Linear Sigma Models for D-branes

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Why study?

1. D-branes wrapping cycles of Calabi-Yau manifolds provide non-trivial examples of curved branes.

2. Can study small CY's i.e., regimes where α' effects are important.

3. One can interpolates betweens various phases – dependence on both Kähler and complex moduli is manifest. Can answer questions such as "What do six-branes look like non-geometric phases?"

4. Walls of marginal stability: Where does a D-brane wrapped on some cycle of a CY decay? What are its end-products? Related questions: what is the worldvolume superpotential of a given D-brane?

Plan of Talk

1. Introduction

   (a) D-brane basics

   (b) The Gauged Linear Sigma Model

2. The D6-brane in the GLSM with boundary

3. Implementing the monad construction in the GLSM

4. Examples

5. Complexes of arbitrary length

6. Summary and Conclusion

What is a D-brane?

It is a conformally invariant boundary condition for an open-string.

Consequences of Conformal Invariance

Closed String	$\beta(g_{\mu\nu}) \propto R_{\mu\nu} = 0$
Open String	$\beta(u^i) \propto h^{ab} K^i_{ab} = 0$

where

$g_{\mu\nu}$ – metric in spacetime M

h_{ab} – induced metric on a submanifold $C \in M$

K^i_{ab} – extrinsic curvature of C

u^i – normal deformations of C in M

Thus, a vanishing beta function implies that

M: Ricci Flat

C: Minimal submanifold of M

Incorporating $(2,2)$ worldsheet supersymmetry

Supersymmetry implies M: Kähler manifold.

In six dimensions, a Ricci-flat Kähler manifold is a Calabi-Yau manifold.

Consider boundary conditions (D-branes) that preserve half of the four supersymmetries. There are two inequivalent possibilities:

A-branes		$G_+ = \bar{G}_-$
Mirror	$\updownarrow$	Symmetry
B-branes		$G_+ = G_-$

where

$G_\pm$ and $\bar{G}_\pm$: supersymmetry generators.

$\pm$: left and right-movers.

A-branes wrap special Lagrangian submanifolds on CY threefolds while B-branes wrap holomorphic cycles.

Metrics on Calabi-Yau manifolds are not known. The non-linear sigma model is thus impossible to work with. Is there a way out?

The strategy

Construct a simpler model (with the right amount of supersymmetry) whose IR fixed point is the correct conformally invariant model. In some limit, one should recover atleast some of the characteristics of the Calabi-Yau manifold.

This model is the Gauged Linear Sigma Model

The Gauged Linear Sigma Model

Ingredients:

- It has $(2,2)$ worldsheet supersymmetry (can be obtained by dimensional reduction of $d=4$, $\mathcal{N}=1$ supersymmetry)

- Chiral Multiplets with charges Q_i^a:

$$\Phi_i = (\phi_i, \psi_\pm, \bar{\psi}_\pm, F_i)$$

- Vector Multiplets:

$$V_a = (v_\mu^a, \sigma^a, \bar{\sigma}^a, \lambda_\pm^a, \bar{\lambda}_\pm^a, D^a)$$

- Twisted Chiral Multiplets ('Field Strength'):

$$\Sigma^a = (\sigma^a, \lambda_\pm^a, D^a + i v_{01}^a)$$

- The "Calabi-Yau condition": $\sum_i Q_i^a = 0$

The Lagrangian

The action is the sum of four terms:

$$S = S_{ch} + S_{gauge} + S_W + S_{FI}$$

where

$$\begin{aligned} S_{ch} &= \int d^2x d^4\theta \; \bar{\Phi}_i \Phi_i \\ S_W &= \int d^2x d^2\theta \; W(\Phi) + h.c. \\ S_{gauge} &= \frac{1}{e^2} \int d^2x d^4\theta \; \bar{\Sigma}^a \Sigma^a \\ S_{FI} &= -r_a \int d^2x \; D^a + \frac{\theta_a}{2\pi} \int d^2x \; v^a_{01} \; . \end{aligned}$$

