}}$$

$\Rightarrow \varphi_\alpha(p) = \text{Const.}$





Uncertainty relation