Summing up Open-String Instantons

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Based on work with T. Jayaraman and T. Sarkar

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Motivation

- Summing up closed-string instantons Morrison – Plesser summed up closed-string instantons in the A-twisted topological version of the gauged linear sigma model(GLSM). Can one carry over these methods to the open-string case?
  Issues: Need to construct a GLSM for A-branes first.

- Extended Mirror Symmetry Mirror symmetry in its extended form includes D-branes on both sides. For the closed string case – the contributions of closed-string instantons is naturally summed up in the mirror where the classical computation is exact. The instanton expansion then arises from the change of variables called the *closed string mirror map*.
  Is there an analogous open-string mirror map?

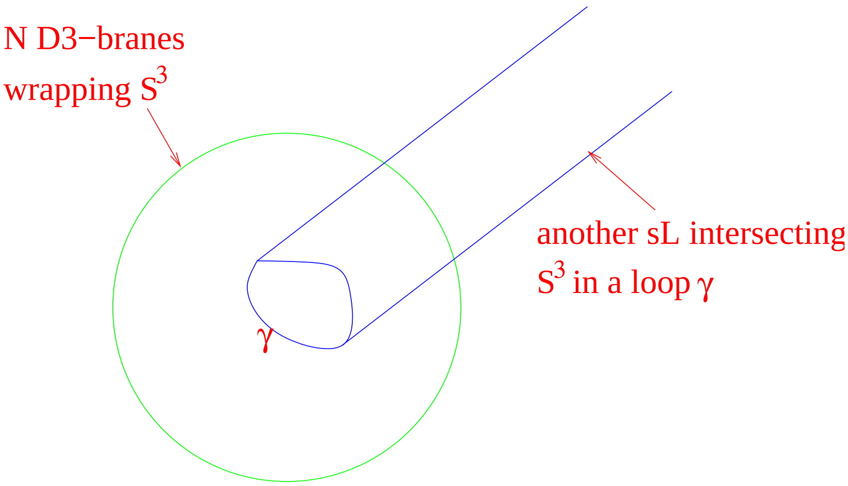
- Open-Closed Duality Gopakumar – Vafa proposed a duality between $SU(N)_k$ Chern-Simons gauge theory on S^3 [obtained as the worldvolume (topological) theory of N D3-branes wrapping S^3 of the deformed conifold] and the closed topological string on the resolved conifold. One can show that the CS partition function can be written as in the large-N 't Hooft expansion
$g_s = 2\pi i/(k+N)$ and $t = 2\pi i N/(k+N)$

$$Z_{CS}(S^3) = \exp\left[\sum_{g=0}^{\infty} g_s^{2g-2} F_g(t)\right]$$

$F_g(t)$ – the g-loop string amplitude on the resolved conifold with Kähler modulus t.

$$F_0(t) = P_3(t) + \sum_{n=1}^{\infty} \frac{q^n}{n^3} \quad , \ q = e^{-t}$$

Ooguri – Vafa generalised this correspondence for knot invariants.