F_i and D^a are auxiliary fields and their equations of motion are

$$\begin{aligned} D^a &= -e^2 \left(\sum_i Q^a_i |\phi_i|^2 - r^a \right) \\ F^*_i &= \frac{\partial W}{\partial \phi_i} \; , \end{aligned}$$

The GLSM for the quintic is obtained by considering

- Five chiral superfields Φ_i of $U(1)$ charge $+1$ and one chiral superfield P of $U(1)$ charge -5

- A superpotential $W = PG(\Phi)$, where G is a homogeneous fifth-order polynomial in Φ_i which is degenerate when all Φ_i simultaneously vanish.

The bosonic potential is

$$
\begin{aligned}
U &= \sum_i \left| p \frac{\partial G}{\partial \phi_i} \right|^2 + |G|^2 + \\
&\quad (\sum_i |\phi_i|^2 - 5|p|^2 - r)^2 + \\
&\quad 2|\sigma|^2 \left(|\phi_i|^2 + 25|p|^2 \right) \ .
\end{aligned}
$$

The Calabi-Yau Phase

Consider $r \gg 0$. The ground state condition is given by

- $p = \sigma = 0$

- $\sum_i |\phi_i|^2 = r$: implies ϕ_i are coordinates on $\mathbb{CP}^4$.

- $G = 0$: implies that ϕ_i are coordinates of the hypersurface $G = 0$ in $\mathbb{CP}^4$.

- $U(1)$ is completely broken.

- The massless fluctuations are given by a Non-Linear Sigma Model on M.

