The Leigh-Strassler theory and the AdS-CFT correspondence

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The Leigh-Strassler theory

- This theory is a CFT obtained by perturbing the $\mathcal{N}=4$ Super Yang-Mills theory by all marginal perturbations that preserve $\mathcal{N}=1$ supersymmetry.
- It is, arguably, one of simplest $\mathcal{N}=1$ superconformal field theories given its close relationship to $\mathcal{N}=4$ Super Yang-Mills theory.
- Clearly, this makes it a worthwhile candidate for the gauge theory side of the AdS-CFT correspondence.
- The gravity/string dual is known only for the β-deformation and not for the more general deformations.
- One major difficulty is that it has no continuous symmetries beyond the $U(1)_R$ symmetry.

LS theory – details

- In terms of $\mathcal{N} = 1$ multiplets, the theory has one $SU(N)$ ($N > 2$) vector multiplet and three adjoint valued chiral multiplets Φ_i.
- The cubic $\mathcal{N} = 4$ superpotential is given by

$$W = \frac{f}{6}\,\epsilon^{ijk}\,\mathrm{Tr}_F\left(\Phi_i \Phi_j \Phi_k\right) \qquad .$$

LS theory – details

- In terms of $\mathcal{N}=1$ multiplets, the theory has one $SU(N)$ ($N>2$) vector multiplet and three adjoint valued chiral multiplets Φ_i.
- The marginal deformations are parametrized by a totally symmetric third-rank tensor of $SU(3)$ – c^{ijk} – as follows

$$W_{LS} = \frac{f}{6}\,\epsilon^{ijk}\,\mathrm{Tr}_F(\Phi_i\Phi_j\Phi_k) + \frac{c^{ijk}}{6}\,\mathrm{Tr}_F(\Phi_i\Phi_j\Phi_k)\ ,$$

- Using linear redefinitions ($SL(3,\mathbb{C})$) of the three fields, we can set eight of the components to zero. The non-zero components are, say,

$$c^{123} = c_0 \text{ and } c^{111} = c^{222} = c^{333} = c_1\ .$$

Conformal Invariance of the LS theory

- Conformal invariance of the LS theory requires the various beta functions should vanish.
- Holomorphy implies that the only quantum corrections to the superpotential occur through wavefunction renormalisation. Thus the beta functions vanish when the (matrix of) anomalous dimensions (γ_l^k) of the three chiral superfields vanish.

$$\beta(c_{ijk}) \sim c_{ljk}\ \gamma_i^l + c_{ilk}\ \gamma_j^l + c_{ijl}\ \gamma_k^l\ .$$

- The NSVZ beta function for the gauge coupling g is

$$\beta^{NSVZ}(g) = -\frac{g^3}{32\pi^2}\Big[\frac{2N\ \gamma_k^k}{1-g^2N(16\pi^2)^{-1}}\Big]\ .$$

- Thus for conformal invariance, we need $\gamma_l^k = 0$.

Conformal invariance

- Naively, it appears that we have nine conditions while the space of couplings, (g, f, c_0, c_1) is four dimensional.
- However, symmetries of the theory require $\gamma_l^k \propto \delta_l^k$.
- Thus, the vanishing of the matrix of anomalous dimensions imposes only one condition. [Leigh-Strassler]
- At one-loop, this is the condition

$$\boxed{|f|^2 + \left(|c_0|^2 + \frac{|c_1|^2}{2}\right)\frac{N^2-4}{N^2} = 4g^2}$$

- Thus, for $\mathcal{N}=4$ SYM with $f = 2g$, we recover the usual SYM Lagrangian with one coupling constant.
- Thus the LS theory is parametrised by two extra couplings, c_0 and c_1.

LS theory – Symmetries

- Recall that the R-symmetry for $\mathcal{N} = 4$ SYM theory is $SU(4)$.
- In the beta-deformed theory, this is broken to $U(1)^3$.
- In the LS theory, the symmetry is further broken down to $\left[U(1)_R \times \Delta(27)\right]/\mathbb{Z}_3$.
- $\Delta(27)$ is the sub-group of $SU(3) \subset SL(3, \mathbb{C})$ that is the invariance of the LS superpotential.
- It is this symmetry that implies $\gamma_l^k \propto \delta_l^k$.
- The center of $\Delta(27)$ is a $\mathbb{Z}_3$ sub-group of $U(1)_R$. This identification provides a neat relationship between irreps of $\Delta(27)$ and the $U(1)_R$ charges.