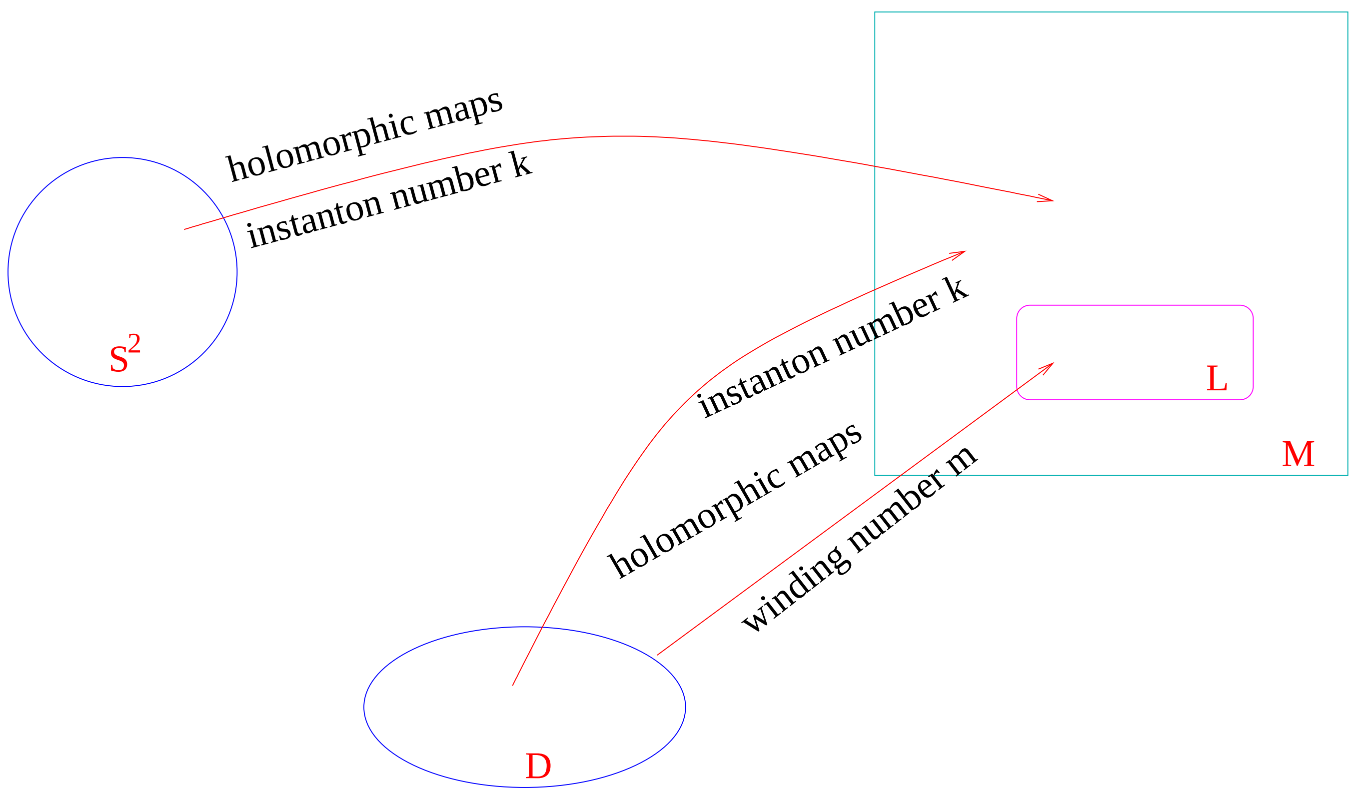
The CS partition function can be rewritten

$$\exp\left[\sum_{g,h} (g_s)^{2g-2+h} F_{g,h}(s,t)\right]$$

s is the holonomy of the gauge field on sL and $F_{g,h}$ is the string amplitude on the resolved conifold. In particular,

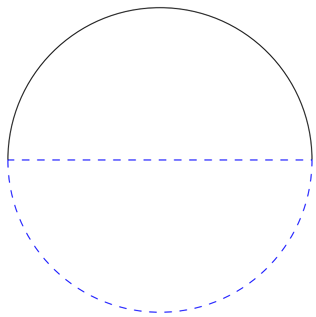
$$F_{disc} = F_{0,1} = \sum_{n=1}^{\infty} \sum_{m,k} \frac{d_{k,m}}{n^2} q^{nk} s^{mn}$$

There are strong integrality conditions on the $d_{k,m}$.


holomorphic maps
instanton number k
S^2
instanton number k
holomorphic maps
winding number m
L
M
D

GETTING CLOSED OBJECTS FROM A DISC

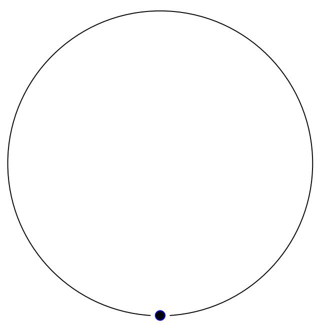
S^2



DOUBLING

(Katz−Liu)

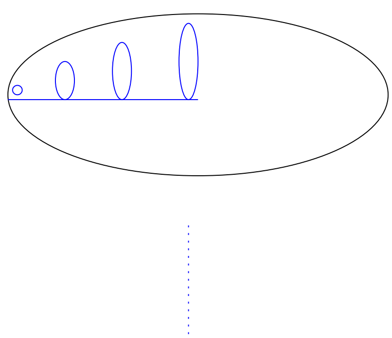
S^2



ONE−POINT COMPACTIFICATION

(SG−Jayaraman−Sarkar)

S^3



CIRCLE FIBRATION

(Aganagic−Vafa; Mayr)

M−theory lift

Plan

- Motivation
- A-branes in $\mathbb{C}^n$ and non-compact Calabi-Yau threefolds
- A gauged linear sigma model (GLSM) with boundary for A-branes
- The topological A-model and the open-string instanton moduli space
- Finding the open-string mirror map and the superpotential
- Conclusion and Outlook

Special Lagrangian submanifolds of $\mathbb{C}^n$

Harvey-Lawson; Aganagic-Vafa

$\underbrace{\omega \equiv i \sum_i dz_i \wedge d\bar{z}_i}_{\text{Kahler form}}$ and $\underbrace{\Omega = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n}_{\text{hol. (n,0) form}}$