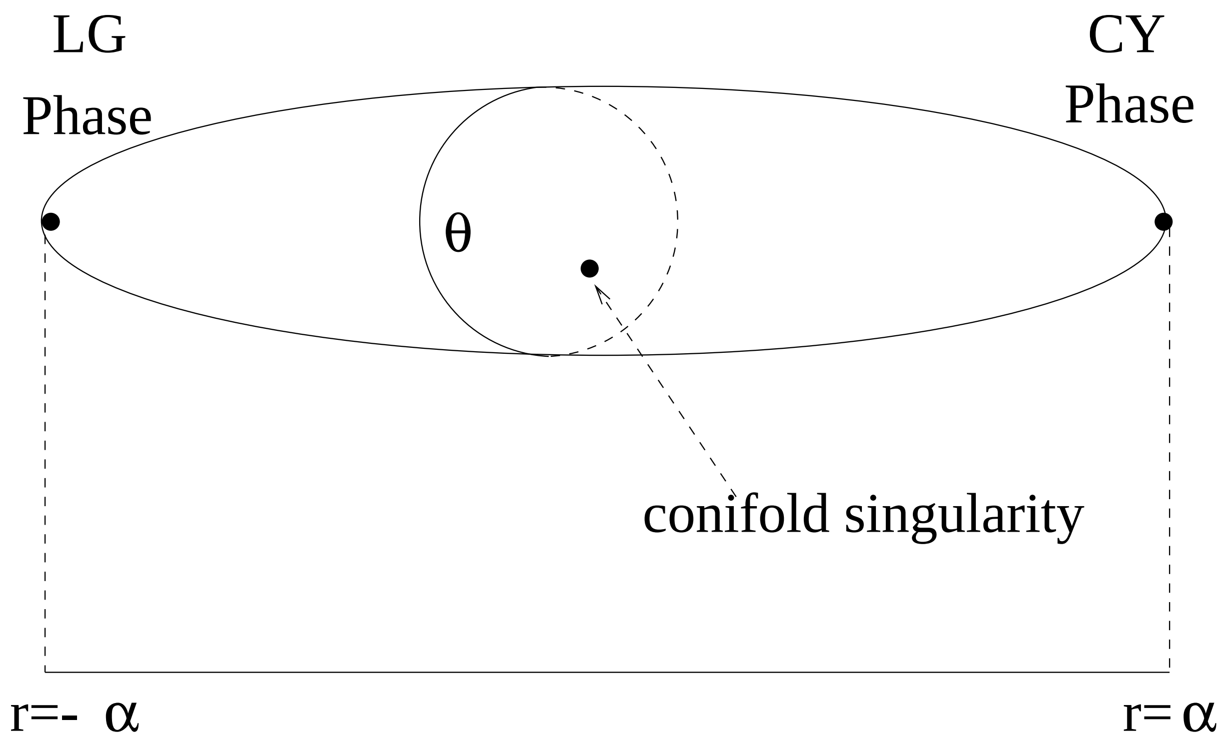
The Conifold Singularity

Let $r = 0$. In the ground state both p and ϕ_i vanish and σ is unconstrained. The CY manifold has shrunk to zero-size.

The Landau-Ginzburg Phase

Consider $r \ll 0$. The ground state condition implies

- $\phi_i = 0$.
- $\sigma = 0$.
- $|p| = |r|/\sqrt{5}$. The $U(1)$ is broken to $\mathbb{Z}_5$.
- Massless fluctuations about the minimum correspond to a $\mathbb{C}^5/\mathbb{Z}_5$ orbifold – this is an LG orbifold.



Kahler moduli space for the Quntic

The Six-Brane in the GLSM

One of the simplest B-branes is the six-brane which wraps the full Calabi-Yau manifold. What are the boundary conditions that one needs to choose for this brane?

Natural guess: Since ϕ_i are coordinates of the CY, choose Neumann boundary conditions on these coordinates.

Issues to consider

However . . . need $G = 0$ and $p = 0$

Do boundary conditions close under the unbroken supersymmetry?

What are the boundary conditions on the fields in the vector multiplet?

Requirements:

- The bulk equations of motion are not modified.

- Boundary conditions should form a closed set under B-type supersymmetry.

- The boundary conditions should have a consistent non-linear sigma model (NLSM) limit – this fixes boundary conditions on the fields in the vector multiplet.

- The θ term is dealt with properly – *large-volume monodromy* is correctly implemented.

- Require some boundary conditions in the NLSM be realised in the GLSM as *low-energy conditions* – this is one reason to introduce boundary fermions.

The Six-Brane (Part I: $\theta = 0$)

Take the worldsheet to be the upper half-plane with coordinates $(x^0, x^1 \geq 0)$

Neumann boundary conditions (at $x^1 = 0$) on the ϕ_i

$$\begin{aligned} \xi_i \equiv (\psi_{+i} + \psi_{-i}) &= 0 \\ D_1\phi_i - i(\frac{\sigma - \bar{\sigma}}{\sqrt{2}})\phi_i &= 0 \\ F_i = \bar{p}\ \bar{\partial}_i \bar{G}(\bar{\phi}) &= 0 \end{aligned}$$

Choose $p = 0$ to take care of the last condition. Then, one has

$$\begin{aligned} p &= 0 \\ \tau_p \equiv (\psi_{+p} - \psi_{-p}) &= 0 \end{aligned}$$

Leave $G = 0$ to arise from continuity in the bulk.

Boundary conditions for the vector multiplet

In the limit $e^2 \rightarrow 0$, the fields in the vector multiplet are Lagrange multipliers. Their values are determined completely in terms of the fields in chiral multiplets.

We can thus use the boundary conditions on the chiral multiplets to fix this. One obtains

$$\sigma - \bar{\sigma} = 0$$

This gives additional conditions under the action of the unbroken supersymmetry. This can be summarised by

$$(\Sigma - \overline{\Sigma}) = 0 \ ,$$

where the boundary in superspace is given by $x^1 = 0$ and $\theta^+ = \theta^-$.