Plan of the talk

- Introduction
- Part I: Chiral primaries in the LS theory
  - Organising chiral operators using $\Delta(27)$
  - A conjecture on the spectrum of chiral primaries
  - Towards proving the conjecture
- Part II: Searching for the gravity dual to the CFT
  - Realising the LS theory as the worldvolume theory of D-branes – an example.
  - Elliptic algebras and solutions to F-term equations
  - Implications for the gravity dual

Part I

Chiral primaries in the LS theory

with K. Madhu and Pramod Dominic

Chiral primaries in the Leigh-Strassler deformed N=4 SYM - a perturbative study.

Kallingalthodi Madhu & SG   JHEP 05 (2007) 038. [hep-th/0703020]

Chiral primaries at planar one-loop

- We will focus on operators that arise in the scalar sub-sector of the theory.
- In particular, we will consider only single-trace operators such as $\mathrm{Tr}(\phi_1\phi_2^2\phi_3^4)$.
- We extract the anomalous dimension of linear combinations of such operators at planar ($N \to \infty$) one-loop from the two-point functions. We then identify operators for which the anomalous dimension vanishes.

- The numbers of operators grow exponentially with the number of fields (L) and it becomes difficult to carry out computations even for low values of L, like 4 or 5.
- The trihedral symmetry, $\Delta(27)$, provided considerable simplification in identifying protected operators.

The trihedral symmetry

- The trihedral groups are defined to be finite sub-groups of $SU(3,\mathbb{C})$ of the form $A \rtimes \mathcal{C}_3$. In the fundamental representation, A is a diagonal Abelian group and $\mathcal{C}_3$ is the group of cyclic permutations.
- $\Delta(27) \sim [(\mathbb{Z}_3)_R \times \mathbb{Z}_3] \rtimes \mathcal{C}_3$ is generated by

$$\begin{array}{rll} (\mathbb{Z}_3)_R & g: & \Phi_1 \longrightarrow \omega\Phi_1 \,, \Phi_2 \longrightarrow \omega\Phi_2 \,, \Phi_3 \longrightarrow \omega\Phi_3 \\ \mathbb{Z}_3 & h: & \Phi_1 \longrightarrow \Phi_1 \,, \Phi_2 \longrightarrow \omega\Phi_2 \,, \Phi_3 \longrightarrow \omega^2\Phi_3 \\ \mathcal{C}_3 & \tau: & \Phi_1 \longrightarrow \Phi_2 \longrightarrow \Phi_3 \longrightarrow \Phi_1 \end{array}$$

  with ω, a non-trivial cube-root of unity. Note that $g = h\tau^{-1}h^2\tau$.
- The R-charge assignments is $2/3$ for all three chiral fields. Thus, the $(\mathbb{Z}_3)_R$ charge equals $\frac{3R}{2}$ mod 3.

Representations of $\Delta(27)$

- $\Delta(27)$ has nine one-dimensional representations $\mathcal{L}_{Q,j}$ and two three-dimensional representations $\mathcal{V}_1$ and $\mathcal{V}_2$.
- The generators h and τ act on the one-dimensional representation v as follows:

$$h \cdot v = \omega^Q \, v \;, \quad \tau \cdot v = \omega^j \, v \tag{1}$$

where $v \in \mathcal{L}_{Q,j}$. The 'charges' $Q = 0, 1, 2$ and $j = 0, 1, 2$ both are clearly valued modulo three. The singlet corresponds to $\mathcal{L}_{0,0}$ in this notation.

Representations of $\Delta(27)$

- $\Delta(27)$ has nine one-dimensional representations $\mathcal{L}_{Q,j}$ and two three-dimensional representations $\mathcal{V}_1$ and $\mathcal{V}_2$.
- On three-dimensional representation $\mathcal{V}_a$ with $a = 1, 2$

$$h \cdot \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^a & 0 \\ 0 & 0 & \omega^{2a} \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix}$$

$$\tau \cdot \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix}$$

Polynomials as irreps of $\Delta(27)$

- Consider three complex variables (z_1, z_2, z_3) that transform in the three-dimensional representation $\mathcal{V}_1$ of $\Delta(27)$. (These three complex variables will be eventually replaced by the three chiral superfields.)
- Polynomials in these three complex variables organise into irreducible representations according to their degrees Δ_0:
- For $\Delta_0 = 0$ mod 3, polynomials are in the one-dimensional representations $\mathcal{L}_{Q,j}$. For example,

$$(z_1^3 + \omega^j z_2^3 + \omega^{2j} z_3^3) \in \mathcal{L}_{0,j} \ ,$$

  with $j = 0$ giving rise to a singlet. Similarly, $z_1 z_2 z_3$ is also a singlet.