Consider the submanifold L given by

$$\sum_{i=1}^{n} q_i^\alpha |\phi_i|^2 = c^\alpha \quad \alpha = 1, \ldots, r$$

$$\sum_{i=1}^{n} v_\beta^i \theta_i = 0 \quad \beta = 1, \ldots, (n-r)$$

L is *Lagrangian* ($\omega|_L = 0$) if

$$\sum_i q_i^\alpha v_\beta^i = 0$$

and *special Lagrangian* ($\mathrm{Im}\Omega|_L$ or $\mathrm{Re}\Omega|_L = 0$) when one of the angle conditions is $\sum_i \theta_i = 0$ which implies

$$\sum_i q_i^\alpha = 0$$

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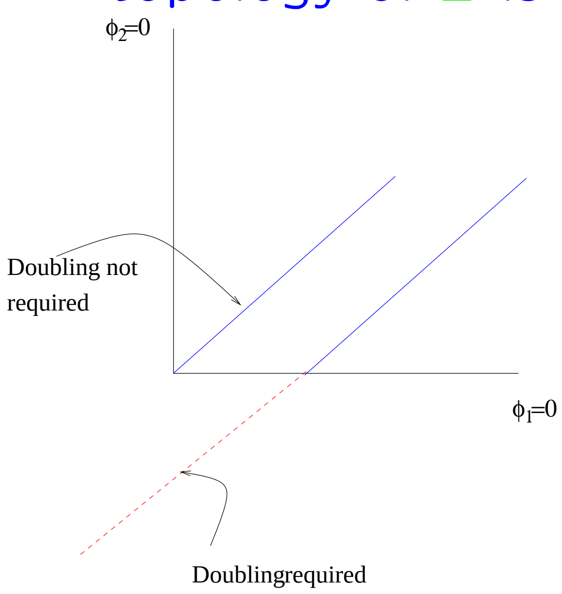
A-branes in $\mathbb{C}^2$

Consider the submanifold L given by

$$|\phi_1|^2 - |\phi_2|^2 = c^1 \text{ and } \theta_1 + \theta_2 = 0 \; .$$

1. When $c^1 \neq 0$, L is a non-compact manifold with boundary – it is the circle of radius $\sqrt{c^1}$ when $\phi_2 = 0$. The boundary can be removed by *doubling* i.e., by including $\theta_1 + \theta_2 = \pi$. The topology of L is $\mathbb{R} \times S^1$.

2. When $c^1 = 0$, there is no boundary and the topology of L is $\mathbb{R}^+ \times S^1$.



Non-compact CY3^s in the GLSM

Ingredients:

- $(n-3)$ $U(1)$ Vector Multiplets:

$$V_a = (v_\mu^a, \sigma^a, \bar{\sigma}^a, \lambda_\pm^a, \bar{\lambda}_\pm^a, D^a)$$

  The field strengths are part of twisted chiral multiplets:

$$\Sigma^a = (\sigma^a, \lambda_\pm^a, D^a + i v_{01}^a)$$

- n Chiral Multiplets with charges Q_i^a:

$$\Phi_i = (\phi_i, \psi_\pm, \bar{\psi}_\pm, F_i)$$

- The "Calabi-Yau condition": $\sum_i Q_i^a = 0$

The bosonic potential is

$$U = \frac{1}{2}\sum_a \frac{(D^a)^2}{e^2} + \sum_i |F_i|^2 + 2\sum_{a,b}\sigma_a\sigma_b\sum_i Q_i^a Q_i^b |\phi_i|^2$$

We will consider examples where there is no superpotential and thus $F_i = 0$.

Further, let us assume that the F.I. parameters r^a are such that $\sigma_a = 0$ at the minimum of the potential.

Thus, the ground state is given by ϕ_i subject to the D-term condition(s)