Under this set, one can verify that all boundary terms which arise under the variations of the action vanish as required.

The Six-Brane: (Part II $\theta \neq 0$)

The θ-term in the Lagrangian is

$$S_\theta = \frac{\theta}{2\pi} \int d^2x \; v_{01} \ .$$

It corresponds to turning on a B-field in the NLSM limit given by

$$B_{i\bar{j}} = \frac{i\theta}{2\pi r} \delta i\bar{j}$$

In the presence of a B-field, Neumann boundary conditions get modified. Thus one expects something of the form

$$D_1\phi_i + \frac{i\theta}{2\pi r} D_0\phi_i = 0$$

This clearly requires us to expect modified boundary conditions in the GLSM as well.

In order to make this work, one finds the need to add contact interactions on the boundary of the form

$$S_{\mathrm{C}} = \int dx^0 \left(\frac{i\theta}{4\pi r} \sum_i (\phi_i \tilde{D}_0 \overline{\phi}_i - \overline{\phi}_i \tilde{D}_0 \phi_i) + \frac{\sigma - \bar{\sigma}}{\sqrt{2} i} \frac{D}{e^2} \right)$$

This term can be derived by keeping track of total derivatives that one usually discards!

When $\theta = 2\pi n$, for some integer n, this contact term can be rewritten as (in the NLSM limit where $D = 0$) as the pull-back of a holomorphic connection associated with the line-bundle $\mathcal{O}(n)$ on $\mathbb{CP}^4$!

$\mathcal{O}$ is a six-brane. What we are seeing here is the action under large-volume monodromy. Under $\theta \to \theta + 2\pi$,

$$E \to E \otimes \mathcal{O}(1)$$

Thus, the contact term is essential in capturing this non-trivial behaviour.

Summary of what we have achieved:

- We have been able to construct boundary conditions corresponding to a six-brane (the line-bundle $\mathcal{O}$) in the GLSM.

- We needed to introduce θ-dependent boundary interactions to obtain the correct large-volume monodromy.

There are two problems with the construction we have done so far.

- We do not quite realise the boundary conditions as low-energy conditions. We just impose the NSLM conditions directly on the fields in the chiral multiplet.

- Things get worse, when one tries to construct a four-brane – given by say, setting $\phi_1 = 0$. Then, one is unable to find appropriate boundary conditions for the fields in the vector multiplet that have a nice NLSM limit.

We find that the introduction of fields living on the boundary – especially boundary fermions – enable us to solve both problems.

It also fits in naturally with the general setting of D-branes associated with vector bundles (coherent sheaves).

Digression: Boundary Fermions for Chan-Paton factors

$$P\left(\exp\left[\int dx^0 \partial_0 \phi^\mu A^r_\mu(\phi) T^r\right]\right)_{\bar{a}b}$$
$$= \int [D\pi][D\bar{\pi}]\bar{\pi}_a e^{\int dx^0 (\bar{\pi}_a D_0 \pi_a)} \pi_b$$

- A is the connection on a vector bundle E and T^r are in the fundamental representation.

- $D_0 \pi_a = (\partial_0 + \partial_0 \phi^\mu A^r_\mu(\phi) T^r)\pi.$

- The path-integral is restricted to one-particle states.

This first appeared in the context of index theorems for vector bundles on manifolds and supersymmetric quantum mechanics in the early 80's. (Alvarez-Gaume; Friedan & Windey)

So if we obtain massless fermions which are sections of the appropriate bundle, their path-integral should lead to the right sort of Chan-Paton factors.

The $(0, 2)$ construction of vector bundles for heterotic compactifications is quite similar to this.

One considers a set of fermions and impose gauge invariances as well as holomorphic constraints on them.