Polynomials as irreps of $\Delta(27)$

- Consider three complex variables (z_1, z_2, z_3) that transform in the three-dimensional representation $\mathcal{V}_1$ of $\Delta(27)$. (These three complex variables will be eventually replaced by the three chiral superfields.)
- Polynomials in these three complex variables organise into irreducible representations according to their degrees Δ_0:
- For $\Delta_0 \neq 0$ mod 3, polynomials are in the three dimensional representations $\mathcal{V}_a$, where $a = \Delta_0$ mod 3. For example,

$$\begin{pmatrix} z_2 z_3 \\ z_3 z_1 \\ z_1 z_2 \end{pmatrix} \in \mathcal{V}_2 \ .$$

From polynomials to operators

- The relevance of the discussion just concluded can be seen by the replacement $z_i \to \Phi^i$.
- The main change being that the Φ^i are matrix valued since they transform in the adjoint of $SU(N)$. We can obtain $SU(N)$ invariant combinations by taking the trace.
- However, ordering becomes important here. For example, the singlet $(z_1 z_2 z_3)$ leads to two different operators Tr$(\Phi^1\Phi^2\Phi^3)$ as well as Tr$(\Phi^1\Phi^3\Phi^2)$.
- The degree Δ_0 of the polynomial becomes the naive scaling dimension of the operator.
- It is now easy to see that the superpotential for the LS consists of linear combinations of three singlets of $\Delta(27)$. This is another way of seeing the invariance of the superpotential under $\Delta(27)$.

Consequences for the LS theory

- We just saw that the naive scaling dimension of an operator, Δ_0, decides the set of irreps of $\Delta(27)$ to which the operator can belong.
- When $\Delta_0 \neq 0$ mod 3, the answer is unique.
- When $\Delta_0 = 0$ mod 3, the operators can belong to any of the nine one-dimensional irreps, $\mathcal{L}_{Q,j}$.
- Since $\Delta(27)$ is a symmetry of the theory, we are guaranteed that there will be no mixing among operators belonging to different irreps. This brings about an enormous simplification.

The conjecture

- The simplification obtained enabled us to find protected operators up to $\Delta_0 = 6$. Based on this, we were lead to a sharp conjecture for the spectrum of chiral primaries for generic values of couplings.
- When the dimension $\Delta_0 > 2$, we conjectured [SG, Madhu]

Scaling dim.	$\Delta_0 = 3r$	$\Delta_0 = a$ mod 3
$\mathcal{N} = 4$ theory	$\mathcal{L}_{0,0} \oplus \frac{r(r+1)}{2} \left[\oplus_{i,j} \mathcal{L}_{i,j} \right]$	$\frac{(\Delta_0+1)(\Delta_0+2)}{6} \mathcal{V}_a$
β-def. theory	$\mathcal{L}_{0,0} \oplus_j \mathcal{L}_{0,j}$	$\mathcal{V}_a$
LS theory	$2\ \mathcal{L}_{0,0}$	$\mathcal{V}_a$

- The first two lines are the rewriting of known results in terms of irreps of $\Delta(27)$.

An example

- It is useful to think of the LS chiral primaries as linear combinations of $\mathcal{N}=4$ primaries.
- Consider the most general dimension $\Delta_0 = 4$ operator.
- It appears in the three dimensional representation $\mathcal{V}_1$ and it suffices to consider one of the three terms.

$$\mathcal{O}_4^1 = \mathrm{Tr}\,\Big(\phi_2^4 + b\,\phi_1^3\phi_2 + c\,\phi_1^2\phi_3^2 + c_1\phi_1\phi_3\phi_1\phi_3 \\ + d\,\phi_1\phi_2^2\phi_3 + \; d_1\phi_1\phi_2\phi_3\phi_2 + d_2\phi_3\phi_2^2\phi_1 + f\,\phi_3^3\phi_2\Big)$$

- The above operator is a linear combination of five $\mathcal{N}=4$ chiral primaries – we indicate them by choosing the same letter of the alphabet for the undetermined coefficients.