$$\sum_i Q_i^a |\phi_i|^2 = r^a$$

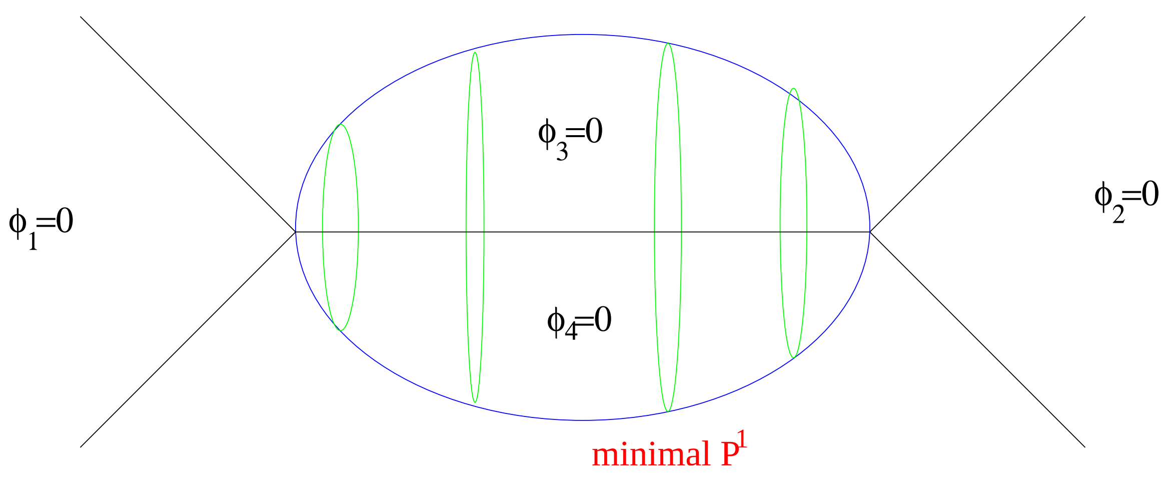
modulo the gauge invariances. Equivalently, the space of ground states is given by the Kähler quotient : $\mathbb{C}^n/(\mathbb{C}^*)^{n-3}$

Two examples of non-compact CY3's

1. Consider four fields with charges $(1, 1, -1, -1)$. The D-term constraint is

$$|\phi_1|^2 + |\phi_2|^2 - |\phi_3|^2 - |\phi_4|^2 = r$$

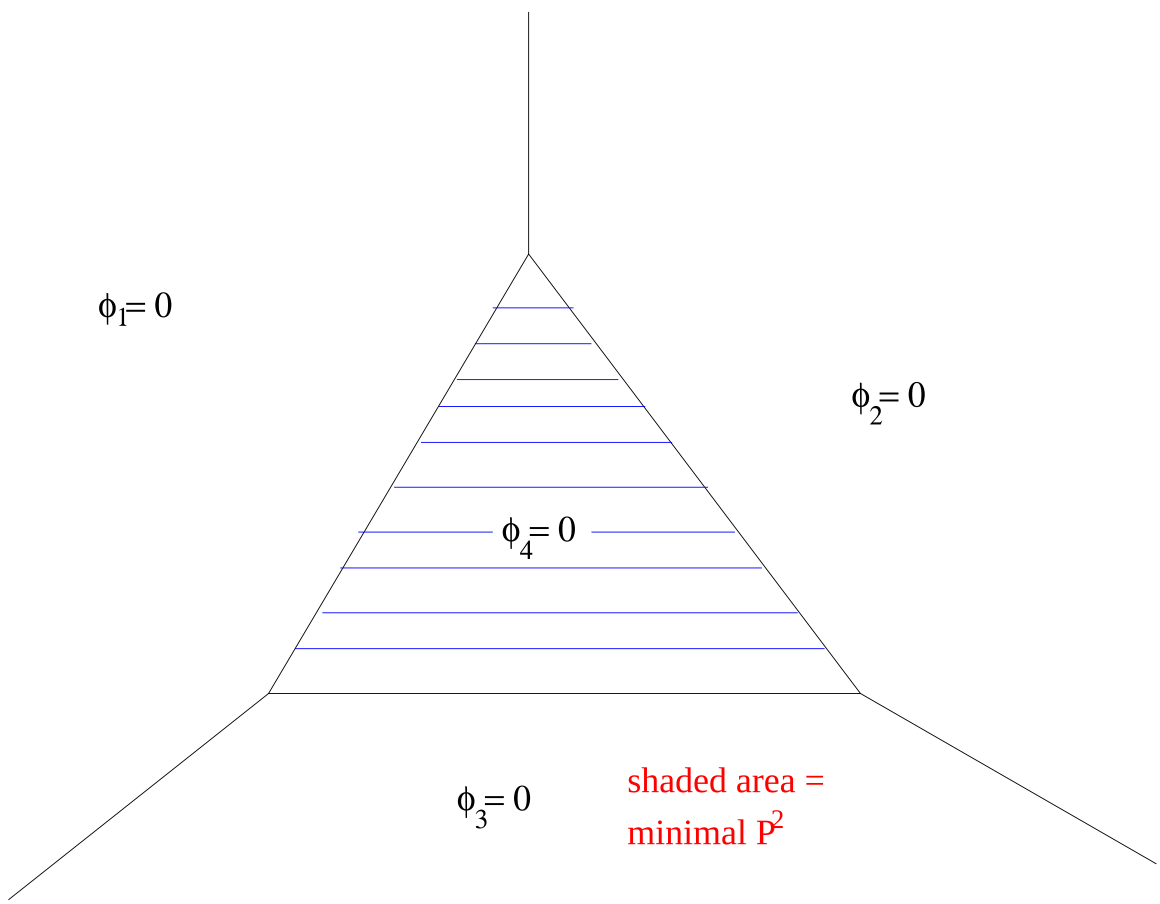
For $r > 0$, the fields ϕ_3 and ϕ_4 with negative $U(1)$ charge become (sections of) line bundles $\mathcal{O}(-1)$ on a $\mathbb{P}^1$. The minimal $\mathbb{P}^1$ is given by $\phi_3 = \phi_4 = 0$. The toric skeleton is