The remaining massless fermions will be sections of a bundle given by a particular sequence – called the *monad*

The Monad Construction

Consider the following complex of holomorphic vector bundles A, B and C

$$0 \rightarrow A \xrightarrow{a} B \xrightarrow{b} C \rightarrow 0 \quad ,$$

- It is exact at A and C.
- The holomorphic vector bundle

  $$E = \text{ker } b/\text{Im } a$$

  is the cohomology of the monad.
- ch$E$ =ch$B-$ch$C-$ch$A$.
- Chern classes $\leftrightarrow$ RR-charges (six-brane charge $=$ rank)

Its field-theoretic construction

- Consider fermions π_a $(a = 1, \ldots, \text{rk } B)$

- The map a is realised as the gauge invariance

$$\pi_a \sim \pi_a + E^i_a(\phi)\kappa_i \quad ,$$

where κ_i are sections of A $(i = 1, \ldots, \text{rk } A)$.

- This gauge-invariance is fixed by the condition(s)

$$\overline{E}^i_a \pi_a = 0$$

.

- The map b is implemented by the holomorphic constraint

$$J^a_m(\phi)\pi_a = 0 \quad (m = 1, \ldots, \text{rk } C) \quad .$$

Multiplets of B-type supersymmetry

Boundary superspace with coordinates

$$x^0 \ , \ \frac{\theta}{\sqrt{2}} = \theta^+ = \theta^- \ , \ \frac{\bar{\theta}}{\sqrt{2}} = \bar{\theta}^+ = \bar{\theta}^-$$

The bulk multiplets decompose as

- A $(2,2)$ chiral multiplet Φ decomposes into a scalar chiral multiplet $\Phi' = (\phi, \tau)$ and a Fermi chiral multiplet $\Xi = (\xi, F)$ respectively.

- A twisted chiral multiplet Σ becomes an unconstrained complex multiplet.

- The singlet combination $\tilde{v}_0 = v_0 + \eta \frac{\sigma + \overline{\sigma}}{\sqrt{2}}$ is the boundary gauge field.

Monads in the GLSM

Introduce boundary Fermi multiplets $\Pi_a = (\pi_a, l_a)$ satisfying

$$\overline{\mathcal{D}}\Pi_a = \sqrt{2}\Sigma'_i E^i_a(\Phi')$$

where Σ'_i are B-type chiral multiplets. This takes care of the gauge-invariance associated with the fermions in a supersymmetric way.

The holomorphic constraint is achieved by the interaction

$$S_J = -\frac{1}{\sqrt{2}}\int dx^0 d\theta (\Pi_a P'^m J^a_m(\Phi'))|_{\overline{\theta}=0} - \text{h.c.}$$

where we P'^m are B-type chiral multiplets.

By studying the component form of the action, one can see that suitable combinations of fermions pick up masses as required after eliminating the auxiliary fields l_a.

Differences with the heterotic construction

- Need

$$\sigma'_i E^i_a(\phi) p'^m J^a_m(\phi) = W = pG(\phi)$$

in order to have closure of boundary conditions under supersymmetry.

- This is taken care of by introducing a single fermi multiplet $\hat{\Pi}$ with $E = 1$ and $J = PG$ and *no boundary scalar multiplets* such as σ' and p'.

- The rest of the multiplets satisfy

$$E^i_a(\phi) J^a_m(\phi) = 0$$

as in the heterotic case.

- Use first-order kinetic terms for the bosonic multiplets Σ' and P'.

Examples

- The Six-brane: Introduce a single-fermion with $J = P'P$. The fermion has support where $p = 0$ and $G = 0$ is achieved by continuity from the bulk.

- A four-brane given by the holomorphic equation $f(\phi) = 0$. Choose $J = P'P + P_1' f(\Phi')$ for a single-fermion.

- $\Omega^1(1)$ is given by the sequence

$$0 \to \Omega^1(1) \to \mathcal{O}^{\oplus 5} \to \mathcal{O}(1) \to 0$$

  Consider five Fermi multiplets Π^i with one holomorphic constraint $J_i = P'\Phi_i'$. The restriction to the hypersurface $G = 0$ arises from continuity in the bulk.