An example

- Below we rewrite the couplings as follows: $f = h(q+\bar{q})$, $c_0 = h(q-\bar{q})$ and $c_1 = 2h'$.
- The vanishing of the anomalous dimension at planar one-loop is given by

$$
\begin{aligned}
&|4\bar{h}' + \bar{h}\bar{q}d - \bar{h}qd_2|^2 \;+\; |\bar{h}\bar{q}d - \bar{h}qd_1 + \bar{h}'b|^2 \\
&+|\bar{h}\bar{q}d - \bar{h}qd_1 + \bar{h}'f|^2 \;+\; |\bar{h}\bar{q}d_1 - \bar{h}qd_2 + \bar{h}'b|^2 \\
&+|\bar{h}\bar{q}d_1 - \bar{h}qd_2 + \bar{h}'f|^2 \;+\; |\bar{h}\bar{q}c - 2\bar{h}qc_1 - \bar{h}'d|^2 \\
&+|\bar{h}qc - 2\bar{h}\bar{q}c_1 + \bar{h}'d_2|^2 \;+\; |\bar{h}(q-\bar{q})b - \bar{h}'c|^2 \\
&+|\bar{h}(q-\bar{q})f - \bar{h}'c|^2 = 0
\end{aligned}
$$

- A solution exists for generic values of couplings. This is in agreement with the conjecture.

Towards proving the conjecture

- The conjecture has not been verified when $\Delta_0 > 6$ – direct calculations become very hard.
- At $\Delta = 6$, the computation involves 18 operators that mix quantum mechanically and the anomalous dimensions are given by the eigenvalues of a 18×18 matrix – this after using all symmetries else it would have been 58 dimensional.
- So we are pursuing a different approach. For the $\mathcal{N} = 4$ theory, the spectrum of a certain spin-chain reproduces the spectrum of anomalous dimensions to four-loops.

  [Minahan-Zarembo, Beisert et. al.]
- A similar spin-chain Hamiltonian has been proposed for the one-loop anomalous dimensions of the LS theory.

  [Bundzik-Mansson]
- The length of the spin-chain is equal to Δ_0.

The spin-chain for anomalous dimensions

- Due to supersymmetry, the energy eigenvalues of the Hamiltonian are bounded from below by zero.
- The operators with vanishing one-loop anomalous dimension get mapped to eigenvectors of the spin-chain Hamiltonian that have zero eigenvalue.
- We are currently using the trihedral symmetry to organise the computation. [SG. Pramod Dominic]
- For instance, one prediction for spin-chains of length $3L$ is that there are no zero-energy states in the irreps $\mathcal{L}_{0,j}$ when $j \neq 0$.
- Starting from the beta-deformed theory, one can treat the c_1 dependent part as a perturbation and show that the zero eigenvalue becomes non-zero at first-order.

Is there a Bethe Ansatz?

- A much harder problem is to obtain the full spectrum of the spin-chain.
- Equivalently, Is there a Bethe ansatz for this spin-chain? Yes, for the beta-deformation but the answer seems to be no in general. [Beisert-Roiban;Bundzik-Mansson]
- An intuitive way to see this is that the operator $\mathrm{Tr}(\phi_1^k)$ is a chiral primary in the beta deformed theory. This is the ground state for the Bethe ansatz and all other states are created as excitations over this ground state.
- There are no such chiral primaries in the LS theory as this family of operators that we just considered now mix with other operators.

Part II

Searching for the gravity dual to the LS theory

Effective superpotentials for B-branes in Landau-Ginzburg models.

SG and Hans Jockers. JHEP 10 (2006) 060. [hep-th/0608027]

What is known?

- The gravity dual for the beta-deformation has been constructed. [Lunin- Maldacena]
- A perturbative study of the supergravity equations of motion (to third-order) about the $AdS_5 \times S^5$ background does show precisely two marginal deformations as well as the $\Delta(27)$ symmetry. [Aharony-Kol-Yankelowicz].
- Using the pure spinor approach to generalized geometry, the NS sector of the Lunin-Maldacena solution has been obtained. [Halmaygi-Tomasiello]

The hope is that there exists a NS background to which one adds D-branes and takes the near horizon limit (as Maldacena originally did) to get the gravity dual to the LS theory. Is this possible?