2. Consider four fields with charges $(1,1,1,-3)$. The D-term constraint is

$$|\phi_1|^2+|\phi_2|^2+|\phi_3|^2-3|\phi_4|^2=r$$

For $r>0$, the field ϕ_4 with negative $U(1)$ charge become a (section of a) line bundle $\mathcal{O}(-3)$ on a $\mathbb{P}^2$. The minimal $\mathbb{P}^2$ is given by $\phi_4=0$. The toric skeleton is



sL submanifolds for non-compact CY3^s

- First, construct sL for $\mathbb{C}^n$. Compatibility with the Kähler quotient implies that we must choose $(n-3)$ of the conditions to be the D-term constraints.

- The $U(1)$ invariance of the conditions on phases (given by v_β) is $\sum_i Q_i^a v_\beta^i = 0$. This is the same as the Lagrangian condition!

- The CY condition $\sum_i Q_i^a = 0$ is identical to the sL condition.

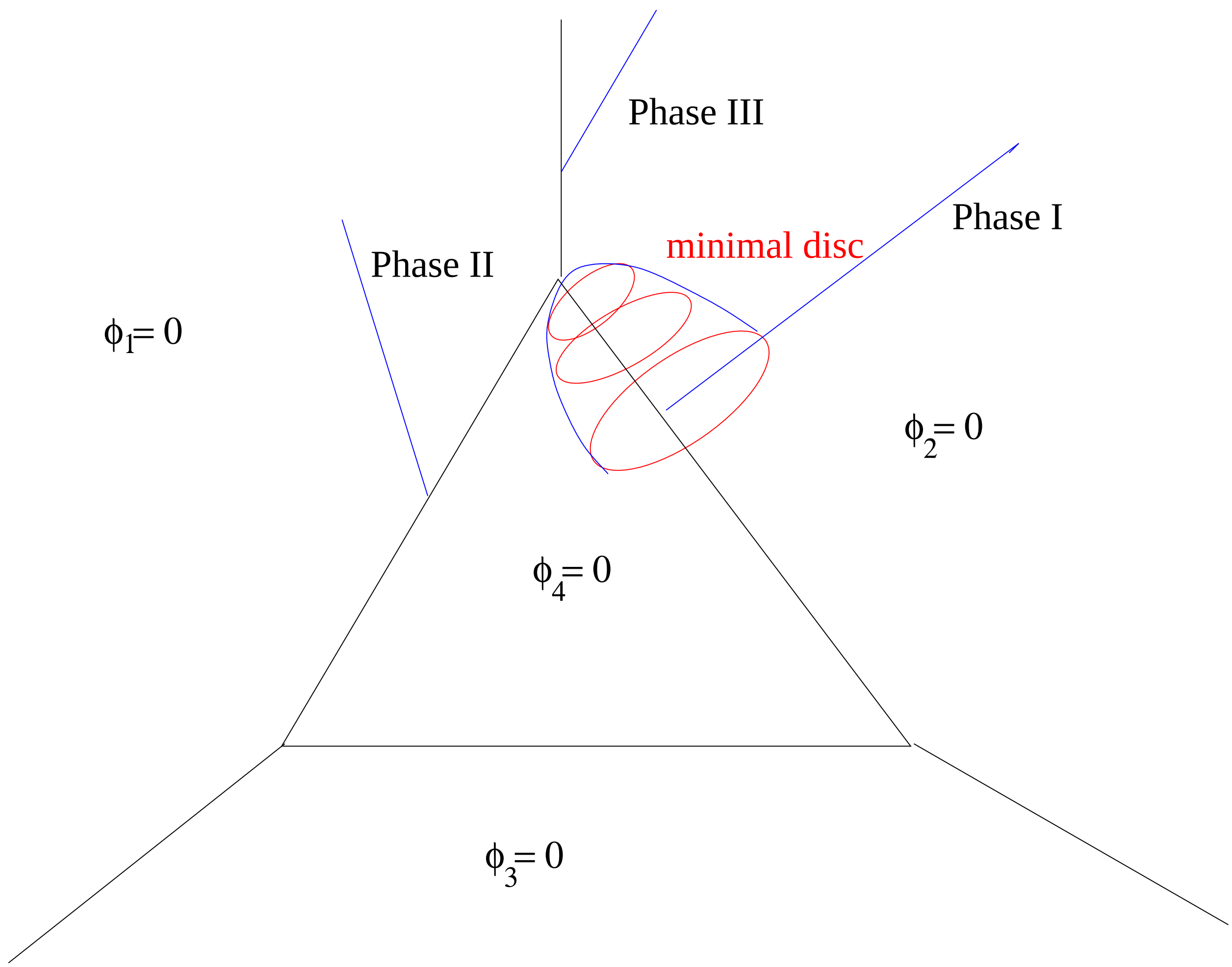
Consider the second example: $\mathcal{O}(-3)$ on $\mathbb{P}^2$ – Consider the following sL submanfolds:

$$\begin{array}{rl} q^1 = (1,0,0,-1) & |\phi_1|^2 - |\phi_4|^2 = c^1 \\ q^2 = (0,1,0,-1) & |\phi_2|^2 - |\phi_4|^2 = c^2 \\ v = (1,1,1,1) & \sum_i \theta_i = 0 \end{array}$$

For generic values of c^1 and c^2, one needs to double to obtain an sL without boundary. However, there are three phases where no doubling is necessary and the topology of the sL is $\mathbb{C} \times S^1$.

$$\begin{array}{rll} \text{Phase I} & c^2 = 0 & 0 < c^1 < r \\ \text{Phase II} & c^1 = 0 & 0 < c^2 < r \\ \text{Phase III} & c^1 = c^2 & c^1 < 0 \end{array}$$

c^1 is the area of a minimal disc whose boundary ends on the sL in phase I.



Phase III
Phase I
minimal disc
Phase II
φ1= 0
φ2= 0
φ4= 0
φ3= 0

The sL as b.c.'s in the GLSM w/ boundary

We work on a worldsheet with coordinates (x_0, x_1) and boundary (at $x_1 = 0$) preserving A-type supersymmetry. The boundary superspace has coordinates: $(x^0, \theta, \bar{\theta})$ with $\bar{D} = -\partial/\partial\bar{\theta} + i\theta\partial_0$.

We add complex boundary multiplets

$$U_\alpha = u_\alpha + \theta v_\alpha - \bar{\theta}\xi_\alpha + \theta\bar{\theta}(D'_\alpha + iR_\alpha)$$

subject to a gauge-invariance

$$\begin{aligned} U_\alpha &\rightarrow U_\alpha + L_\alpha \\ \text{Im}\log\Phi'_i &\rightarrow \text{Im}\log\Phi'_i + q_i^\alpha\left(\frac{L_\alpha + \bar{L}_\alpha}{2}\right) \end{aligned} \quad .$$

where L_α are chiral superfields($\bar{D}L_\alpha = 0$). This gauge-invariance identifies Re (u_α) with the 'angle' conjugate to the b.c. given by q_α. R_α transforms like a gauge field.

The gauge-invariant 'field-strength' is given by a fermi multiplet

$$\Xi_\alpha \equiv \bar{D}U_\alpha = \xi_\alpha + \theta[D'_\alpha + i(R_\alpha + \partial_0 u_\alpha)]$$

The boundary action is given by the sum of three terms: S_1 – the boundary-bulk interaction, S_2 – the boundary K.E. and S_3 – the boundary F.I. term.

$$\begin{aligned}
S_1 &= \int dx^0 d^2\theta \left(\sum_i q_i^\alpha \bar{\Phi}'_i \Phi'_i \right) \times \\
&\quad \times \left[\left(\frac{U_\alpha + \bar{U}_\alpha}{2} \right) - \sum_j \tilde{q}^j_\alpha \mathrm{Im} \log \Phi'_j \right] \\
S_2 &= -\frac{1}{2e_b^2} \int dx^0 d^2\theta \ \bar{\Xi}_\alpha \Xi_\alpha \\
S_3 &= -\mathrm{Re} \left(w^\alpha \int dx^0 d\theta \ \Xi_\alpha \right) \\
&= -\int dx^0 \left(c^\alpha D'_\alpha - \frac{\theta^\alpha}{2\pi} (R_\alpha + \partial_0 u_\alpha) \right)
\end{aligned}$$

where $w^\alpha = c^\alpha + i\frac{\theta^\alpha}{2\pi}$ is a complex parameter.

D'_α is an auxiliary field and its eqn. of motion is

$$-\frac{D'_\alpha}{e_b^2} = \sum_i q_i^\alpha |\phi_i|^2 - c^\alpha$$

The topological GLSM w/ boundary

We Wick rotate the worldsheet to an Euclidean one: $x_0 \to -ix_2$, $R \to iR$. The worldsheet is now topologically a disc D with its boundary ∂D a circle.

The bulk theta term in the non-linear sigma model is related to the B-field. The complex combination $t^a = r^a - i\theta/2\pi$ is thus related to the complexification of the Kähler class.