A bound state

The bound state $\mathcal{B}$ of the brane associated with $\Omega^1(1)$ and the anti-brane $\mathcal{O}$ is given by

$$0 \to \mathcal{B} \to \Omega^1(1) \to \mathcal{O} \to 0$$

This is done by considering five Fermi multiplets Π^i as we did for $\Omega^1(1)$ and then imposing an additional constraint of degree zero:

$$a_i \pi^i = 0$$

where a_i are five constants. This has four moduli – the five a_i subject to an overall scaling.

This state is the large-volume analogue of a Recknagel-Schomerus boundary state constructed in the Gepner model associated with the quintic.

Implementing large-volume monodromy

In the monad, $E \to E(n)$ is achieved by replacing the vector bundles A, B, C with $A(n), B(n), C(n)$.

In the field theory, this is done by *shifting the charges* of π (as well as σ' and p') by n-units and adding a θ dependent contact term.

$$S_{\rm C} = \int dx^0 \left\{ \frac{i\Theta}{2\pi r} \sum_i (\phi_i \widetilde{D}_0 \overline{\phi}_i - \overline{\phi}_i \widetilde{D}_0 \phi_i) \right\}$$

where

$$\frac{\Theta}{2\pi r} \equiv \left[\frac{\theta_f}{2\pi r} + \frac{[\theta/2\pi]}{2r} \left(\overline{\pi}_a \pi_a - |\sigma'|^2 + |p'|^2 \right) \right] \quad .$$

Note the we need to use *first-order* actions for the bosonic multiplets Σ' and P'!

Complexes of length > 2

Not all vector bundles can be constructed from monads which are complexes of length two. How does one deal with such situations?

This appears more or less through nested gauge invariances. Either, the Σ' or P' multiplets might have some extra invariances associated with them.

Ignoring these invariances leads to more massless fermions than necessary. So we gauge fix them. As an example, consider the monad for $\Omega^2(2)$, which is given by

$$0 \to \Omega^2(2) \to \mathcal{O}^{\oplus 10} \xrightarrow{J^k_{[ij]}} \Omega^1(2) \to 0$$

where $J^k_{[ij]}(\phi) = (\phi_i \delta^k_j - \phi_j \delta^k_i)$.

We introduce ten Fermi multiplets $\Pi^{[ij]}$ as well as five scalar multiplets P'_k with the superpotential

$$S_J = -\frac{1}{\sqrt{2}} \int dx^0 d\theta \left(\Pi^{[ij]} J^k_{[ij]}(\Phi') P'_k \right) |_{\overline{\theta}=0} - \text{h.c.}$$

The identity $\phi_k J^k_{[ij]}(\phi) = 0$ implies a gauge invariance for the superpotential

$$p'_k \sim p'_k + b\phi_k$$

which we fix by the constraint

$$\overline{\mathcal{D}} P'_k = \sqrt{2} N \Phi'_k$$

where N is a new Fermi multiplet. Thus, in the GLSM, one is implementing a complex of length 3

$$0 \to \Omega^2(2) \to \mathcal{O}^{\oplus 10} \to \mathcal{O}^{\oplus 5}(1) \to \mathcal{O}(2) \to 0$$

Summary

We have seen that one can construct D-branes associated with vector bundles/coherent sheaves given by as the cohomology of arbitrary complexes in the GLSM with boundary.

Future issues

The examples we have considered are vector bundles on $\mathbb{CP}^4$ which are restricted to the CY hypersurface. Useful to study other examples.

Are there any restrictions on the vector bundles that are allowed? (need to study conformal invariance in the quantum theory)

Applications for the heterotic string: Need to understand how to deal with first-order actions here. Are there any conditions beyond the usual ones involving c_1 and c_2?

Need to study boundary phases in the GLSM.