Realising the LS theory on D-branes

- We will now see an example of the LS superpotential arising from D-branes. This has been obtained using three distinct approaches:
  (i) summing up worldsheet instantons [Cremades-Ibáñez-Marchesano]
  (ii) matrix factorizations [Brunner-Herbst-Lerche-Scheuner]
  (iii) a topological B-model computation [SG-Jockers]
- The LS couplings f, c_0 and c_1, in our notation are given in terms of theta-functions with characteristic:

$$c_1 = \kappa\, \theta_0(\tau, \eta)\ ,\ (c_0 - f) = \kappa\, \theta_1(\tau, \eta)\ ,\ (c_0 + f) = \kappa\, \theta_2(\tau, \eta)\ ,$$

  where $\theta_a(\tau, \eta) = \sum_{n\in\mathbb{Z}} q^{\frac{3}{2}(n+\frac{a}{3})^2}\, e^{2\pi i(n+\frac{a}{3})(\eta-\eta_0)}$ and $\eta_0 = (1+\tau)/2$.
- τ is the complex structure of the elliptic curve $\mathcal{E} \in \mathbb{CP}^2$ and η is a point on it(in the B-model).

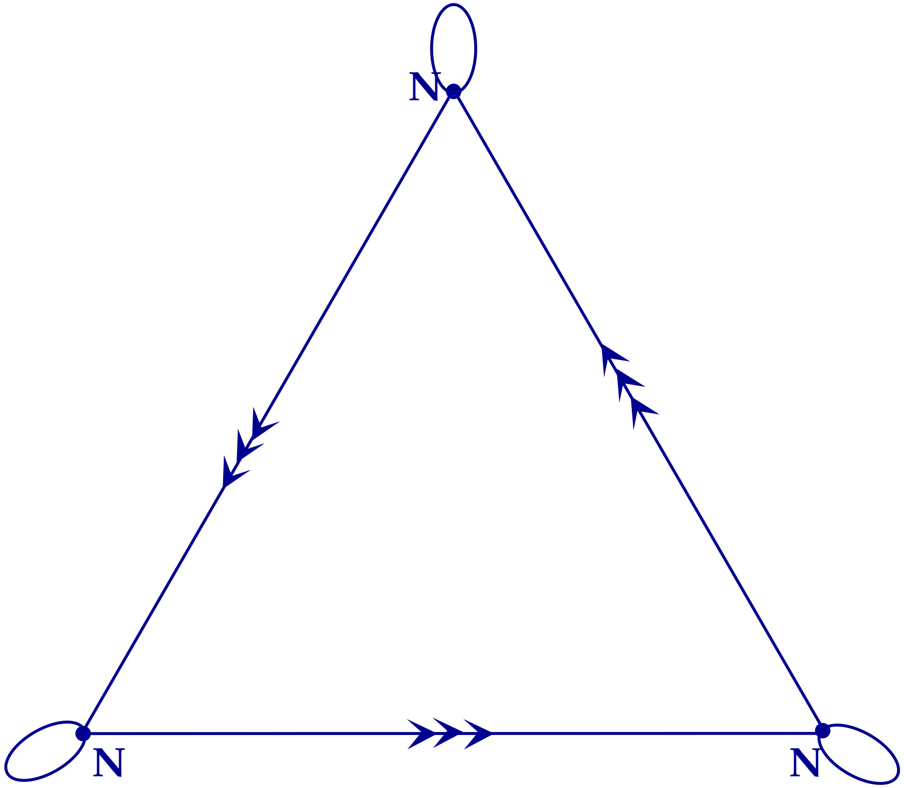
More details

- The D-brane of interest appears in the $(2,2)$ LG orbifold associated with the cubic torus.
- The LG model consists of three chiral superfields X_i and a worldsheet superpotential

$$W_{LG}(X) = X_1^3 + X_2^3 + X_3^3 - 3a(\tau) X_1 X_2 X_3 \ ,$$

  along with an $\mathbb{Z}_3$ orbifold action. In the absence of the superpotential, the model is the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold.
- The LG orbifold has three kinds of 'long' branes and the bound state is formed by taking equal numbers, N, of all three long branes.
- Recall that, in the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold, this is the zero-brane that can move off the singularity unlike the 'fractional' branes.

The quiver



Note that the NS background is described by only τ – this appeared implicitly in the LG superpotential. η appears in this configuration as an open-string modulus!

Features

- The scale of the superpotential is determined by conformal invariance which relates it to the YM coupling constant g.
- This superpotential captures both the marginal deformations of the LS theory.
- However, the values they take are not generic. The beta deformed theory appears when $c_1 = 0$. The theta function has three zeros and the beta deformation associated with cube roots of unity can only appear.
- One of the parameters, τ, arises as a closed string modulus and the other, η, arises as an open-string modulus. This suggests that, maybe, the original hope that there exists a NS background for the LS theory may not be possible.