The boundary theta term also complexifies c^α (the size of a minimal disc) in a similar fashion to $w^\alpha = c^\alpha - i\theta^\alpha/2\pi$. In the non-linear sigma model, it is related to Wilson lines on compact one-cycles of the sL.

The integers $n_a \equiv -\int_D d^2x\ v^a_{12}/2\pi$ and $m_\alpha \equiv -\int_{\partial D} dx_2(R_\alpha + \partial_2 u_\alpha)$ are the first Chern class of the bulk gauge field and the winding number of the boundary gauge field R_α respectively.

The localisation eqns for the bulk fields are:

$$\begin{aligned}(D_1 + iD_2)\phi_i &= 0 \\ D^a + v^a_{12} &= 0 \\ \sigma^a &= 0\end{aligned}$$

The localisation eqns for the bdry fields are:

$$\begin{aligned}D'_\alpha + (R_\alpha + \partial_2 u_\alpha) &= 0 \\ \xi_\alpha &= 0\end{aligned}$$

The path-integral of the topological theory reduces to a finite-dimensional integral over the moduli space of solutions of the above equations modulo gauge invariances – bulk and boundary.

In nice (stable) situations, this moduli space is equivalent to the moduli space of the eqn. $(D_1 + iD_2)\phi_i = 0$ modulo complexified gauge transformations for the bulk D-terms and scalings for the boundary D-terms.

The Instanton moduli space

Closed-string case

The localisation equations imply that ϕ_i are holomorphic sections of $\mathcal{O}(n_i \equiv \sum_a Q_i^a n_a)$. This implies that whenever $n_i < 0$, $\phi_i = 0$ since line bundles of negative degree do not admit holomorphic solutions. Holomorphic line bundles of positive degree n_i can be written as $(z = x_1 + ix_2)$

$$\phi_i(z) = \sum_{j=0}^{n_i} \phi_{i,j} z^j$$

Thus, the moduli space is the space of $\phi_{i,j}$ modulo complex gauge transformations.

In the two examples that we considered, when the F.I. parameter $r \gg 0$, then one requires $n \geq 0$ and for $r \ll 0$, $n \leq 0$.

Open-string case

When the disc is very large, one can treat it as the complex plane. Stability imposes some conditions on how this is done. The only changes one needs to do is to let $n_i = \sum_a Q_i^a n_a + \sum_\alpha q_i^\alpha m_\alpha$. The instanton moduli space is now the $\phi_{i,j}$ modulo complex gauge transformations for the bulk D-terms and real scalings for the boundary D-terms.

In the $n = m = 0$ sector, the moduli space is the sL manifold (as expected).

It looks as if we can treat the boundary and the bulk conditions on a equal footing in the topological theory.

The next step would be to compute the partition function on the disc. But ...

The closed-string mirror map

The genus-zero partition function $F_0(t)$ can be computed in the topological GLSM by summing up the contributions from all instanton sectors.

$$F_0(t) = \sum_n (\cdots) e^{-nt}$$

where $(\cdots)$ refers to the result from the correlation function in the sector with instanton number n.

However, these are not the coordinates for which one has integrality predictions. The change of variable is the closed-string mirror map. It includes the relevant quantum corrections and takes the form

$$\widehat{t} = t + \text{constant} + \text{instanton corrections}$$

The GKZ differential equation

The closed string mirror map as well as the partition function are obtained from solutions (periods) of the following differential equations, the GKZ system:

$$\mathcal{L}_{Q^a}\ \tilde{\pi}(a) = 0$$

$$\mathcal{L}_{Q^a} = \prod_{Q_i^a > 0} \left(\frac{\partial}{\partial a_i}\right)^{Q_i^a} - \prod_{Q_i^a < 0} \left(\frac{\partial}{\partial a_i}\right)^{-Q_i^a}$$

There are too many variables in the above equation. Instead let the periods be functions of the variables (algebraic coordinates)

$$z_a = \prod_i (a_i)^{Q_i^a} \quad .$$

These are to be identified with e^{-t_a}. This leads to a system of differential equations satisfied by the periods associated with three-cycles on the mirror.

The solutions of interest near $z \to 0$ are

1. The non-singular solution – $\pi_0(z)$
2. Those with logarithmic behaviour as $z \to 0$ – $\pi_1(z)$.
3. Those that behave as $(\log z)^2$ as $z \to 0$ – $\pi_2(t)$.