Elliptic Algebras

- The same couplings that was obtained on the brane appear in a different context.
- Consider (one of) the F-term equations, $dW_{LS} = 0$, for the LS theory:

$$(c_0 + f)\ \phi_2\phi_3 + (c_0 - f)\ \phi_3\phi_2 + \ c_1\phi_1^2 = 0$$

- Interpreting the equations as operator equations of an elliptic algebra, Odesskii has constructed an interesting class of solutions to the F-term equations which we now describe. [Odesskii, Wijnholt]
- The data that determine the solution are the following: an elliptic curve $\mathcal{E} \subset \mathbb{CP}^2$ with modular parameter τ and a point $\eta \in \mathcal{E}$.

Elliptic Algebras

- Let v_α ($\alpha = 0, 1, 2, \ldots$) be the elements of the module, $\mathcal{V}$, on which u and e are generators with the following action:

$$u\ v_\alpha = (u_0 + \alpha\,\eta)\ v_\alpha \quad , \quad e\ v_\alpha = v_{\alpha+1}\ .$$

  Thus e is a upper-triangular (sparse) matrix and u is a diagonal matrix.

- It follows that

$$e\ u = (u - \eta)\ e\ .$$

- In terms of these two generators, the fields ϕ_1, ϕ_2 and ϕ_3 are represented by the following operators acting on the module $\mathcal{V}$:

$$\phi_a \sim \theta_a(u)\ e\ .$$

Elliptic Algebras

The F-term relations

$$(c_0 + f)\ \phi_2\phi_3 + (c_0 - f)\ \phi_3\phi_2 + c_1\ \phi_1^2 = 0$$

become the following theta-function identity

$$\begin{aligned}\theta_i(u)\theta_{i+1}(u-\eta)\theta_2(\eta) + \theta_{i+1}(u)\theta_i(u-\eta)\theta_1(\eta)\\ +\theta_{i+2}(u)\theta_{i+2}(u-\eta)\theta_0(\eta) = 0\ ,\end{aligned}$$

with the couplings:

$$c_1 = \kappa\ \theta_0(\eta)\ ,\ (c_0 - f) = \kappa\ \theta_1(\eta)\ ,\ (c_0 + f) = \kappa\ \theta_2(\eta)\ ,$$

This is exactly the couplings obtained in our D-brane construction!

Implications for the gravity dual

- It appears that there is something special happening when the couplings are given in terms of the theta functions.
- In the context of Matrix Factorizations(MF) in the cubic LG model, there exists an exceptional 4×4 MF. This appears as the singular limit of a 3×3 MF. The beta deformation associated with cube roots of unity also appears as a special limit on the theta functions. The gravity dual with the theta function couplings parallels the 3×3 MF.
- For generic values of η, the module is clearly infinite dimensional. Interpreting the eigenvalue of u on the module as the position of a $D0$-brane, we see that the full module maps to zero-branes located at points $(u_0 + \alpha\eta)$ (with $\alpha = 0, 1, 2, \ldots$) on the elliptic curve.

Implications for the gravity dual

- This is reminiscent of the 'puffing up' of zero-branes via the Myers effect to form the elliptic curve $\mathcal{E}$. Of course, the relevant NS background will have only one modulus and the other modulus should appear as an open-string modulus.
- Does the LG model help? Maybe. As the LG model does not give a geometric background, one can attempt to use the Hori-Vafa map, in particular, carry out two T-dualities (and not three) to obtain something geometric. This generically leads to worldsheet model which has both chiral and twisted chiral fields (and maybe semi-chiral fields). It would be interesting to see if looks like the generalized Kähler potential of Halmaygi-Tomasiello.

Concluding remarks

- It would be interesting to see if the theta function couplings have some implications for the spectrum of anomalous dimensions. Will it help solve the eigenvalues of the spin-chain?
- An open question in the CFT is to obtain the precise locus in the space of coupling constants where the theory is conformal. Further, most of the analysis ignore the theta term. Do we need to include it?
- The trihedral symmetry has not been used in obtaining the gravity dual. Maybe one should attempt to include it in one's considerations.

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- An open question in the CFT is to obtain the precise locus in the space of coupling constants where the theory is conformal. Further, most of the analysis ignore the theta term. Do we need to include it?
- The trihedral symmetry has not been used in obtaining the gravity dual. Maybe one should attempt to include it in one's considerations.
- I hope to have convinced you that the simplest $\mathcal{N} = 1$ SCFT and its dual needs to be understood better! Thank you.