The closed string mirror map is given by the identification

$$\hat{t} = \pi_1/\pi_0$$

and the free-energy is given by

$$\partial_{\hat{t}} F_0 = \pi_2/\pi_0 \quad .$$

For the case of $\mathcal{O}(-3)$ on $\mathbb{P}^2$, one has $z = a_1 a_2 a_3/(a_4)^3$ and the differential equation $(\theta_z = z\partial_z)$

$$\left[\Theta_z^3 + z(3\Theta_z + 2)(3\Theta_z + 1)3\Theta_z\right] \pi = 0$$

The basic solutions are

$$\begin{aligned}
\pi_0 &= 1 \\
\pi_1 &= \log z + \sum_{n=1}^{\infty} (-)^n \frac{3n!}{(n!)^3} \frac{z^n}{n} \\
\pi_2 &= \frac{1}{2}\pi_1^2 + \mathcal{C}^2 \\
&\quad + \sum_{n=1}^{\infty} (-)^n \frac{3n!}{(n!)^3} \left(-\frac{1}{n} + \sum_{j=n+1}^{3n} \frac{1}{j} \right) \frac{z^n}{n}
\end{aligned}$$

where $\mathcal{C} = \sum_{n=1}^{\infty} (-)^n \frac{3n!}{(n!)^3} \frac{z^n}{n}$.

The closed-string mirror map is $[q = \exp(-\hat{t})]$

$$z = q + 6q^2 + 9q^3 + 56q^4 - 300q^5 + \cdots$$

The GKZ diff eqn for open-string moduli

We saw that it made sense to think of the boundary and bulk D-terms on par. This suggests adding the following differential operators to the bulk GKZ system:

$$\mathcal{L}_{q^\alpha} = \prod_{q_i^\alpha > 0} \left(\frac{\partial}{\partial a_i} \right)^{q_i^\alpha} - \prod_{q_i^\alpha < 0} \left(\frac{\partial}{\partial \alpha_i} \right)^{-q_i^\alpha}$$

The open-string moduli are then given by

$$s_\alpha = \prod_i (a_i)^{q_i^\alpha} \quad .$$

For instance, in phase I for the $\mathcal{O}(-3)$ on $\mathbb{P}^2$ example, the GKZ equation becomes

$$\left[\Theta_z^2 (\Theta_z + \Theta_s) + z \prod_{i=0}^{2} (3\Theta_z + \Theta_s + i) \right] \pi = 0$$

$$\left[(\Theta_z + \Theta_s) + s(3\Theta_z + \Theta_s) \right] \pi = 0$$

Identifying the solutions of interest

Following Aganagic, Klemm and Vafa, we choose a solution with the property

$$\widehat{w} = \log s + \Delta(\vec{z})$$

for the open-string mirror map i.e., the quantum corrections are only functions of close-string moduli.

The above ansatz is not compatible with $\mathcal{L}_q$ since Δ is not a function of s.

Require that it be a solution of $\mathcal{L}_Q$ – the extension of the bulk GKZ by the open-string moduli.

The solution associated with the superpotential is obtained by an ansatz of the form

$$\partial_{\widehat{w}} W(\widehat{w}) = \sum_{\vec{k},m} a(\vec{k},m)\ z^{\vec{k}} s^m$$

such that

- the zero instanton contribution vanish i.e., $a(0,0) = 0$

- The only non-zero contributions come from sectors with allowed instanton and winding numbers.

- It should solve the full set of boundary GKZ differential equations.

Does this method work?

For phase I of $\mathcal{O}(-3)$ on $\mathbb{P}^2$, the open-string mirror map is

$$s = -\hat{s}(1+2q-q^2+20q^3-177q^4+1980q^5+\cdots)$$

where $\hat{s}=e^{-\hat{w}}$; $s=e^{-w}$ and $q=e^{-\hat{t}}$.

The superpotential is given by The solution is given by

$$\partial_{\hat{w}}W = \sum_{k=0}^{\infty}\sum_{m=-k}^{\infty}\frac{(-1)^{k+m}\Gamma(3k+m)}{\Gamma(k+m+1)\Gamma(k+1)^2}z^k s^m$$

One needs to implement the change of variables $(z,s)\to(q,\hat{s})$ to verify the integrality properties.

One has

$$\partial_{\widehat{w}} W = \sum_{k,m,n} \frac{m}{n} d_{k,m} \left(q^k \widehat{s}^m\right)^n \quad .$$

The folowing table lists the degeneracies $d_{k,m}$ of disc instantons.

$m \backslash k$	0	1	2	3	4	5
−5	0	0	0	0	0	5
−4	0	0	0	0	−2	28
−3	0	0	0	1	−10	102
−2	0	0	−1	4	−32	326
−1	0	1	−2	12	−104	1085
1	1	−1	5	−40	399	−4524
2	0	−1	7	−61	648	−7661
3	0	−1	9	−93	1070	−13257
4	0	−1	12	−140	1750	−22955
5	0	−1	15	−206	2821	−39315
6	0	−1	19	−296	4450	−66213
7	0	−1	23	−416	6868	−109367

This agrees with the results of Aganagic, Klemm and Vafa.